# Direct guaranteed lower eigenvalue bounds with optimal a priori convergence rates for the bi-Laplacian 

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#### Abstract

An extra-stabilised Morley finite element method (FEM) directly computes guaranteed lower eigenvalue bounds with optimal a priori convergence rates for the bi-Laplace Dirichlet eigenvalues. The smallness assumption $\min \left\{\lambda_{h}, \lambda\right\} h_{\max }^{4}$ $\leqslant 184.9570$ in 2D (resp. $\leqslant 21.2912$ in 3D) on the maximal mesh-size $h_{\max }$ makes the computed $k$-th discrete eigenvalue $\lambda_{h} \leqslant \lambda$ a lower eigenvalue bound for the $k$-th Dirichlet eigenvalue $\lambda$. This holds for multiple and clusters of eigenvalues and serves for the localisation of the bi-Laplacian Dirichlet eigenvalues in particular for coarse meshes. The analysis requires interpolation error estimates for the Morley FEM with explicit constants in any space dimension $n \geqslant 2$, which are of independent interest. The convergence analysis in 3D follows the Babuška-Osborn theory and relies on a companion operator for the Morley finite element method. This is based on the Worsey-Farin 3D version of the Hsieh-Clough-Tocher macro element with a careful selection of center points in a further decomposition of each tetrahedron into 12 sub-tetrahedra. Numerical experiments in 2D support the optimal convergence rates of the extra-stabilised Morley FEM and suggest an adaptive algorithm with optimal empirical convergence rates.


Keywords. biharmonic eigenvalue problem, direct guaranteed lower eigenvalue bounds, Morley finite element, conforming companion, nonconforming interpolation, Hsieh-CloughTocher, Worsey-Farin, a priori error estimates, adaptive mesh-refinement

## 1 Introduction

The biharmonic eigenvalue problem $\Delta^{2} u=\lambda u$ allows upper bounds from the Rayleigh-Ritz (or) min-max principle for conforming finite element methods (FEMs) [BO91, Bof10]. Guaranteed lower eigenvalue bounds (GLB) can be even more relevant in a safety analysis in computational mechanics, for the detection of spectral gaps, or for valid bounds of the Sobolev embedding $H_{0}^{2}(\Omega) \hookrightarrow L^{2}(\Omega)$. There is a rich literature on lower eigenvalue bounds for the Laplace operator, cf., e.g., [SV14, $\mathrm{CDM}^{+} 18$, HXYZ18] and the references therein. Throughout this paper the focus is on the biharmonic eigenvalue problem with former contribution in [YLBL12, CG14a, HHL14, Liu15, YLB16, LSL19].

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### 1.1 Motivation

A post-processing for the nonconforming Morley FEM allows for GLBs for the bi-Laplacian in [CG14a]. The $k$-th discrete eigenvalue $\lambda_{M}(k)$ computed from the Morley FEM (displayed in (2.12) below) leads to a guaranteed lower bound

$$
\begin{equation*}
\operatorname{GLB}(k):=\frac{\lambda_{M}(k)}{1+\lambda_{M}(k) \kappa_{2}^{2} h_{\max }^{4}} \leqslant \lambda_{k} \tag{1.1}
\end{equation*}
$$

for the exact $k$-th Dirichlet eigenvalue $\lambda_{k}$ of the bi-Laplacian. The explicit analytical parameter $\kappa_{2}=0.25746$ for $n=2$ is known from [CG14a] and $\kappa_{2}=0.21672$ for $n=3$ is provided in Theorem 2.1.b below. The numerical experiments in this paper utilize the improved computational bound $\kappa_{2}=0.07353$ from [LSL19] for $n=2$. The maximal mesh-size $h_{\text {max }}$ enters as a global parameter in (1.1) and may cause a significant underestimation for adaptive mesh-refinement, when local mesh-refining leaves a few simplices coarse and $h_{\max }$ large. This leads to a dramatic underestimation in the following motivational example with convergence history plot Fig. 1.1 and useless post-processed bound (1.1). The new method is an extra-stabilised Morley FEM with an additional piecewise quadratic variable and the finetuned parameter $\kappa_{2}=0.07353$. The new method allows for an optimal empirical convergence rate one (with respect to the number $|\mathcal{T}|$ of triangles in the triangulation $\mathcal{T}$ ) with adaptive mesh-refinement. The dumbbell domain with a slit (see Fig. 4.1.a below for the initial triangulation) is an extreme example. The adaptive refinement occurs in one of the two cells with minimal coupling, so that $h_{\max }$ is not reduced. The first and fourth Morley eigenvalue $\lambda_{M}(k)<\lambda_{k}$ for $k=1,4$ in Fig. 1.1 are smaller than the approximation $\lambda_{1}=80.93261350$ and $\lambda_{4}=386.80177939$ of the exact eigenvalues, but this is not guaranteed in general, cf. [CG14a, Sec. 2] for a counter example. The eigenvalues of the Morley FEM converge only asymptotically from below [YLBL12, YLB16] and it remains unclear whether a given triangulation belongs to the asymptotic regime. Since $\lambda_{h}(k) h_{\max }^{4} \leqslant \kappa_{2}^{-2}$ holds for all levels in Fig. 1.1, Theorem 1.1 below implies that the discrete eigenvalue $\lambda_{h}(k) \leqslant \lambda_{k}$ for $k=1,4$ is a guaranteed lower eigenvalue bound under the hypothesis of the exact solve of the algebraic eigenvalue problem. (The discussion of interval arithmetic and perturbation analysis for inexact solve in numerical linear algebra is beyond this paper - the focus here is on the understanding of the discretization error.) The numerical results for GLB $\leqslant \lambda_{h} \leqslant \lambda_{M}$ are almost indistinguishable for uniform mesh-refinement in Fig. 1.1 and result in one line with empirical convergence rate 1 for the principal and $1 / 2$ for the fourth eigenvalue.
In short, if the GLB relies exclusively on (1.1), a naive adaptive mesh-refinement appears useless in this example, while the new bound displays optimal empirical convergence rates.

### 1.2 Eigenvalue problems and main results

The continuous eigenvalue problem seeks eigenpairs $(\lambda, u) \in \mathbb{R}^{+} \times V$ with

$$
\begin{equation*}
a(u, v)=\lambda b(u, v) \quad \text { for all } v \in V \quad \text { and } \quad\|u\|_{L^{2}(\Omega)}=1 \tag{1.2}
\end{equation*}
$$

in the Hilbert space $V:=H_{0}^{2}(\Omega)$ with the energy scalar product $a(\bullet, \bullet):=\left(D^{2} \bullet, D^{2} \bullet\right)_{L^{2}(\Omega)}$ for the Hessian $D^{2}$ and the $L^{2}$ scalar product $b(\bullet, \bullet):=(\bullet, \bullet)_{L^{2}(\Omega)}$; the infinite but countable many eigenvalues $0<\lambda_{1} \leqslant \lambda_{2} \leqslant \ldots$ with $\lim _{j \rightarrow \infty} \lambda_{j}=\infty$ in (1.2) are enumerated in ascending order counting multiplicities. For any shape-regular triangulation $\mathcal{T}$ of $\Omega \subset \mathbb{R}^{n}$ into simplices, the piecewise constant mesh-size function $h_{\mathcal{T}} \in P_{0}(\mathcal{T})$ is defined by $\left.h_{\mathcal{T}}\right|_{T}=h_{T}:=\operatorname{diam}(T)$


Figure 1.1: Convergence history plot for the error in the principal and in the fourth Dirichlet eigenvalue of the bi-Laplacian on uniform $(\theta=1$, solid) and adaptive $(\theta=0.5$, dashed) triangulations of the dumbbell-slit domain from Fig. 4.1.a.
in each simplex $T \in \mathcal{T}$ and $h_{\max }:=\max _{T \in \mathcal{T}} h_{T}$. The discrete space $\boldsymbol{V}_{\boldsymbol{h}}:=P_{2}(\mathcal{T}) \times M(\mathcal{T}) \subset$ $P_{2}(\mathcal{T}) \times P_{2}(\mathcal{T})$ consists of piecewise quadratic polynomials. The Morley space $M(\mathcal{T})$ is well established for two-dimensional plate problems [Mor68] and generalized in [MX06] for any space dimension (cf. Subsection 2.1 below for details). The algebraic eigenvalue problem of the extra-stabilised method seeks discrete eigenpairs $\left(\lambda_{h}, \boldsymbol{u}_{\boldsymbol{h}}\right) \in \mathbb{R}^{+} \times\left(\boldsymbol{V}_{\boldsymbol{h}} \backslash\{0\}\right)$ with

$$
\begin{equation*}
\boldsymbol{a}_{\boldsymbol{h}}\left(\boldsymbol{u}_{\boldsymbol{h}}, \boldsymbol{v}_{\boldsymbol{h}}\right)=\lambda_{h} \boldsymbol{b}_{\boldsymbol{h}}\left(\boldsymbol{u}_{\boldsymbol{h}}, \boldsymbol{v}_{\boldsymbol{h}}\right) \quad \text { for all } \boldsymbol{v}_{\boldsymbol{h}} \in \boldsymbol{V}_{\boldsymbol{h}} . \tag{1.3}
\end{equation*}
$$

The discrete scalar product $\boldsymbol{a}_{\boldsymbol{h}}$ contains the scalar product $a_{\mathrm{pw}}(\bullet, \bullet):=\left(D_{\mathrm{pw}}^{2} \bullet, D_{\mathrm{pw}}^{2} \bullet\right)_{L^{2}(\Omega)}$ for the piecewise Hessian of the Morley functions in $M(\mathcal{T})$ and some stabilisation; the bilinear form $\boldsymbol{b}_{\boldsymbol{h}}$ is the $L^{2}$ scalar product of the piecewise quadratic components in $P_{2}(\mathcal{T})$,

$$
\begin{aligned}
\boldsymbol{a}_{\boldsymbol{h}}\left(\boldsymbol{v}_{\boldsymbol{h}}, \boldsymbol{w}_{\boldsymbol{h}}\right) & :=a_{\mathrm{pw}}\left(v_{M}, w_{M}\right)+\kappa_{2}^{-2}\left(h_{\mathcal{T}}^{-4}\left(v_{\mathrm{pw}}-v_{M}\right), w_{\mathrm{pw}}-w_{M}\right)_{L^{2}(\Omega)} \\
\boldsymbol{b}_{\boldsymbol{h}}\left(\boldsymbol{v}_{\boldsymbol{h}}, \boldsymbol{w}_{\boldsymbol{h}}\right) & :=\left(v_{\mathrm{pw}}, w_{\mathrm{pw}}\right)_{L^{2}(\Omega)} \quad \text { for all } \boldsymbol{v}_{\boldsymbol{h}}=\left(v_{\mathrm{pw}}, v_{M}\right), \boldsymbol{w}_{\boldsymbol{h}}=\left(w_{\mathrm{pw}}, w_{M}\right) \in \boldsymbol{V}_{\boldsymbol{h}}
\end{aligned}
$$

Since $\left(\boldsymbol{V}_{\boldsymbol{h}}, \boldsymbol{a}_{\boldsymbol{h}}\right)$ is a Hilbert space and $\boldsymbol{b}_{\boldsymbol{h}}$ is a semi-scalar product with kernel $\{0\} \times M(\mathcal{T}) \subset \boldsymbol{V}_{\boldsymbol{h}}$, the algebraic eigenvalue problem (1.3) has $M:=\operatorname{dim}\left(P_{2}(\mathcal{T})\right)=\binom{2+n}{n}|\mathcal{T}|$ finite and positive algebraic eigenvalues $0<\lambda_{h}(1) \leqslant \cdots \leqslant \lambda_{h}(M)<\infty$ enumerated in ascending order counting multiplicities. The new method (1.3) directly computes guaranteed lower eigenvalue bounds for any space dimension $n \geqslant 2$. The a priori and a posteriori smallness assumption is explicit in terms of the maximal mesh-size $h_{\max }$, but surprisingly robust with respect to the shapes of the simplices in the triangulation $\mathcal{T}$. The interpolation estimates of Theorem 2.1 below define the global parameter

$$
\begin{equation*}
\kappa_{1}:=\sqrt{\frac{1}{\pi^{2}}+\frac{1}{2 n(n+1)(n+2)}} \quad \text { and } \quad \kappa_{2}:=\frac{\kappa_{1}}{\pi}+\sqrt{\frac{n \kappa_{1}^{2}+2 \kappa_{1}}{2(n-1)(n+1)(n+2)}} . \tag{1.4}
\end{equation*}
$$

Theorem 1.1 (GLB). For any $k=1, \ldots, M$, the $k$-th eigenvalue $\lambda_{k}$ from (1.2) and the $k$-th eigenvalue $\lambda_{h}(k)$ from (1.3) satisfy that $\min \left\{\lambda_{h}(k), \lambda_{k}\right\} \kappa_{2}^{2} h_{\max }^{4} \leqslant 1$ implies $\lambda_{h}(k) \leqslant \lambda_{k}$.

Notice that $\min \left\{\lambda_{h}(k), \lambda_{k}\right\} \kappa_{2}^{2} h_{\max }^{4} \leqslant 1$ in Theorem 1.1 means that each of the conditions (i) $\lambda_{k} \kappa_{2}^{2} h_{\max }^{4} \leqslant 1$ (a priori) or (ii) $\lambda_{h}(k) \kappa_{2}^{2} h_{\max }^{4} \leqslant 1$ (a posteriori) implies the GLB property $\lambda_{h}(k) \leqslant \lambda_{k}$. Remarks 2.3-2.4 below explain that the choice $\kappa_{2}=0.07353$ in 2D (resp. $\kappa_{2}=0.21672$ in 3D) is possible in the discrete system (1.3) and in Theorem 1.1 and leads to the condition $\min \left\{\lambda_{h}, \lambda\right\} h_{\max }^{4} \leqslant 184.9570$ in 2D (resp. $\leqslant 21.2912$ in 3D) sufficient for $\lambda \leqslant \lambda_{h}$.

Theorem 1.1 leads under some condition at least a posteriori to GLB and hence the next question is the quality of those. To describe optimal a priori convergence rates, let $\mathbb{T}$ denote the set of uniformly shape-regular triangulation of a fixed bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^{3}$ into tetrahedra with respect to a global shape-regularity constant $C_{\text {sr }}>0$ : Any tetrahedron $T \in \mathcal{T} \in \mathbb{T}$ with diameter $h_{T}$ and volume $|T|$ satisfies $|T|^{1 / 3} \leqslant h_{T} \leqslant C_{\text {sr }}|T|^{1 / 3}$. The subset $\mathbb{T}(\delta) \subset \mathbb{T}$ denotes the triangulations with maximal mesh-size $h_{\max } \leqslant \delta$. Let $\sigma:=\min \left\{1, \sigma_{\text {reg }}\right\}$ denote the minimum of one and the index of elliptic regularity $\sigma_{\mathrm{reg}}>0$ from (1.5) below.

Theorem 1.2 (a priori convergence). Suppose $\lambda$ is an eigenvalue of (1.2) of multiplicity $\mu$ with eigenspace $E(\lambda) \subset H^{2+t}(\Omega) \cap V$ for some $t$ with $\sigma \leqslant t \leqslant 1$. Then there exist $\delta, C>0$ such that any triangulation $\mathcal{T} \in \mathbb{T}(\delta)$ and the discrete space $\boldsymbol{V}_{\boldsymbol{h}}:=P_{2}(\mathcal{T}) \times M(\mathcal{T})$ lead in (1.3) to exactly $\mu$ algebraic eigenvalues $\lambda_{h, 1}, \ldots, \lambda_{h, \mu}$ of (1.3) (counting multiplicities), that converge to $\lambda$ as $h_{\max } \rightarrow 0$. Let $E_{h}:=\operatorname{span}\left\{\boldsymbol{u}_{h} \in E_{h}\left(\lambda_{h, k}\right): k=1, \ldots, \mu\right\}$ abbreviate the span of the discrete eigenspaces $E_{h}\left(\lambda_{h, k}\right) \subset \boldsymbol{V}_{\boldsymbol{h}}$ of $\lambda_{h, k}$ for $k=1, \ldots, \mu$. Then

$$
\begin{aligned}
& h_{\max }^{-t} \max _{k=1, \ldots, \mu}\left|\lambda-\lambda_{h, k}\right|+h_{\max }^{-\sigma} \max _{u \in E(\lambda)} \min _{u_{h}=\left(u_{\mathrm{pw}}, u_{M}\right) \in E_{h}}\left\|u-u_{\mathrm{pw}}\right\|_{L^{2}(\Omega)} \\
& \|u\|_{L^{2}(\Omega)}=1 \quad\left\|u_{\mathrm{pw}}\right\|_{L^{2}(\Omega)}=1 \\
& +h_{\max }^{-\sigma} \max _{\substack{\left.u_{h}=u_{\mathrm{pw}}, u_{M}\right) \in E_{h} \\
\left\|u_{\mathrm{pw}}\right\|_{L^{2}(\Omega)}=1}} \min _{\substack{u \in E(\lambda) \\
\|u\|_{L^{2}(\Omega)}=1}}\left\|u-u_{\mathrm{pw}}\right\|_{L^{2}(\Omega)} \leqslant C h_{\max }^{t} .
\end{aligned}
$$

The results of Theorem 1.1 and 1.2 assume exact solve of the algebraic eigenvalue problem (1.3), but standard perturbation results in numerical linear algebra [Par98] can be added to obtain rigorous bounds in practical applications.

### 1.3 Outline

Section 2 analyses the discrete eigenvalue problem (1.3) and proves Theorem 1.1 in any space dimension $n \geqslant 2$. Subsection 2.1 recalls the Morley finite element (FE) and presents interpolation error estimates in Theorem 2.1. The interpolation constant $\kappa_{2}$ in (1.4) leads to the guaranteed lower bound property from Theorem 1.1 in Subsection 2.2 and a generalization of [CG14a] in Theorem 2.5. Subsection 2.3 introduces a reduced formulation for the new method, remarks on the relation to the standard Morley eigenvalue problem, and introduces a related extra-stabilised Crouzeix-Raviart method. The a priori convergence analysis in 3D of Section 3 is based on a conforming companion operator, i.e., a right-inverse of the interpolation operator in $M(\mathcal{T})$ with the extra properties in Theorem 3.1. This operator relies on the conforming Hsieh-Clough-Tocher finite element in 3D suggested by WorseyFarin (WF) in [WF87] and allows for $L^{2}$ error estimates of separate interest. The analysis of the conforming companion contains some technical details like the correct scaling of the WF basis functions, which extends [Cia78, § 6.1, p.340ff] to $n=3$ and is explained in the selfcontained supplement to this paper. The preparations in Subsections 3.1-3.2 allow the proof of Theorem 1.2 in Subsection 3.3. Since the method is new in any space dimension, the 2D
numerical experiments in Section 4 confirm the theoretical results, present details on Fig. 1.1, and provide striking numerical evidence for the superiority of adaptive mesh-refinement for the bi-Laplace Dirichlet eigenvalue problem.

### 1.4 Notation

Standard notation on Lebesgue and Sobolev spaces applies throughout this paper; $(\bullet, \bullet)_{L^{2}(\Omega)}$ abbreviates the $L^{2}$ scalar product and $H^{2}(T)$ abbreviates $H^{2}(\operatorname{int}(T))$ for a compact set $T$ with non-void interior $\operatorname{int}(T)$. The vector space $H^{2}(\mathcal{T}):=\left\{v \in L^{2}(\Omega):\left.v\right|_{T} \in H^{2}(T)\right\}$ consists of piecewise $H^{2}$ functions and is equipped with the semi-norm $\|\bullet \bullet\|_{\mathrm{pw}}^{2}:=\left(D_{\mathrm{pw}}^{2} \bullet, D_{\mathrm{pw}}^{2} \bullet\right)_{L^{2}(\Omega)}$. The piecewise Hessian $D_{\mathrm{pw}}^{2}$ is understood with respect to the non-displayed regular triangulation $\mathcal{T}$ of $\Omega \subset \mathbb{R}^{n}$ into simplices. The context-depending notation $|\cdot|$ denotes the euclidean length of a vector, the cardinality of a finite set, as well as the non-trivial $n-,(n-1)$-, or $(n-2)$ - dimensional Lebesgue measure of a subset of $\mathbb{R}^{n}$. Let $P_{2}(T)$ denote the space of quadratic functions on $T \in \mathcal{T}$ and $P_{2}(\mathcal{T}):=\left\{v \in L^{2}(\Omega):\left.v\right|_{T} \in P_{2}(T)\right.$ for all $\left.T \in \mathcal{T}\right\}$ the space of piecewise quadratic functions. Given a function $v \in L^{2}(\omega)$, define the integral mean $f_{\omega} v \mathrm{~d} x:=1 /|\omega| \int_{\omega} v \mathrm{~d} x$. The $L^{2}$ projection $\Pi_{0}$ onto the piecewise constant functions $P_{0}(\mathcal{T})$ reads $\left.\Pi_{0}(f)\right|_{T}:=f_{T} f \mathrm{~d} x$ for all $f \in L^{2}(\Omega)$ and $T \in \mathcal{T}$. For any $A \in P_{0}\left(\mathcal{T} ; \mathbb{R}^{\ell \times \ell}\right)$ $\mathrm{SPD},(\bullet, \bullet)_{A}:=(A \bullet, \bullet)_{L^{2}(\Omega)}$ abbreviates the weighted $L^{2}$ scalar product with induced $A$ weighted $L^{2}$ norm $\|\cdot\|_{A}:=\left\|A^{1 / 2} \cdot\right\|_{L^{2}(\Omega)}$. Let $\sigma:=\min \left\{1, \sigma_{\mathrm{reg}}\right\}$ denote the minimum of one for the approximation property and the positive index of elliptic regularity $\sigma_{\text {reg }}>0$ for the source problem of the bi-Laplacian $\Delta^{2}$ in $H_{0}^{2}(\Omega)$ on the bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^{n}:$ Given any right-hand side $f \in L^{2}(\Omega)$, the weak solution $u \in V$ to $\Delta^{2} u=f$ satisfies

$$
\begin{equation*}
u \in H^{2+\sigma}(\Omega) \text { and }\|u\|_{H^{2+\sigma}(\Omega)} \leqslant C(\sigma)\|f\|_{L^{2}(\Omega)} \tag{1.5}
\end{equation*}
$$

The Sobolev space $H^{2+s}(\Omega)$ is defined for $0<s<1$ by complex interpolation of $H^{2}(\Omega)$ and $H^{3}(\Omega)$. Notice $E(\lambda) \subset H^{2+\sigma}(\Omega)$ in Theorem 1.2 follows from (1.5) but $E(\lambda) \subset H^{2+t}(\Omega)$ is possible for $t \geqslant \sigma$ for some eigenvalues $\lambda$. Throughout this paper, $a \lesssim b$ abbreviates $a \leqslant C b$ with a generic constant $C$ only dependent on $\sigma$ in (1.5) and the shape-regularity constant $C_{\text {sr }}$ of $\mathcal{T} \in \mathbb{T} ; a \approx b$ stands for $a \lesssim b \lesssim a$.

## 2 Eigensolver for guaranteed lower bounds in any dimension

### 2.1 The Morley finite element

Given a shape-regular triangulation $\mathcal{T}$ of a bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^{n}$ into $n$-simplices (tetrahedra in 3 D ) in the sense of Ciarlet [BS08, Bra13, BBF13], let $\mathcal{V}$ (resp. $\mathcal{V}(\Omega)$ or $\mathcal{V}(\partial \Omega)$ ) denote the set of all (resp. interior or boundary) vertices, let $\mathcal{F}$ (resp. $\mathcal{F}(\Omega)$ or $\mathcal{F}(\partial \Omega)$ ) denote the set of all (resp. interior or boundary) ( $n-1$ )-subsimplices (faces in 3D), and let $\mathcal{E}$ (resp. $\mathcal{E}(\Omega)$ or $\mathcal{E}(\partial \Omega)$ ) denote the set of all (resp. interior or boundary) $(n-2)$ subsimplices (edges in 3D and vertices in 2D) in $\mathcal{T}$. The degrees of freedom for the Morley element [MX06, Def. 1] on an $n$-simplex $T \in \mathcal{T}$ are the integral means of the function $f$ along any $(n-2)$-subsimplex $E \in \mathcal{E}(T)$ of $T$ and of the normal derivative $\partial f / \partial \nu$ for each $(n-1)$ subsimplex $F \in \mathcal{F}(T)$ of $T$. Let the integral mean over a node $z \in \mathcal{V}$ be the point evaluation, to see that this reduces to the classical definition [Mor68] for $n=2$. The $m:=|\mathcal{F}|+|\mathcal{E}|$ global degrees of freedom are labelled, for any $f \in H^{2}(\Omega)$, by

$$
L_{E}(f):=\oint_{E} f \mathrm{~d} s \text { for any } E \in \mathcal{E} \quad \text { and } \quad L_{F}(f):=\int_{F} \nabla f \cdot \nu_{F} \mathrm{~d} \sigma \text { for any } F \in \mathcal{F}
$$

where $\nu_{F}$ denotes the unit normal for any side $F \in \mathcal{F}$ with a fixed orientation. Section 6 in [CGH14] introduces the dual basis for the Morley finite element in 2D and specifies an implementation in 30 lines of matlab. The nodal basis in [MX06, Thm. 1] reads for $n \geqslant 3$ as follows.
$\underline{\text { Given }}$ any side $F \in \mathcal{F}$ with unit normal $\nu_{F}$ of fixed orientation, the support $\operatorname{supp}\left(\phi_{F}\right)=$ $\overline{\omega(F)}$ of the basis function $\Phi_{F} \in P_{2}(\mathcal{T})$ dual to $L_{F}$ consists of all adjacent $n$-simplices. For any interior side $F=\partial T_{+} \cap \partial T_{-} \in \mathcal{F}(\Omega)$ the side-patch $\overline{\omega(F)}:=T_{+} \cup T_{-}$consists of the neighbouring simplices $T_{ \pm} \in \mathcal{T}$ with $\nu_{F}=\left.\nu_{T_{+}}\right|_{F}=-\nu_{T_{-}} \mid F$. For any boundary side $F=$ $\partial T_{+} \cap \partial \Omega \in \mathcal{F}(\partial \Omega)$ set $\overline{\omega(F)}:=T_{+}$with $\nu_{F}=\left.\nu_{T_{+}}\right|_{F}$. Suppose that $F=F_{j}$ is the side opposite the vertex $P_{j}$ with barycentric coordinate $\lambda_{j}$ in $T_{ \pm} \subset \overline{\omega(F)}$, then

$$
\begin{equation*}
\left.\phi_{F}\right|_{T_{ \pm}}:= \pm\left(\lambda_{j}\left(n \lambda_{j}-2\right)\right) /\left(2\left|\nabla \lambda_{j}\right|\right) \tag{2.1}
\end{equation*}
$$

