**Problem 1**

What kind of singularities does a generic smooth function \( f : \mathbb{R}^2 \to \mathbb{R} \) have? Give an example, and one for the non-generic case.

**Problem 2**

Let \( M \) and \( N \) be smooth manifolds of dimensions \( m \) and \( n = m + 1 \). What is the smallest value \( m \) such that a generic smooth map \( f : M \to N \) can have corank 2 singularities? What about corank 3?

**Problem 3**

Let \( M \) be a smooth manifold. A smooth function \( f : M \to \mathbb{R} \) is called a **Morse function** if

- it has only non-degenerate critical points (a critical point \( x \) of \( f \) is non-degenerate if the Hessian matrix \( \left( \frac{\partial^2 f}{\partial x_i \partial x_j} (x) \right)_{i,j=1,\ldots,n} \) is invertible - with respect to some, and hence any, set of local coordinates),
- no two critical points have the same critical value.

Show that the set of Morse functions is dense in \( C^\infty(M, \mathbb{R}) \).

**Problem 4**

Consider two maps \( f_\pm : \mathbb{R}^4 \to \mathbb{R}^3 \), given by

\[
    f_+(x_1, x_2, x_3, x_4) := (x_1, x_2, x_3^2 + x_4^2 + x_1 x_3 + x_2 x_4)
\]

and

\[
    f_-(x_1, x_2, x_3, x_4) := (x_1, x_2, x_3^2 - x_4^2 + x_1 x_3 + x_2 x_4).
\]

Compute and classify their singularities. Try to argue why \( f_+ \) and \( f_- \) are not equivalent.

**Problem 5**

Discuss the singularities of \( f : \mathbb{R}^2 \to \mathbb{R}^3 \), \((x, y) \mapsto (xy, y, x^2)\) and sketch the image of this map.