SINGULAR INTEGRALS. EXERCISE SHEET NO. 3

Solutions to the exercises will be discussed in class on 18.12.18.

1

Let $X = \mathbb{C}^2$, $S = \{(x, y) \mid x = 0\}$ and $\omega = \frac{f(x,y)}{x} dx \wedge dy \in \Omega^*(X \setminus S)$ with f holomorphic on X. For $\sigma \in C_*(S)$ a chain in S its *coboundary* $\delta(\sigma)$ is given by

$$\delta(\sigma) = \{(x, e^{it}) \mid x \in \sigma, t \in [0, 2\pi]\} \subset X.$$

Compute the residue of ω and use this to express the integral

$$\langle \omega, \delta(\sigma) \rangle = \int_{\delta(\sigma)} \omega$$

as an integral over σ.

2

Show that the Poincaré residue (see the notes for Lecture 7) is independent of the choice of $j \in \{1, \ldots, n\}$ such that $\frac{\partial s}{\partial z_j} \neq 0$.

3

For the following functions $f_i : \mathbb{C} \to \mathbb{C}$ find their "branch points" and discuss their discontinuities as one crosses the corresponding "branch cuts" (*we haven't defined these notions in class rigorously, so you'll have to do some library work or look online... as in most cases, Wikipedia is a good starting point*):

1) $f_1 : z \longmapsto z^{\frac{1}{3}}$
2) $f_2 : z \longmapsto z^{\frac{1}{2}} \log z$
3) $f_3 : z \longmapsto (z - a)^{\frac{1}{2}} (z - b)^{\frac{1}{2}}$
4) $f_4 : z \longmapsto (z - a)^{\frac{1}{2}} (z - b)^{-\frac{1}{2}}$

Extra question for some Christmas candy: What happens if we replace in f_1 the exponent $\frac{1}{3}$ by $\frac{p}{q}$ for $p, q \in \mathbb{Z}$ or by $\lambda \in \mathbb{R} \setminus \mathbb{Q}$?