Let $E_{j k}:=\operatorname{conv}\left\{P_{1}, \ldots, P_{j-1}, P_{j+1}, \ldots, P_{k-1}, P_{k+1}, \ldots, P_{n+1}\right\} \in \mathcal{E}(T)$ denote the $(n-2)$ subsimplex of $T:=\operatorname{conv}\left\{P_{1}, \ldots, P_{n+1}\right\} \in \mathcal{T}$ in the intersection $E \in \partial F_{j} \cap \partial F_{k}$ of the sides $F_{j}, F_{k} \in \mathcal{F}(T)$. Given a $(n-2)$-subsimplex $E \in \mathcal{E}$, the support $\operatorname{supp}\left(\phi_{E}\right)=\bigcup \mathcal{T}(E)$ of the basis function $\Phi_{E} \in P_{2}(\mathcal{T})$ dual to $L_{E}$ consists of all adjacent $n$-simplices $\mathcal{T}(E):=\{T \in \mathcal{T}$ : $E \in \mathcal{E}(T)\}$. Suppose that $E=E_{j k}$ is the $(n-2)$-subsimplex in the intersection $E \in \partial F_{j} \cap \partial F_{k}$ of the sides $F_{j}, F_{k} \in \mathcal{F}(T)$ in $T \in \mathcal{T}(E)$ with barycentric coordinates $\lambda_{j}$ and $\lambda_{k}$ in $T$ (associated with the opposite vertices $P_{j}$ and $\left.P_{k}\right)$. Then

$$
\begin{equation*}
\left.\phi_{E}\right|_{T}=1-(n-1)\left(\lambda_{j}+\lambda_{k}\right)+n(n-1) \lambda_{j} \lambda_{k}-(n-1) \nabla \lambda_{j} \cdot \nabla \lambda_{k} \sum_{\ell \in\{j, k\}} \frac{\lambda_{\ell}\left(n \lambda_{\ell}-2\right)}{2\left|\nabla \lambda_{\ell}\right|^{2}} \tag{2.2}
\end{equation*}
$$

This defines the nodal basis functions: $L_{G}\left(\left.\phi_{F}\right|_{T}\right)=\delta_{F G}, L_{D}\left(\left.\phi_{E}\right|_{T}\right)=\delta_{D E}$, and $L_{F}\left(\left.\phi_{E}\right|_{T}\right)=$ $0=L_{E}\left(\left.\phi_{F}\right|_{T}\right)$ follows for any $n$-simplex $T \in \mathcal{T}$, any $(n-1)$-simplices $F, G \in \mathcal{F}(T)$, and any ( $n-2$ )-simplices $D, E \in \mathcal{E}(T)$. The Morley finite element space with homogeneous boundary conditions reads

$$
M(\mathcal{T}):=\operatorname{span}\left\{\phi_{F}: F \in \mathcal{F}(\Omega)\right\} \oplus \operatorname{span}\left\{\phi_{E}: E \in \mathcal{E}(\Omega)\right\} \subset P_{2}(\mathcal{T})
$$

Given the dual basis for the Morley FEM in (2.1)-(2.2), define the interpolation operator $I_{M}: V \rightarrow M(\mathcal{T})$ for any $v \in V:=H_{0}^{2}(\Omega)$ by

$$
\begin{equation*}
I_{M}(v):=\sum_{F \in \mathcal{F}(\Omega)} f_{F} \nabla v \cdot \nu_{F} \mathrm{~d} \sigma \phi_{F}+\sum_{E \in \mathcal{E}(\Omega)} f_{E} v \mathrm{~d} s \phi_{E} \tag{2.3}
\end{equation*}
$$

This interpolation operator has the following important properties with the explicit constants $\kappa_{1}$ and $\kappa_{2}$ from (1.4), which are not quantified in [MX06].
Theorem 2.1 (properties of $\boldsymbol{I}_{\boldsymbol{M}}$ ). (a) Any $v \in V$ satisfies $\Pi_{0} D^{2} v=D_{\mathrm{pw}}^{2} I_{M} v$, in particular $a_{\mathrm{pw}}\left(v-I_{M} v, w_{M}\right)=0$ for all $w_{M} \in M(\mathcal{T})$ and $v \in V ; I_{M}$ is the $a_{\mathrm{pw}}$-orthogonal projection onto $M(\mathcal{T})$ with $\left\|v-I_{M} v\right\|_{\mathrm{pw}}=\min _{v_{M} \in M(\mathcal{T})}\left\|v-v_{M}\right\|_{\mathrm{pw}}$ for any $v \in V$. (b) Any $v \in H^{2}(T)$ in $T \in \mathcal{T}$ satisfies $\left|v-I_{M} v\right|_{H^{2-\ell}(T)} \leqslant \kappa_{\ell} h_{T}^{\ell}\left|v-I_{M} v\right|_{H^{2}(T)}$ for $\ell=1,2$.

Proof. This is known for $n=2$ from [CG14a], so let $n \geqslant 3$ in the sequel.
Proof of (a). The definition of $I_{M}$ implies $f_{F} \nabla I_{M} v \cdot \nu_{F} \mathrm{~d} \sigma=f_{F} \nabla v \cdot \nu_{F} \mathrm{~d} \sigma$ for any $F \in \mathcal{F}$. This and an integration by parts prove $\Pi_{0} D^{2} v=D_{\mathrm{pw}}^{2} I_{M} v$. Since $w_{M} \in M(\mathcal{T}) \subset P_{2}(\mathcal{T})$, this concludes the proof of $a_{\mathrm{pw}}\left(v-I_{M} v, w_{M}\right)=\left(\left(1-\Pi_{0}\right) D^{2} v, D_{\mathrm{pw}}^{2} w_{M}\right)_{L^{2}(\Omega)}=0$.

Proof of (b) for $\ell=1$. A key observation is that the piecewise gradient $\nabla_{\mathrm{pw}} v_{M} \in C R_{0}^{1}(\mathcal{T})^{n}$ of any $v_{M} \in M(\mathcal{T})$ is a Crouzeix-Raviart function in $n$ components. (This follows from $f_{F} \nabla I_{M} v \mathrm{~d} \sigma=f_{F} \nabla v \mathrm{~d} \sigma$, since for any $v \in C^{1}(T)$ in $T \in \mathcal{T}$ the Morley degrees of freedom uniquely determine $f_{F} \nabla v \mathrm{~d} \sigma$ for any $F \in \mathcal{F}(T)$ [MX06, Lem. 1].) Moreover, the CrouzeixRaviart interpolation operator $I_{C R}$ [CR73] (applied component-wise) satisfies $\nabla_{\mathrm{pw}} I_{M} v=$ $I_{C R} \nabla v$ for any $v \in H^{2}(T)$ in $T \in \mathcal{T}$. Lemma A. 1 in [CZZ20] shows $\left\|f-I_{C R} f\right\|_{L^{2}(T)} \leqslant$ $\kappa_{1} h_{T}\left|f-I_{C R} f\right|_{H^{1}(T)}$ for $f \in H^{1}(T)$ and any $n \geqslant 2$. The choice $f=\partial v / \partial x_{j}$ for $j=1, \ldots, n$ concludes the proof of $\left|v-I_{M} v\right|_{H^{1}(T)} \leqslant \kappa_{1} h_{T}\left|v-I_{M} v\right|_{H^{2}(T)}$ for $v \in H^{2}(T)$.

Proof of (b) for $\ell=2$. Let $g:=v-I_{M} v \in H^{2}(T)$ and set $I_{C R}(g):=\sum_{F \in \mathcal{F}}\left(f_{F} g \mathrm{~d} \sigma\right) \psi_{F}$ with the side-oriented Crouzeix-Raviart basis function $\psi_{F} \in C R^{1}(\mathcal{T})$ with $\psi_{F}(\operatorname{mid}(G))=\delta_{F G}$ for all $F, G \in \mathcal{F}$. The local mass matrix $M(T) \in \mathbb{R}^{(n+1) \times(n+1)}$ for $\mathcal{F}(T):=\left\{F_{1}, \ldots, F_{n+1}\right\}$ reads

$$
M(T):=\left(\int_{T} \psi_{F_{j}} \psi_{F_{k}} \mathrm{~d} \sigma\right)_{j, k=1, \ldots, n+1}=\left(\frac{|T|\left(2-n+n^{2} \delta_{j k}\right)}{(n+1)(n+2)}\right)_{j, k=1, \ldots, n+1}
$$

The eigenvalue $|T| /(n+1)$ of $M(T)$ has the eigenvector $(1, \ldots, 1) \in \mathbb{R}^{n+1}$. The eigenvalue $|T| n^{2} /((n+1)(n+2))$ has the $n$-dimensional eigenspace of vectors in $\mathbb{R}^{n+1}$ perpendicular to $(1, \ldots, 1)$. Hence the coefficient vector $x:=\left(f_{F} g \mathrm{~d} \sigma: F \in \mathcal{F}(T)\right) \in \mathbb{R}^{n+1}$ of $I_{C R} g$ satisfies

$$
\begin{align*}
\left\|I_{C R} g\right\|_{L^{2}(T)}^{2} & =\int_{T}\left(\sum_{F \in \mathcal{F}(T)} f_{F} g \mathrm{~d} \sigma \psi_{F}\right)^{2} \mathrm{~d} x=x \cdot M(T) x \\
& \leqslant \frac{|T| n^{2}}{(n+1)(n+2)}|x|^{2}=\frac{|T| n^{2}}{(n+1)(n+2)} \sum_{F \in \mathcal{F}(T)}\left(f_{F} g \mathrm{~d} \sigma\right)^{2} . \tag{2.4}
\end{align*}
$$

If the $m$-simplex $F$ has the $(m-1)$-subsimplex $E$ opposite to the vertex $P$ in $F$ for $m \geqslant 2$, then any $v \in H^{1}(F)$ satisfies the trace identity

$$
\begin{equation*}
f_{E} v \mathrm{~d} s=f_{F} v \mathrm{~d} \sigma+m^{-1} f_{F}(x-P) \cdot \nabla v \mathrm{~d} \sigma \tag{2.5}
\end{equation*}
$$

(This follows from an integration by parts [CGR12, CH17].) In each ( $n-1$ )-subsimplex $F \in$ $\mathcal{F}(T)$ with midpoint $\operatorname{mid}(F)$, the set of $(n-2)$-subsimplices (edges in 3 D$) \mathcal{E}(F):=\{E \in \mathcal{E}(T)$ : $E \subset \partial F\}=\left\{E_{1}, \ldots, E_{n}\right\}$ defines the sub-triangulation of $F$ into $F_{j}:=\operatorname{conv}\left(\operatorname{mid}(F), E_{j}\right) \subset F$ for $j=1, \ldots, n$. Since the function $\left.g\right|_{F}:=\left.\left(v-I_{M} v\right)\right|_{F} \in H^{1}(F)$ satisfies $f_{E_{j}} g \mathrm{~d} s=0$ for any $j=1, \ldots, n$, the trace identity (2.5) for each ( $n-1$ )-simplex $F_{j} \subset F$ proves

$$
f_{F} g \mathrm{~d} \sigma=-\frac{1}{|F|(n-1)} \int_{F}(x-\operatorname{mid}(F)) \cdot \nabla g \mathrm{~d} \sigma \leqslant \frac{1}{|F|(n-1)}\|\bullet-\operatorname{mid}(F)\|_{L^{2}(F)}\|\nabla g\|_{L^{2}(F)}
$$

with the Cauchy-Schwarz inequality in the last step. For the $(n-1)$-dimensional simplex $F=\operatorname{conv}\left\{P_{1}, \ldots, P_{n}\right\}$ assume without loss of generality that $\operatorname{mid}(F)=\frac{1}{n} \sum_{j=1}^{n} P_{j}=0$. An estimation of the mass-matrix for the Courant basis functions associated with the vertices $P_{1}, \ldots, P_{n}$ of $F$ leads to

$$
\|\bullet-\operatorname{mid}(F)\|_{L^{2}(F)}^{2}=\int_{F}|x|^{2} \mathrm{~d} \sigma=\frac{|F|}{n(n+1)} \sum_{\ell=1}^{n}\left|P_{\ell}\right|^{2} .
$$

Similar to [CZZ20, Lem. A.1], elementary algebra with $\frac{1}{n} \sum_{j=1}^{n} P_{j}=0$ and $\left|P_{j}-P_{k}\right| \leqslant h_{F}$ for all $j, k=1, \ldots, n$ with $j \neq k$ lead to

$$
\sum_{\ell=1}^{n}\left|P_{\ell}\right|^{2}=1 /(2 n) \sum_{j, k=1}^{n}\left|P_{j}-P_{k}\right|^{2} \leqslant h_{F}^{2}(n-1) / 2
$$

The combination of the last three displayed estimates reads

$$
\left(f_{F} g \mathrm{~d} \sigma\right)^{2} \leqslant \frac{1}{(n-1)^{2}|F|^{2}} \frac{h_{F}^{2}|F|(n-1)}{2 n(n+1)}\|\nabla g\|_{L^{2}(F)}^{2} \leqslant \frac{h_{T}^{2}}{|F|}(2 n(n-1)(n+1))^{-1}\|\nabla g\|_{L^{2}(F)}^{2}
$$

The trace inequality $\|v\|_{L^{2}(F)}^{2} \leqslant(|F| /|T|)\|v\|_{L^{2}(T)}\left(\|v\|_{L^{2}(T)}+2 h_{T} / n\|\nabla v\|_{L^{2}(T)}\right)$ for $v \in H^{1}(T)$ and $F \in \mathcal{F}(T)$ is a direct consequence of the trace identity (2.5) for the $n$-simplex $T \in \mathcal{T}$ with ( $n-1$ )-subsimplex $F \in \mathcal{F}(T)$. This and Young's inequality show

$$
\frac{|T|}{|F|}\|\nabla g\|_{L^{2}(F)}^{2} \leqslant\left(1+\left(\kappa_{1} n\right)^{-1}\right)\|\nabla g\|_{L^{2}(T)}^{2}+h_{T}^{2} \kappa_{1} n^{-1}\left\|D^{2} g\right\|_{L^{2}(T)}^{2}
$$

The proven Theorem 2.1.b for $\ell=1$ shows $\|\nabla g\|_{L^{2}(T)} \leqslant \kappa_{1} h_{T}\left\|D^{2} g\right\|_{L^{2}(T)}$. In combination with the last two displayed inequalities and (2.4), this reads

$$
\begin{equation*}
\left\|I_{C R} g\right\|_{L^{2}(T)}^{2} \leqslant \frac{n \kappa_{1}^{2}+2 \kappa_{1}}{2(n-1)(n+1)(n+2)} h_{T}^{4}\left\|D^{2} g\right\|_{L^{2}(T)}^{2} \tag{2.6}
\end{equation*}
$$

On the other hand, the Crouzeix-Raviart interpolation operator $I_{C R}$ satisfies $\left\|\left(1-I_{C R}\right) g\right\|_{L^{2}(T)}$ $\leqslant \kappa_{1} h_{T}\left\|\nabla\left(1-I_{C R}\right) g\right\|_{L^{2}(T)}$ and $\nabla I_{C R} g=\Pi_{0} \nabla g$ as in [CG14a, CG14b] for $n=2$ and in [CZZ20] for $n \geqslant 3$. Hence, the Poincaré inequality with Payne-Weinberger constant [PW60, Beb03] shows

$$
\left\|\left(1-I_{C R}\right) g\right\|_{L^{2}(T)} \leqslant \kappa_{1} h_{T}\left\|\nabla\left(1-I_{C R}\right) g\right\|_{L^{2}(T)}=\kappa_{1} h_{T}\left\|\left(1-\Pi_{0}\right) \nabla g\right\|_{L^{2}(T)} \leqslant \frac{\kappa_{1}}{\pi} h_{T}^{2}\left\|D^{2} g\right\|_{L^{2}(T)}
$$

The combination of this with (2.6) and a triangle inequality concludes the proof of $\| v-$ $I_{M} v \|_{L^{2}(T)} \leqslant \kappa_{2} h_{T}^{2}\left|v-I_{M} v\right|_{H^{2}(T)}$ for any $v \in H^{2}(T)$.

The constant $\kappa_{2}=0.25746$ for $n=2$ from [CG14a, Thm. 3] is recovered, if the Poincaré constant $1 / \pi$ is replaced by the optimal $1 / j_{1,1}$ in 2 D [LS10]. The computational bounds $\kappa_{1} \leqslant 0.1893$ from [Liu15] and $\kappa_{2} \leqslant 0.07353$ [LSL19] improve the analytical bounds from [CG14a] in 2D.

Corollary 2.2 (further properties). (a) Any $v \in H^{2+s}(\Omega)$ with $0 \leqslant s \leqslant 1$ satisfies

$$
\left\|\left(1-I_{M}\right) v\right\|_{\mathrm{pw}} \leqslant\left(h_{\max } / \pi\right)^{s}\|v\|_{H^{2+s}(\Omega)}
$$

(b) Any $v, w \in V$ and $v_{M} \in M(\mathcal{T})$ satisfy $a_{\mathrm{pw}}\left(v, v_{M}\right)=a_{\mathrm{pw}}\left(I_{M} v, v_{M}\right)$ and

$$
\begin{aligned}
a_{\mathrm{pw}}\left(v,\left(1-I_{M}\right) w\right) & =a_{\mathrm{pw}}\left(\left(1-I_{M}\right) v,\left(1-I_{M}\right) w\right) \\
& \leqslant \min _{v_{M} \in M(\mathcal{T})}\left\|v-v_{M}\right\|_{\mathrm{pw}} \min _{w_{M} \in M(\mathcal{T})}\left\|w-w_{M}\right\|_{\mathrm{pw}}
\end{aligned}
$$

Proof. Theorem 2.1.a and a piecewise Poincaré inequality (as above from [PW60, Beb03]) show for $s=0$ and $s=1$ that

$$
\begin{equation*}
\left\|\left(1-I_{M}\right) v\right\|_{\mathrm{pw}}=\left\|D^{2} v-\Pi_{0} D^{2} v\right\|_{L^{2}(\Omega)} \leqslant\left(h_{\max } / \pi\right)^{s}\|v\|_{H^{2+s}(\Omega)} \tag{2.7}
\end{equation*}
$$

Since $H^{2+s}(\Omega)$ is defined by complex interpolation of $H^{2}(\Omega)$ and $H^{3}(\Omega)$, each component of the Hessian $D^{2} v$ belongs to $H^{s}(\Omega) \in\left[L^{2}(\Omega), H^{1}(\Omega)\right]_{s}$ in the complex interpolation space between $L^{2}(\Omega)$ and $H^{1}(\Omega)$ [Tar07]. The interpolation of (2.7) concludes the proof of (a). Theorem 2.1.a implies the first claim in (b). The combination with the Cauchy-Schwarz inequality implies the second.

### 2.2 Guaranteed lower bounds

This section proves that the discrete method (1.3) indeed provides GLBs for the continuous eigenvalues in any space dimension $n \geqslant 2$.

Proof of Theorem 1.1. Abbreviate $\lambda=\lambda_{k}$ from (1.2) and $\lambda_{h}=\lambda_{h}(k)$ from (1.3). Let $\phi_{1}, \ldots, \phi_{k}$ denote the first $k b$-orthonormal eigenfunctions of (1.2); the min-max principle [SF08, Bof10] guarantees $\|\phi\|^{2} \leqslant \lambda$ for any $\phi \in \operatorname{span}\left\{\phi_{1}, \ldots, \phi_{k}\right\}$ with $\|\phi\|_{L^{2}(\Omega)}=1$. Let $\Pi_{2}$ denote $L^{2}$ projection onto $P_{2}(\mathcal{T})$.

Case 1. Assume the $L^{2}$-projections $\Pi_{2} \phi_{1}, \ldots, \Pi_{2} \phi_{k}$ are linear dependent. Then there exists some $\phi \in \operatorname{span}\left\{\phi_{1}, \ldots, \phi_{k}\right\}$ with $\|\phi\|_{L^{2}(\Omega)}=1$ and $\Pi_{2} \phi=0$. Let $\kappa_{2}^{\prime}$ denote the best possible constant in

$$
\begin{equation*}
\left\|\left(1-\Pi_{2}\right) \psi\right\|_{L^{2}(\Omega)} \leqslant \kappa_{2}^{\prime} h_{\max }^{2}\left\|\left(1-I_{M}\right) \psi\right\|_{\mathrm{pw}} \quad \text { for all } \psi \in H_{0}^{2}(\Omega) \tag{2.8}
\end{equation*}
$$

The approximation property of $\Pi_{2}$ and $M(\mathcal{T}) \subset P_{2}(\mathcal{T})$ imply (2.8) with $\kappa_{2}^{\prime} \leqslant \kappa_{2}$. The above $\phi$ therefore satisfies

$$
1=\|\phi\|_{L^{2}(\Omega)}=\left\|\left(1-\Pi_{2}\right) \phi\right\|_{L^{2}(\Omega)} \leqslant \kappa_{2}^{\prime} h_{\max }^{2}\left\|\left(1-I_{M}\right) \phi\right\|_{\mathrm{pw}} .
$$

The Pythagoras theorem from Theorem 2.1.a and $\|\phi\|^{2} \leqslant \lambda$ from the min-max principle [SF08, Bof10] for (1.2) show

$$
\left\|\left(1-I_{M}\right) \phi\right\|_{\mathrm{pw}}^{2}+\left\|I_{M} \phi\right\|_{\mathrm{pw}}^{2}=\|\phi\|^{2} \leqslant \lambda .
$$

The combination of the last two displayed inequalities reads $1 \leqslant \lambda\left(\kappa_{2}^{\prime}\right)^{2} h_{\max }^{4}$. Throughout this paper, the values used for $\kappa_{2}$ satisfy $\kappa_{2}^{\prime}<\kappa_{2}$ (see Remarks 2.3-2.4 below), whence $\lambda \kappa_{2}^{2} h_{\max }^{4}>1$. In other words, the a priori condition $\lambda \kappa_{2}^{2} h_{\max }^{4} \leqslant 1$ fails and (the remaining hypothesis of Theorem 1.1) $\lambda_{h} \kappa_{2}^{2} h_{\max }^{4} \leqslant 1$ holds. This proves $\lambda_{h} \kappa_{2}^{2} h_{\max }^{4} \leqslant 1<\lambda \kappa_{2}^{2} h_{\max }^{4}$ and concludes the proof of $\lambda_{h} \leqslant \lambda$ in the first case.

Case 2. Assume that the projections $\Pi_{2} \phi_{1}, \ldots, \Pi_{2} \phi_{k}$ are linear independent. Set $\boldsymbol{S}_{k}:=$ $\operatorname{span}\left\{\left(\Pi_{2} \phi_{1}, I_{M} \phi_{1}\right), \ldots,\left(\Pi_{2} \phi_{k}, I_{M} \phi_{k}\right)\right\} \subset \boldsymbol{V}_{\boldsymbol{h}}$ with $\operatorname{dim}\left(\boldsymbol{S}_{\boldsymbol{k}}\right)=k$. Since $\boldsymbol{b}_{\boldsymbol{h}}$ is positive definite on $\boldsymbol{S}_{k} \times \boldsymbol{S}_{k}$, the min-max principle [SF08, Bof10] for (1.3) shows

$$
\begin{equation*}
\lambda_{h} \leqslant \max _{\boldsymbol{v}_{\boldsymbol{h}} \in \boldsymbol{S}_{\boldsymbol{k}} \backslash\{0\}} \frac{\boldsymbol{a}_{\boldsymbol{h}}\left(\boldsymbol{v}_{\boldsymbol{h}}, \boldsymbol{v}_{\boldsymbol{h}}\right)}{\boldsymbol{b}_{\boldsymbol{h}}\left(\boldsymbol{v}_{\boldsymbol{h}}, \boldsymbol{v}_{\boldsymbol{h}}\right)} . \tag{2.9}
\end{equation*}
$$

Let $\boldsymbol{v}_{\boldsymbol{h}}=\left(\Pi_{2} \phi, I_{M} \phi\right) \in \boldsymbol{S}_{k} \backslash\{0\}$ be some maximizer in the upper bound of (2.9) with $\|\phi\|_{L^{2}(\Omega)}=$ 1 and deduce $\|\phi\|^{2} \leqslant \lambda$ from the min-max principle [SF08, Bof10] for (1.2). The inequality (2.9) ensures for $\boldsymbol{v}_{\boldsymbol{h}}=\left(\Pi_{2} \phi, I_{M} \phi\right) \in \boldsymbol{S}_{k} \backslash\{0\}$ that

$$
\lambda_{h}\left\|\Pi_{2} \phi\right\|_{L^{2}(\Omega)}^{2}=\lambda_{h} \boldsymbol{b}_{\boldsymbol{h}}\left(\boldsymbol{v}_{\boldsymbol{h}}, \boldsymbol{v}_{\boldsymbol{h}}\right) \leqslant \boldsymbol{a}_{\boldsymbol{h}}\left(\boldsymbol{v}_{\boldsymbol{h}}, \boldsymbol{v}_{\boldsymbol{h}}\right)=\left\|I_{M} \phi\right\|_{\mathrm{pw}}^{2}+\kappa_{2}^{-2}\left\|h_{\mathcal{T}}^{-2}\left(\Pi_{2}-I_{M}\right) \phi\right\|_{L^{2}(\Omega)}^{2} .
$$

Since the piecewise constant mesh-size $h_{\mathcal{T}}$ does not interact with the piecewise $L^{2}$ projections, the Pythagoras theorem shows

$$
\begin{aligned}
\left\|h_{\mathcal{T}}^{-2}\left(\Pi_{2}-I_{M}\right) \phi\right\|_{L^{2}(\Omega)}^{2} & =\left\|h_{\mathcal{T}}^{-2}\left(1-I_{M}\right) \phi\right\|_{L^{2}(\Omega)}^{2}-\left\|h_{\mathcal{T}}^{-2}\left(1-\Pi_{2}\right) \phi\right\|_{L^{2}(\Omega)}^{2} \\
& \leqslant \kappa_{2}^{2}\left\|\left(1-I_{M}\right) \phi\right\|_{\mathrm{pw}}^{2}-h_{\max }^{-4}+h_{\max }^{-4}\left\|\Pi_{2} \phi\right\|_{L^{2}(\Omega)}^{2}
\end{aligned}
$$

with Theorem 2.1.b for $\ell=2$ and $1-\left\|\Pi_{2} \phi\right\|_{L^{2}(\Omega)}^{2} \leqslant h_{\max }^{4}\left\|h_{\mathcal{T}}^{-2}\left(1-\Pi_{2}\right) \phi\right\|_{L^{2}(\Omega)}^{2}$ in the last step. The combination of the previous two displayed estimates leads to

$$
\begin{equation*}
\left(\lambda_{h}-\kappa_{2}^{-2} h_{\max }^{-4}\right)\left\|\Pi_{2} \phi\right\|_{L^{2}(\Omega)}^{2}+\kappa_{2}^{-2} h_{\max }^{-4} \leqslant\left\|I_{M} \phi\right\|_{\mathrm{pw}}^{2}+\left\|\left(1-I_{M}\right) \phi\right\|_{\mathrm{pw}}^{2}=\|\phi\|^{2} \leqslant \lambda \tag{2.10}
\end{equation*}
$$

with the Pythagoras theorem from Theorem 2.1.a in the equality and $\|\phi\|^{2} \leqslant \lambda$ from above in the last step. The inequality (2.10) implies

$$
\begin{equation*}
1-\lambda \kappa_{2}^{2} h_{\max }^{4} \leqslant\left(1-\lambda_{h} \kappa_{2}^{2} h_{\max }^{4}\right)\left\|\Pi_{2} \phi\right\|_{L^{2}(\Omega)}^{2} \tag{2.11}
\end{equation*}
$$

Without loss of generality assume $\lambda \kappa_{2}^{2} h_{\max }^{4} \leqslant 1$, otherwise the remaining hypothesis $\lambda_{h} \kappa_{2}^{2} h_{\max }^{4}$ $\leqslant 1<\lambda \kappa_{2}^{2} h_{\max }^{4}$ proves the claim. Since the linear independence of $\Pi_{2} \phi_{1}, \ldots, \Pi_{2} \phi_{k}$ shows $\left\|\Pi_{2} \phi\right\|_{L^{2}(\Omega)}>0$, (2.11) implies $\lambda_{h} \kappa_{2}^{2} h_{\max }^{4} \leqslant 1$. Hence $\left\|\Pi_{2} \phi\right\|_{L^{2}(\Omega)} \leqslant\|\phi\|_{L^{2}(\Omega)}=1$ and (2.11) imply $1-\lambda \kappa_{2}^{2} h_{\text {max }}^{4} \leqslant 1-\lambda_{h} \kappa_{2}^{2} h_{\text {max }}^{4}$. This proves $\lambda_{h} \leqslant \lambda$ in the second case.

