# SINGULARITY THEORY, HOMEWORK SHEET NO. 4

MARKO BERGHOFF, HU BERLIN, SUMMER 2020

## Problem 1

What kind of singularities does a generic smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ have? Give an example, and one for the non-generic case.

## Problem 2

Let $M$ and $N$ be smooth manifolds of dimensions $m$ and $n = m + 1$. What is the smallest value $m$ such that a generic smooth map $f : M \to N$ can have corank 2 singularities? What about corank 3?

## Problem 3

Let $M$ be a smooth manifold. A smooth function $f : M \to \mathbb{R}$ is called a *Morse function* if

- it has only non-degenerate critical points (a critical point $x$ of $f$ is *non-degenerate* if the Hessian matrix $\big(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\big)_{i,j=1,\ldots,n}$ is invertible - with respect to some, and hence any, set of local coordinates),
- no two critical points have the same critical value.

Show that the set of Morse functions is dense in $C^\infty(M, \mathbb{R})$.

## Problem 4

Consider two maps $f_\pm : \mathbb{R}^4 \to \mathbb{R}^3$, given by

$$f_+(x_1, x_2, x_3, x_4) := (x_1, x_2, x_3^2 + x_4^2 + x_1x_3 + x_2x_4)$$

and

$$f_-(x_1, x_2, x_3, x_4) := (x_1, x_2, x_3^2 - x_4^2 + x_1x_3 + x_2x_4).$$

Compute and classify their singularities. Try to argue why $f_+$ and $f_-$ are not equivalent.

## Problem 5

Discuss the singularities of $f : \mathbb{R}^2 \to \mathbb{R}^3$, $(x, y) \mapsto (xy, y, x^2)$ and sketch the image of this map.