Recap:

- for good aka. 1-generic maps
  \[ \Sigma^i(f) \subset M \] smooth submanifolds of \( M \)
  of codim \( i(n-m+1) \)

- \( \text{Imm}(M,N) \) dense & open in \( C^\infty(M,N) \)
  \[ \text{if } n \geq 2m \]
  e.g. \( M=\mathbb{R} \quad N=\mathbb{R}^2 \)

- \( \text{Emb}(M,\mathbb{R}^{2m+1}) \) dense in \( C^\infty(M,\mathbb{R}^{2m+1}) \)
  \( M=S^1 \quad N=\mathbb{R}^2 \)
  \( N=\mathbb{R}^3 \quad f_e(S^1) \)
Recall the Whitney map (Exerc. 2, Problems 283)

\[ w: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad (x_1, x_2) \mapsto (w_1, w_2) = (x_1^3 + x_2 x_1, x_2) \]

\[ \text{Crit}(w) = \left\{ x_2 = -3 x_1^2 \right\} = \Sigma'(w) \]

**idea:**

\[ \Sigma'(w) \text{ is smooth m.f., consider} \]

\[ w|_{\Sigma'(w)} \] and its singularities:

\[ w|_{\left\{ x_2 = -3 x_1^2 \right\}}: x_1 \mapsto (-2 x_1^3, -3 x_1^2) \]

\[ \Rightarrow \text{Crit pts} = \left\{ x_1 = 0 \right\} \]
i.e. we can write

\[ \Sigma'(w) = \Sigma^0(w|\Sigma'(w)) \cup \Sigma'(w|\Sigma'(w)) \]

↑
fold pts
↑
pleat pt.

If possible, repeat...

\[ V = x^4 + ax^2 + bx \quad a, b \text{ control parameters} \]

Critical pts / singularities :

\[ \frac{dV}{dx} = 4x^3 + 2ax + b = 0 \]

\[ \Leftrightarrow b = -4x^3 - 2ax \]

connection to catastrophe theory.
This is equivalent to studying the singularities of $w$.

Consider the graph $\Gamma(w)$

\[ \Gamma(w) = \{(x_1, x_2, x_1^3 + x_2x_1)\} \]

and its projection to $\mathbb{R}^2$ with coordinates $(x_2, x_3)$

$\text{proj}_w : \Gamma(w) \rightarrow \mathbb{R}^2$
we find $\text{Crit}(p_w) = \text{Crit}(w)$.

For 3-parameter catastrophe

$$V = x^5 + ax^3 + bx^2 + cx$$

$$\frac{dV}{dx} = 5x^4 + 3ax^2 + 2bx + c = 0$$

"swallowtail catastrophe" (see below)

1. Definition

Let $\Sigma[f] = \Sigma \Gamma_i^k(f)$ be a smooth mf. Then define

$$\Sigma^{i_1, \ldots, i_k, i_{k+1}}(f) := \Sigma \Gamma_{i_{k+1}}(f | \Sigma[\Gamma_i^k(f)])$$

as the set of points where $\text{corank} (f | \Sigma[\Gamma_i^k(f)]) = i_{k+1}$.
Q: Is this a well-defined definition?

A:

2. Conj (Thom) / Thm (Boardman)

For a residual set of maps in $C^\infty(M,N)$ this definition makes sense.

Proof (sketch).

1. For $I = \{ i_1, \ldots, i_k \}$, $i_1 \geq i_2 \geq i_3 \ldots$ define "appropriate" submanifolds

$$ S^I \subset J^k(M,N) $$

2. Call $f$ good / $k$-generic if

$$ j^k f \pitchfork S^I \quad \forall I : |I| = k $$

3. Show that $f$ good $\Rightarrow$ $\Sigma^I(f) \circ (j^k f)^I(S^I)$

4. Show that

$$ T_{S^I} = \{ f \in C^\infty(M,N) / j^k f \pitchfork S^I \} $$

is a residual set in $C^\infty(M,N)$. 

$\square$
The hard part is 3., 4. follows then by Thm 5.9.

\[ \begin{align*}
&\text{e.g. } \quad f: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \\
&(x, y, z) \mapsto (x, y, z^4 + xz^2 + yz) \\
&\text{C} \cap \{ f \} = \left\{ 4z^3 + 2xz + y = 0 \right\} \\
&\begin{align*}
&= \Sigma'(f) \\
&= \left\{ (x, -4z^3 - 2xz, z) \right\} \\
&\cong \Gamma\left( (x, z) \mapsto \omega_1(x, z) \right)
\end{align*}
\end{align*} \]

According to Def. 1

\[ f|\Sigma'(f): (x, z) \mapsto (x, -4z^3 - 2xz, -3z^4 - xz^2) \]

\[ d(f|\Sigma'(f))(x, z) = \begin{pmatrix}
1 & 0 \\
-2z & -12z^2 - 2x \\
-z^2 & -12z^2 - 2xz
\end{pmatrix} \]

\[ \Rightarrow \Sigma''(f) = \Sigma'(f|\Sigma'(f)) = \left\{ x = -6z^2 \right\} \subset \Sigma'(f) \]
So \( \Sigma''(\xi) = \{ (-6\xi^2, -16\xi^3, \xi) \mid \xi \in \mathbb{R} \} \)

Repeat again,

\[
\Sigma'''(\xi) : \xi \mapsto (-6\xi^2, -16\xi^3, -21\xi^4)
\]

and thus \( \Sigma''''(\xi) = \Sigma' (f|_{\Sigma'''(\xi)}) = \{ (0,0,0) \} \).

For maps between 3-mfds this is the generic case, i.e. only singularities of type \( \Sigma', \Sigma'', \Sigma''' \) occur.

In dimension 4 a \( \Sigma^{2,0} \) appears (see exercise)!

However,

3. Thm (Morin)

Let \( m = n \) and \( f : M \to N \) good. Then for \( x_0 \in \Sigma''''(\xi) \) there exists coordinates \( x_1, \ldots, x_n \) s.t. locally \( f \) is given by

\[
(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{m-1}, x_m + x_m \cdot x_{k-1} + \ldots + x_m \cdot x_2 + x_m \cdot x_1)
\]
How to visualize where is the swallowtail?

1. As $\Gamma(w_0)$

$$\Sigma'''(f) = \begin{cases} \Sigma'''(f) = \end{cases}$$

$$\Sigma'(f) = \Gamma(w_1) \text{ (see above)}$$

2. Critical values of $f$

$$f(\Sigma'(f)) = \text{im}(f|_{\Sigma'(f)})$$

$$= \left\{ \left(x, -4x^3 - 2x^2, -3x^4 - x^2 \right) \right\}$$

$\begin{align*}
  & x > 0 \\
  & x = 0 \\
  & x < 0
\end{align*}$
put together to get the swallowtail surface

# preimages of pts under f