Remark 2.3 (choice of $\kappa_{2}=0.07353$ in $2 D$ ) The computational bound $\kappa_{2}=0.07353>\kappa_{2}^{\star}$ in 2 D from [LSL19] is a guaranteed upper bound for some optimal $\kappa_{2}^{\star}$ with $\left\|v-I_{M} v\right\|_{L^{2}(T)} \leqslant$ $\kappa_{2}^{\star} h_{T}^{2}\left|v-I_{M} v\right|_{H^{2}(T)}$ for all $v \in H^{2}(T), T \in \mathcal{T}$. Hence $\kappa_{2}^{\prime}:=\kappa_{2}^{\star}<\kappa_{2}$ and Theorem 1.1 holds for this improved choice of $\kappa_{2}$.

Remark 2.4 (choice of $\kappa_{2}$ in (1.4)) Standard arguments including successive (piecewise) Poincaré inequalities [PW60, Beb03] eventually imply $\kappa_{2}^{\prime} \leqslant 1 / \pi^{2}$ and the analytical bound in (1.4) satisfies $1 / \pi^{2}<\kappa_{2}$, hence the claim $\kappa_{2}^{\prime}<\kappa_{2}$ follows for $n \geqslant 3$ as well.

The standard Morley eigenvalue problem seeks $\left(\lambda_{M}, \phi_{M}\right) \in \mathbb{R}^{+} \times(M(\mathcal{T}) \backslash\{0\})$ with

$$
\begin{equation*}
a_{\mathrm{pw}}\left(\phi_{M}, v_{M}\right)=\lambda_{M}\left(\phi_{M}, v_{M}\right)_{L^{2}(\Omega)} \quad \text { for all } v_{M} \in M(\mathcal{T}) \tag{2.12}
\end{equation*}
$$

Assume the $N:=\operatorname{dim}(M(\mathcal{T}))$ algebraic eigenvalues $0<\lambda_{M}(1) \leqslant \lambda_{M}(2) \leqslant \cdots \leqslant \lambda_{M}(N)<\infty$ of (2.12) are enumerated in ascending order counting multiplicities. The following Theorem 2.5 refines [CG14a] for $n=2$ by removing an unnecessary separation condition [Liu15, Rem. 2.2] and generalizes it to $n \geqslant 3$.

Theorem 2.5 (GLB in (1.1)). For any $k=1, \ldots, N$ the $k$-th eigenvalue $\lambda_{M}(k)$ of (2.12) leads in (1.1) to a guaranteed lower bound $\operatorname{GLB}(k) \leqslant \lambda_{k}$ for the $k$-th eigenvalue $\lambda_{k}$ of (1.2).

Proof. Let $\phi_{1}, \ldots, \phi_{k}$ denote the first $k L^{2}$-orthonormal eigenfunctions in (1.2). Assume first, that the interpolations $I_{M} \phi_{1}, \ldots, I_{M} \phi_{k}$ are linear dependent. Let $\phi \in \operatorname{span}\left\{\phi_{1}, \ldots, \phi_{k}\right\}$ with $\|\phi\|_{L^{2}(\Omega)}=1$ and $I_{M} \phi=0$. Theorem 2.1.b and the min-max principle for (1.2) show

$$
1=\|\phi\|_{L^{2}(\Omega)}^{2}=\left\|\left(1-I_{M}\right) \phi\right\|_{L^{2}(\Omega)}^{2} \leqslant \kappa_{2}^{2} h_{\max }^{4}\left\|\left(1-I_{M}\right) \phi\right\|_{\mathrm{pw}}^{2}=\kappa_{2}^{2} h_{\max }^{4}\|\phi\|^{2} \leqslant \lambda_{k} \kappa_{2}^{2} h_{\max }^{4} .
$$

It follows $\kappa_{2}^{-2} h_{\max }^{-4} \leqslant \lambda_{k}$ and $\operatorname{GLB}(k)=\lambda_{M}(k) /\left(1+\lambda_{M}(k) \kappa_{2}^{2} h_{\max }^{4}\right) \leqslant \kappa_{2}^{-2} h_{\max }^{-4} \leqslant \lambda_{k}$ concludes the proof. Theorem 2 in [CG14a] states (1.1) under a separation condition $h_{\max }^{4} \kappa_{2}^{2}<$ $\left(\sqrt{1+k^{-1}}-1\right) / \sqrt{\lambda_{k}}$ (with adapted notation) in 2D. If $I_{M} \phi_{1}, \ldots, I_{M} \phi_{k}$ are linearly independent, the proof in [CG14a] does not need the separation condition to show (1.1) and holds with Theorem 2.1.b for $n \geqslant 3$. The remaining details in [CG14a] apply verbatim and are therefore omitted.

### 2.3 Comments

This section introduces a reduced problem (2.13) as a disturbed nonconforming eigenvalue problem and compares the new method with the standard Morley formulation.

### 2.3.1 Reduced eigenvalue problem

Under the condition $\lambda_{h} \kappa_{2}^{2} h_{\max }^{4}<1$, the algebraic eigenvalue problem (1.3) is equivalent to a reduced form that seeks $\left(\lambda_{h}, u_{M}\right) \in \mathbb{R}^{+} \times(M(\mathcal{T}) \backslash\{0\})$ with

$$
\begin{equation*}
a_{\mathrm{pw}}\left(u_{M}, v_{M}\right)=\lambda_{h} b\left(\frac{u_{M}}{1-\lambda_{h} \kappa_{2}^{2} h_{\mathcal{T}}^{4}}, v_{M}\right) \quad \text { for all } v_{M} \in M(\mathcal{T}) \tag{2.13}
\end{equation*}
$$

The formulation (2.13) is reduced in that the additional variables $u_{\mathrm{pw}}$ and $v_{\mathrm{pw}}$ in (1.3) are condensed out; but (2.13) is a (simple) rational eigenvalue problem with the same dimension and sparsity as the Morley eigenvalue problem (2.12). A solution $\left(\lambda_{h}, u_{M}\right)$ to (2.13) is also called an eigenpair and the (geometric) multiplicity $\geqslant 1$ is the dimension of the eigenspace of all $u_{M} \in M(\mathcal{T}) \backslash\{0\}$ so that $\left(\lambda_{h}, u_{M}\right)$ solves (2.13). The numerical treatment of the rational eigenvalue problem (2.13) is left as a topic for future research in numerical linear algebra.
Proposition 2.6 (equivalence). (a) If the eigenpair ( $\lambda_{h}, \boldsymbol{u}_{\boldsymbol{h}}$ ) of (1.3) satisfies $\lambda_{h}<\kappa_{2}^{-2} h_{\max }^{-4}$ with $\boldsymbol{u}_{\boldsymbol{h}}=\left(u_{\mathrm{pw}}, u_{M}\right) \in \boldsymbol{V}_{\boldsymbol{h}} \backslash\{0\}$, then $\left(\lambda_{h}, u_{M}\right)$ solves (2.13) and $u_{\mathrm{pw}}=\left(1-\lambda_{h} \kappa_{2}^{2} h_{\mathcal{T}}^{4}\right)^{-1} u_{M}$.
(b) If $\left(\lambda_{h}, u_{M}\right)$ is a solution to (2.13) with $0<\lambda_{h}<\kappa_{2}^{-2} h_{\max }^{-4}$, then $\lambda_{h}$, $u_{M}$, and $u_{\mathrm{pw}}=$ $\left(1-\lambda_{h} \kappa_{2}^{2} h_{\mathcal{T}}^{4}\right)^{-1} u_{M}$ in $\boldsymbol{u}_{\boldsymbol{h}}=\left(u_{\mathrm{pw}}, u_{M}\right)$ form an eigenpair $\left(\lambda_{h}, \boldsymbol{u}_{\boldsymbol{h}}\right)$ of (1.3).
(c) The set of eigenvalues of (1.3) in the open interval $\left(0, \kappa_{2}^{-2} h_{\max }^{-4}\right)$ is equal to the set of solutions $\lambda_{h}$ to (2.13) in ( $0, \kappa_{2}^{-2} h_{\max }^{-4}$ ) counting (geometric) multiplicities.
Proof. Throughout this proof with a specific and fixed $\lambda_{h}$ with $0<\lambda_{h}<\kappa_{2}^{-2} h_{\max }^{-4}$, abbreviate

$$
\begin{equation*}
\delta:=\frac{1}{1-\lambda_{h} \kappa_{2}^{2} h_{\mathcal{T}}^{4}}-1=\frac{\lambda_{h} \kappa_{2}^{2} h_{\mathcal{T}}^{4}}{1-\lambda_{h} \kappa_{2}^{2} h_{\mathcal{T}}^{4}}=h_{\mathcal{T}}^{4} \kappa_{2}^{2} \lambda_{h}(1+\delta) \in P_{0}(\mathcal{T}) . \tag{2.14}
\end{equation*}
$$

Proof of (a). Suppose that $\left(\lambda_{h}, \boldsymbol{u}_{\boldsymbol{h}}\right) \in \mathbb{R}^{+} \times \boldsymbol{V}_{\boldsymbol{h}}$ is an eigenpair of (1.3). For $v_{\mathrm{pw}} \in P_{2}(\mathcal{T})$, the test function $\left(v_{\mathrm{pw}}, 0\right) \in \boldsymbol{V}_{\boldsymbol{h}}$ in (1.3) shows $\kappa_{2}^{-2} h_{\mathcal{T}}^{-4}\left(u_{\mathrm{pw}}-u_{M}\right)=\lambda_{\boldsymbol{h}} u_{\mathrm{pw}}$. This is equivalent to $u_{\mathrm{pw}}=(1+\delta) u_{M}$. The test function $\boldsymbol{v}_{\boldsymbol{h}}=\left(v_{M}, v_{M}\right) \in M(\mathcal{T}) \times M(\mathcal{T}) \subset \boldsymbol{V}_{\boldsymbol{h}}$ in (1.3) leads to

$$
a_{\mathrm{pw}}\left(u_{M}, v_{M}\right)=\lambda_{h}\left(u_{\mathrm{pw}}, v_{M}\right)_{L^{2}(\Omega)}=\lambda_{h}\left((1+\delta) u_{M}, v_{M}\right)_{L^{2}(\Omega)}
$$

Since this holds for all $v_{M} \in M(\mathcal{T}),\left(\lambda_{h}, u_{M}\right)$ is a solution to (2.13).

Proof of (b). Suppose $\left(\lambda_{h}, u_{M}\right) \in \mathbb{R}^{+} \times M(\mathcal{T})$ is a solution to (2.13) with $0<\lambda_{h}<\kappa_{2}^{-2} h_{\max }^{-4}$ and $\delta$ in (2.14). Then $u_{\mathrm{pw}}:=(1+\delta) u_{M}$ and $\boldsymbol{u}_{\boldsymbol{h}}:=\left(u_{\mathrm{pw}}, u_{M}\right)$ in (2.13) lead to

$$
\boldsymbol{a}_{\boldsymbol{h}}\left(\boldsymbol{u}_{\boldsymbol{h}}, \boldsymbol{v}_{\boldsymbol{h}}\right)=\lambda_{h}\left((1+\delta) u_{M}, v_{M}\right)_{L^{2}(\Omega)}+\left(\kappa_{2}^{-2} h_{\mathcal{T}}^{-4} \delta u_{M}, v_{\mathrm{pw}}-v_{M}\right)_{L^{2}(\Omega)}
$$

for all $\boldsymbol{v}_{\boldsymbol{h}}=\left(v_{\mathrm{pw}}, v_{M}\right) \in \boldsymbol{V}_{\boldsymbol{h}}$. Recall $\kappa_{2}^{-2} h_{\mathcal{T}}^{-4} \delta=\lambda_{h}(1+\delta)$ from (2.14) to verify

$$
\boldsymbol{a}_{\boldsymbol{h}}\left(\boldsymbol{u}_{\boldsymbol{h}}, \boldsymbol{v}_{\boldsymbol{h}}\right)=\lambda_{h}\left((1+\delta) u_{M}, v_{\mathrm{pw}}\right)_{L^{2}(\Omega)}=\lambda_{h} \boldsymbol{b}_{\boldsymbol{h}}\left(\boldsymbol{u}_{\boldsymbol{h}}, \boldsymbol{v}_{\boldsymbol{h}}\right)
$$

Hence $\left(\lambda_{h}, \boldsymbol{u}_{\boldsymbol{h}}\right)$ is an eigenpair of (1.3).
Proof of (c). The combination of (a)-(b) proves the equality of the solution sets for $\lambda_{h}$ in (1.3) resp. (2.13) in the open interval $\left(0, \kappa_{2}^{-2} h_{\max }^{-4}\right)$. It remains to prove equality of the multiplicities. Let $p$ resp. $q$ denote the multiplicities of the eigenvalue $\lambda_{h}$ in (1.3) resp. (2.13). The point is that $u_{M, 1}, \ldots, u_{M, \gamma} \in M(\mathcal{T})$ are linear independent if and only if $((1+$ ס) $\left.\left.\left.u_{M, 1}, u_{M, 1}\right)\right), \ldots,\left((1+\delta) u_{M, \gamma}, u_{M, \gamma}\right)\right)$ are linear independent in $\boldsymbol{V}_{\boldsymbol{h}}$ for the fixed $\delta$ from (2.14) for $\lambda_{h}$ with $0<\lambda_{h}<\kappa_{2}^{-2} h_{\max }^{-4}$. This and (a) resp. (b) imply $p \leqslant q$ resp. $q \leqslant p$. Consequently $p=q$ concludes the proof.

### 2.3.2 Comparison with Morley eigenvalues

For $k \leqslant N:=\operatorname{dim}(M(\mathcal{T}))$, the following Lemma 2.7 allows the placement of $\lambda_{h}(k)$ from (1.3) in $\operatorname{GLB}(k) \leqslant \lambda_{h}(k) \leqslant \lambda_{M}(k)$ between $\operatorname{GLB}(k)$ from (1.1) and $\lambda_{M}(k)$ from (2.12).

Lemma 2.7 (comparison). For any $k=1, \ldots, N$, the $k$-th algebraic eigenvalues $\lambda_{h}(k)$ of (2.13) and $\lambda_{M}(k)$ of (2.12) satisfy $\lambda_{h}(k) \leqslant \lambda_{M}(k)$. If $\lambda_{h}(k)$ satisfies $\lambda_{h}(k) \kappa_{2}^{2} h_{\max }^{4}<1$, then $\operatorname{GLB}(k) \leqslant \lambda_{h}(k)$ holds. For a uniform triangulation $\mathcal{T}$ with $h_{\max }=h_{\mathcal{T}}$ a.e. in $\Omega$ follows equality $\operatorname{GLB}(k)=\lambda_{h}(k)$.

Proof. The first result is a straightforward modification of [CZZ20, Thm. 6.2]. Since the Morley eigenfunctions are linearly independent, the pairs $\left(\phi_{M}(1), \phi_{M}(1)\right), \ldots,\left(\phi_{M}(k), \phi_{M}(k)\right)$ form a $k$-dimensional subspace of $\boldsymbol{V}_{\boldsymbol{h}}$. Hence the min-max principle proves the claim. The test functions $\left(v_{M}, v_{M}\right) \in M(\mathcal{T}) \times M(\mathcal{T}) \subset \boldsymbol{V}_{\boldsymbol{h}}$ and $\left(v_{\mathrm{pw}}, 0\right) \in \boldsymbol{V}_{\boldsymbol{h}}$ in (1.3) show for the first $k$ eigenpairs $\left(\lambda_{h}, \boldsymbol{u}_{\boldsymbol{h}}\right) \in \mathbb{R}_{+} \times \boldsymbol{V}_{\boldsymbol{h}}$ with $\boldsymbol{u}_{\boldsymbol{h}}=\left(u_{\mathrm{pw}}, u_{M}\right) \in P_{2}(\mathcal{T}) \times M(\mathcal{T})$ of (1.3), that $a_{\mathrm{pw}}\left(u_{M}, v_{M}\right)=\lambda_{h} b\left(u_{\mathrm{pw}}, v_{M}\right)$ for all $v_{M} \in M(\mathcal{T})$ and $u_{M}=\left(1-\lambda_{h} \kappa_{2}^{2} h_{\mathcal{T}}^{4}\right) u_{\mathrm{pw}}$. Hence the arguments for the second inequality are analogue to [CZZ20, Thm. 6.4] with $\varepsilon$ replaced by $\kappa_{2}^{2} h_{\max }^{4}$ and therefore further details are omitted. On a uniform mesh with $h_{\max }=h_{\mathcal{T}}$ a.e. in $\Omega$ the scaling on the right-hand side of (2.13) is constant, thus (2.13) and (2.12) are equivalent with $\lambda_{h}(k)=\lambda_{M}(k) /\left(1+\kappa_{2}^{2} \lambda_{M}(k) h_{\max }^{4}\right)=G L B(k)$.

Remark 2.8 (verification of the mesh-size condition) Lemma 2.7 and $\lambda_{M}(k)<\kappa_{2}^{-2} h_{\max }^{-4}$ guarantee that the discrete eigenvalue $\lambda_{h}(k)$ satisfies $\lambda_{h}(k)<\kappa_{2}^{-2} h_{\text {max }}^{-4}$. This is an a priori test sufficient for the applicability of Proposition 2.6.

### 2.3.3 An extra-stabilized Crouzeix-Raviart FEM

The arguments of this paper allow for an eigenvalue solver of the Dirichlet eigenvalue of the Laplacian with guaranteed lower eigenvalue bound. For the Laplace eigenvalue problem
$-\Delta u=\lambda u$ in $H_{0}^{1}(\Omega)$ an extra-stabilised Crouzeix-Raviart FEM comparable to (1.3) seeks $\left(\lambda_{h},\left(u_{\mathrm{pw}}, u_{C R}\right)\right) \in \mathbb{R}_{+} \times\left(P_{1}(\mathcal{T}) \times C R_{0}^{1}(\mathcal{T})\right) \backslash\{0\}$, such that

$$
\begin{equation*}
\left(\nabla_{\mathrm{pw}} u_{C R}, \nabla_{\mathrm{pw}} v_{C R}\right)_{L^{2}(\Omega)}+\kappa_{1}^{-2}\left(h_{\mathcal{T}}^{-2}\left(u_{\mathrm{pw}}-u_{C R}\right), v_{\mathrm{pw}}-v_{C R}\right)_{L^{2}(\Omega)}=\lambda_{h} b\left(u_{\mathrm{pw}}, v_{\mathrm{pw}}\right) \tag{2.15}
\end{equation*}
$$

for any $\left(v_{\mathrm{pw}}, v_{C R}\right) \in P_{1}(\mathcal{T}) \times C R_{0}^{1}(\mathcal{T})$. The eigenvalue problem (2.15) is for $n=2$ the lowest-order skeleton method in [CZZ20]; for $n \geqslant 3$ it is a completely different method. The standard interpolation operator (see e.g. [CG14b, CP20]) for the Crouzeix-Raviart FE $C R_{0}^{1}(\mathcal{T}) \subset P_{1}(\mathcal{T})[\mathrm{CR} 73]$ satisfies the conditions in Theorem 2.1.a-b (with the Hessian $D^{2}$ replaced by the gradient $\nabla, V=H^{2}$ replaced by $H^{1}, h_{T}^{2}$ by $h_{T}$, and $P_{2}(\mathcal{T})$ by $\left.P_{1}(\mathcal{T})\right)$. Hence the results analogue to those of Subsection 2.2-2.3 hold for the Dirichlet eigenvalues of the Laplacian and the discrete eigenpairs of (2.15). (A conforming companion with the properties in Theorem 3.1 (again $D^{2}$ replaced by $\nabla, V=H^{2}$ by $H^{1}, h_{\mathcal{T}}^{2}$ by $h_{\mathcal{T}}$, and $P_{2}(\mathcal{T})$ by $P_{1}(\mathcal{T})$ ) is designed in [CGS15, Prop. 2.3] for $n=2$; a generalization for $n \geqslant 3$ is straight-forward.)

## 3 Convergence rates in 3D

This section presents a conforming companion in 3D to apply the Babuška-Osborn convergence analysis [BO91] for the discrete eigenvalue problem (1.3) and the standard Morley eigenvalue problem (2.12). For the latter the paper [YLB16] for $n \geqslant 2$ follows [Ran79] for $n=2$ and utilizes the trace inequality for second order derivatives $\partial^{\alpha} u / \partial x_{\alpha}$ for $|\alpha|=2$ under the regularity assumption $u \in W^{3, p}(\Omega)$ for $4 / 3<p \leqslant 2$. Those terms arise in an integration by parts in the classical a priori error analysis of the Morley FEM. The present paper circumvents this by using the companion operator $J_{M}$ following [CGS13, Gal15a, CN21]. This allows results for a general $u \in H^{2+\sigma}(\Omega)$ even for small $\sigma$ with $0<\sigma \leqslant 1$.

### 3.1 Conforming companion

The conforming companion operator $J_{M}$ in this paper is seen as a right-inverse of the Morley interpolation operator $I_{M}: V \rightarrow M(\mathcal{T})$ from (2.3) with an additional $L^{2}$ orthogonality.

Theorem 3.1 (properties of $\boldsymbol{J}_{\boldsymbol{M}}$ ). There exists a constant $M_{2} \approx 1$ (that exclusively depends on $\mathbb{T}$ ) and a conforming companion $J_{M} v_{M} \in V:=H_{0}^{2}(\Omega)$ for any $v_{M} \in M(\mathcal{T})$ with
(a) $J_{M}$ is a right inverse to the interpolation $I_{M}$ in that $I_{M} \circ J_{M}=\operatorname{id}$ in $M(\mathcal{T})$,
(b) $\left\|h_{\mathcal{T}}^{-2}\left(1-J_{M}\right) v_{M}\right\|_{L^{2}(\Omega)}+\left\|\left(1-J_{M}\right) v_{M}\right\|_{\mathrm{pw}} \leqslant M_{2} \min _{v \in V}\left\|v_{M}-v\right\|_{\mathrm{pw}}$,
(c) the orthogonality $\left(1-J_{M}\right)(M(\mathcal{T})) \perp P_{2}(\mathcal{T})$ holds in $L^{2}(\Omega)$.

Outline of the proof. The design can follow the 2D discussions in [Gal15a, VZ19] in the spirit of [CGS15]: one subtle issue is the scaling of the nodal basis functions for the $W F$ FEM as a generalization of [Cia78, Thm. 6.1.3] to 3D. While the technical details of the proof are provided in the supplement, an outline of the design will follow here.
WF partition. Unlike the $H C T$ partition of each triangle in 3 subtriangles, the subdivision in the 3D WF finite element scheme [WF87, Sor09] depends on the triangulation $\mathcal{T} \in \mathbb{T}$. Each tetrahedron $T \in \mathcal{T}$ is divided into 12 sub-tetrahedra with respect to a careful selection of center points $c_{F}$ on each facet $F \in \mathcal{F}(T)$ of $T$ and $c_{T}$ inside the tetrahedron $T \in \mathcal{T}: c_{T}$ is the midpoint of the incircle of $T$ and $c_{F}$ is the intersection of $F \in \mathcal{F}(\Omega)$ with the straight line through $c_{T_{+}}$and $c_{T_{-}}$for $T_{ \pm} \in \mathcal{T}$ aligned to $F=\partial T_{+} \cap \partial T_{-}$, while $c_{F}:=\operatorname{mid}(F)$ is simply the
center of gravity for a triangle $F \in \mathcal{F}(\partial \Omega)$ on the boundary. Theorem A. 3 of the supplement guarantees the (uniform) shape-regularity of the resulting subtriangulation $\widehat{\mathcal{T}}$ for $\mathcal{T} \in \mathbb{T}$ and that the distance of each center point $c_{T}$ (resp. $c_{F}$ ) to the boundary $\partial T$ (resp. the relative boundary $\partial F$ ) is bounded from below by some global constant times $h_{T}$ (resp. $h_{F}$ ).
WF finite element. The 28 local degrees of freedom for any $K=\operatorname{conv}\left\{Q_{1}, Q_{2}, Q_{3}, Q_{4}\right\} \in \mathcal{T}$ are the evaluation of the function $f \in H^{2}(K)$ and its gradient $\nabla f$ at the vertices $Q_{1}, \ldots, Q_{4}$ of $K$ and the evaluation of the gradient $\tau_{E} \times \nabla f(\operatorname{mid}(E))$ at each edge midpoint $\operatorname{mid}(E)$ for $E \in \mathcal{E}(T)$ with unit tangent vector $\tau_{E}$. This determines $\nabla f(\operatorname{mid}(E))$ in the non-tangential directions $\nu_{E, 1}, \nu_{E, 2}$ with $\operatorname{span}\left\{\tau_{E}, \nu_{E, 1}, \nu_{E, 2}\right\}=\mathbb{R}^{3}$. Those 28 degrees of freedom define a finite element $\left(K, C^{1}(K) \cap P_{3}(\widehat{\mathcal{T}}(K)),\left\{L_{1}, \ldots, L_{28}\right\}\right)$ in the sense of Ciarlet. Since the explicit proof of this is not included in [WF87], Theorem A. 1 provides it in the supplement. The facet center point $c_{F}$ of an interior facet $F=\partial T_{+} \cap \partial T_{-} \in \mathcal{F}(\Omega)$ shared by the two neighbouring tetrahedra $T_{ \pm} \in \mathcal{T}$ belongs to the same straight line as their center points $c_{T_{+}}$and $c_{T_{-}}$and then [WF87] implies $C^{1}$ conformity of $W F(\mathcal{T}):=P_{3}(\widehat{\mathcal{T}}) \cap V$. Theorem A. 2 in the supplement provides a comprehensive proof, that is supposed to be readable without profound a priori knowledge of Bernstein polynomials [dB87] in multivariate $C^{1}$ splines.

