Locally, $J^1(M,N)$ looks like $U \times V \times M_{m,n}(R)$ and $S^r := U \times V \times M_{m,n}^r(R)$ is a submanifold. Given $f$ smooth, we have

$$\Sigma^i(f) = \left( j^1 f \right)^{-1}(S^{m-i}) \quad \text{if } m \geq n,$$

or $(S^{n-i})$ if $m \leq n$.

Recap: Jets & Jet spaces/bundles

- $f_{\circ} g$ at $x \iff f(x) = g(x)$
- $f_{\circ,1} g$ at $x \iff f_{\circ} g$ at $x$ and $df_x = dg_x$
- $f_{\circ,2} g$ at $x \iff f_{\circ} g$ at $x$ and $df \circ dg$ at $(x,y) \in T_x M$
  
  $\quad f_{\circ,2} g$ at $x$ : Equivalence relation

- $J^k(M,N)_{x,y} := \{ f \in \mathcal{C}^0(M,N) \mid f(x) = y \} / \sim_k$ at $x$

- $J^k(M,N) := \bigcup_{x,y \in M \times N} J^k(M,N)_{x,y}$

- $\text{$h$-jet of } f: M \to N : j^h f : M \to J^h(M,N)$
  
  $x \mapsto [f] \in J^h(M,N)_{x,f(x)}$
locally, \( j^k f(x) = (x_1, \ldots, x_n, \frac{\partial f_i}{\partial x_j})_{i=1, \ldots, n, \ j=1, \ldots, m} \)

Thus: \( j^k f : M \to J^k(M, N) \) is smooth

\[ j^k f : M \to J^k(M, N) \text{ is smooth} \]

\[ m+n\left(\frac{(m+k)!}{m! \cdot k!}\right) \]

We need one last ingredient to state the (strong) transversality theorem.

My generalizations:

- \( J^\infty(M, N) \)

- Algebraic geometry: \( J^k(M) \) allows to study singular points of \( M \)

- Jets of sections of vector bundles: study differential operators \( \Gamma(\mathcal{E}_M) \to \Gamma(\mathcal{E}_L^M) \)
IV. Whitney topologies

For $U \in J^k(M,N)$ let

$$M(U) := \{ f \in C^\infty(M,N) \mid j^k f(M) \subset U \}.$$

1. Def

- The Whitney $C^k$ topology is the topology on $C^\infty(M,N)$ generated by the basis

$$\{M(U)\} \cup J^k(M,N)$$

Write $W^k$ for the set of open sets in this topology.

Recall: Basis of topology on $X$ is $B \subseteq \mathcal{P}(X)$

s.t.

1. $\cup B = X$

   $B \in B$

2. $B_1, B_2 \in B \implies \forall x \in B_1 \cap B_2 : \exists B_3 \in B$ with $x \in B_3 \land B_3 \subset B_1 \cap B_2$

This generates a topology by open sets := unions of elements of $B$.
The Whitney $C^0$-topology on $C^0(M,N)$ is generated by

$$W^e := \bigcup_{k=0}^{\infty} W^e_k .$$

(this is well-defined because

$$e \leq k \Rightarrow W^e_e \subset W^e_k$$

How do open neighborhoods look like in the Whitney topologies?

Let $f \in C^0(M,N)$ and let $d$ a metric on $J^k(M,N)$ inducing its topology ("every mf is metrizable") and $\delta : M \to \mathbb{R}_+$ continuous.

Then

$$B_d (f) := \left\{ g \in C^0(M,N) \mid \forall x \in M : d\left( j^k_x (f(x)), j^k_x (g(x)) \right) < \delta (x) \right\}$$

is open:

$$\Delta : J^k(M,N) \to \mathbb{R} \ , \ \sigma \mapsto \delta (s(\sigma)) - d\left( j^k_s (f(\sigma)), \sigma \right)$$

is continuous. Set $U := \Delta^{-1}(\mathbb{R}_+)$. 

