SINGULAR INTEGRALS. EXERCISE SHEET NO. 2

Solutions to the exercises will be discussed in class on 27.11.18.

1

Let X be a complex manifold that is one-dimensional and compact. Show that every holomorphic function $f : X \to \mathbb{C}$ is constant.

Hint: We haven't covered this in class but you might have heard about the maximum principle...

What about the map $\iota : \mathbb{C}^* \to \mathbb{C}, z \mapsto z$ - why is this not a counterexample?

2

Let $v_1, v_2 \in \mathbb{R}^2$ be linearly independent and let

$$\Gamma := \langle v_1, v_2 \rangle_{\mathbb{Z}} = \{\lambda v_1 + \mu v_2 \mid \lambda, \mu \in \mathbb{Z}\}.$$

Show that $X := \mathbb{R}^2/\Gamma$ can be equipped with a complex structure. What space is this homeomorphic to? How does the complex structure depend on the choice of vectors v_1, v_2?

3

Let $f : X \to Y$ be a submersion between two smooth manifolds (this means for every $x \in X$ the differential $df_x : T_xX \to T_{f(x)}Y$ is surjective). Show that each fiber $f^{-1}(y)$ is a submanifold of X.

4

Show that the tangent bundle TX of a smooth manifold X is a vector bundle. Furthermore, given a smooth map $f : X \to Y$, what is the induced natural map of vector bundles $TX \to TY$?

5

For $n \in \mathbb{N}$ recall the definition of complex projective space:

$$\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{0\})/_\sim$$

where

$$z \sim z' \iff \exists \lambda \in \mathbb{C} \setminus \{0\} : z = \lambda z'.$$

Show that for all $n \in \mathbb{N}$ the spaces $\mathbb{CP}^n$ are complex manifolds.

Extra question: Using homogenous coordinates as introduced in the lecture, determine the complement of a chart domain U_i. What set/space is it homeomorphic to? This allows to find a recursive *cellular decomposition* of $\mathbb{CP}^n$. Moreover, if you know *cellular homology* you may use this to compute the homology groups of $\mathbb{C}P^n$...

6

Use the language of (co-)homology to prove the standard fact from complex analysis that $\int_\gamma f = 0$ whenever f is holomorphic on $U \subset \mathbb{C}$ and γ homologous to zero, i.e. $[\gamma] = [0] \in H_1(U)$.

7

Consider the family $C = (C_*, d_*)$ with

$$C_i = \mathbb{Z},\ d_i : z \longmapsto \begin{cases} 2z & \text{if } i \text{ is even} \\ 0 & \text{else.} \end{cases}$$

Show that C is a chain complex and compute its homology groups (over $\mathbb{Z}$).

8

Consider a sequence of vector spaces X, Y, Z,

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

Show that exactness of this sequence implies that $Y \cong X \oplus Z$.

9

Consider a (semi-)simplicial decomposition (Δ-*complexes are also referred to as semi-simplicial complexes since they generalise the notion of simplicial complexes*) of the Klein bottle K as pictured below. Form the associated chain complex (C_*, d_*) with $\mathbb{Z}$ coefficients. Which of the $\mathbb{Z}$-modules C_i are non-trivial? Determine the corresponding non-trivial boundary maps d_i? Use this to compute the homology groups $H_*(K) = \ker d / \text{im}\ d$.