Scaling of the WF basis functions. Let $\varphi_{z, 1}, \ldots \varphi_{z, 4}$ and $\varphi_{E, 1}, \varphi_{E, 2}$ for $z \in \mathcal{V}(\Omega)$ and $E \in \mathcal{E}(\Omega)$ denote the nodal $W F$ basis functions dual to the global degrees of freedom for $z \in \mathcal{V}, j=1,2,3$, $E \in \mathcal{E}$, and $\mu=1,2$,

$$
L_{z, 1} f:=f(z), \quad L_{z, j+1}:=\frac{\partial f}{\partial x_{j}}(z), \quad \text { and } \quad L_{E, \mu} f:=\frac{\partial f}{\partial \nu_{E, \mu}}(\operatorname{mid}(E))
$$

(such that $L_{z, j}\left(\varphi_{a, k}\right)=\delta_{z a} \delta_{j k}, L_{E, \mu}\left(\varphi_{F, \kappa}\right)=\delta_{E F} \delta_{\mu \kappa}$, and $L_{z, j}\left(\varphi_{E, \mu}\right)=0=L_{E, \mu}\left(\varphi_{z, j}\right)$ for any $a, z \in \mathcal{V}, E, F \in \mathcal{E}, j, k=1, \ldots, 4$, and $\mu, \kappa=1,2)$. Theorem A. 4 in the supplement generalizes a conclusion of [Cia78, Thm. 6.1.3] to $W F$ and asserts the expected scaling of the nodal basis functions. For $s=0,1,2$

$$
\begin{equation*}
h_{\ell}\left\|\varphi_{z, 1}\right\|_{H^{s}(\Omega)}+\left\|\varphi_{z, j+1}\right\|_{H^{s}(\Omega)}+\left\|\varphi_{E, \mu}\right\|_{H^{s}(\Omega)} \lesssim h_{\ell}^{5 / 2-s} \tag{3.1}
\end{equation*}
$$

holds with the volume $h_{\ell}^{3}:=\left|\operatorname{supp}\left(\varphi_{\ell}\right)\right|$ of the nodal patch $\overline{\omega(z)}:=\bigcup \mathcal{T}(z), \mathcal{T}(z):=\{T \in$ $\mathcal{T}: z \in \mathcal{V}(T)\}$ for $\varphi_{\ell}=\varphi_{z, 1}$ or $\varphi_{\ell}=\varphi_{z, j+1}$, and of the edge patch $\overline{\omega(E)}:=\bigcup \mathcal{T}(E)$, $\mathcal{T}(E):=\{T \in \mathcal{T}: E \in \mathcal{E}(T)\}$ for $\varphi_{\ell}=\varphi_{E, \mu}$. The point is that the constants in (3.1) are uniformly bounded in terms of the uniform shape-regularity of $\mathbb{T}$.
The $W F$ allows the four-step design of $J_{M} \equiv J_{4}$ below. Details of the proofs are provided in Supplement B.
Definition of $J_{1}$. The enrichment operator $J_{1}: M(\mathcal{T}) \rightarrow W F(\mathcal{T})$ with homogeneous boundary conditions is defined by averaging of degrees of freedom of $W F(\mathcal{T})$ The scaling of the $W F$ basis function (3.1) is a key argument in the proof of the local approximation property

$$
\begin{equation*}
h_{T}^{-4}\left\|v_{M}-J_{1} v_{M}\right\|_{L^{2}(T)}^{2} \lesssim \sum_{z \in \mathcal{V}(T)} \sum_{F \in \mathcal{F}(z)} h_{F}\left\|\left[D^{2} v_{M}\right]_{F} \times \nu_{F}\right\|_{L^{2}(F)}^{2} \tag{3.2}
\end{equation*}
$$

for any $T \in \mathcal{T} ; \mathcal{F}(z):=\{F \in \mathcal{F}: z \in \partial F\}$ denotes the set of faces with vertex $z \in \mathcal{V}$ in (3.2) and $\left[D^{2} v_{M}\right]_{F} \times \nu_{F}$ denotes the tangential components of the jump $\left[D^{2} v_{M}\right]_{F}$ across a side $F \in \mathcal{F}(z)$ with the row-wise cross product $\left[D^{2} v_{M}\right]_{F} \times \nu_{F}$ with the unit normal $\nu_{F} \in \mathbb{R}^{3}$.

The estimate (3.2) and the 2D arguments from [Gal15a, Prop. 2.3] modified with the Curl operator in 3D as in [CBJ02] lead to

$$
\left\|h_{\mathcal{T}}^{-2}\left(1-J_{1}\right) v_{M}\right\|_{L^{2}(\Omega)} \lesssim \min _{v \in V}\left\|v_{M}-v\right\|_{\mathrm{pw}}
$$

Definition of $J_{2}$. For each edge $E \in \mathcal{E}(\Omega)$ define below a function $\xi_{E} \in H_{0}^{2}(\widehat{\omega}(E)) \subset H_{0}^{2}(\Omega)$ with $f_{G} \xi_{E} \mathrm{~d} s=\delta_{G E}$ for all edges $G \in \mathcal{E}$, such that the support $\operatorname{supp}\left(\xi_{E}\right) \subset \widehat{\omega}(E)$ is contained in the edge patch $\widehat{\omega}(E):=\operatorname{int}(\bigcup \widehat{\mathcal{T}}(E))$ of $E$ in the $W F$ partition $\hat{\mathcal{T}}$. Then

$$
J_{2}\left(v_{M}\right):=J_{1} v_{M}+\sum_{E \in \mathcal{E}(\Omega)}\left(f_{E}\left(v_{M}-J_{1} v_{M}\right) \mathrm{d} s\right) \xi_{E} \in V
$$

The shape-regularity of $\hat{\mathcal{T}}$ (from Theorem A.3) allows the choice of a ball $B:=B\left(\operatorname{mid}(E), R_{E}\right)$ $\subset \widehat{\omega}(E)$ with midpoint $\operatorname{mid}(E)$ and radius $R_{E}$ such that $h_{T} \approx R_{E} \approx h_{E}$ in the definition of $\xi_{E} \in C^{1}\left(\mathbb{R}^{3}\right) \cap H_{0}^{2}(B)$ by

$$
\xi_{E}(x):=\frac{|E|}{R_{E}}\left(1-3 \frac{|y|^{2}}{R_{E}^{2}}+2 \frac{|y|^{3}}{R_{E}^{3}}\right) \quad \text { for } x \in \bar{B} \text { and } y:=x-\operatorname{mid}(E) .
$$

Definition of $J_{3}$. For each side $F \in \mathcal{F}(\Omega)$ define below a function $\zeta_{F} \in H_{0}^{2}(\omega(F)) \subset H_{0}^{2}(\Omega)$ with $f_{G} \nabla \zeta_{F} \cdot \nu_{G} \mathrm{~d} s=\delta_{G F}$ for all sides $G \in \mathcal{F}$ and support $\operatorname{supp}\left(\zeta_{F}\right) \subset \overline{\omega(F)}$ in the face patch $\omega(F):=\operatorname{int}\left(T_{+} \cup T_{-}\right)$of the neighbouring tetrahedra $T_{ \pm} \in \mathcal{T}(F)$ with $F=\partial T_{+} \cap \partial T_{-}$and with unit normal vectors of a fixed orientation $\nu_{F}=\left.\nu_{T_{+}}\right|_{F}=-\left.\nu_{T_{-}}\right|_{F}$ of $F$ in $\mathcal{T}$. Then

$$
J_{3}\left(v_{M}\right):=J_{2} v_{M}+\sum_{F \in \mathcal{F}(\Omega)}\left(f_{F} \nabla\left(v_{M}-J_{2} v_{M}\right) \cdot \nu_{F} \mathrm{~d} s\right) \zeta_{F} \in V .
$$

Suppose $F=\operatorname{conv}\left\{z_{1}, z_{2}, z_{3}\right\} \in \mathcal{F}\left(T_{ \pm}\right)$is the common face of $T_{ \pm}=\operatorname{conv}\left\{z_{1}, \ldots, z_{4}^{ \pm}\right\} \in \mathcal{T}$ opposite the vertex $z_{4}^{ \pm} \in \mathcal{V}\left(T_{ \pm}\right)$. Let $\lambda_{k}^{ \pm}$denote the barycentric coordinate in $T_{ \pm}$associated with the vertex $z_{k}^{ \pm} \in \mathcal{V}\left(T_{ \pm}\right)$for $k=1, \ldots, 4$. Then $\zeta_{F} \in P_{7}(\mathcal{T}) \cap C^{1}(\Omega) \cap H_{0}^{2}(\omega(F))$ reads

$$
\left.\zeta_{F}\right|_{T_{ \pm}}:= \pm \frac{7!}{2} \operatorname{dist}\left(z_{4}^{ \pm}, F\right)\left(\lambda_{1} \lambda_{2} \lambda_{3}\right)^{2} \lambda_{4}^{ \pm} \in P_{7}\left(T_{ \pm}\right)
$$

in $T_{ \pm} \in \mathcal{T}(F)$ (and vanishes outside the face patch $\omega(F)$ ). The integral mean corrections guarantee that $J_{3} v_{M}$ satisfies (a).
Definition of $J_{4}$. The correction $J_{4} v_{M} \in V$ is designed such that its $L^{2}$ projection $\Pi_{2}\left(J_{4} v_{M}\right)$ onto $P_{2}(\mathcal{T})$ coincides with $v_{M} \in M(\mathcal{T}) \subset P_{2}(\mathcal{T})$, i.e., $J_{4} \equiv J_{M}$ satisfies (c). For any $T \in \mathcal{T}$, recall the barycentric coordinate $\lambda_{z}$ associated with the vertex $z \in \mathcal{V}(T)$ in $T$, and define the scaled squared volume-bubble function $b_{T}:=4^{8} \prod_{z \in \mathcal{V}(T)} \lambda_{z}^{2} \in P_{8}(T) \cap H_{0}^{2}(T) \subset H_{0}^{2}(\Omega)$ with $\left\|b_{T}\right\|_{L^{\infty}(T)}=1$. Let $v_{T} \in P_{2}(T)$ denote the Riesz representation of the linear functional $w_{T} \mapsto$ $\int_{T}\left(v_{M}-J_{3} v_{M}\right) w_{T} \mathrm{~d} x$ in the Hilbert space $P_{2}(T)$ endowed with the weighted $L^{2}$ scalar product $\left(b_{T} \bullet \cdot \bullet\right)_{L^{2}(\Omega)}$, such that $\left(v_{M}-J_{3} v_{M}, w_{T}\right)_{L^{2}(T)}=\left(b_{T} v_{T}, w_{T}\right)_{L^{2}(T)}$ for all $w_{T} \in P_{2}(T)$. Set

$$
J_{4} v_{M}:=J_{3} v_{M}+\sum_{T \in \mathcal{T}} v_{T} b_{T} .
$$

Outline of the proof of $(a)-(c)$. Since $\left.\left.\left(v_{T} b_{T}\right)\right|_{\partial T} \equiv 0 \equiv\left(\nabla\left(v_{T} b_{T}\right)\right)\right|_{\partial T}$ vanishes along the boundary $\partial T$ of $T \in \mathcal{T}$ and $\left.\zeta_{F}\right|_{\partial F}=0$ vanishes along the boundary $\partial F$ of $F \in \mathcal{F}(\Omega), J_{M} \equiv J_{4}$ satisfies (a) and (c). The above correction functions satisfy $\left\|\xi_{E}\right\|_{L^{2}(T)} \approx h_{T}^{3 / 2},\left\|\zeta_{F}\right\|_{L^{2}(T)} \approx h_{T}^{5 / 2}$, and $\left\|v_{T} b_{T}\right\|_{L^{2}(T)} \lesssim\left\|v_{M}-J_{3} v_{M}\right\|_{L^{2}(T)}$ for any tetrahedron $T \in \mathcal{T}$ with edge $E \in \mathcal{E}(T)$ and face $F \in \mathcal{F}(T)$. This and a combination of inverse estimates [BS08], Cauchy-Schwarz, and discrete trace inequalities ensure $\left\|\left(1-J_{M}\right) v_{M}\right\|_{L^{2}(T)} \lesssim\left\|\left(1-J_{1}\right) v_{M}\right\|_{L^{2}(T)}$. Hence (b) follows from the local analysis of the averaging operator $J_{1}$ above. The details on the universal constant $M_{2} \approx 1$ are provided in Supplement B.

Corollary 3.2 (further properties). Any $w \in V$ and any $v_{M} \in M(\mathcal{T}), \mathcal{T} \in \mathbb{T}$, satisfy

$$
\text { (a) } \begin{aligned}
& b\left(v_{M}-J_{M} v_{M}, w\right)=b\left(v_{M}-J_{M} v_{M}, w-I_{M} w\right) \leqslant\left\|v_{M}-J_{M} v_{M}\right\|_{L^{2}(\Omega)}\left\|w-I_{M} w\right\|_{L^{2}(\Omega)} \\
& \leqslant h_{\max }^{4} \kappa_{2}^{2} M_{2} \min _{v \in V}\left\|v_{M}-v\right\|_{\mathrm{pw}} \min _{w_{M} \in M(\mathcal{T})}\left\|w-w_{M}\right\|_{\mathrm{pw}} ; \\
& \text { (b) } \begin{aligned}
a_{\mathrm{pw}}\left(v_{M}-J_{M} v_{M}, w\right) & =a_{\mathrm{pw}}\left(v_{M}-J_{M} v_{M}, w-I_{M} w\right) \leqslant\left\|v_{M}-J_{M} v_{M}\right\|_{\mathrm{pw}}\left\|w-I_{M} w\right\|_{\mathrm{pw}} \\
& \leqslant M_{2} \min _{v \in V}\left\|v-v_{M}\right\|_{\mathrm{pw}} \min _{w_{M} \in M(\mathcal{T})}\left\|w-w_{M}\right\|_{\mathrm{pw}} .
\end{aligned}
\end{aligned}
$$

Proof of (a). The identity follows from the orthogonality in Theorem 3.1.c and the CauchySchwarz inequality implies the first inequality. The first term is controlled by Theorem 3.1.b and the second term by Theorem 2.1 for the Morley interpolation ( $(\mathrm{b})$ for $\ell=2$ ).

Proof of (b). The identity follows from the orthogonality in Theorem 2.1.a, which also allows to bound the second term resulting from the Cauchy-Schwarz inequality. This and Theorem 3.1.b conclude the proof of (b).

Remark 3.3 (Guaranteed upper eigenvalue bounds) The companion operator $J_{M}$ can be employed in a postprocessing for guaranteed upper eigenvalue bounds as follows. Given $m \in \mathbb{N}$, let $\left(\lambda_{h}(j), \boldsymbol{u}_{h, j}\right)$ with $\boldsymbol{u}_{h, j}=\left(u_{\mathrm{pw}, j}, u_{M, j}\right) \in V_{h} \backslash\{0\}$ denote the $j$-th eigenpair of (1.3) (or alternatively $\left(\lambda_{M}(j), u_{M, j}\right)$ the $j$-th eigenpair of (2.12)). If $u_{M, 1}, \ldots, u_{M, m}$ are linearly independent, then $J_{M} u_{M, 1}, \ldots, J_{M} u_{M, m}$ are linear independent vectors in $V$ as well, because $I_{M} J_{M} u_{M, j}=u_{M, j}$ from Theorem 3.1.a. For the linear independence $u_{M, 1}, \ldots, u_{M, m}$ in (2.13) the mesh-size condition $\lambda_{h}(m) \kappa_{2}^{2} h_{\max }^{4}<1$ is sufficient according to Proposition 2.6 (in (2.12) the condition $m \leqslant \operatorname{dim}(M(\mathcal{T}))$ is sufficient). Then an $m \times m$ generalized algebraic eigenvalue problem with $A:=\left(a\left(J_{M} u_{M, j}, J_{M} u_{M, k}\right): j, k=1, \ldots, m\right)$ and $B:=\left(b\left(J_{M} u_{M, j}, J_{M} u_{M, k}\right):\right.$ $j, k=1, \ldots, m)$ leads to algebraic eigenvalues $\mu_{1} \leqslant \mu_{2} \leqslant \cdots \leqslant \mu_{m}$. The exact eigenvalue $\lambda_{j} \leqslant \mu_{j}$ of (1.2) has the guaranteed upper bound $\mu_{j}$ by the min-max principle [SF08, Bof10]. The same strategy applies to the $C R$-eigenvalue problem as well [CG14b]; cf. [LLX12] for an alternative post-processing.

### 3.2 Convergence analysis for the source problem

Recall $a(\bullet, \bullet):=\left(D^{2} \bullet, D^{2} \bullet\right)_{L^{2}(\Omega)}$ and its piecewise version $a_{\mathrm{pw}}(\bullet, \bullet):=\left(D_{\mathrm{pw}}^{2} \bullet, D_{\mathrm{pw}}^{2} \bullet\right)_{L^{2}(\Omega)}$. Given $f \in L^{2}(\Omega)$, let $u \in V \equiv H_{0}^{2}(\Omega)$ solve

$$
\begin{equation*}
a(u, v)=(f, v)_{L^{2}(\Omega)} \quad \text { for all } v \in V . \tag{3.3}
\end{equation*}
$$

Let $u_{M} \in M(\mathcal{T})$ denote the discrete solution to the Morley source problem

$$
\begin{equation*}
a_{\mathrm{pw}}\left(u_{M}, v_{M}\right)=\left(f, v_{M}\right)_{L^{2}(\Omega)} \quad \text { for all } v_{M} \in M(\mathcal{T}) . \tag{3.4}
\end{equation*}
$$

Given the $L^{2}$ projection $\Pi_{2}$ onto $P_{2}(\mathcal{T})$, the second order oscillation of $f \in L^{2}(\Omega)$ reads $\operatorname{osc}_{2}(f, \mathcal{T}):=\left\|h_{\mathcal{T}}^{2}\left(1-\Pi_{2}\right) f\right\|_{L^{2}(\Omega)}$. Recall $0<\sigma \leqslant 1$ from (1.5).

Lemma 3.4 (discrete error estimate in $\boldsymbol{M}(\mathcal{T})$ ). There exist constants $C_{1}, C_{2} \approx 1$ (that exclusively depend on $\mathbb{T}$ ) such that given any $f \in L^{2}(\Omega)$, the exact solution $u \in H^{2+\sigma}(\Omega) \cap V$ to (3.3) and the discrete solution $u_{M} \in M(\mathcal{T})$ to (3.4) for $\mathcal{T} \in \mathbb{T}$ with maximal mesh-size $h_{\max }$ satisfy (a)-(b).
(a) If $u \in H^{2+s}(\Omega)$ for some $s$ with $\sigma \leqslant s \leqslant 1$, then

$$
\left\|u-u_{M}\right\|_{\mathrm{pw}}+h_{\max }^{-\sigma}\left\|u-u_{M}\right\|_{L^{2}(\Omega)} \leqslant C_{1}\left(h_{\max }^{s}\|u\|_{H^{2+s}(\Omega)}+\operatorname{osc}_{2}(f, \mathcal{T})\right) .
$$

(b) Given any eigenvalue $\lambda$ of (1.2) with eigenspace $E(\lambda) \subset H^{2+t}(\Omega) \cap V$ for some $t$ with $\sigma \leqslant t \leqslant 1$, suppose $f, g \in E(\lambda)$. Then

$$
\left|\left(u-u_{M}, g\right)_{L^{2}(\Omega)}\right| \leqslant C_{2}\left(\lambda^{-1}+\kappa_{2} h_{\max }^{4}\right)^{2} h_{\max }^{2 t}\|f\|_{H^{2+t}(\Omega)}\|g\|_{H^{2+t}(\Omega)}
$$

Proof of (a). Theorem 2.1 shows that the interpolation operator $I_{M}$ satisfies the utilized properties [Gal15a, Eqn. (2.3),(2.5)]. The 3D companion operator $J_{M}$ in Theorem 3.1 satisfies in particular the orthogonalities $\Pi_{0}\left(v_{M}-J_{M} v_{M}\right)=0$ and $\Pi_{0}\left(D_{\mathrm{pw}}^{2}\left(v_{M}-J v_{M}\right)\right)=0$ for all $v_{M} \in M(\mathcal{T})$ and the approximation property [Gal15a, Eqn. (2.7),(2.8)]. Hence the arguments in [Gal15a, Prop. 2.9-2.10] apply verbatim to $n=3$ and further details are omitted.

Proof of (b). Since $u \in V$ solves (3.3) with the right-hand side $f \in E(\lambda), u=f / \lambda \in H^{2+t}(\Omega) \cap$ $V$. Remark 2.4 and Corollary 2.2.a prove $\operatorname{osc}_{2}(f, \mathcal{T}) \leqslant h_{\max }^{4+t} / \pi^{2+t}\|f\|_{H^{2+t}(\Omega)}$. This and (a) show

$$
\begin{equation*}
\left\|u-u_{M}\right\|_{\mathrm{pw}} \leqslant C_{1} h_{\max }^{t}\left(1 / \lambda+h_{\max }^{4} / \pi^{2+t}\right)\|f\|_{H^{2+t}(\Omega)} . \tag{3.5}
\end{equation*}
$$

Since $g \in E(\lambda)$ is an eigenvector in (1.2), $\lambda\left(u-J_{M} u_{M}, g\right)_{L^{2}(\Omega)}=a\left(u-J_{M} u_{M}, g\right)$. Corollary 2.2.b implies $a_{\mathrm{pw}}\left(u_{M}, g\right)=a_{\mathrm{pw}}\left(u_{M}, I_{M} g\right)$. This shows the first equality in

$$
\begin{aligned}
\lambda\left(u-J_{M} u_{M}, g\right)_{L^{2}(\Omega)} & =a(u, g)-a_{\mathrm{pw}}\left(u_{M}, I_{M} g\right)+a_{\mathrm{pw}}\left(u_{M}-J_{M} u_{M}, g\right) \\
& =\left(f, g-I_{M} g\right)_{L^{2}(\Omega)}+a_{\mathrm{pw}}\left(u_{M}-J_{M} u_{M}, g\right) .
\end{aligned}
$$

The second equality follows because $u$ solves (3.3) and $u_{M} \in M(\mathcal{T})$ solves (3.4) with righthand side $f$. The term $\left(f, g-I_{M} g\right)_{L^{2}(\Omega)}=\left(f, J_{M} I_{M} g-I_{M} g\right)_{L^{2}(\Omega)}+\left(f, g-J_{M} I_{M} g\right)_{L^{2}(\Omega)}$ is split into two. Corollary 3.2.a controls the first contribution

$$
\left(f, J_{M} I_{M} g-I_{M} g\right)_{L^{2}(\Omega)} \leqslant M_{2} h_{\max }^{4} \kappa_{2}^{2}\left\|f-I_{M} f\right\|_{\mathrm{pw}}\left\|g-I_{M} g\right\|_{\mathrm{pw}}
$$

Corollary 3.2.b ensures

$$
a_{\mathrm{pw}}\left(u_{M}-J_{M} u_{M}, g\right) \leqslant M_{2}\left\|u-u_{M}\right\|_{\mathrm{pw}}\left\|g-I_{M} g\right\|_{\mathrm{pw}} .
$$

Since $f \in E(\lambda)$ is an eigenvector in (1.2), $\lambda\left(f, g-J_{M} I_{M} g\right)_{L^{2}(\Omega)}=a\left(f, g-J_{M} I_{M} g\right)$. The right-inverse property Theorem 3.1.a and the orthogonality in Theorem 2.1.a show $a(f, g-$ $\left.J_{M} I_{M} g\right)=a_{\mathrm{pw}}\left(f-I_{M} f, g-J_{M} I_{M} g\right)$. A triangle inequality and Theorem 3.1.b ensure $\| g-$
$J_{M} I_{M} g\left\|_{\mathrm{pw}} \leqslant\right\| g-I_{M} g\left\|_{\mathrm{pw}}+\right\| g-J_{M} I_{M} g\left\|_{\mathrm{pw}} \leqslant\left(1+M_{2}\right)\right\| g-I_{M} g \|_{\mathrm{pw}}$. This and a CauchySchwarz inequality verify
$\lambda\left(f, g-J_{M} I_{M} g\right)_{L^{2}(\Omega)} \leqslant\left\|f-I_{M} f\right\|_{\mathrm{pw}}\left\|g-J_{M} I_{M} g\right\|_{\mathrm{pw}} \leqslant\left(1+M_{2}\right)\left\|f-I_{M} f\right\|_{\mathrm{pw}}\left\|g-I_{M} g\right\|_{\mathrm{pw}}$.
The combination of the last four displayed estimates reads
$\lambda\left(u-J_{M} u_{M}, g\right)_{L^{2}(\Omega)} \leqslant\left\|g-I_{M} g\right\|_{\mathrm{pw}} M_{2}\left(\left(\frac{1+M_{2}^{-1}}{\lambda}+h_{\max }^{4} \kappa_{2}^{2}\right)\left\|f-I_{M} f\right\|_{\mathrm{pw}}+\left\|u-u_{M}\right\|_{\mathrm{pw}}\right)$.
This, Corollary 2.2.a, and (3.5) verify that
$\lambda\left(u-J_{M} u_{M}, g\right)_{L^{2}(\Omega)} \leqslant h_{\max }^{2 t}\|f\|_{H^{2+t}(\Omega)}\|g\|_{H^{2+t}(\Omega)} \frac{M_{2}}{\pi^{t}}\left(\frac{1+M_{2}^{-1}}{\lambda \pi^{t}}+\frac{h_{\max }^{4} \kappa_{2}}{\pi^{t}}+C_{1}\left(\frac{1}{\lambda}+\frac{h_{\max }^{4}}{\pi^{2+t}}\right)\right)$.
For the term $\left(J_{M} u_{M}-u_{M}, g\right)_{L^{2}(\Omega)}$, Corollary 3.2.a followed by Corollary 2.2.a and (3.5) show

$$
\begin{aligned}
\left(J_{M} u_{M}-u_{M}, g\right)_{L^{2}(\Omega)} & \leqslant M_{2} \kappa_{2} h_{\max }^{4}\left\|g-I_{M} g\right\|_{\mathrm{pw}}\left\|u-u_{M}\right\|_{\mathrm{pw}} \\
& \leqslant h_{\max }^{2 t}\|f\|_{H^{2+t}(\Omega)}\|g\|_{H^{2+t}(\Omega)} \frac{C_{1} M_{2} \kappa_{2} h_{\max }^{4}}{\pi^{t}}\left(\frac{1}{\lambda}+\frac{h_{\max }^{4}}{\pi^{2+t}}\right) .
\end{aligned}
$$

The combination of the last two displayed inequalities proves that $\left(u-u_{M}, g\right)_{L^{2}(\Omega)}=\left(J_{M} u_{M}-\right.$ $\left.u_{M}, g\right)_{L^{2}(\Omega)}+\left(u-J_{M} u_{M}, g\right)_{L^{2}(\Omega)} \lesssim h_{\text {max }}^{2 t}\|f\|_{H^{2+t}(\Omega)}\|g\|_{H^{2+t}(\Omega)}$. The bookkeeping of the multiplicative constants concludes the proof.

