Recap

- $\Sigma^i(f)$, corank of $\nabla f + x = \min(m,n) - \text{rank } df_x$

- corank product formula (Thm 8)

For a generic $f$

$\Sigma^i(f)$ is a subaf of $M$ with

codim $\Sigma^i(f) = i \cdot (|n-m|+i)$

corank at target

\[\text{Proof:} \quad 1. \text{ linear case} \]

\[2. \text{ smooth case: transversality} \]

\[X, Y \subset V \text{ transverse if } \begin{cases} X \cap Y = \emptyset, \\ X+Y = V \end{cases} \]

\[\mathbb{R}^3, \quad \begin{array}{c} \downarrow \text{ but} \end{array} \]

\[\begin{array}{c} \times \\ \times \end{array} \]

\[\text{WTThm (Thm 12)} \]

$M$ closed (comp. & $0$), SCN submfd closed, then $\exists f : S\subset C^0(M,N)$ open & dense.
How do we Thm 12 to prove Thm R? 

**Sketch:** \( f: \mathbb{R}^m \to \mathbb{R}^n \)
\[ df: M \to \text{Hom} (\mathbb{R}^m, \mathbb{R}^n) \times \to df_x \]

**Lemma 9:** \( \text{Hom}^r (\mathbb{R}^m, \mathbb{R}^n) = \{ \text{rank } r \text{ lin. maps } \in \text{Hom}(\mathbb{R}^m, \mathbb{R}^n) \} \)

is a **submanifold of** codim. \((m-r)(n-r)\)

**WT Thm:** Generically, \( df \times \text{Hom}^r (\mathbb{R}^m, \mathbb{R}^n) \)

**Prop 11:** \( \text{df}^{-1} (\text{Hom}^r (\mathbb{R}^m, \mathbb{R}^n)) \) is a **submanifold of** codim. \((m-r)(n-r)\) -

Q: What's the problem? 

- openness of submanifold

- Thm 8: For a **generic** map \( f \),

  We can't translate an arbitrary perturbation \( df \) into the differential of a map \( fe: M \to N \)!

We need another way of looking at differentials of maps:
III. Jet bundles

1. Def: For $M, N$ manifolds, $f, g : M \to N$ smooth, $x \in M$, $y = f(x) = g(x)$, we say

1. $f$ has first order contact with $g$ at $x$
   if $df_x = dg_x : T_x M \to T_y N$

2. $f$ has $k$-th order contact with $g$ at $x$
   if $df : TM \to TN$ has $(k-1)$st order contact with $dg$ at every point in $T_x M$.
   This defines an equivalence relation, denoted by $f \sim_k g$ at $x$. (Exercise)

3. $J^k(M, N)_{x,y} :=$ set of equivalence classes under $\sim_k$ at $x$ 
   on $\{ f \in C^k(M, N) | f(x) = y \}$
4. \( J^k(M,N) := \bigcup_{x,y} J^k(M,N)_{x,y} \) for \((x,y) \in M \times N \)

An element \( \sigma \) in \( J^k(M,N) \) is called a \textit{k-jet} (of maps) from \( M \) to \( N \).

5. Let \( \sigma \in J^k(M,N) \). Then there is a pair \((x,y)\) with \( \sigma \in J^k(M,N)_{x,y} \).

\( x \) is the source of \( \sigma \), \( y \) the target, \( s : J^k(M,N) \to M \) the source map and \( t : J^k(M,N) \to N \) the target map.

6. The canonically defined \( (k) \) map \( \text{(for f:} M \to N \text{ smooth)} \)

\( j^k f : M \to J^k(M,N) \), \( x \to \{ f \} \in J^k(M,N)_{x,f(x)} \)

is called the \textit{k-jet} (extension) of \( f \).

Q: - Germ vs k-jet of maps? What's the relation?
- Why the recursive definition? Covariance...
- What's a 0-jet? \( f \circ g \) at \( x \iff f(x) = g(x) \)
- \( J^0(M,N) \)? \( J^0(M,N) = M \times N \)
A 1-jet from \( \mathbb{R} \) to \( \mathbb{R} \) is given by
\[
(x, y, l) \quad \text{so} \quad J'(\mathbb{R}, \mathbb{R}) = \bigcup_{x \in \mathbb{R}} J'(\mathbb{R}, \mathbb{R})_{x, y} \quad (x, y) \in \mathbb{R}^2
\]
\[\subseteq \mathbb{R}^3\]

more generally in local coordinates a bi-jet may be represented by the Taylor polynomial of degree \( k \), i.e.
\[
 f, g : M \subset \mathbb{R}^m \to \mathbb{R}^n \text{ smooth , then}
\]
\[
 f_{\text{near } x} \iff \frac{\partial|f|}{\partial x^i}(x) = \frac{\partial|g|}{\partial x^i}(x)
\]
Induction
for all \( 0 \leq |x| \leq k \)
and \( i = 1, \ldots, n \).

\[
 J'(M, N) \cong \text{Hom}(TM, TN) \quad \text{as vector bundles over } M \times N. \text{ The fiber over } (x, y) \text{ is}
\]
\{ \sigma \in \mathcal{J}(M,N) \mid s(c) = x, t(c) = y \}\). If \( f \) represents \( \sigma \), then \( df_x \in \text{Hom}(T_xM, T_{f(x)}N) \).