$\text{eg} \ \text{Intervals on} \ \mathbb{R}$
This is open and \( B_8(f) = M(U) \)
\[ \exists g \mid j^k g(M) c U \]

If \( M \) is compact, then \( B_{\frac{1}{n}}(f) \) defines a countable neighborhood basis (each \( S \) is bounded below by some \( u \) \( \frac{1}{n} \)...)

and \( f_n \rightarrow f \) in Whitney \( C^k \)-topology on \( C^\infty(M,N) \)

\( \Leftrightarrow \) \( j^k f_n \rightarrow j^k f \) uniformly in \( C^\infty(M, J^k(M,N)) \)

If \( M \) is not compact, then there is no countable nbh basis and

\( f_n \rightarrow f \) in Whitney \( C^k \)-topology on \( C^\infty(M,N) \)

\( \Leftrightarrow \) \( \exists K \subset M \) compact such that

on \( K \) : \( j^k f_n \rightarrow j^k f \) uniformly

off \( K \) : \( \exists n_0 \in N : \forall n > n_0 : f_n(x) = f(x) \ \forall x \in M \setminus K \)
Proof: \( \Leftarrow \Rightarrow \)

Let \( f_n \to f \) in Whitney \( C^4 \)-topology and assume that there is no \( K \) s.t.

Take a sequence of compacta \( \{K_i\}_{i \in \mathbb{N}} \) s.t.

\[ K_i \subset K_{i+1} \quad \text{and} \quad M = \bigcup_{i \in \mathbb{N}} K_i \]

\( \exists n_1 \) with \( f_{n_1} \neq f \) \( \Rightarrow \) \( \exists x_i \in M : d(j^k f_{n_1}(x_i), j^k f(x_i)) = a_i \)

\( \exists m_i \) with \( x \in K_{m_i} \).

Set \( S |_{K_{m_i}} \equiv a_i \).

Repeat to get after \( s \) steps.

\( n_1 < \ldots < n_s \), \( K_{m_s} \), \( S : K_{m_s} \to \mathbb{R}_+ \)

and \( x_1, \ldots, x_s \in K_{m_s} \) with

\[ \forall i \leq s : d( j^k f_{n_i}(x_i), j^k f(x_i) ) > S(x_i) . \]

Now choose \( f_{n_{s+1}} \) for \( n_{s+1} > n_s \)

such that \( f_{n_{s+1}} \neq f \) off \( K_{m_{s+1}} \).
Let $x_{s+1}$ lie outside $K_{m_{s+1}}$ with
$$d(j^k f_{n_{s+1}}(x_{s+1}), j^k f(x_{s+1})) =: a_{s+1} > 0.$$ Choose $m_{s+1}$ such that $x_{s+1} \in K_{m_{s+1}}$ and extend $\delta$ (continuously) to $K_{m_{s+1}}$
by $\delta\big|_{K_{m_{s+1}} \setminus K_{m_{s+1}}} = a_{s+1}.$

This procedure gives a subsequence $f_{n_i}$ and $\delta: M \to \mathbb{R}_+$ contin. such that
$$\forall i: f_{n_i} \notin B_{\delta}(f)$$ which contradicts $f_{n_i} \to f$. 
Why are these topologies useful?

Because

2. Def X top. space.

- \( Y \subseteq X \) is residual if it is a countable intersection of open and dense subsets of \( X \).

- \( X \) is a Baire space if every residual set is dense.

\[ \begin{align*}
\text{eg} & \quad \text{Q c R is not residual} \\
& \quad \text{R \cap Q c R is residual} \\
& \quad q : \mathbb{N} \to \mathbb{Q} \text{ bij. then } R \setminus Q = \bigcap_{n \in \mathbb{N}} (R \setminus q, s) \\
& \quad \text{Cantor set C is not a residual subset of R, but R \cap C is} \\
& \quad \text{R is a Baire space} \\
& \quad \text{C is a Baire space}
\end{align*} \]
3. Def. \( X \) (Baire) space. \( P: X \to \{0,1,\infty\} \) a "property". \( P \) is \underline{generic} if \( P^{-1}(1) \) contains a residual set in \( X \).

\[ \text{Eg. Morse functions: } X = C^\infty(M,\mathbb{R}), \ P(f) = f \text{ is Morse. Then } P \text{ is generic (of course, this needs to be shown)} \]

4. Prop.:

Let \( M, N \) smooth mfs. Then \( C^\infty(M,N) \)

is a Baire space in the Whitney \( C^\infty \) topology.

\[ \text{Proof: Lit. (G&G)} \]

\[ \text{Eg. } \{\text{Morse functions}\} \subset C^\infty(M,\mathbb{R}) \text{ residual } \]

\& \( C^\infty(M,\mathbb{R}) \) is Baire, thus \( \{\text{Morse functions}\} \)

is a dense subset.