Given any right-hand side $f \in L^{2}(\Omega)$, let $\boldsymbol{u}_{\boldsymbol{h}}=\left(u_{\mathrm{pw}}, u_{M}\right) \in \boldsymbol{V}_{\boldsymbol{h}}$ denote the discrete solution to the extra-stabilised source problem

$$
\begin{equation*}
\boldsymbol{a}_{\boldsymbol{h}}\left(\boldsymbol{u}_{\boldsymbol{h}}, \boldsymbol{v}_{\boldsymbol{h}}\right)=\left(f, v_{\mathrm{pw}}\right)_{L^{2}(\Omega)} \text { for all } \boldsymbol{v}_{\boldsymbol{h}}=\left(v_{\mathrm{pw}}, v_{M}\right) \in \boldsymbol{V}_{\boldsymbol{h}} . \tag{3.6}
\end{equation*}
$$

The analysis of (3.6) reduces to that of Lemma 3.4 plus perturbation terms.
Lemma 3.5 (discrete error estimate in $\boldsymbol{V}_{\boldsymbol{h}}$ ). There exists a constant $C_{\mathrm{pw}}>0$ (that exclusively depends on $\mathbb{T}$ ), such that given any $f \in L^{2}(\Omega)$, the exact solution $u \in H^{2+\sigma}(\Omega) \cap V$ to (3.3) and the discrete solution $\boldsymbol{u}_{\boldsymbol{h}}=\left(u_{\mathrm{pw}}, u_{M}\right) \in \boldsymbol{V}_{\boldsymbol{h}}$ to (3.6) for $\mathcal{T} \in \mathbb{T}$ with maximal mesh-size $h_{\max }$ satisfy (a)-(b).
(a) If $u \in H^{2+s}(\Omega)$ for some $s$ with $\sigma \leqslant s \leqslant 1$, then

$$
\left\|u-u_{\mathrm{pw}}\right\|_{\mathrm{pw}}+h_{\max }^{-\sigma}\left\|u-u_{\mathrm{pw}}\right\|_{L^{2}(\Omega)} \leqslant C_{\mathrm{pw}}\left(h_{\max }^{s}\|u\|_{H^{2+s}(\Omega)}+\operatorname{osc}_{2}(f, \mathcal{T})\right) .
$$

(b) Given any eigenvalue $\lambda$ of (1.2) with eigenspace $E(\lambda) \subset H^{2+t}(\Omega) \cap V$ for some $t$ with $\sigma \leqslant t \leqslant 1$, suppose $f, g \in E(\lambda)$ and $C_{2}$ from Lemma 3.4. Then

$$
\left|\left(u-u_{\mathrm{pw}}, g\right)_{L^{2}(\Omega)}\right| \leqslant\left(C_{2}\left(\lambda^{-1}+\kappa_{2} h_{\max }^{4}\right)^{2}+\kappa_{2}^{2} h_{\max }^{2}\right) h_{\max }^{2 t}\|f\|_{H^{2+t}(\Omega)}\|g\|_{H^{2+t}(\Omega)} .
$$

Proof of (a). For $v_{\mathrm{pw}} \in P_{2}(\mathcal{T})$, the test-function $\left(v_{\mathrm{pw}}, 0\right) \in \boldsymbol{V}_{\boldsymbol{h}}$ in (3.6) leads to $\kappa_{2}^{-2} h_{\mathcal{T}}^{-4}\left(u_{\mathrm{pw}}-\right.$ $\left.u_{M}\right)=\Pi_{2} f$. Thus $u_{\mathrm{pw}}=u_{M}+\kappa_{2}^{2} h_{\mathcal{T}}^{4} \Pi_{2} f$ and a triangle inequality shows

$$
\left\|u-u_{\mathrm{pw}}\right\|_{\mathrm{pw}}+h_{\max }^{-\sigma}\left\|u-u_{\mathrm{pw}}\right\|_{L^{2}(\Omega)}
$$

$$
\begin{equation*}
\leqslant\left(\left\|u-u_{M}\right\|_{\mathrm{pw}}+h_{\max }^{-\sigma}\left\|u-u_{M}\right\|_{L^{2}(\Omega)}\right)+\kappa_{2}^{2}\left(\left\|h_{\mathcal{T}}^{4} \Pi_{2} f\right\|_{\mathrm{pw}}+h_{\max }^{-\sigma}\left\|h_{\mathcal{T}}^{4} \Pi_{2} f\right\|_{L^{2}(\Omega)}\right) . \tag{3.7}
\end{equation*}
$$

The test-function $\left(v_{M}, v_{M}\right) \in M(\mathcal{T}) \times M(\mathcal{T}) \subset \boldsymbol{V}_{\boldsymbol{h}}$ shows $a_{\mathrm{pw}}\left(u_{M}, v_{M}\right)=\left(f, v_{M}\right)_{L^{2}(\Omega)}$. In other words, the solution component $u_{M} \in M(\mathcal{T})$ solves the Morley source problem (3.4). Lemma 3.4.a controls the first term in (3.7),

$$
\left\|u-u_{M}\right\|_{\mathrm{pw}}+h_{\max }^{-\sigma}\left\|u-u_{M}\right\|_{L^{2}(\Omega)} \leqslant C_{1}\left(h_{\max }^{s}\|u\|_{H^{2+s}(\Omega)}+\operatorname{osc}_{2}(f, \mathcal{T})\right) .
$$

An inverse estimate for $P_{2}(\mathcal{T})$ with constant $c_{\text {inv }}>0$ and the boundedness of $\Pi_{2}$ show

$$
\left\|h_{\mathcal{T}}^{4} \Pi_{2} f\right\|_{\mathrm{pw}}+h_{\max }^{-\sigma}\left\|h_{\mathcal{T}}^{4} \Pi_{2} f\right\|_{L^{2}(\Omega)} \leqslant\left(c_{\mathrm{inv}}+h_{\max }^{2-\sigma}\right)\left\|h_{\mathcal{T}}^{2} \Pi_{2} f\right\|_{L^{2}(\Omega)} \leqslant\left(c_{\mathrm{inv}}+h_{\max }^{2-\sigma}\right)\left\|h_{\mathcal{T}}^{2} f\right\|_{L^{2}(\Omega)} .
$$

The efficiency estimate $\left\|h_{\mathcal{T}}^{2} f\right\|_{L^{2}(\Omega)} \lesssim\left\|u-u_{M}\right\|_{\text {pw }}+\operatorname{osc}_{2}(f, \mathcal{T})$ follows from the bubblefunction methodology due to [Ver13], cf. [BdVNS07, Thm. 2]. The combination with Lemma 3.4.a shows

$$
\left\|h_{\mathcal{T}}^{2} f\right\|_{L^{2}(\Omega)} \lesssim h_{\max }^{s}\|u\|_{H^{2+s}(\Omega)}+\operatorname{osc}_{2}(f, \mathcal{T})
$$

The last two displayed inequalities bound the second term in (3.7) and that concludes the proof of (a).

Proof of (b). The substitution of $u_{\mathrm{pw}}=u_{M}+\kappa_{2}^{2} h_{\mathcal{T}}^{4} \Pi_{2} f$ from part (a) leads to

$$
\left(u-u_{\mathrm{pw}}, g\right)_{L^{2}(\Omega)}=\left(u-u_{M}, g\right)_{L^{2}(\Omega)}-\kappa_{2}^{2}\left(h_{\mathcal{T}}^{4} \Pi_{2} f, g\right)_{L^{2}(\Omega)} .
$$

Since $u_{M}$ solves the Morley source problem (3.4), Lemma 3.4.b controls $\left(u-u_{M}, g\right)_{L^{2}(\Omega)}$. This and $\left(\Pi_{2} f, g\right)_{L^{2}(\Omega)} \leqslant\|f\|_{L^{2}(\Omega)}\|g\|_{L^{2}(\Omega)}$ conclude the proof.

### 3.3 Convergence rates for the eigenvalue problem

The preparations in Subsection 3.1-3.2 allow the proof of the optimal a priori convergence rates in Theorem 1.2 with fundamental arguments from [BO91].

Proof of Theorem 1.2. Given any right-hand side $f \in L^{2}(\Omega)$ let $S(f):=u \in V$ denote the continuous solution to (3.3) and let $S_{h}(f):=u_{\mathrm{pw}} \in P_{2}(\mathcal{T})$ denote the first component of the solution $\boldsymbol{u}_{\boldsymbol{h}}=\left(u_{\mathrm{pw}}, u_{M}\right) \in \boldsymbol{V}_{\boldsymbol{h}}$ to (3.4). This defines solution operators $S: L^{2}(\Omega) \rightarrow L^{2}(\Omega)$ and $S_{h}: L^{2}(\Omega) \rightarrow L^{2}(\Omega)$. Lemma 3.5.a implies the convergence $S_{h} \rightarrow S$ in the operator norm of $\mathcal{L}\left(L^{2}(\Omega)\right)$ as $h_{\text {max }} \rightarrow 0$. Suppose $(\lambda, \phi) \in \mathbb{R}^{+} \times V$ denotes an eigenpair of (1.2) and $\left(\lambda_{h}, \boldsymbol{u}_{\boldsymbol{h}}\right) \in \mathbb{R}^{+} \times \boldsymbol{V}_{\boldsymbol{h}}$ with $\boldsymbol{u}_{\boldsymbol{h}}=\left(u_{\mathrm{pw}}, u_{M}\right)$ denotes an eigenpair of (1.3), then (1/ג, $\phi$ ) is an eigenpair of $S$ and $\left(1 / \lambda_{h}, u_{\mathrm{pw}}\right)$ is an eigenpair of $S_{h}$. Hence, the Babuška-Osborn theory [BO91] (see also [Bof10, Sec. 9] or [SZ17, Sec. 1.4.2]) implies for any non-zero eigenvalue $1 / \lambda$ of $S$ with eigenspace $E(\lambda)=\operatorname{ker}\left(\lambda^{-1}-S\right)$ of dimension $\mu=\operatorname{dim}(E(\lambda)) \in \mathbb{N}$, that there exist exactly $\mu$ eigenvalues $1 / \lambda_{h, 1}, \ldots, 1 / \lambda_{h, \mu}$ of $S_{h}$, which converge to $1 / \lambda$ as $h_{\max } \rightarrow 0$. The error estimates for the selfadjoint operator $S$ in [BO91, Rem. 7.5] read (with a generic constant which depends on $\lambda)\left\|u-u_{\mathrm{pw}, k}\right\|_{L^{2}(\Omega)} \lesssim\left\|\left.\left(S-S_{h}\right)\right|_{E(\lambda)}\right\|_{\mathcal{L}\left(E(\lambda) ; L^{2}(\Omega)\right)}$ and

$$
\begin{aligned}
\max _{1 \leqslant k \leqslant \mu}\left|\lambda-\lambda_{h, k}\right| & \lesssim \max _{1 \leqslant k \leqslant \mu}\left|\lambda^{-1}-\lambda_{h, k} k^{-1}\right| \\
& \lesssim \sup _{\phi, \psi \in E(\lambda) \backslash\{0\}} \frac{\left|\left(\left(S-S_{h}\right) \phi, \psi\right)_{L^{2}(\Omega)}\right|}{\|\phi\|_{L^{2}(\Omega)}\|\psi\|_{L^{2}(\Omega)}}+\left\|\left.\left(S-S_{h}\right)\right|_{E(\lambda)}\right\|_{\mathcal{L}\left(E(\lambda) ; L^{2}(\Omega)\right)}^{2}
\end{aligned}
$$

For the finite-dimensional eigenspace $E(\lambda) \subset H^{2+t}(\Omega)$ there exists $C_{t}:=\sup _{\phi \in E(\lambda)} \frac{\|\phi\|_{H^{2+t}(\Omega)}}{\|\phi\|_{L^{2}(\Omega)}}$ $<\infty$ so that $\|\phi\|_{H^{2+t}(\Omega)} \leqslant C_{t}\|\phi\|_{L^{2}(\Omega)}$ for all $\phi \in E(\lambda)$. The combination of Lemma 3.5.a with Remark 2.4 and Corollary 2.2.a shows (as an analog to (3.5)) that

$$
\left\|\left.\left(S-S_{h}\right)\right|_{E(\lambda)}\right\|_{\mathcal{L}\left(E(\lambda) ; L^{2}(\Omega)\right)} \leqslant h_{\max }^{t+\sigma} C_{t} C_{\mathrm{pw}}\left(1 / \lambda+h_{\max }^{4} / \pi^{2+t}\right)
$$

Lemma 3.5.b bounds the remaining term

$$
\sup _{\phi, \psi \in E(\lambda) \backslash\{0\}} \frac{\left|\left(\left(S-S_{h}\right) \phi, \psi\right)_{L^{2}(\Omega)}\right|}{\|\phi\|_{L^{2}(\Omega)}\|\psi\|_{L^{2}(\Omega)}} \leqslant h_{\max }^{2 t} C_{t}^{2}\left(C_{2}\left(\lambda^{-1}+\kappa_{2} h_{\max }^{4}\right)^{2}+\kappa_{2}^{2} h_{\max }^{2}\right) .
$$

This concludes the proof.
Unlike [Ran79, YLB16], the following Theorem 3.6 specifies the convergence rates for (2.12) directly in terms of $\sigma=\min \left\{1, \sigma_{\text {reg }}\right\}$ for the index of elliptic regularity $\sigma_{\text {reg }}$ from (1.5) and the Sobolev regularity $t$ of $E(\lambda)$.

Theorem 3.6 (a priori convergence for (2.12)). Given a non-zero eigenvalue $\lambda$ of (1.2) of multiplicity $\mu$, suppose that $E(\lambda) \subset H^{2+t}(\Omega) \cap V$ holds for some $t$ with $\sigma \leqslant t \leqslant 1$. Then there exist $\delta, C>0$ such that any $\mathcal{T} \in \mathbb{T}(\delta)$ and the discrete space $M(\mathcal{T})$ lead in (2.12) to exactly $\mu$ algebraic eigenvalues $\lambda_{M, 1}, \ldots, \lambda_{M, \mu}$ that converge to $\lambda$ as $h_{\max } \rightarrow 0$. Let $E_{M}:=\operatorname{span}\left\{u_{M} \in E_{M}\left(\lambda_{M, k}\right): k=1, \ldots, \mu\right\}$ denote the union of the discrete eigenspaces $E_{M}\left(\lambda_{M, k}\right) \subset M(\mathcal{T})$ for $\lambda_{M, 1}, \ldots, \lambda_{M, \mu}$. Then the convergence results in Theorem 1.2 hold with $\lambda_{h, k}$ replaced by $\lambda_{M, k}$, and $\boldsymbol{u}_{\boldsymbol{h}}=\left(u_{\mathrm{pw}}, u_{M}\right) \in E_{h}$ with $\left\|u_{\mathrm{pw}}\right\|_{L^{2}(\Omega)}=1$ replaced by $\phi_{M} \in E_{M}$ with $\left\|\phi_{M}\right\|_{L^{2}(\Omega)}=1$.

Proof. Define the solution operator $S_{M}: L^{2}(\Omega) \rightarrow L^{2}(\Omega)$ with $S_{M}(f):=u_{M} \in M(\mathcal{T})$ the Morley finite element solution of (3.4) for right-hand side $f \in L^{2}(\Omega)$. The proof is verbatim to the proof of Theorem 1.2 with the results in Lemma 3.4 instead of Lemma 3.5 and so the details are omitted here.

## 4 Numerical experiments in 2D

This section presents numerical evidence for the superiority of the new GLB in Theorem 1.1 over (1.1) and the asymptotic convergence rates from Theorem 1.2 in 2D.


Figure 4.1: Initial triangulation $\mathcal{T}_{0}$ of dumbbell-slit (a), L-shaped (b), and four-slit domain (c).

### 4.1 Implementation

The implementation in this paper is realized in MATLAB based on the data structure and assembling from [CB18, Sec. 7.8]. Fig. 4.1 displays the initial triangulations $\mathcal{T}_{0}$ for the numerical experiments below. The $k$-th eigenpair $\left(\lambda_{h}(k), \boldsymbol{u}_{\boldsymbol{h}}(k)\right) \in \mathbb{R}^{+} \times \boldsymbol{V}_{\boldsymbol{h}}$ of (1.3) with $\boldsymbol{u}_{\boldsymbol{h}}(k)=\left(u_{\mathrm{pw}}, u_{M}(k)\right) \in P_{2}(\mathcal{T}) \times M(\mathcal{T})$ and for comparison the post-processed Morley bound $\operatorname{GLB}(k)$ in (1.1) from [CG14b] are computed with the MATLAB routine eigs exactly; the termination and round-off errors are small and neglected for simplicity. Under the condition $\kappa_{2}^{2} h_{\max }^{4} \lambda_{h}(k) \leqslant 1$, Theorem 1.1 guarantees $\lambda_{h}(k) \leqslant \lambda_{k}$ for the $k$-th eigenvalue $\lambda_{k}$ of (1.2). Otherwise (if $1<\kappa_{2}^{2} h_{\max }^{4} \lambda_{h}(k)$ on a coarser mesh) the value $\lambda_{h}(k)$ is set zero, but the point is that this never occurs in all the examples displayed in this paper. The adaptive algorithm [Dör96, MNS02, CFPP14, CR17] is based on the refinement indicator $\eta(T)$ defined in (4.1) below for any triangle $T \in \mathcal{T}$. Given the discrete solution $\left(\lambda_{h}, \boldsymbol{u}_{\boldsymbol{h}}\right) \in \mathbb{R}^{+} \times \boldsymbol{V}_{\boldsymbol{h}}$ of (1.3) of number $k, \lambda_{h}:=\lambda_{h}(k)$, the local contribution $\eta^{2}(T)=(\eta(T))^{2}$ for any $T \in \mathcal{T}$ with area $|T|$ and set of edges $\mathcal{F}(T)$ solely depends on the Morley component $u_{M} \in M(\mathcal{T})$ of $\boldsymbol{u}_{\boldsymbol{h}}=\left(u_{\mathrm{pw}}, u_{M}\right) \in \boldsymbol{V}_{\boldsymbol{h}}$ and reads

$$
\begin{equation*}
\eta^{2}(T)=|T|^{2}\left\|\lambda_{h} u_{M}\right\|_{L^{2}(T)}^{2}+|T|^{1 / 2} \sum_{F \in \mathcal{F}(T)}\left\|\left[D^{2} u_{M}\right]_{F} \times \nu_{F}\right\|_{L^{2}(F)}^{2} \tag{4.1}
\end{equation*}
$$

with the tangential components $\left[D^{2} v\right]_{F} \times \nu_{F}$ of the jump $\left[D^{2} v\right]_{F}$ along any edge $F \in \mathcal{F}$ and the (piecewise) Hessian $D^{2}$. The respective convergence history plots in Fig. 1.1 and Fig. 4.2 display the difference $\lambda_{k}-\lambda_{h}(k)$ and $\lambda_{k}-\operatorname{GLB}(k)$ of the exact eigenvalue $\lambda_{k}$ and guaranteed lower bounds $\lambda_{h}(k)$ and $\operatorname{GLB}(k)$ for uniform red-mesh-refinement $\theta=1$ (solid line and filled markers) and adaptive mesh-refinement with a bulk parameter $\theta=0.5$ in the Dörfler marking algorithm and newest vertex bisection (dashed line and striped markers) plotted against the number of triangles $|\mathcal{T}|$. The computational bound $\kappa_{2}=0.07353$ from [LSL19] improves the analytical bound from [CG14a] and the effect is investigated in Fig. 4.2.a with a comparison between the bounds computed with $\kappa_{2}=0.07353$ from [LSL19] (line color orange/blue) and $\kappa_{2}=0.25746$ from [CG14a] (line color red/green). On uniform meshes GLB $(k)$ (line color blue) and $\lambda_{h}(k)$ (line color orange) coincide by Lemma 2.7 and are visible in orange only in Fig. 1.1 in the introduction and in Fig. 4.2 below.

### 4.2 Dumbbell-slit domain

The principal and fourth eigenvalue $\lambda_{1}=80.93261350$ and $\lambda_{4}=386.80177939$ on the nonconvex dumbbell domain with a slit $\Omega:=(-1,1) \times(-1,5) \backslash([0,1) \times\{0\} \cup[1,3] \times[-0.75,1])$ of Fig. 4.1.a are approximated with Bogner-Fox-Schmidt and Aitken extrapolation as in [CG14a]. Fig. 1.1 has been discussed in the introduction as an example where uniform mesh-refining leads to a better GLB from (1.1) than adaptive refinement. The complex geometry suggests a large computational pre-asymptotic regime, but the new method converges systematically even for course triangulations. The example provides striking numerical evidence for the superiority of the adaptive version of the extra-stabilized Morley eigensolver.

### 4.3 L-shaped domain

The principal eigenvalue $\lambda_{1}=418.97504246688220$ on the non-convex L-shaped domain $\Omega:=$ $(-1,1)^{2} \backslash[0,1) \times(-1,0]$ of Fig. 4.1.b is approximated in [CG14a]. The associated eigenfunction in $H_{0}^{2}(\Omega) \backslash H^{3}(\Omega)$ results in the reduced empirical convergence rate 0.66 for uniform meshrefinement in Fig. 4.2.a. The adaptive mesh-refinement with (4.1) allows to recover the
optimal convergence rate one (with respect to the number of triangles $|\mathcal{T}|$ in the triangulation $\mathcal{T}$ ) for $\lambda_{h}(1)$. The choice of $\kappa_{2}=0.07353$ (line color blue) instead of $\kappa_{2}=0.25746$ (line color green) improves the guaranteed lower bound $\operatorname{GLB}(k)$ significantly. The bound computed with $\kappa_{2}=0.25746$ suffers from the involvement of $h_{\max }$ visible in form of steps, while the choice $\kappa_{2}=0.07353$ leads to a straight line in the convergence history plot. Undisplayed experiments on graded meshes [CB18] of the L-shaped domain, e.g., with grading parameter $\beta=10 / 7$, recover the optimal convergence rates and confirm Lemma 2.7 as well.
(a)

(b)


Figure 4.2: Comparison of the distance between $\lambda_{k}$ and $\lambda_{h}(k)$ (resp. GLB $(k)$ ) computed on uniform $(\theta=1$, solid) and adaptive $(\theta=0.5$, dashed) meshes of the L-shaped domain for $k=1$ in (a) and the four-slit domain for $k=1,3,4$ in (b).

### 4.4 Four-slit domain

The principal eigenvalue $\lambda(1)=830.21478777$ and the double eigenvalue $\lambda(3)=1125.1279=$ $\lambda(4)$ on the four-slit domain $\Omega:=(-1,1)^{2} \backslash([0,0.5) \times\{0\} \cup[0,-0.5) \times\{0\} \cup\{0\} \times[0,0.5) \cup$ $\{0\} \times[0,-0.5))$ of Fig. 4.1.c are approximated as in [CG14a]. The associated eigenfunctions on the non-convex domain seem to belong to $H_{0}^{2}(\Omega) \backslash H^{3}(\Omega)$ because uniform mesh-refinement leads to the reduced convergence rates 0.5 for the first and 0.55 for the third and fourth in Fig. 4.2.b. The AFEM algorithm with bulk parameter $\theta=0.5$ driven by the estimator (4.1) allows to recover the optimal convergence rate one. The GLB in Fig. 4.2.b are computed with $\kappa_{2}=0.07353$. Undisplayed comparison with $\kappa_{2}=0.25746$ lead to worse GLB. A clustering adaptive algorithm as in [Gal15b] was not necessary for the double eigenvalue $\lambda_{3}=\lambda_{4}$.

### 4.5 Comments and Conclusions

The empirical observations of the numerical experiments in Subsection 4.3-4.4 show:
(i) All experiments confirm the a priori convergence rates of Theorem 1.2. The empirical convergence rate depends only on the smoothness of the approximated eigenfunction. For instance Fig. 1.1 displays for uniform refinement the optimal convergence rate one for the principal eigenvalue despite the reduced empirical convergence rate for the fourth eigenvalue.
(ii) Theorem 1.2 predicts a convergence for a sufficiently fine initial mesh. In all examples the convergence rate is visible even for moderately fine triangulations, so this restriction does not affect the numerical examples much.
(iii) If the condition on the mesh-size is satisfied, the method (1.3) provides indeed guaranteed lower eigenvalue bounds in all numerical experiments and so confirms Theorem 1.1.
(iv) The constant $\kappa_{2}=0.07353$ from [LSL19] leads to a significant improvement of the known bound (1.1) in examples with adaptive mesh-refinement.
(v) The (undisplayed) improvement factor $q:=\left(\lambda_{k}-\lambda_{h}(k)\right) /\left(\lambda_{k}-\operatorname{GLB}(k)\right)$ was computed with $\kappa_{2}=0.07353$ on the adaptive triangulations. For the principal eigenvalue of the Lshaped and four-slit domain the improvement with the new method is marginal and the ratio $q$ oscillates between 0.6 and 1 . In the remaining examples the improvement is more significant. For the fourth eigenvalue of the four-slit domain the ratio $q$ oscillates between 0.05 and 0.25 for triangulations with more than 4500 triangles. For the first (resp. fourth) eigenvalue of the dumbbell-slit domain the ratio $q$ decreases from 0.16 (resp. 0.15) to 0.0008 (resp. 0.005) for triangulations with more than 600 (resp. 3800) triangles.
(vi) The new method increases the number of degrees of freedom by a factor four in the 2D numerical benchmarks. The equivalent rational problem (2.13) from Proposition 2.6 could be efficiently addressed by a Newton scheme, so the final comparison is beyond this paper. As it stands, the new method is favourable at least for the examples in Subsection 4.2 and the fourth eigenvalue in Subsection 4.4.
(vii) The adaptive algorithm driven by the estimator (4.1) recovers the optimal empirical convergence rates in all examples for the extra stabilised method. The analysis of optimal convergence rates and further details of the proposed adaptive algorithm shall appear in [CP22].
(viii) The overall conclusion of the numerical experiments is that there exist examples, where the post-processed GLB (1.1) may fail completely for localized triangulations. In contrast, the new scheme is compatible with adaptive mesh-refining and leads to GLB that cannot be reached with (1.1).
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# A Primer in the Worsey-Farin 3D FEM and a conforming companion for Morley FEM in 3D 

Supplement material to the paper 'Direct guaranteed lower eigenvalue bounds with optimal a priori convergence rates for the bi-Laplacian' by Carsten Carstensen and Sophie Puttkammer

## A. The Worsey-Farin FEM in 3D

This section aims at an elementary self-contained introduction to the Worsey-Farin (WF) FEM and the analysis of the associated nodal basis functions to generalize [Cia78, Thm. 6.1.3]. The Clough-Tocher finite element has been proposed for any space dimension $n \geqslant 3$ in [WF87]. More than two decades later, [Sor09] observed additional constraints on the choice of the subtriangulation of the macro element for $n \geqslant 4$ necessary for $C^{1}$ conformity, i.e., [WF87] is wrong for $n \geqslant 4$. It appears still an open problem, whether all conditions for $n=4$ can be satisfied simultaneously [Sor09, p.42, l.40ff]. This underlines that the details are technical and explains the restriction to $n=3$ in this paper. This supplement summarizes the necessities on Bernstein polynomials [dB87] and the Algorithm 1 from [WF87] for $n=3$ for a general audience.

