SINGULAR INTEGRALS. EXERCISE SHEET NO. 4

Homework: Solutions to be handed in on Friday 11.01.19.

1

The *dilogarithm* is defined by

$$\mathrm{Li}_2(t) = \int_{0\le z_1\le z_2\le 1} \frac{t dz_1 dz_2}{(1-tz_1)z_2} = \int_0^1 \frac{dz_2}{z_2}\int_0^{z_2} \frac{t dz_1}{1-tz_1}.$$

(1) Find the branch points and cuts of Li_2 and determine the corresponding discontinuities. **Hint:** One is only present on non-principal sheets.
(2) Show that $\mathrm{Li}_2(1) = \zeta(2)$ for ζ the Riemann zeta function.

2

For

$$S = \{z_1^2 + z_2^2 = 1\} \subset \mathbb{C}^2$$

let σ denote the real part of S, $\sigma = S \cap \mathbb{R}^2$.

(1) Show that σ is a cycle (what topological object is it homeomorphic to?) and therefore defines a homology class $[\sigma] \in H_*(S)$.
(2) Compute the Leray coboundary $\delta_*[\sigma] \in H_*(X \setminus S)$. What object is $\delta\sigma$ homeomorphic to?

3

(1) Show that $f(t) = \int_\gamma \frac{dz}{z^2-t}$ where γ is a circle of radius $\frac{1}{2}$ around $z=1$ is holomorphic in a neighborhood of $t=1$. Find the maximal domain $\Omega \subset \mathbb{C}$ to which f can be analytically continued by deforming γ continuously.
(2) Discuss the analytic structure of the function $f(t) = \int_{\sigma_t} dz$ where $\sigma_t = [-\sqrt{t}, \sqrt{t}] \subset \mathbb{C}$ without solving the Integral.