The proof of Mathers theorem goes via a chain of equivalent notions of stability. Especially useful is the following.

6. Def

For \( f \in C^\infty(M, N) \) denote by \( [f]_x \) the germ of \( f \) at \( x \).

\( f \) is locally infinitesimally stable at \( x \) if for every germ of a vector field along \( f \)

(= germ of a section \( M \to f^*TN \)), \( [v]_x \), there exist germs of vector fields \( [s]_x \) in \( C^\infty(TM)_x \) and \( [t]_x \) in \( C^\infty(TN)_{f(x)} \),

s.t.

\[
[v]_x = [df \cdot s]_x + [t \cdot f]_x .
\]

In local coordinates \((x_1, \ldots, x_m)\) around \( x \) and \((y_1, \ldots, y_n)\) around \( f(x) \) this amounts to solving
\[ V_i = \sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j} s_j + t_i (f_1, \ldots, f_n) \quad i=1, \ldots, n \]

7. Thm: \( f \) is loc. inf. stable at \( x \) if these equations can be solved to order \( n = \dim N \), i.e.,

\[ V_i = \sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j} s_j + t_i (f_1, \ldots, f_n) + O(\|x - x_0\|^m) \]

Proof: Generalized Malgrange Preparation Theorem (see [Gr])

Note that

1. loc. inf. stability at \( x \) is determined by \( j^{*+i}f(x) \! \)!

2. this reduces the problem to finite dimensions (can use implicit function theorem!)

E.g.: (almost) every singularity we had so far, \( x \mapsto x^k \), \((x, y) \mapsto (x, y^2)\), \ldots
For a vector bundle $\overset{\pi}{E} \to M$ consider

$$\pi_* : J^k(M,E) \to J^k(M,M)$$

and $I \subset J^k(M,M)$, defined by

$$I := \{ \sigma = [id_M] \} .$$

This is a submanifold (and $\pi_*$ submersion), so

$$J^k(E) := \pi_*^{-1}(I)$$

is a submanifold of $J^k(M,E)$, the $k$-jet bundle of sections of $E$ (a vector bundle over $M$).

\[ \text{Corollary:} \quad f \text{ loc. inf. stable at } x \Rightarrow J^n(f^*TN)_x = df(x) J^n(TM)_x + f^*J^n(TN)_{f(x)} \]

If $f$ is injective this is equivalent to global (inf.) stability.

In the general case we define for $y \in V$

$$S = \{ x_1, \ldots, x_k \} \subset f^{-1}(y)$$

$$J^k(E)_S := \bigoplus_{i=1}^k J^k(E)_{x_i} .$$
$f$ induces maps

- $f^*: J^l(TN)_y \rightarrow J^l(f^*TN)_{x_i} \quad \forall i=1,...,k$

and

$$f^*: J^l(TN)_y \rightarrow J^l(f^*TN)_s$$

$$f^* [\zeta]_y := (\left[ t \circ f \zeta_{x_1}, ..., t \circ f \zeta_{x_k} \right]_{x_i})$$

- $df: J^l(TM)_{x_i} \rightarrow J^l(f^*TN)_{x_i}$

and

$$df: J^l(TM)_s \rightarrow J^l(f^*TN)_s$$

$$df(\left[ s_i \right]_{x_1}, ..., \left[ s_k \right]_{x_k}) := (\left[ df \cdot s_i \right]_{x_1}, ..., \left[ df \cdot s_k \right]_{x_k})$$

We call $f$ simultaneously locally infinitesimally stable at $x_1, ..., x_k$ if for all germs of vector fields along $f$, $\left[ v_1 \right]_{x_1}, ..., \left[ v_k \right]_{x_k}$, there exist germs of vector fields $\left[ s_1 \right]_{x_1}, ..., \left[ s_k \right]_{x_k}$ and $[\zeta]_y$ s.t.
\[
[v_i]_{x_i} = [df s_i]_{x_i} + (t \circ f)_{x_i} \quad i=1, \ldots, k
\]

9. Thm: \( f \) is inf. stable if and only if 

\[\forall y \in N \text{ and every } S \subset f^{-1}(y) \text{ with } |S| \leq n+1 \]

\[J^u(\xi^*TN)_S = df(J^u(TM)_S) + f^*J^u(TN)_y \]

Proof: G\&G.