## A.1. Bernstein polynomials in a simplex

Given $m \in\{1,2,3,4\}$ points $P_{1}, P_{2}, \ldots, P_{m} \in \mathbb{R}^{3}$ with linearly independent differences $P_{2}$ $P_{1}, \ldots, P_{m}-P_{1}$, their convex hull $T=\operatorname{conv}\left\{P_{1}, \ldots, P_{m}\right\}$ is an $(m-1)$-simplex (of positive ( $m-1$ )-dimensional Hausdorff measure). The set of vertices $\mathcal{V}(T)=\left\{P_{1}, \ldots, P_{m}\right\}$ of $T$ is the set of extremal points and so uniquely defined. We fix an ordering and so identify the ( $m-1$ )-simplex $T$ with an ordered list $\left(P_{1}, \ldots, P_{m}\right)$ of vertices. We will encounter a finite union of simplices and decouplings of $(m-1)$-simplices and need to keep track of the vertex set. Below $\left(P_{1}, \ldots, P_{m}\right)$ will be replaced by $\left(P_{\sigma(1)}, \ldots, P_{\sigma(m)}\right)$ for global numbers $\sigma(1), \ldots, \sigma(m)$ in a finite list of all vertices $\mathcal{V}$ in a regular triangulation $\mathcal{T}$ of $\Omega \subset \mathbb{R}^{3}$ into tetrahedra in the sense of Ciarlet. Let $\langle T\rangle:=P_{1}+\operatorname{span}\left\{P_{2}-P_{1}, \ldots, P_{m}-P_{1}\right\}$ denote the unique $(m-1)$ dimensional affine subspace $\langle T\rangle$ of $\mathbb{R}^{3}$ that contains $T \equiv\left(P_{1}, \ldots, P_{m}\right)$. The barycentric coordinates $\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ of a point $x \in\langle T\rangle$ solve the $4 \times m$ linear system of equations

$$
\left(\begin{array}{cccc}
1 & 1 & \ldots & 1  \tag{A.1}\\
P_{1} & P_{2} & \ldots & P_{m}
\end{array}\right)\left(\begin{array}{c}
\lambda_{1} \\
\vdots \\
\lambda_{m}
\end{array}\right)=\binom{1}{x} .
$$

The dependence on $x$ in $\lambda_{j}(x)=\lambda_{j}$ from (A.1) is typically not written out explicitly for brevity. It holds $x \in T$ if and only if $x \in \mathbb{R}^{3}$ satisfies (A.1) with $\lambda_{1}, \ldots, \lambda_{m} \geqslant 0$. The relative interior of $T$ reads

$$
\operatorname{relint}(T):=\left\{\sum_{\mu=1}^{m} \lambda_{\mu} P_{\mu} \mid \sum_{\mu=1}^{m} \lambda_{\mu}=1 \text { and } \lambda_{1}, \ldots, \lambda_{m}>0\right\}
$$

Hence $T=\overline{\operatorname{relint}(T)}$ and the relative boundary $\operatorname{relbdy}(T)=\partial T:=T \backslash \operatorname{relint}(T)$ of $T$ reads

$$
\partial T=\left\{\sum_{\mu=1}^{m} \lambda_{\mu} P_{\mu} \mid \sum_{\mu=1}^{m} \lambda_{\mu}=1, \lambda_{1}, \ldots, \lambda_{m} \geqslant 0, \text { and } \lambda_{j}=0 \text { for at least one } 1 \leqslant j \leqslant m\right\}
$$

The space $P_{k}(T)$ of algebraic polynomials of total degree at most $k \in \mathbb{N}_{0}$ is seen as a subspace of $C^{\infty}(T)$ and allows many representations. The Bernstein polynomials form a natural basis of $P_{k}(T)$ with standard multi-index notation for $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{N}_{0}^{m}$ with length $|\alpha|=$ $\alpha_{1}+\cdots+\alpha_{m}$ and

$$
\begin{equation*}
\frac{\lambda^{\alpha}}{\alpha!}:=\frac{\lambda_{1}^{\alpha_{1}} \lambda_{2}^{\alpha_{2}} \cdots \lambda_{m}^{\alpha_{m}}}{\alpha_{1}!\alpha_{2}!\cdots \alpha_{m}!} \quad \text { at } \quad x=\sum_{\mu=1}^{m} \lambda_{\mu} P_{\mu} \in T \tag{A.2}
\end{equation*}
$$

(We follow the convention $\lambda^{\alpha} / \alpha!\equiv 0$ if at least one of the indices $\alpha_{1}, \ldots, \alpha_{m}$ of $\alpha$ is negative.) If the length $m \in\{1,2,3,4\}$ is fixed, abbreviate $A_{k}:=\left\{\alpha \in \mathbb{N}_{0}^{m}| | \alpha \mid=k\right\} \subset \mathbb{N}_{0}^{m}$. The Bernstein polynomials of degree $k \in \mathbb{N}_{0}$ are all $\lambda^{\alpha} / \alpha$ ! from (A.2) for $\alpha \in A_{k}$. In fact, any (real) polynomial $f \in P_{k}(T)$ has unique (real) coefficients $\left(c(\alpha) \mid \alpha \in A_{k}\right)$ called ordinates such that

$$
\begin{equation*}
f=k!\sum_{\alpha \in A_{k}} c(\alpha) \frac{\lambda^{\alpha}}{\alpha!} \quad \text { at } \quad x=\sum_{\mu=1}^{m} \lambda_{\mu} P_{\mu} \in T \tag{A.3}
\end{equation*}
$$

The definition (A.3) makes sense also in the entire affine subspace $\langle T\rangle:=P_{1}+\operatorname{span}\left\{P_{2}-\right.$ $\left.P_{1}, \ldots, P_{m}-P_{1}\right\} \ni x$ and immediately extends any $f \in P_{k}(T)$ to $f \in P_{k}(\langle T\rangle)$. The ordinates of $f$ form a family $\left(c(\alpha) \mid \alpha \in A_{k}\right.$ ) and can be arranged in an $m$-dimensional array. Any ordinate $c(\alpha)$ for $\alpha \in A_{k}$ is associated with the position $x=\sum_{\mu=1}^{m} \frac{\alpha_{\mu}}{k} P_{\mu} \in T$ as illustrated in Fig. A. 1 for $m=3$ and $k=2,3$. If the length of $m$ is fixed (and there is no need to highlight the length $m$ of the multi-indices from the context), then $e_{j}:=(0, \ldots, 0,1,0, \ldots, 0) \in \mathbb{N}_{0}^{m}$ denotes the $j$-th canonical unit vector with the coefficients $e_{j}(\ell)=\delta_{j \ell}$ for $j, \ell \in\{1, \ldots, m\}$. It cannot be overemphasized that the ordinates $\left(c(\alpha) \mid \alpha \in A_{k}\right)$ of the polynomial $f$ in (A.3) are neither coefficients of a monomic tensor basis $x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}} \ldots x_{n}^{\alpha_{n}}$ nor the values of the polynomial in general. However, at the $m$ vertices, the ordinates are the values of the polynomial $f$ : For any $j=1, \ldots, m$, the vertex $P_{j}$ is identified with the multi-index $k e_{j}$ and $c\left(k e_{j}\right)=f\left(P_{j}\right)$ follows from (A.3) and the barycentric coordinates $\left(\lambda_{1}, \ldots, \lambda_{m}\right)=e_{j}$ of $P_{j}$.

Remark A.1 (gradient $\nabla f)$. The derivative of $f$ from (A.3) reads at $x=\sum_{\mu=1}^{m} \lambda_{\mu} P_{\mu} \in T$

$$
\begin{equation*}
\nabla f=k!\sum_{\alpha \in A_{k}} \sum_{\mu=1}^{m} c_{j}(\alpha) \frac{\lambda^{\left(\alpha-e_{\mu}\right)}}{\left(\alpha-e_{\mu}\right)!} \nabla \lambda_{\mu}=k!\sum_{\beta \in A_{k-1}} \frac{\lambda^{\beta}}{\beta!} \sum_{\mu=1}^{m} c\left(\beta+e_{\mu}\right) \nabla \lambda_{\mu} \tag{A.4}
\end{equation*}
$$



Figure A.1.: Illustration of the position associated with the ordinates $c(\alpha)$ for $\alpha \in A_{k}$ with $m=3$ and $k=2(\mathrm{a})$ and $k=3(\mathrm{~b})$.

Some examples on the evaluation of $f$ and $\nabla f$ on the boundary $\partial T$ of a triangle ( $m=3$ ) and tetrahedron $(m=4)$ conclude this subsection for the sake of an illustration as well as future reference. The reader may skip the proof at first reading, but requires details in the analysis of Algorithm 1 later in Subsection A.4.

Example A. 1 (triangle, $\boldsymbol{k}=\mathbf{2}, \boldsymbol{m}=\mathbf{3}$ ). For $m=3$ consider a triangle $T$ identified with its vertices $\left(P_{1}, P_{2}, P_{3}\right)$. The first edge $E_{1}=\operatorname{conv}\left\{P_{2}, P_{3}\right\}$ of the triangle $T$ lies opposite to the first vertex $P_{1}$ and has midpoint $\operatorname{mid}\left(E_{1}\right):=P_{23}:=\left(P_{2}+P_{3}\right) / 2$. In the two-dimensional affine space $\langle T\rangle=P_{1}+\operatorname{span}\left\{P_{2}-P_{1}, P_{3}-P_{1}\right\}$, the edge $E_{1}$ belongs to the zero set $\left\{\lambda_{1}=0\right\} \supset E_{1}$ of the first barycentric coordinate $\lambda_{1}$, where $\lambda_{2}+\lambda_{3}=1$ for $\lambda_{2}, \lambda_{3} \geqslant 0$. The polynomial $\lambda^{\alpha}$ vanishes at $x=\lambda_{2} P_{2}+\lambda_{3} P_{3} \in E_{1}$, if $\alpha_{1} \geqslant 1$ and so (A.3) reduces to

$$
\begin{equation*}
f=k!\sum_{\substack{\alpha \in A_{k} \\ \alpha_{1}=0}} c(\alpha) \lambda^{\alpha} / \alpha!\quad \text { at } \quad x=\lambda_{2} P_{2}+\lambda_{3} P_{3} \in E_{1} . \tag{A.5}
\end{equation*}
$$

For $k=2$ and with $c(0,2,0)=f\left(P_{2}\right)$ and $c(0,0,2)=f\left(P_{3}\right)$, the formula (A.5) shows at $P_{23}=\operatorname{mid}\left(E_{1}\right)$ that

$$
4 f\left(P_{23}\right)=f\left(P_{2}\right)+2 c(0,1,1)+f\left(P_{3}\right) .
$$

The derivative $\nabla f$ of $f$ along the edge $E_{1} \subset\left\{\lambda_{1}=0\right\}$ follows from (A.4),

$$
\begin{equation*}
\nabla f=k!\sum_{\substack{\alpha \in A_{k-1} \\ \alpha_{1}=0}}\left(\sum_{\mu=1}^{3} c\left(\alpha+e_{\mu}\right) \nabla \lambda_{\mu}\right) \frac{\lambda^{\alpha}}{\alpha!} \quad \text { at } x=\lambda_{2} P_{2}+\lambda_{3} P_{3} \in E_{1} . \tag{A.6}
\end{equation*}
$$

Since $k=2$, there are solely two summands in the sum of the formula (A.6) for $\left(\alpha_{2}, \alpha_{3}\right)=$ $(1,0)$ and $\left(\alpha_{2}, \alpha_{3}\right)=(0,1)$. Hence

$$
\begin{aligned}
\nabla f= & 2\left(c(1,1,0) \nabla \lambda_{1}+c(0,2,0) \nabla \lambda_{2}+c(0,1,1) \nabla \lambda_{3}\right) \lambda_{2} \\
& +2\left(c(1,0,1) \nabla \lambda_{1}+c(0,1,1) \nabla \lambda_{2}+c(0,0,2) \nabla \lambda_{3}\right) \lambda_{3} \quad \text { at } x \in E_{1} .
\end{aligned}
$$

This defines an affine vector function along $E_{1}$. In conclusion, the data $\left.f\right|_{E_{1}}$ and $\left.\nabla f\right|_{E_{1}}$ determine the ordinates $c(0,0,2), c(0,2,0), c(1,1,0), c(1,0,1), c(0,1,1)$ and vice versa.

Example A. 2 (tetrahedron, $\boldsymbol{k}=\mathbf{2}, \boldsymbol{m}=\mathbf{4}$ ). For $m=4$ consider a tetrahedron $T$ identified with its vertices $\left(P_{1}, P_{2}, P_{3}, P_{4}\right)$ in $\mathbb{R}^{3}$ with edge midpoints $P_{k \ell}=\left(P_{k}+P_{\ell}\right) / 2$ for $k, \ell=$ $1, \ldots, 4$ and $k \neq \ell$. The ordinates $\left(c(\alpha) \mid \alpha \in A_{2}\right)$ define the quadratic polynomial $f=$ $2 \sum_{\alpha \in A_{2}} c(\alpha) \lambda^{\alpha} / \alpha!\in P_{2}(T)$, which assumes, for all $j, k, \ell=1, \ldots, 4$ and $k \neq \ell$, the following values.
(a) $f\left(P_{j}\right)=c\left(2 e_{j}\right)$,
(b) $4 f\left(P_{k \ell}\right)=c\left(2 e_{k}\right)+2 c\left(e_{k}+e_{\ell}\right)+c\left(2 e_{\ell}\right)=f\left(P_{k}\right)+2 c\left(e_{k}+e_{\ell}\right)+f\left(P_{\ell}\right)$.

Proof of (a). The first formula (a) follows with $\left(\lambda_{1}, \ldots, \lambda_{4}\right)=e_{j}$ at $x=P_{j}$ and the evaluation of the non-zero $\lambda^{\alpha} / \alpha$ ! for $\alpha=2 e_{j}$.

Proof of (b). At the edge midpoint $x=P_{k \ell}$ in (b), we have $\left(\lambda_{1}, \ldots, \lambda_{4}\right)=\left(e_{k}+e_{\ell}\right) / 2$ and $\lambda^{\alpha} / \alpha$ ! is possibly non-zero only for $\alpha=2 e_{k}, e_{k}+e_{\ell}$, and $2 e_{\ell}$. The polynomial $\lambda^{\alpha} / \alpha$ ! assumes the respective values $\lambda^{\alpha} / \alpha!=1 / 8,1 / 4,1 / 8$. This leads to the formulas in (b).

Example A. 3 (tetrahedron, $\boldsymbol{k}=\mathbf{3}, \boldsymbol{m}=\mathbf{4}$ ). In the notation of Example A.3, the ordinates $\left(c(\alpha) \mid \alpha \in A_{3}\right)$ define the cubic polynomial $f=6 \sum_{\alpha \in A_{3}} c(\alpha) \lambda^{\alpha} / \alpha!\in P_{3}(T)$ and its gradient $\nabla f \in P_{2}\left(T ; \mathbb{R}^{3}\right)$. For all $\{j, k, \ell, m\}=\{1,2,3,4\}$ it follows
(a) $f\left(P_{j}\right)=c\left(3 e_{j}\right)$,
(b) $8 f\left(P_{k \ell}\right)=f\left(P_{k}\right)+3 c\left(2 e_{k}+e_{\ell}\right)+3 c\left(e_{k}+2 e_{\ell}\right)+f\left(P_{\ell}\right)$,
(c) $\left(P_{j}-P_{k}\right) \cdot \nabla f\left(P_{j}\right)=3\left(c\left(3 e_{j}\right)-c\left(2 e_{j}+e_{k}\right)\right)$,
(d) $\frac{4}{3}\left(P_{j}-P_{k \ell}\right) \cdot \nabla f\left(P_{k \ell}\right)=c\left(2 e_{\ell}+e_{j}\right)+2 c\left(e_{\ell}+e_{k}+e_{j}\right)+c\left(2 e_{k}+e_{j}\right)-4 f\left(P_{k \ell}\right)$,
(e) $\frac{8}{3}\left(P_{k}-P_{k \ell}\right) \cdot \nabla f\left(P_{k \ell}\right)=f\left(P_{k}\right)+c\left(2 e_{k}+e_{\ell}\right)-c\left(2 e_{\ell}+e_{k}\right)-f\left(P_{\ell}\right)$.

Proof of (a)-(b). The formulas (a)-(b) follow with $\left(\lambda_{1}, \ldots, \lambda_{4}\right)=e_{j}$ at $x=P_{j}$ and the only possible $\alpha=3 e_{j}$ and $\left(\lambda_{1}, \ldots, \lambda_{4}\right)=\left(e_{k}+e_{\ell}\right) / 2$ at the midpoint $x=P_{k \ell}$ with possible $\alpha=3 e_{k}, 2 e_{k}+e_{\ell}, 2 e_{\ell}+e_{k}, 3 e_{\ell}$ and respective values $\lambda^{\alpha} / \alpha!=1 / 48,1 / 16,1 / 16,1 / 48$.
Proof of (c). The polynomial $\lambda^{\beta} / \beta$ ! vanishes at $x=P_{j}$ for all $\beta \neq 2 e_{j}$, while $\lambda^{2 e_{j}} /\left(2 e_{j}\right)$ ! $=1 / 2$ at $x=P_{j}$. Consequently,

$$
\begin{equation*}
\nabla f\left(P_{j}\right)=3 \sum_{\mu=1}^{4} c\left(2 e_{j}+e_{\mu}\right) \nabla \lambda_{\mu} \tag{A.7}
\end{equation*}
$$

The barycentric coordinates satisfy $\left(P_{j}-P_{k}\right) \cdot \nabla \lambda_{\mu}=\lambda_{\mu}\left(P_{j}\right)-\lambda_{\mu}\left(P_{k}\right)=\delta_{j \mu}-\delta_{k \mu}$ and so (c) follows from (A.7).
Proof of (d). At the edge midpoint $x=P_{k \ell}$, Example A. 2 shows $\lambda^{\beta} / \beta!=1 / 8,1 / 4,1 / 8$ for $\beta=2 e_{k}, e_{k}+e_{\ell}, 2 e_{\ell}$. Hence

$$
\begin{equation*}
\nabla f\left(P_{k \ell}\right)=\frac{3}{4} \sum_{\mu=1}^{4}\left(c\left(2 e_{\ell}+e_{\mu}\right)+2 c\left(e_{\ell}+e_{k}+e_{\mu}\right)+c\left(2 e_{k}+e_{\mu}\right)\right) \nabla \lambda_{\mu} \tag{A.8}
\end{equation*}
$$

Since $\left(P_{j}-P_{k \ell}\right) \cdot \nabla \lambda_{\mu}=\delta_{j \mu}-\frac{1}{2}\left(\delta_{k \mu}+\delta_{\ell \mu}\right)$ is one for $\mu=j,-1 / 2$ for $\mu=k$, $\ell$, and vanishes otherwise, (d) follows from (b).
Proof of (e). Since $\left(P_{k}-P_{k \ell}\right) \cdot \nabla \lambda_{\mu}=\delta_{k \mu}-\frac{1}{2}\left(\delta_{k \mu}+\delta_{\ell \mu}\right)$ is $1 / 2$ for $\mu=k$ and $-1 / 2$ for $\mu=\ell$ and vanishes otherwise, (A.8) leads to (e).


Figure A.2.: Illustration of two neighbouring tetrahedra $T_{1}=\operatorname{conv}\left\{P_{1}, P_{2}, P_{3}, P_{4}\right\}$ in red and $T_{2}=$ $\operatorname{conv}\left\{P_{1}, P_{2}, P_{3}, P_{5}\right\}$ in blue with common face $F=\operatorname{conv}\left\{P_{1}, P_{2}, P_{3}\right\}$ in violet.

## A.2. Smoothness across an interface

This subsection characterizes the function space $C^{1}\left(T_{1} \cup T_{2}\right) \cap P_{k}\left(\left\{T_{1}, T_{2}\right\}\right)$ for two tetrahedra $T_{1}, T_{2}$ with a common face $F=\partial T_{1} \cap \partial T_{2}$. Fig. A. 2 illustrates the following situation. Suppose $T_{1}=\operatorname{conv}\left\{P_{1}, P_{2}, P_{3}, P_{4}\right\}$ and $T_{2}=\operatorname{conv}\left\{P_{1}, P_{2}, P_{3}, P_{5}\right\}$ share the face $F=\partial T_{1} \cap$ $\partial T_{2}=\operatorname{conv}\left\{P_{1}, P_{2}, P_{3}\right\}$ with unit normal $\nu_{F}$. Recall $A_{k}:=\left\{\alpha \in \mathbb{N}_{0}^{4}| | \alpha \mid=k\right\} \subset \mathbb{N}_{0}^{4}$ and set $B_{k}:=\left\{\alpha \in \mathbb{N}_{0}^{3}| | \alpha \mid=k\right\} \subset \mathbb{N}_{0}^{3}$. In this context the notation $c_{j}(\beta ; 0)$ in (A.10)-(A.11) below abbreviates $c_{j}(\beta ; 0):=c_{j}\left(\left(\beta_{1}, \beta_{2}, \beta_{3}, 0\right)\right)$ for some $\beta \in B_{k}$ or $\beta \in B_{k-1}$ and $j=1,2$. Identify $T_{1}$ with $\left(P_{1}, P_{2}, P_{3}, P_{4}\right)$ and $T_{2}$ with $\left(P_{1}, P_{2}, P_{3}, P_{5}\right)$. The function $g \in P_{k}\left(\left\{T_{1}, T_{2}\right\}\right)$ is given in $T_{1}$ resp. $T_{2}$ as the polynomial (A.3) and with ordinates $\left(c_{1}(\alpha) \mid \alpha \in A_{k}\right)$ resp. $\left(c_{2}(\alpha) \mid \alpha \in A_{k}\right)$,

$$
\begin{equation*}
g_{j}=\left.g\right|_{T_{j}}=k!\sum_{\alpha \in A_{k}} c_{j}(\alpha) \frac{\lambda^{\alpha}}{\alpha!} \quad \text { at } \quad x=\lambda_{1} P_{1}+\lambda_{2} P_{2}+\lambda_{3} P_{3}+\lambda_{4} P_{\ell} \in T_{j} \tag{A.9}
\end{equation*}
$$

for $j=1,2$ with $\ell=\ell(1)=4$ for $j=1$ and $\ell=\ell(2)=5$ for $j=2$.
Lemma A.1. The polynomials $g_{j} \in P_{k}\left(T_{j}\right)$ with (A.9) for $j=1,2$ form a function $g \in$ $P_{k}\left(\left\{T_{1}, T_{2}\right\}\right)$ by $\left.g\right|_{T_{j}}=g_{j}$ for $j=1,2$. Then
(a) $g \in C^{0}\left(T_{1} \cup T_{2}\right)$ if and only if

$$
\begin{equation*}
c_{1}(\beta ; 0)=c_{2}(\beta ; 0) \quad \text { for all } \beta \in B_{k} ; \tag{A.10}
\end{equation*}
$$

(b) $g \in C^{1}\left(T_{1} \cup T_{2}\right)$ if and only if (A.10) and

$$
\begin{equation*}
\sum_{\mu=1}^{4} c_{1}\left((\beta ; 0)+e_{\mu}\right)\left(\left.\nu_{F} \cdot \nabla \lambda_{\mu}\right|_{T_{1}}\right)=\sum_{\mu=1}^{4} c_{2}\left((\beta ; 0)+e_{\mu}\right)\left(\nu_{F} \cdot \nabla \lambda_{\mu} \mid T_{2}\right) \text { for all } \beta \in B_{k-1} \tag{A.11}
\end{equation*}
$$

Proof of (a). Recall that the common face $F=\partial T_{1} \cap \partial T_{2}$ satisfies $F \subset\left\{\lambda_{4}=0\right\}$. Hence (A.9) implies

$$
\left.g_{j}\right|_{F}=k!\sum_{\beta \in B_{k}} c_{j}(\beta ; 0) \frac{\lambda^{\beta}}{\beta!} \quad \text { at } x=\lambda_{1} P_{1}+\lambda_{2} P_{2}+\lambda_{3} P_{3} \in F \text { and } j=1,2
$$

with the abbreviation $c_{j}(\beta ; 0):=c_{j}\left(\left(\beta_{1}, \beta_{2}, \beta_{3}, 0\right)\right)$ and $\lambda^{\beta} / \beta!=\frac{\lambda_{1}^{\beta_{1}} \lambda_{2}^{\beta_{2}} \lambda_{3}^{\beta_{3}}}{\beta_{1}!\beta_{2}!\beta_{3}!}$. Therefore, $\left.g_{1}\right|_{F}=$ $\left.g_{2}\right|_{F}$ for $g \in C^{0}\left(T_{1} \cup T_{2}\right)$ is equivalent to

$$
0=\sum_{\beta \in B_{k}}\left(c_{1}(\beta ; 0)-c_{2}(\beta ; 0)\right) \lambda^{\beta} / \beta!\quad \text { at } x=\lambda_{1} P_{1}+\lambda_{2} P_{2}+\lambda_{3} P_{3} \in F
$$

Since the Bernstein polynomials $\left(\lambda^{\beta} / \beta!\mid \beta \in B_{k}\right)$ form a basis of $P_{k}(F)$ the last statement is equivalent to (A.10). This proves (a).

