SINGULARITY THEORY, HOMEWORK SHEET NO. 6

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Problem 1

Show that the Whitney fold and pleat maps $f, p : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$f : (x_1, x_2) \mapsto (x_1^2, x_2) \text{ and } p : (x_1, x_2) \mapsto (x_1^3 + x_1x_2, x_2),$$

are locally infinitesimally stable at the origin.

Problem 2

Consider the map-germ $[f]_0$ of $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \exp(-x^{-2})$. Is it stable?

Problem 3

Let $f \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ be given by $x \mapsto (x^2, x^3)$. Sketch the image of the deformation $F : \mathbb{R} \times I_\epsilon \to \mathbb{R}^2 \times I_\epsilon$, $(x, t) \mapsto (x^2, x^3 + tx, t)$ for $t < 0$, $t = 0$ and $t > 0$. What geometric operation is described? Can you realise the "other" operations of this type also by deformations?

Problem 4

Let M be compact and $f \in C^\infty(M, N)$. Given a deformation $F : M \times I \to N \times I$ (with $I = I_\epsilon$) of f we define a vector field ν_F along F by

$$\nu_F := dF(\frac{\partial}{\partial t}) - F^*(\frac{\partial}{\partial t}),$$

where $\frac{\partial}{\partial t}$ is viewed as a vector field on $M \times I$ or $N \times I$, respectively.
Using the two projections $\pi_M : M \times I \to M$ and $\pi_I : M \times I \to I$ and that $T(M \times I) = \pi_M^*(TM) \oplus \pi_I^*(TI)$ we can decompose any element ξ in $T(M \times I)$ as

$$\xi = \xi_M + \xi_I,$$

where $\xi_M \in \pi_M^*(TM)$ and $\xi_I \in \pi_I^*(TI)$.

(1) Show that $F = f \times id_I$ if and only if ν_F is the zero-section, $\nu_F \equiv 0$.
(2) Use the following theorem by Thom and Levine to show that submersions and embeddings are stable under deformations.

Theorem: *Let M be compact, $f \in C^\infty(M, N)$ and F a deformation of f. Then F is trivial if and only if there exists $\delta \leq \epsilon$ and vector fields σ on $M \times I_\delta$ and τ on $N \times I_\delta$ such that*

$$\sigma_{I_\delta} = 0 = \tau_{I_\delta} \text{ and } \nu_F = dF(\sigma) + F^*(\tau) \text{ on } M \times I_\delta.$$

(For a proof of this theorem see the book by Golubitsky and Guillemin, pp. 124)