Another notion of stability uses homotopies:

10. Def \( \text{Let } f \in C^\infty(M,N), \ I_{\varepsilon} := (-\varepsilon, \varepsilon) \subset \mathbb{R} \)

1. A deformation/unfolding of \( f \) is a smooth map 

\[F: M \times I_{\varepsilon} \rightarrow N \times I_{\varepsilon}
(\xi, \varepsilon) \mapsto (F_{\varepsilon}(\xi), \varepsilon)
\]

with \( F_0 = f \).

2. A deformation/unfolding \( F \) of \( f \) is trivial if
there is $0 < \delta \leq \varepsilon$ and

$$G \in \text{Diff}(M \times I_\varepsilon) \cap \{ \text{deformations of } \text{id}_M \}$$

$$H \in \text{Diff}(N \times I_\varepsilon) \cap \{ 1 \} - \text{id}_N$$

S.T.

$$M \times I_\varepsilon \xrightarrow{F} N \times I_\varepsilon$$

$$G \downarrow \sigma \downarrow H$$

$$M \times I_\varepsilon \longrightarrow N \times I_\varepsilon$$

$$f \times \text{id}_{I_\varepsilon}$$

3. $f$ is stable under deformations/unfoldings/homotopically stable if every deformation of $f$ is trivial.

\[ f(x) = x^2 \quad F_t(x) = x^2 + t \cdot x \quad \text{is trivial} \]

\[ \text{take} \quad G_t(x) = x + \frac{t}{2} \]

\[ H_t(y) = y + \frac{t^2}{4} \]

\[ \Rightarrow H_t \circ F_t \circ G_t^{-1}(x) = \frac{t^2}{4} + (x-\frac{t}{2})^2 + t \cdot (x-\frac{t}{2}) = x^2 \]
This defn is motivated by the following idea:

A deformation of $f$ is a map

$$C : I \rightarrow C^\infty(M, N),$$

a curve in $C^\infty(M, N)$ through $f$ ($C(0) = f$). Recall the map

$$\gamma : \text{Diff}(M) \times \text{Diff}(N) \rightarrow C^\infty(M, N)$$

$$(g, h) \mapsto h \circ f \circ g^{-1}.$$

We argued that $f$ is int. stable iff $\gamma_f(e)$ is onto.

This can be rephrased by: For every curve $c : t \mapsto E_t$ with $c(0) = f$ there is a curve $\tilde{c} : t \mapsto (G_t, H_t)$ in $\text{Diff}(M) \times \text{Diff}(N)$ with $\tilde{c}(0) = e$ s.t.

$$\gamma_f (G_t, H_t) = H_t \circ f \circ G_t^{-1} = E_t$$

which just means $f$ is homotopically stable!
This can be generalized to $k$-parameter unfoldings by replacing $I_e$ with $B_e \subset \mathbb{R}^k$.

Q: If not stable (e.g. $x^3, x^4, x^5 \ldots$), how “many” non-trivial unfoldings are there?

A: For $\dim M \leq 2$ and $k=4$ we find Thom’s list of seven catastrophes:

\[
\begin{align*}
\text{- Fold: } & \ x^3 + t x \\
\text{- Cusp: } & \ x^4 + t_1 x^2 + t_2 x \\
\text{- Swallowtail: } & \ x^5 + t_1 x^3 + t_2 x^2 + t_3 x \\
\text{...}
\end{align*}
\]

There are even more notions of stability needed for the proof of Mather’s theorem, including “transverse stability” which is formulated using $\text{Diff}(M) \times \text{Diff}(N)$ invariant submanifolds of $\mathcal{O}^k(M, N)$. For the details, see the books by Arnold or G&G.
Propositions 3, 4 and 5 suggest that stable mappings are always dense in $C^\infty(M,N)$, but in general this is wrong (it is true for $C^0$-stability).

Mather showed that stable maps are dense in $C^\infty(M,N)$ if and only if for

\[ k := \dim N - \dim M = n-m \] we have

- $n < 7k + 8$ \quad \text{when} \quad k \geq 4
- $n < 7k + 9$ \quad \text{when} \quad 3 \geq k \geq 0
- $n < 8$ \quad \text{when} \quad k = -1
- $n < 6$ \quad \text{when} \quad k = -2
- $n < 7$ \quad \text{when} \quad k \leq -3