Proof of (b). Since $g \in C^{1}\left(T_{1} \cup T_{2}\right)$ includes $g \in C^{0}\left(T_{1} \cup T_{2}\right)$, (A.10) has to hold. Additionally we need continuity of the gradient at $F$. The gradient $\left.\nabla g_{j}\right|_{F}$ from (A.4) reads

$$
\left.\nabla g_{j}\right|_{F}=\left.\left.k!\sum_{\beta \in A_{k-1}} \frac{\lambda^{\beta}}{\beta!}\right|_{F} \sum_{\mu=1}^{4} c_{j}\left(\beta+e_{\mu}\right) \nabla \lambda_{\mu}\right|_{T_{j}}=\left.k!\sum_{\beta \in B_{k-1}} \frac{\lambda^{\beta}}{\beta!} \sum_{\mu=1}^{4} c_{j}\left((\beta ; 0)+e_{\mu}\right) \nabla \lambda_{\mu}\right|_{T_{j}}
$$

at $x=\lambda_{1} P_{1}+\lambda_{2} P_{2}+\lambda_{3} P_{3} \in F$. Hence the continuity of the gradients $\left.\nabla g_{1}\right|_{F}=\left.\nabla g_{2}\right|_{F}$ is equivalent to

$$
0=\sum_{\beta \in B_{k-1}} \frac{\lambda^{\beta}}{\beta!} \sum_{\mu=1}^{4}\left(\left.c_{1}\left((\beta ; 0)+e_{\mu}\right) \nabla \lambda_{\mu}\right|_{T_{1}}-\left.c_{2}\left((\beta ; 0)+e_{\mu}\right) \nabla \lambda_{\mu}\right|_{T_{2}}\right) \quad \text { at } x=\sum_{\mu=1}^{3} \lambda_{\mu} P_{\mu} \in F
$$

Since the Bernstein polynomials $\left(\lambda^{\beta} / \beta!\mid \beta \in B_{k-1}\right)$ form a basis of $P_{k-1}(F)$ the last statement is equivalent to

$$
\begin{equation*}
0=\sum_{\mu=1}^{4}\left(\left.c_{1}\left((\beta ; 0)+e_{\mu}\right) \nabla \lambda_{\mu}\right|_{T_{1}}-\left.c_{2}\left((\beta ; 0)+e_{\mu}\right) \nabla \lambda_{\mu}\right|_{T_{2}}\right) \in \mathbb{R}^{3} \quad \text { for all } \beta \in B_{k-1} \tag{A.12}
\end{equation*}
$$

It remains to show, that (A.12) is equivalent to (A.11). In other words, to verify that solely the normal derivatives $\nu_{F} \cdot \nabla \lambda_{\mu} \mid T_{j}$ for $j=1,2$ and $\mu=1,2,3$ are of interest. Fix two tangential directions $\tau_{1}, \tau_{2}$ of unit length parallel to $\langle F\rangle$, such that $\tau_{1}, \tau_{2}, \nu_{F}$ form an orthonormal basis of $\mathbb{R}^{3}$. Then

$$
\left.\nabla \lambda_{\mu}\right|_{T_{j}}=\left(\nabla \lambda_{\mu} \mid T_{j} \cdot \tau_{1}\right) \tau_{1}+\left(\left.\nabla \lambda_{\mu}\right|_{T_{j}} \cdot \tau_{2}\right) \tau_{2}+\left(\left.\nabla \lambda_{\mu}\right|_{T_{j}} \cdot \nu_{F}\right) \nu_{F}
$$

The components of the difference $\left.\nabla \lambda_{\mu}\right|_{T_{1}}-\left.\nabla \lambda_{\mu}\right|_{T_{2}}$ for $\mu=1,2,3$ in $\mathbb{R}^{3}$ in the direction $\tau_{1}, \tau_{2} \in \operatorname{span}\left\{P_{2}-P_{1}, P_{3}-P_{1}\right\}$ parallel to $\langle F\rangle$ vanish owing to the Hadamard jump condition: The piecewise gradient $\nabla g$ of $g \in C^{0}\left(T_{1} \cup T_{2}\right) \cap P_{k}\left(\left\{T_{1}, T_{2}\right\}\right)$ jumps across the interface $F$ and the jump $[\nabla g]_{F} \| \nu_{F}$ exclusively points in the normal direction. For $\mu=1,2,3$ and the tangential direction $\tau_{j} \in \operatorname{span}\left\{P_{2}-P_{1}, P_{3}-P_{1}\right\}$ for $j=1,2$, we have

$$
\left.\tau_{j} \cdot \nabla \lambda_{\mu}\right|_{T_{1}}=\left.\frac{\partial}{\partial s} \lambda_{\mu}\left(x_{0}+s \tau\right)\right|_{s=0}=\left.\tau_{j} \cdot \nabla \lambda_{\mu}\right|_{T_{2}}=: \tau_{j} \cdot \nabla \lambda_{\mu} \quad \text { at any } x_{0} \in \operatorname{relint}(F)
$$

For $\mu=4$, notice that $P_{r}-P_{1} \perp \nu_{F} \|\left.\nabla \lambda_{4}\right|_{T_{j}}$ for $r=2,3$ and $j=1,2$. In fact, suppose $\varrho_{\ell}>0$ denotes the height of $P_{\ell}$ over the plane $\langle F\rangle$, then $\left.\varrho_{\ell} \nabla \lambda_{4}\right|_{T_{j}}=(-1)^{j} \nu_{F}$ for $\ell=4,5, j=1,2$ provided the orientation of $\nu_{F}$ is fixed from $P_{4}$ to $P_{5}$ in that $\nu_{F}$ is the outer unit normal in $F \subset \partial T_{1}$. In particular this means $\left.\tau_{j} \cdot \nabla \lambda_{4}\right|_{T_{1}}=0=\left.\tau_{j} \cdot \nabla \lambda_{4}\right|_{T_{2}}$ for $j=1,2$. Hence, (A.12) is recast for all $\beta \in B_{k-1}$ as the first identity in

$$
-\nu_{F} \sum_{\mu=1}^{4}\left(c_{1}\left((\beta ; 0)+e_{\mu}\right)\left(\left.\nabla \lambda_{\mu}\right|_{T_{1}} \cdot \nu_{F}\right)-c_{2}\left((\beta ; 0)+e_{\mu}\right)\left(\left.\nabla \lambda_{\mu}\right|_{T_{2}} \cdot \nu_{F}\right)\right)
$$

$$
\left.=\sum_{j=1}^{2} \tau_{j} \sum_{\mu=1}^{3}\left(c_{1}\left((\beta ; 0)+e_{\mu}\right)-c_{2}\left((\beta ; 0)+e_{\mu}\right)\right) \nabla \lambda_{\mu} \cdot \tau_{j}\right)=0
$$

with the equality of ordinates from (A.10) in the last step. The last identity is equivalent to (A.11).

The coefficient relations (A.10)-(A.11) concern the factors one in (A.10) or geometric factors $\left.\nu_{F} \cdot \nabla \lambda\right|_{T_{j}}$ for $j=1,2$ in (A.11). However they do not explicitly display $k$ and this gives rise to a factorization (cf. [dB87, §14] in terms of B-forms). Suppose ( $c_{j}(\alpha) \mid \alpha \in A_{k}$ ) defines $g_{j}$ in (A.9) for $F=\partial T_{1} \cap \partial T_{2}=\operatorname{conv}\left\{P_{1}, P_{2}, P_{3}\right\}$ and $j=1,2$. Define for a fixed index $\xi \in\{1,2,3\}$ suppressed in the notation

$$
\begin{equation*}
\left.g^{\prime}\right|_{T_{j}}:=g_{j}^{\prime}:=(k-1)!\sum_{\beta \in A_{k-1}} c_{j}\left(e_{\xi}+\beta\right) \lambda^{\beta} / \beta!\quad \text { for } j=1,2 \tag{A.13}
\end{equation*}
$$

Then the composition $g^{\prime} \in P_{k-1}\left(\left\{T_{1}, T_{2}\right\}\right)$ from (A.13) is a piecewise polynomial of degree at most $k-1$.

Example A. $4\left(\boldsymbol{g}^{\prime}\right.$ for $\left.\boldsymbol{k}=\mathbf{0}, \ldots, 3\right)$. Abbreviate $\sum_{\mu \leqslant \ell}:=\sum_{\substack{\mu, \ell=1 \\ \mu \leqslant \ell}}^{4}$.
(a) If $k=0$ and $g_{j}=c_{j}((0,0,0,0))$, then $g_{j}^{\prime}=0$ by definition.
(b) If $k=1$ and $g_{j}=\sum_{\mu=1}^{4} c_{j}\left(e_{\mu}\right) \lambda^{e_{\mu}}$, then $g_{j}^{\prime}=c_{j}\left(e_{\xi}\right)$.
(c) If $k=2$ and $g_{j}=2 \sum_{\mu \leqslant \ell} c_{j}\left(e_{\mu}+e_{\ell}\right) \frac{\lambda^{e} \mu+e_{\ell}}{\left(e_{\mu}+e_{\ell}\right)!}$, then $g_{j}^{\prime}=\sum_{\mu=1}^{4} c_{j}\left(e_{\xi}+e_{\mu}\right) \lambda_{\mu}$; e.g., $g^{\prime}=c_{j}((2,0,0,0)) \lambda_{1}+c_{j}((1,1,0,0)) \lambda_{2}+c_{j}((1,0,1,0)) \lambda_{3}+c_{j}((1,0,0,1)) \lambda_{4}$ holds for $\xi=1$.
(d) If $k=3$ and $g_{j}=6 \sum_{\alpha \in A_{3}} c_{j}(\alpha) \frac{\lambda^{\alpha}}{\alpha!}$, then $g_{j}^{\prime}=2 \sum_{\mu \leqslant \ell} c_{j}\left(e_{\xi}+e_{\mu}+e_{\ell}\right) \frac{\lambda^{e} \mu+e_{\ell}}{\left(e_{\mu}+e_{\ell}\right)!}$; e.g.,

$$
\begin{aligned}
g_{j}^{\prime}= & 2 \lambda_{1}\left(c_{j}((2,1,0,0)) \lambda_{2}+c_{j}((2,0,1,0)) \lambda_{3}+c_{j}((2,0,0,1)) \lambda_{4}\right) \\
& +2\left(c_{j}(1,1,1,0) \lambda_{2} \lambda_{3}+c_{j}(1,1,0,1) \lambda_{2} \lambda_{4}+c_{j}(1,0,1,1) \lambda_{3} \lambda_{4}\right) \\
& +c_{j}((3,0,0,0)) \lambda_{1}^{2}+c_{j}((1,2,0,0)) \lambda_{2}^{2}+c_{j}((1,0,2,0)) \lambda_{3}^{2}+c_{j}((1,0,0,2)) \lambda_{4}^{2}
\end{aligned}
$$

holds for $\xi=1$.
Recall that the two tetrahedra $T_{1}$ and $T_{2}$ in Fig. A. 2 share the face $F=\partial T_{1} \cap \partial T_{2}=$ $\operatorname{conv}\left\{P_{1}, P_{2}, P_{3}\right\}$ with fixed unit normal $\nu_{F}$. Suppose $\left.g\right|_{T_{j}}:=g_{j}$ in (A.9) for $j=1,2$ and define $g^{\prime}$ in (A.13) with respect to a fixed index $\xi \in\{1,2,3\}$. Let $\ell, m$ denote the two distinct indices with $\{\xi, \ell, m\}=\{1,2,3\}$ and $E:=\operatorname{conv}\left\{P_{\ell}, P_{m}\right\}$.
Lemma A.2. Under the present notation $g \in C^{1}\left(T_{1} \cup T_{2}\right)$ is equivalent to the three conditions

$$
g^{\prime} \in C^{1}\left(T_{1} \cup T_{2}\right) \text { as well as } g \text { and } \frac{\partial g}{\partial \nu_{F}} \text { are continuous at } E \text {. }
$$

Proof. Without loss of generality fix $\xi=1$ and $E=\operatorname{conv}\left\{P_{2}, P_{3}\right\}$ throughout this proof. Lemma A. 1 characterizes $g \in C^{1}\left(T_{1} \cup T_{2}\right)$ in (A.10)-(A.11); but it also applies to $g^{\prime} \in$ $P_{k-1}\left(\left\{T_{1}, T_{2}\right\}\right)$ and shows that $g^{\prime} \in C^{1}\left(T_{1} \cup T_{2}\right)$ is equivalent to (A.14)-(A.15),

$$
\begin{equation*}
c_{1}\left(e_{1}+(\beta ; 0)\right)=c_{2}\left(e_{1}+(\beta ; 0)\right) \quad \text { for all } \beta \in B_{k-1}:=\left\{\alpha \in \mathbb{N}_{0}^{3}| | \alpha \mid=k-1\right\} \tag{A.14}
\end{equation*}
$$

$$
\begin{equation*}
\left.\sum_{\mu=1}^{4} c_{1}\left(e_{1}+(\beta ; 0)+e_{\mu}\right) \nu_{F} \cdot \nabla \lambda_{\mu}\right|_{T_{1}}=\left.\sum_{\mu=1}^{4} c_{2}\left(e_{1}+(\beta ; 0)+e_{\mu}\right) \nu_{F} \cdot \nabla \lambda_{\mu}\right|_{T_{2}} \quad \text { for all } \beta \in B_{k-2} . \tag{A.15}
\end{equation*}
$$

Observe that the conditions (A.14)-(A.15) concern $e_{1}+(\beta ; 0)$ for $\beta \in B_{k-1}$ and $\beta \in B_{k-2}$ and are included in (A.10)-(A.11). In other words (A.10)-(A.11) imply (A.14)-(A.15).
" $\Rightarrow$ " The first implication assumes $g \in C^{1}\left(T_{1} \cup T_{2}\right)$ and Lemma A. 1 guarantees (A.10)-(A.11) and so (A.14)-(A.15). Recall from the very beginning of this proof that (A.14)-(A.15) imply $g^{\prime} \in C^{1}\left(T_{1} \cup T_{2}\right)$. Finally $g \in C^{1}\left(T_{1} \cup T_{2}\right)$ and $\nabla g \in C^{0}\left(T_{1} \cup T_{2} ; \mathbb{R}^{3}\right)$ imply that $g$ and $\frac{\partial g}{\partial \nu_{F}}$ are continuous along $E=\operatorname{conv}\left\{P_{2}, P_{3}\right\}$.
$» \Leftarrow$ " The converse implication that assumes $g^{\prime} \in C^{1}\left(T_{1} \cup T_{2}\right)$, whence (A.14)-(A.15), and that $g$ and $\frac{\partial g}{\partial \nu_{F}}$ are continuous along $E \subset\left\{\lambda_{1}=0=\lambda_{4}\right\}$. First (A.14) implies some conditions of (A.10) and the remaining conditions read

$$
c_{1}\left(0, \beta_{2}, \beta_{3}, 0\right)=c_{2}\left(0, \beta_{2}, \beta_{3}, 0\right) \text { for all } \beta_{2}, \beta_{3} \in \mathbb{N}_{0} \text { with } \beta_{2}+\beta_{3}=k .
$$

The remaining conditions concern $\beta \in B_{k}$ with $\beta_{1}=0$. On the other hand, the continuity of $g$ along $E$ implies $\left.g_{1}\right|_{E}=\left.g_{2}\right|_{E}$. In terms of (A.9), this reads

$$
\begin{equation*}
0=\left.g_{1}\right|_{E}-\left.g_{2}\right|_{E}=k!\sum_{\substack{\beta \in B_{k} \\ \beta_{1}=0}}\left(c_{1}((\beta ; 0))-c_{2}((\beta ; 0))\right) \frac{\lambda^{\beta}}{\beta!} \quad \text { at } x=\lambda_{2} P_{2}+\lambda_{3} P_{3} \in E . \tag{A.16}
\end{equation*}
$$

Since the Bernstein polynomials $\left(\lambda^{\beta} / \beta!\mid \beta \in B_{k}, \beta_{1}=0\right)$ form a basis of $P_{k}(E)$, (A.16) implies the remaining condition in (A.10). Consequently, (A.14) and $\left.g\right|_{E} \in C^{0}(E)$ prove $g \in C^{0}\left(T_{1} \cup\right.$ $T_{2}$ ). Second, (A.15) implies the conditions of (A.11) except those for $\beta \in B_{k-1}$ with $\beta_{1}=0$. Recall that (A.9) and Remark A. 1 show

$$
\left.\nu_{F} \cdot \nabla g_{j}\right|_{E}=k!\sum_{\substack{\beta \in B_{k-1} \\ \beta_{1}=0}} \frac{\lambda^{\beta}}{\beta!} \sum_{\mu=1}^{4} c_{j}\left((\beta ; 0)+e_{\mu}\right)\left(\nu_{F} \cdot \nabla \lambda_{\mu} \mid T_{j}\right) \text { at } x=\lambda_{2} P_{2}+\lambda_{3} P_{3} \in E
$$

for $j=1,2$. Deduce that $\frac{\partial\left(g_{1}-g_{2}\right)}{\partial \nu_{F}}=\nu_{F} \cdot \nabla\left(g_{1}-g_{2}\right)=0$ in $E$ implies $c_{1}((\beta ; 0))=c_{2}((\beta ; 0))$ for all $\beta \in B_{k-1}$ with $\beta_{1}=0$. Consequently, (A.14)-(A.15) and $\left.g\right|_{E},\left.\frac{\partial g}{\partial \nu_{F}}\right|_{E} \in C^{0}(E)$ imply (A.10)-(A.11).

## A.3. Piecewise quadratic polynomials in a WF partition of a tetrahedron

Given a tetrahedron $K=\operatorname{conv}\left\{Q_{1}, Q_{2}, Q_{3}, Q_{4}\right\} \subset \mathbb{R}^{3}$ of a positive volume, select five center points $c_{K}, c_{1}, c_{2}, c_{3}, c_{4}$ in $K$ as follows
(a) $c_{K} \in \operatorname{int}(K)$ (i.e. $\left.\operatorname{dist}\left(c_{K}, \partial K\right)>0\right)$
(b) $c_{m} \in \operatorname{relint}\left(F_{m}\right)$ (i.e. $c_{m}=\sum_{\mu=1}^{4} \lambda_{\mu} Q_{\mu}$ for $\lambda_{1}+\cdots+\lambda_{4}=1$ and $\lambda_{m}=0$ and all other $\left.\lambda_{j}>0\right)$ for the face $F_{m}$ of $K$ that is opposite to the vertex $Q_{m}$ for $m=1,2,3,4$.


Figure A.3.: Illustration of the division of $K=\operatorname{conv}\left\{Q_{1}, \ldots, Q_{4}\right\}$ with center $c_{K} \in \operatorname{int}(K)$ into $T(1), \ldots, T(4)$ in (a) with $T(1)=\operatorname{conv}\left(c_{K}, F_{1}\right)$ marked in red. Further subdivision of $T(1)$ with center $c_{1} \in \operatorname{int}\left(F_{1}\right)$ of face $F_{1}$ in the $W F$ partition in (b) with one subsimplex $T=\operatorname{conv}\left(c_{k}, F\right) \in W F 3 D(K)$ with face $F=\operatorname{conv}\left\{c_{1}, Q_{2}, Q_{4}\right\} \subset F_{1}$ marked in red.

Definition A. 1 ( WF partition in 3D). Given $c_{K}, c_{1}, c_{2}, c_{3}, c_{4}$ with (a)-(b) in the tetrahedron $K=\operatorname{conv}\left\{Q_{1}, Q_{2}, Q_{3}, Q_{4}\right\}$, let the regular triangulation $W F 3 D(K)=\widehat{\mathcal{T}}$ be the set of 12 subtetrahedra obtained as the convex hull $\operatorname{conv}\left\{Q_{j}, Q_{k}, c_{K}, c_{m}\right\}$ of two distinct vertices $Q_{j}$ and $Q_{k}, 1 \leqslant j<k \leqslant 4$, the center point $c_{K}$ of the tetrahedron, and the center $c_{m} \in \operatorname{relint}\left(F_{m}\right)$ of the face $F_{m}$ of $K$ opposite to the vertex $Q_{m}$ for $m \in\{1,2,3,4\} \backslash\{j, k\}$.

Each face $F_{m} \in \mathcal{F}(K)$ is a triangle partitioned in $W F 3 D(K)$ by connecting the center $c_{m} \in$ $\operatorname{relint}\left(F_{m}\right)$ with its vertices $\mathcal{V}\left(F_{m}\right)$ as in the 2D $H C T$ partition of the triangle $F_{m}$. In fact there are two steps in the partition illustrated in Fig. A.3. First, the four tetrahedra $T(m):=$ $\operatorname{conv}\left(c_{K}, F_{m}\right)$ for $m=1,2,3,4$ partition $K$ as displayed in Fig. A.3.a. Second, $\widehat{\mathcal{T}}\left(c_{m}\right):=$ $\left\{T \in \widehat{\mathcal{T}}(K) \mid c_{m}\right.$ vertex of $\left.T\right\}$ partitions each of the tetrahedra $T(1), T(2), T(3), T(4)$ in three subtetrahedra; Fig. A.3.b illustrates the partition of $T(1)$. The following Lemma A. 3 adopts the notation of a $W F$ triangulation $\widehat{\mathcal{T}}=W F 3 D(K)$ and considers each $T(m)=\operatorname{conv}\left(c_{K}, F_{m}\right)$ for $m \in\{1, \ldots, 4\}$ with its decomposition $\widehat{\mathcal{T}}(T(m))=\{T \in \widehat{\mathcal{T}} \mid T \subset T(m)\}=: \widehat{\mathcal{T}}\left(c_{m}\right)$ into three subtetrahedra.
Lemma A.3. Any $f \in P_{k}\left(\widehat{\mathcal{T}}\left(c_{m}\right)\right)$ for $k=0,1,2$ and $m=1,2,3,4$ satisfies $f \in C^{1}(T(m))$ if and only if $f \in P_{k}(T(m))$ is a global polynomial in $T(m)$.

Proof. Without loss of generality let $m=1$ and $T(1)=\operatorname{conv}\left\{c_{K}, Q_{2}, Q_{3}, Q_{4}\right\}$ as displayed in Fig. A.3.b. The assertion is trivial for $k=0$ and immediate for $k=1$ (because continuity means $\nabla f\left(c_{1}\right)=\left.\nabla f\right|_{T}\left(c_{1}\right)$ for all $T \in \hat{\mathcal{T}}\left(c_{1}\right)$ and leads to a global constant vector $\left.\nabla f\right)$. Suppose $k=2$ and $f \in C^{1}(T(1)) \cap P_{2}\left(\widehat{\mathcal{T}}\left(c_{1}\right)\right)$. The function values $f(z)$ at the vertices $z \in\left\{c_{K}, Q_{2}, Q_{3}, Q_{4}\right\}$ of $T(1)$ and at the edge midpoints $z \in\left\{Q_{j k}:=\left(Q_{j}+Q_{k}\right) / 2: 2 \leqslant j<\right.$ $k \leqslant 4\} \cup\left\{\left(c_{K}+Q_{j}\right) / 2: 2 \leqslant j \leqslant 4\right\}$ of $T(1)$ define the quadratic Lagrange interpolation $I_{2} f \in$ $P_{2}(T(1))$ in the tetrahedron $T(1)$. Subtract this (global) quadratic Lagrange interpolation $I_{2} f \in P_{2}(T(1))$ from $f$. The difference $\left(f-I_{2} f\right)(z)=0$ vanishes for each vertex and each edge midpoint of $T(1)$. Since the assertion $f \in P_{2}(T(1))$ is equivalent to $f-I_{2} f \equiv 0$, we may
and will assume without loss of generality that $f$ vanishes at these 10 distinct points on the boundary $\partial T(1)$ and conclude $f \equiv 0$ below.
Identify any subtetrahedron $T=\operatorname{conv}\left\{c_{1}, c_{K}, Q_{j}, Q_{k}\right\} \in \hat{\mathcal{T}}\left(c_{1}\right)$ for distinct $2 \leqslant j<k \leqslant 4$ with $\left(c_{1}, c_{K}, Q_{j}, Q_{k}\right)$. The Bernstein basis of $\left.f\right|_{T} \in P_{2}(T)$ leads to ordinates $\left(c_{T}(\alpha) \mid \alpha \in A_{2}\right)$ in

$$
\left.f\right|_{T}=2 \sum_{\alpha \in A_{2}} c_{T}(\alpha) \lambda^{\alpha} / \alpha!\text { at } x=\lambda_{1} c_{1}+\lambda_{2} c_{K}+\lambda_{3} Q_{j}+\lambda_{4} Q_{k} \in T \in \hat{\mathcal{T}}\left(c_{1}\right)
$$

Since $\left.f\right|_{T}$ is a quadratic function in each $T \in \widehat{\mathcal{T}}\left(c_{1}\right)$, it is in particular a quadratic function on each face $F_{j k}:=\operatorname{conv}\left\{c_{K}, Q_{j}, Q_{k}\right\}$ for $2 \leqslant j<k \leqslant 4$ in $T$ opposite to $c_{1}$. The quadratic function $\left.f\right|_{F_{j k}} \in P_{2}\left(F_{j k}\right)$ vanishes at the vertices $c_{K}, Q_{j}, Q_{k}$ and at the midpoints $Q_{j k},\left(c_{K}+\right.$ $\left.Q_{j}\right) / 2$, and $\left(c_{K}+Q_{k}\right) / 2$ of the edges of the triangle $F_{j k}$. Hence

$$
\left.\left(\left.f\right|_{T}\right)\right|_{F_{j k}}=2 \sum_{\substack{\alpha \in A_{2} \\ \alpha_{1}=0}} c_{T}(\alpha) \lambda^{\alpha} / \alpha!=0 \quad \text { at } x=\lambda_{2} c_{K}+\lambda_{3} Q_{j}+\lambda_{4} Q_{k} \in F_{j k}
$$

vanishes and $c_{T}(0 ; \beta)=0$ for all $\beta \in \mathbb{N}_{0}^{3}$ with $|\beta|=2$ and $T \in \widehat{\mathcal{T}}\left(c_{1}\right)$. This leads to

$$
\begin{equation*}
\left.f\right|_{T}=2 \lambda_{1} \sum_{\beta \in A_{1}} \frac{c_{T}\left(e_{1}+\beta\right)}{\left(1+\beta_{1}\right)} \frac{\lambda^{\beta}}{\beta!} \quad \text { at } x=\lambda_{1} c_{1}+\lambda_{2} c_{K}+\lambda_{3} Q_{j}+\lambda_{4} Q_{k} \in T \in \widehat{\mathcal{T}}\left(c_{1}\right) \tag{A.17}
\end{equation*}
$$

The continuity conditions in Lemma A. 2 with $\xi=1$ in (A.13) motivate the piecewise affine function

$$
\begin{equation*}
\left.f^{\prime}\right|_{T}:=\sum_{\beta \in A_{1}} c_{T}\left(e_{1}+\beta\right) \frac{\lambda^{\beta}}{\beta!}=\sum_{\mu=1}^{4} c_{T}\left(e_{1}+e_{\mu}\right) \lambda_{\mu} \quad \text { at } x \in T \in \widehat{\mathcal{T}}\left(c_{1}\right) \tag{A.18}
\end{equation*}
$$

with ordinates $\left(c_{T}\left(e_{1}+e_{\mu}\right): \mu=1, \ldots, 4\right)$ of $\left.f\right|_{T}$ for each $T \in \hat{\mathcal{T}}\left(c_{1}\right)$. Since $f \in C^{1}(T(1))$ and any two tetrahedra $T_{+}, T_{-} \in \hat{\mathcal{T}}\left(c_{1}\right)$ share a face $F=\partial T_{+} \cap \partial T_{-}=\operatorname{conv}\left\{c_{1}, c_{K}, Q_{j}\right\}$ for some $2 \leqslant j \leqslant 4$ with common vertex $c_{1}$, Lemma A. 2 shows $f^{\prime} \in C^{1}\left(T_{+} \cup T_{-}\right)$. The repeated application of Lemma A. 2 leads to $f^{\prime} \in C^{1}(T(1))$. Since $f^{\prime} \in P_{1}\left(\widehat{\mathcal{T}}\left(c_{1}\right)\right)$ is piecewise affine, the already proven assertion of this lemma for $k=1$ implies that $f^{\prime} \in C^{1}(T(1)) \cap P_{1}\left(\widehat{\mathcal{T}}\left(c_{1}\right)\right)=$ $P_{1}(T(1))$ is one global affine function in $\mathbb{R}^{3}$. Let $\varphi_{1} \in S^{1}\left(\widehat{\mathcal{T}}\left(c_{1}\right)\right):=P_{1}\left(\hat{\mathcal{T}}\left(c_{1}\right)\right) \cap C(T(1))$ denote the first-order nodal basis function with $\varphi_{1}\left(c_{1}\right)=1$ and $\varphi_{1}\left(c_{K}\right)=0=\varphi_{1}\left(Q_{j}\right)$ for any $2 \leqslant j \leqslant 4$. In other words $\left.\varphi_{1}\right|_{T}=\lambda_{1}$ restricted to any $T \in \widehat{\mathcal{T}}\left(c_{1}\right)$ is the first barycentric coordinate $\lambda_{1}$ in $T$. A comparison of $f$ in (A.17) with $f^{\prime}$ in (A.18) reveals

$$
f=2 f^{\prime} \varphi_{1}-\gamma \varphi_{1}^{2} \quad \text { in } C^{1}(T(1))
$$

where $\left.\gamma\right|_{T}:=c_{T}\left(2 e_{1}\right)$ for any $T \in \widehat{\mathcal{T}}\left(c_{1}\right)$ defines the piecewise constant function $\gamma \in P_{0}\left(\hat{\mathcal{T}}\left(c_{1}\right)\right)$. The piecewise quadratic $f$ in $C^{1}$ and the piecewise affine $\nabla f=2\left(f^{\prime}-\gamma \varphi_{1}\right) \nabla \varphi_{1}+2 \varphi_{1} \nabla f^{\prime}$ are continuous in $T(1)$. In particular $\nabla f$ is continuous at the boundary $\partial T(1) \backslash \partial K:=$ $\bigcup_{1 \leqslant j<k \leqslant 4} F_{j k} \subset\left\{\varphi_{1}=0\right\}$ for $F_{j k}:=\operatorname{conv}\left\{c_{K}, Q_{j}, Q_{k}\right\}$ without the face $F_{1} \in \mathcal{F}(K)$, where $\varphi_{1}=0$ vanishes. Hence $f^{\prime} \nabla \varphi_{1}$ is continuous there as well. Since $\nabla \varphi_{1}$ jumps at $c_{K}$ and $Q_{j} \in \partial T(1)$, this implies $f^{\prime}\left(Q_{j}\right)=0=f^{\prime}\left(c_{K}\right)$ for any $2 \leqslant j \leqslant 4$. Thus the affine function $f^{\prime} \in P_{1}(T(1))$ vanishes and $f=\gamma \varphi_{1}^{2} \in C^{1}(K)$ leads to $\gamma=0$ as well. This shows that $f \in C^{1}(T(1)) \cap P_{2}\left(\widehat{\mathcal{T}}\left(c_{1}\right)\right)$ with $\left.f\right|_{F_{j k}}=0$ for all $1 \leqslant j<k \leqslant 4$ implies $f \equiv 0$.