This defines a diffeomorphism \( \Phi : \mathcal{J}(M,N) \to \text{Hom}(TM,\pi^*TN) \)

with \( \pi \times \ell = \pi \circ \Phi \) where \( \pi : \text{Hom}(TM,\pi^*TN) \to M \times N \).

- For \( k > 1 \), \( \mathcal{J}^k(M,N) \to \mathcal{J}^{k-1}(M,N) \), \( M \), \( N \), \( M \times N \)

are smooth fibrations, but not vector bundles (unless \( N = \mathbb{R}^n \)).

Two natural operations (push forwards & pullbacks):

- \( h : N_1 \to N_2 \) smooth induces a map

\[ h_* : \mathcal{J}^k(M,N_1) \to \mathcal{J}^k(M,N_2) \]

\[ J^k(M,N_1) \ni \sigma \mapsto \left[ \text{holo} J \in J^k(M,N_2) x, h(y) \right] \]

\( f : M \to N_1 \), repr. \( \sigma \) \( / f_* \sigma \)

- \( g : M \to M_2 \) diffeom. induces a map

\[ g_* : \mathcal{J}^k(M_2,N) \to \mathcal{J}^k(M_1,N) \]

\[ J^k(M_2,N) \ni \tau \mapsto \left[ \text{holo} \ g \ J \in \mathcal{J}^k(M_1,N) g_*^\dagger(\tau), \gamma \right] \]

\( f \) repr. \( \tau \) \( / f_* \tau \)
Exercise: both well-defined!

We'll establish some more properties in the exercises. For us most important is

2. Theorem: For $M, N$ mfs

1. $\forall k \in \mathbb{N}: J^k(M,N)$ is a (smooth) mfs.

(Q: What's the dimension?)

2. $J^0(M,N) \xrightarrow{\phi \times \phi} M \times N$ are submersions.

3. If $f: M \to N$ smooth, then $j^k f: M \to J^k(M,N)$ is smooth.

Proof:

1. We sketch the construction of charts:

Let $P_m^k$ be the vector space of polynomials

$$p(t_1, \ldots, t_m) = \sum_{|\alpha| = k} a_{\alpha} t_1^{\alpha_1} \cdots t_m^{\alpha_m}$$

and set $P_{m,n}^k = \bigoplus_{i=1}^n P_m^k$. 

\[\]
Both are real fin. dim. vector spaces, hence smooth mfs. (w. coordinates)

For \( U \subset \mathbb{R}^n \) open and \( f: U \to \mathbb{R} \) define
\[
T_k f: U \to P^k_m \quad \text{by} \quad x_0 \mapsto T_k f(x_0), \quad \text{the degree } k \text{ Taylor polynomial of } f \text{ at } x_0
\]
minus constant term.

If \( V \subset \mathbb{R}^n \) open, then there is a canonical bijection
\[
T_{U,V}: J^k(U,V) \to U \times V \times P^k_{m,n}
\]
\[
\sigma \mapsto (x_0, y_0, T_k f(x_0), \ldots, T_k f(x_0))
\]
where \( x_0 = \sigma(x) \), \( y_0 = \epsilon(x) \) (i.e. \( \sigma \in J^k(U,V)_{x_0,y_0} \)).

\( f: U \to V \) representing \( \sigma \), \( f = (f_1, \ldots, f_n) \).

This is well-def. & bijective.

Now for \( U \subset M, V \subset N \) with charts \( \phi: U \to U' \subset \mathbb{R}^m \)
and \( \psi: V \to V' \subset \mathbb{R}^n \) define
\[
T_{U,V} := T_{U',V'} \circ (\phi')^* \psi^* : J^k(U,V) \to U' \times V' \times P^k_{m,n}
\]
Declare these $T_{u,v}$ to be charts, a tedious but straightforward calculation establishes the smoothness of coord. changes ...

2. Follow by a tedious but straightforward calculation...

3. Locally $f: U \rightarrow \mathbb{R}^n$. Then $j^k f: U \rightarrow J^k(U, \mathbb{R}^n) \ni U \times \mathbb{R}^n \times P_{m,n}^k$ is given by

\[ j^k f(x_0) = (x_0, y_0, \overline{T}_k f_1(x_0), \ldots, \overline{T}_k f_n(x_0)) \]

all smooth in $x_0$ (partial derivatives...)

now use charts...

Locally, $C^\infty(M,N)$ looks like $U \times V \times \text{Map}(\mathbb{R})$ and $S^r := U \times V \times \text{Map}^r(\mathbb{R})$ is a submanifold. Given $f$ smooth, we have

\[ \Sigma^i(f) = (j^i f)^{-1}(S^{m-i}) \quad \text{if} \quad m \geq n \]

or \( S^{n-i} \) if \( m \leq n \).