SINGULAR INTEGRALS. EXERCISE SHEET NO. 1

1

Evaluate the integrals (a) $\int_\gamma |z|^2 dz$ and (b) $\int_\gamma \frac{dz}{z^2}$ along the contour γ that connects the points -1 and i by

(1) a line segment.
(2) an arc of the unit circle.

2

Find $\int_0^\infty \frac{\cos x}{1+x^2} dx$.

Hint: Consider the complex integral $\int_\gamma f$ where $f(z) = \frac{e^{iz}}{1+z^2}$ and γ is a path parametrising the intervall $[-r, r]$ and a half circle $\{re^{i\phi} \mid \phi \in [0, \pi]\}$.

3

Calculate $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$.

4

Show $\int_{-\infty}^\infty e^{-t^2} \cos 2\omega t dt = \sqrt{\pi} e^{-\omega^2}$.

Hint: Consider the complex integral $\int_\gamma f$ for $f(z) = e^{-z^2}$ and γ a path parametrising the rectangle with corners $-r, r, r + i\omega, -r + i\omega$ (and recall $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$).

5

Let $a, b > 0$. Find

$$\int_0^{2\pi} \frac{1}{a^2 \cos^2 t + b^2 \sin^2 t} dt,$$

by relating it to the contour integral of $f(z) = 1/z$ along the ellipse $\gamma(t) = a \cos t + ib \sin t$.

6

Calculate $\int_\gamma f_n$ for the sequence of meromorphic functions

$$f_n(z) = \frac{1}{(z - z_0)^n} + \psi(z), \; n \in \mathbb{N},$$

with ψ holomorphic on $\mathbb{C}$ and γ encircling z_0 counterclockwise. Use this to deduce for any f holomorphic

$$\int_\gamma \frac{f(z)}{(z - z_0)^n} dz = 2\pi i \frac{1}{(n-1)!} f^{(n-1)}(z_0).$$