Lemma A.4. Let $\hat{\mathcal{T}}=W F 3 D(K)$ be a WF partition of $K$ and $k=0,1,2$. Then $f \in P_{k}(\widehat{\mathcal{T}})$ satisfies $f \in C^{1}(K)$ if and only if $f \in P_{k}(K)$ is a global polynomial in $K$.

Proof. The proof is similar to that of Lemma A.3. The assertion is trivial for $k=0$ and immediate for $k=1$. Suppose $k=2$ and $f \in C^{1}(K) \cap P_{2}(\widehat{\mathcal{T}})$. In particular $\left.f\right|_{T(m)} \in$ $C^{1}(T(m)) \cap P_{2}\left(\widehat{\mathcal{T}}\left(c_{m}\right)\right)$ for any $m=1,2,3,4$ and Lemma A. 3 shows $\left.f\right|_{T(m)} \in P_{2}(T(m))$. Hence $f \in P_{2}(\widetilde{\mathcal{T}})$ holds for $\widetilde{\mathcal{T}}:=\{T(1), \ldots, T(4)\}$. The 5 vertices in the triangulation $\tilde{\mathcal{T}}$ are $c_{K}, Q_{1}, Q_{2}, Q_{3}, Q_{4}$ and each subtetrahedra $T(m)=\operatorname{conv}\left\{c_{K}, Q_{j}, Q_{k}, Q_{\ell}\right\} \in \widetilde{\mathcal{T}}$ for $1 \leqslant j<$ $k<\ell \leqslant 4$ and $\{j, k, \ell, m\}=\{1,2,3,4\}$ will be identified with $\left(c_{K}, Q_{j}, Q_{k}, Q_{\ell}\right)$. The Bernstein basis of $\left.f\right|_{T(m)} \in P_{2}(T(m))$ leads to ordinates $\left(c_{m}(\alpha) \mid \alpha \in A_{2}\right)$ in

$$
\left.f\right|_{T(m)}=2 \sum_{\alpha \in A_{2}} c_{m}(\alpha) \lambda^{\alpha} / \alpha!\text { at } x=\lambda_{1} c_{K}+\lambda_{2} Q_{j}+\lambda_{3} Q_{k}+\lambda_{4} Q_{\ell} \in T(m) \in \widetilde{\mathcal{T}}
$$

for $m=1, \ldots, 4$. The subtraction of the global quadratic Lagrange interpolation on $K$ shows that, without loss of generality, $f$ vanishes at the vertices $Q_{1}, \ldots, Q_{4}$ and the edge midpoints $Q_{j k}:=\left(Q_{j}+Q_{k}\right) / 2$ for $1 \leqslant j<k \leqslant 4$. Since $f$ is a quadratic function on $\partial K=F_{1} \cup \cdots \cup F_{4}=\left\{\lambda_{1}=0\right\}$,

$$
\left.f\right|_{F_{m}}=2 \sum_{\substack{\alpha \in A_{2} \\ \alpha_{1}=0}} c_{m}(\alpha) \lambda^{\alpha} / \alpha!=0 \quad \text { at } x=\lambda_{2} Q_{j}+\lambda_{3} Q_{k}+\lambda_{4} Q_{\ell} \in F_{m}
$$

vanishes. Hence $c_{m}(0 ; \beta)=0$ for all $\beta \in \mathbb{N}_{0}^{3}$ with $|\beta|=2$ and $\ell=1, \ldots, 4$. Consequently,

$$
\left.f\right|_{T(m)}=2 \lambda_{1} \sum_{\beta \in A_{1}} \frac{c_{m}\left(e_{1}+\beta\right)}{\left(1+\beta_{1}\right)} \frac{\lambda^{\beta}}{\beta!} \quad \text { at } x=\lambda_{1} c_{K}+\lambda_{2} Q_{j}+\lambda_{3} Q_{k}+\lambda_{4} Q_{\ell} \in T(m) \in \widetilde{\mathcal{T}}
$$

Since all $T(m) \in \widetilde{\mathcal{T}}$ are face connected with common vertex $c_{K}$, define $f^{\prime}$ for $\xi=1$ as in (A.13). Repeated applications of Lemma A. 2 show $f^{\prime} \in C^{1}(K) \cap P_{1}(\hat{\mathcal{T}})$ and therefore $f^{\prime} \in P_{1}(K)$ by the proven assertion of this lemma. The comparison of $f$ and $f^{\prime}$ concludes the proof as that of Lemma A.3. Since the arguments apply verbatim, further details are omitted.

## A.4. The WF finite element

Given a tetrahedron $K$ and its $W F$ partition $\widehat{\mathcal{T}}=W F 3 D(K)$ from Definition A.1, this section characterizes

$$
W F(K):=P_{3}(\widehat{\mathcal{T}}) \cap C^{1}(K)
$$

as a 28 -dimensional vector space such that the 28 degrees of freedom (dof) $L_{1}, \ldots, L_{28}$ in Table A. 1 are linear independent and the triple $\left(K, W F(K),\left(L_{1}, \ldots, L_{28}\right)\right.$ ) forms a finite element in the sense of Ciarlet. To describe the degrees of freedom from Table A. 1 for the tetrahedron $K=\operatorname{conv}\left\{Q_{1}, \ldots, Q_{4}\right\}$, determine, for each edge $E_{j k}=\operatorname{conv}\left\{Q_{j}, Q_{k}\right\}$ of $K$ with $1 \leqslant j<k \leqslant 4$ and with midpoint $Q_{j k}=\left(Q_{j}+Q_{k}\right) / 2$ a unit tangential vector $\tau_{j k} \| Q_{j}-Q_{k}$ and two additional unit vectors $\nu_{j k, 1}, \nu_{j k, 2} \in \mathbb{R}^{3} \operatorname{such}$ that $\operatorname{span}\left\{\tau_{j k}, \nu_{j k, 1}, \nu_{j k, 2}\right\}=\mathbb{R}^{3}$. Without loss of generality, assume that $\left(\tau_{j k}, \nu_{j k, 1}, \nu_{j k, 2}\right)$ is an orthonormal basis of $\mathbb{R}^{3}$. The degrees of freedom with the enumeration in Table A. 1 are the evaluation of a function $f \in C^{1}(K)$ and its gradient $\nabla f$ at the vertices $Q_{1}, \ldots, Q_{4}$ of $K$ and the evaluation of the gradient $\nabla f\left(Q_{j k}\right) \cdot s$ in the direction $s=\nu_{j k, 1}$ and $s=\nu_{j k, 2}$ at each edge midpoint $Q_{j k}$. Throughout this section, the

$$
\begin{array}{ll}
L_{\mu}(f)=f\left(Q_{\mu}\right) & \text { for } \mu=1, \ldots, 4 \\
L_{4 \ell+\mu}(f)=\nabla f\left(Q_{\mu}\right) \cdot e_{\ell} & \text { for } \mu=1, \ldots, 4, \ell=1,2,3 \\
L_{20-2^{4-j}+2 k+m}(f)=\nabla f\left(Q_{j k}\right) \cdot \nu_{j k, m} & \text { for } j=1,2,3, k=j+1, \ldots, 4, m=1,2
\end{array}
$$

Table A.1.: The 28 degrees of freedom $L_{1}, \ldots, L_{28}$ of $W F(K)$ for any $f \in C^{1}(K)$.
four vertices $Q_{1}, \ldots, Q_{4} \in \mathcal{V}$ of the tetrahedron $K$ are opposite to the faces $F_{1}, \ldots, F_{4} \in \mathcal{F}$. The Algorithm 1 below with input $x_{1}, \ldots, x_{28} \in \mathbb{R}$ for the 28 degrees of freedom computes $f \in P_{3}(\widehat{\mathcal{T}})$ in terms of ordinates $\left(c_{T}(\alpha) \mid \alpha \in A_{3}, T \in \widehat{\mathcal{T}}\right)$ in

$$
\begin{equation*}
\left.f\right|_{T}=6 \sum_{\alpha \in A_{3}} c_{T}(\alpha) \frac{\lambda^{\alpha}}{\alpha!} \quad \text { at } x=\sum_{\mu=1}^{4} \lambda_{\mu} P_{\sigma(T, \mu)} \in T \in \widehat{\mathcal{T}} \tag{A.19}
\end{equation*}
$$

with the global enumeration $P_{1}, \ldots, P_{9}$ of all the vertices in $\widehat{\mathcal{T}}$ with $P_{1}=c_{K}$. For each $T \in \hat{\mathcal{T}}$ let $P_{\sigma(T, 1)}=P_{1}=c_{K}, P_{\sigma(T, 2)}=c_{m}$ for the unique $m \in\{1,2,3,4\}$ with $\partial T \cap \partial K \subset F_{m}$, and the remaining vertices $P_{\sigma(T, 3)}, P_{\sigma(T, 4)} \in\left\{Q_{1}, \ldots, Q_{4}\right\} \backslash\left\{Q_{m}\right\}$. The algorithm is contained in [WF87].

```
Algorithm 1 Worsey-Farin-CT3D
Input: \(x_{1}, \ldots, x_{28} \in \mathbb{R}\), a tetrahedron \(K=\operatorname{conv}\left\{Q_{1}, Q_{2}, Q_{3}, Q_{4}\right\}\) with WF partition \(\hat{\mathcal{T}}=\)
\(W F 3 D(K)\) and a local enumeration so that \(T \equiv\left(P_{\sigma(T, 1)}, \ldots, P_{\sigma(T, 4)}\right)\) with \(P_{\sigma(T, 1)}=c_{K}\),
\(P_{\sigma(T, 2)}=c_{m}\), and \(P_{\sigma(T, 3)}, P_{\sigma(T, 4)} \in\left\{Q_{1}, \ldots, Q_{4}\right\} \backslash\left\{Q_{m}\right\}\) for each \(T \in \widehat{\mathcal{T}}\left(c_{m}\right)\) with \(m \in\{1, \ldots, 4\}\)
```

(a) A preprocessing computes $f\left(Q_{j k}\right)$ and $\nabla f\left(Q_{j k}\right)$ with $Q_{j k}:=\left(Q_{j}+Q_{k}\right) / 2$ for any $j, k=$ $1, \ldots, 4$, note $Q_{j j} \equiv Q_{j}$, from the data $x_{1}, \ldots, x_{28}$ representing $L_{1}(f), \ldots, L_{28}(f)$ (cf. Remark A. 2 below for further details).
(b) For all $\ell=1, \ldots, 4$ and $1 \leqslant j<k \leqslant 4$,

$$
\begin{aligned}
g\left(Q_{\ell}\right) & :=f\left(Q_{\ell}\right)-\frac{1}{3}\left(Q_{\ell}-c_{K}\right) \cdot \nabla f\left(Q_{\ell}\right) \\
g\left(Q_{j k}\right) & :=\frac{1}{3}\left(c_{K}-Q_{j k}\right) \cdot \nabla f\left(Q_{j k}\right)+f\left(Q_{j k}\right)
\end{aligned}
$$

Let $g \in P_{2}(K)$ denote the quadratic Lagrange interpolation of this data at the vertices $\left(Q_{1}, \ldots, Q_{4}\right)$ and edge midpoints $\left(Q_{j k}: 1 \leqslant j<k \leqslant 4\right)$ of $K$ and compute the ordinates $\left(c_{T}(\alpha) \mid \alpha \in A_{3}, \alpha_{1} \geqslant 1, T \in \hat{\mathcal{T}}\right)$ of

$$
\left.g\right|_{T}=2 \sum_{\beta \in A_{2}} c_{T}\left(e_{1}+\beta\right) \frac{\lambda^{\beta}}{\beta!} \quad \text { at } x=\sum_{\mu=1}^{4} \lambda_{\mu} P_{\sigma(T, \mu)} \in T \in \widehat{\mathcal{T}} .
$$

(c) For all $m=1, \ldots, 4$ let $f_{m} \in H C T\left(F_{m}\right):=P_{3}\left(\widehat{\mathcal{F}}\left(F_{m}\right)\right) \cap C^{1}\left(F_{m}\right)$ denote the 2D $H C T$ finite element interpolation of the input data (resp. data from (a)) on the face $F_{m}$ of $K$ with unit normal $\nu_{F_{m}}$. The degrees for freedom of $H C T\left(F_{m}\right)$ are the evaluation of
$f$ and the tangential derivatives $\nabla f \times \nu_{F_{m}}$ at the vertices of the triangle $F_{m}$ as well as for each edge $E \in \mathcal{E}\left(F_{m}\right)$ with unit tangent vector $\tau_{E}$ of $F_{m}$ the directional derivative in $\left\langle F_{m}\right\rangle$ perpendicular to the edge $E\left(\nabla f \cdot\left(\tau_{E} \times \nu_{F_{m}}\right)\right)$ evaluated in the edge midpoint. Compute coefficients $\left(c_{T}(\alpha) \mid \alpha \in A_{3}, \alpha_{1}=0, T \in \hat{\mathcal{T}}\right)$ such that

$$
\left.f_{m}\right|_{F_{m} \cap \partial T}=6 \sum_{\substack{\alpha \in A_{3} \\ \alpha_{1}=0}} c_{T}(\alpha) \frac{\lambda^{\alpha}}{\alpha!} \quad \text { at } x=\sum_{\mu=2}^{4} \lambda_{\mu} P_{\sigma(T, \mu)} \in F_{m} \cap \partial T \text { for } T \in \widehat{\mathcal{T}} .
$$

Output: $\left(c_{T}(\alpha) \mid \alpha \in A_{3}, T \in \hat{\mathcal{T}}\right)$ and $f \in P_{3}(\hat{\mathcal{T}})$ with $\left.f\right|_{T}=6 \sum_{\alpha \in A_{3}} c_{T}(\alpha) \lambda^{\alpha} / \alpha$ ! at $x=$ $\underline{\sum_{\mu=1}^{4} \lambda_{\mu} P_{\sigma(T, \mu)} \in T \in \hat{\mathcal{T}} .}$

Remark A. 2 (preprocessing in Algorithm 1.a). Algorithm 1 aims at the computation of some global $C^{1}$ function $f$ with prescribed values $L_{\mu}(f)=x_{\mu}$ for $\mu=1, \ldots, 28$. The step (a) provides $f\left(Q_{j k}\right)$ and $\nabla f\left(Q_{j k}\right)$ for any $Q_{j k}$ as follows.
Step a. 1 for $j=k$. Then $Q_{j j}=Q_{j}$ is a vertex and $x_{j}$ directly represents $f\left(Q_{j}\right)=x_{j}$ and $x_{4 \ell+j}$ represents $\frac{\partial f}{\partial x_{\ell}}\left(Q_{j}\right)=x_{4 \ell+j}$ for $\ell=1,2,3$.
Step a.2 for $j<k$. Then $Q_{j k}$ is the midpoint of the edge $E=\operatorname{conv}\left\{Q_{j}, Q_{k}\right\}$ from $Q_{j}$ to $Q_{k}$ with unit tangent vector $\tau_{j k}$ in the tetrahedron $K$. We consider $\left.f\right|_{E} \in P_{3}(E)$ as a one-dimensional cubic polynomial that satisfies the following interpolation conditions. The function values $f\left(Q_{\ell}\right)=x_{\ell}$ and the one-dimensional derivative $f^{\prime}\left(Q_{\ell}\right)=\nabla f\left(Q_{\ell}\right) \cdot \tau_{j k}$ at the vertices $Q_{j}$ and $Q_{k}$ (for $\ell=j, k$ ) are prescribed in Step a.1. The four one-dimensional Hermite interpolation conditions determine a unique cubic polynomial $\left.f\right|_{E} \in P_{3}(E)$ along the edge $E$. This polynomial provides the values $f\left(Q_{j k}\right)$ and $f^{\prime}\left(Q_{j k}\right)=\nabla f\left(Q_{j k}\right) \cdot \tau_{j k}$. The final components of $\nabla f\left(Q_{j k}\right)$ are prescribed by $x_{20-2^{4-j}+2 k+m}=\nabla f\left(Q_{j k}\right) \cdot \nu_{j k, m}$ for $m=1,2$ and the fixed vectors $\nu_{j k, m}$ with $\operatorname{span}\left\{\tau_{j k}, \nu_{j k, 1}, \nu_{j k, 2}\right\}=\mathbb{R}^{3}$.

Remark A. 3 (feasability of Algorithm 1). The data in Algorithm 1.b define a unique $g \in$ $P_{2}(K)$, which has a unique representation in the Bernstein basis of each $T \in \hat{\mathcal{T}}$. Since the 2D HCT finite element is a finite element in the sense of Ciarlet [Cia78, Thm. 6.1.2], $F_{m} \cap \partial T=: F \in \widehat{\mathcal{F}}\left(F_{m}\right)$ and $\left.f_{m}\right|_{F} \in P_{3}(F)$ allow for unique ordinates in Algorithm 1.c (cf. [Mey12] for details on the implementation of a reduced HCT element in 2D).

Remark A. 4 (parameter dependence of Algorithm 1). Suppose the input $x_{1}, \ldots, x_{28} \in \mathbb{R}$ of Algorithm 1 is fixed, while the center points $c_{K} \in \operatorname{int}(K)$ and $c_{m} \in \operatorname{relint}\left(F_{m}\right)$ for $m=1,2,3,4$ may vary, i.e., the $W F$ partition $\hat{\mathcal{T}}=W F 3 D(K)$ differs. The point is that the output $f \in P_{3}(\tilde{\mathcal{T}}) \cap C^{1}(\Omega)$ depends continuously on these five parameters $c_{K}, c_{1}, \ldots, c_{4} \in \mathbb{R}^{3}$. The preprocessing in Algorithm 1.a depends only on the edges of $K$ (see Remark A.2) and is independent of the subtriangulation $\hat{\mathcal{T}}$. The values of $g \in P_{2}(K)$ in the vertices and edge midpoints in Algorithm 1.b depend directly on the choice of $c_{K}$. This dependence in $g$ is affine, whence continuous. The 2D $H C T$ function $f_{m}$ on the face $F_{m}$ of $K$ for $m=1, \ldots, 4$ in Algorithm 1.c depends on the choice of the face center $c_{m}$. The coefficients of the function $f_{m} \in P_{3}(\hat{\mathcal{T}})$ for a given $c_{m}$ are obtained through the solution of a linear system with an invertible matrix [Cia78, p.345, 1.26ff]; this is already exploited in [Cia78, p.346, l.1ff]. Hence the dependence of the ordinates $\left(c_{T}(\alpha): \alpha \in A_{3}, T \in \hat{\mathcal{T}}\right)$ on the points $c_{K}, c_{1}, \ldots, c_{4}$ that define the subtriangulation $\widehat{\mathcal{T}}$ is continuous.

The $W F$ finite element is a finite element in the sense of Ciarlet as announced in [NW19], Since [WF87, NW19] do not contain the proofs, this is carried out here for convenient reading.

Theorem A. 1 ( $\boldsymbol{W} \boldsymbol{F}$ finite element). (a) For any input $x_{1}, \ldots, x_{28} \in \mathbb{R}$, the output $f \in$ $C^{1}(K) \cap P_{3}(\hat{\mathcal{T}})$ of Algorithm 1 is continuously differentiable and

$$
\begin{equation*}
L_{\mu} f=x_{\mu} \quad \text { for } \mu=1, \ldots, 28 \tag{A.20}
\end{equation*}
$$

(b) There exist at most one function $f \in C^{1}(K) \cap P_{3}(\hat{\mathcal{T}})$ with (A.20).
(c) The triple $\left(K, C^{1}(K) \cap P_{3}(\hat{\mathcal{T}}),\left(L_{1}, \ldots, L_{28}\right)\right)$ is a finite element in the sense of Ciarlet.

Proof of (a). This proof is divided in two steps.
Step 1 ( $L_{\mu} f=x_{\mu}$ for $\left.\mu=1, \ldots, 28\right)$. The output $f \in P_{3}(\hat{\mathcal{T}})$ is a piecewise cubic polynomial. To prove (A.20) fix one $T \in \mathcal{T}$, abbreviate $f_{T}:=\left.f\right|_{T}$, and check $L_{\mu} f_{T}=x_{\mu}$ for any degree of freedom $L_{\mu}$, which can be evaluated for $f_{T}$.
Suppose $T:=\operatorname{conv}\left\{c_{K}, c_{m}, Q_{j}, Q_{k}\right\} \in \hat{\mathcal{T}}$ for $1 \leqslant j<k \leqslant 4$ and $m \in\{1, \ldots, 4\} \backslash\{j, k\}$ and identify $T$ with ( $c_{K}, c_{m}, Q_{j}, Q_{k}$ ). The output $f$ on $T \in \hat{\mathcal{T}}$ reads

$$
f_{T}:=\left.f\right|_{T}=6 \sum_{\alpha \in A_{3}} c_{T}(\alpha) \lambda^{\alpha} / \alpha!\quad \text { at } x=\lambda_{1} c_{K}+\lambda_{2} c_{m}+\lambda_{3} Q_{j}+\lambda_{4} Q_{k} \in T .
$$

We can evaluate 10 degrees of freedom from Table A. 1 in $T:=\operatorname{conv}\left\{c_{K}, c_{m}, Q_{j}, Q_{k}\right\}$, namely $L_{\mu}, L_{4+\mu}, L_{8+\mu}, L_{12+\mu}$ for $\mu=j$ and $\mu=k$, as well as $L_{20-2^{4-j}+2 k+1}$ and $L_{20-2^{4-j}+2 k+2}$.
Proof of (A.20) for $\mu=j, k$. Since the 2D HCT finite element in Algorithm 1.c interpolates the nodal values exactly, Example A.3.a shows

$$
L_{j}\left(f_{T}\right)=f_{T}\left(Q_{j}\right)=c_{T}\left(3 e_{3}\right)=\left.f_{m}\right|_{T}\left(Q_{j}\right)=x_{j} .
$$

The proof of $L_{k}\left(f_{T}\right)=x_{k}$ follows by symmetry of $j$ and $k$. Since $L_{j}\left(f_{T}\right)=x_{j}$ for any $T \in \hat{\mathcal{T}}$, the output function $f \in P_{3}(\widehat{\mathcal{T}})$ is single valued and continuous at every vertex $Q_{j} \in \mathcal{V}$ of $K$ for $j=1, \ldots, 4$ with $f\left(Q_{j}\right)=x_{j}$.

Proof of continuity at $Q_{j k}$. Remark A. 2 explains that the data $x_{1}, \ldots, x_{28}$ allow the computation of a value $x_{j k}$ in the edge midpoint in Algorithm 1.a. Algorithm 1.c interpolates $x_{j k}$ such that $\left.f_{m}\right|_{F_{m}}\left(Q_{j k}\right)=x_{j k}$. For any $T \in \hat{\mathcal{T}}(T(m))$ identified with ( $\left.c_{K}, c_{m}, Q_{j}, Q_{k}\right)$ as above, Example A.3.b shows

$$
\begin{align*}
f_{T}\left(Q_{j k}\right) & =\frac{1}{8}\left(f_{T}\left(Q_{k}\right)+3 c_{T}(0,0,2,1)+3 c_{T}(0,0,1,2)+f_{T}\left(Q_{j}\right)\right) \\
& \left.=\frac{1}{8}\left(x_{k}+3 c_{T}(0,0,2,1)+3 c_{T}(0,0,1,2)+x_{j}\right)=f_{m} \right\rvert\, F_{m}\left(Q_{j k}\right)=x_{j k} . \tag{A.21}
\end{align*}
$$

The result $f_{T}\left(Q_{j k}\right)=x_{j k}$ is independent of $m \in\{1,2,3,4\} \backslash\{j, k\}$ and $T \in \hat{\mathcal{T}}(T(m))$. Hence the output function $f \in P_{3}(\hat{\mathcal{T}})$ is single valued and continuous at every edge midpoint $Q_{j k}=$ $\left(Q_{j}+Q_{k}\right) / 2$ of $K$ for $1 \leqslant j<k \leqslant 4$ with $f\left(Q_{j k}\right)=x_{j k}$.

Proof of (A.20) for $\mu=4+j, 8+j, 12+j, 4+k, 8+k, 12+k$. Abbreviate $X_{j}:=$ $\left(x_{4+j}, x_{8+j}, x_{12+j}\right)^{\top}$, recall $T \in \hat{\mathcal{T}}(T(m))$ is identified with $\left(c_{K}, c_{m}, Q_{j}, Q_{k}\right)$. Observe that the gradient $\nabla f_{T}\left(Q_{j}\right) \in \mathbb{R}^{3}$ is uniquely determined by its scalar product with $Q_{j}-Q_{k}$, $Q_{j}-c_{m}$, and $Q_{j}-c_{K}$. The 2D $H C T$ interpolation in Algorithm 1.c interpolates the tangential derivatives correctly. In other words $\left.\nabla f_{m}\right|_{F_{m}}\left(Q_{j}\right) \times \nu_{F_{m}}=X_{j} \times \nu_{F_{m}}$ for the unit normal $\nu_{F_{m}}$ of $F_{m}$. The first two directions $Q_{j}-Q_{k}$ and $Q_{j}-c_{m}$ are parallel to the plane $\left\langle F_{m}\right\rangle$ and Example A.3.c shows

$$
\begin{aligned}
\left(Q_{j}-Q_{k}\right) \cdot \nabla f_{T}\left(Q_{j}\right) & =3\left(f_{T}\left(Q_{j}\right)-c_{T}\left(2 e_{3}+e_{4}\right)\right)=3\left(x_{j}-c_{T}\left(2 e_{3}+e_{4}\right)\right) \\
& =\left.\left(Q_{j}-Q_{k}\right) \cdot \nabla f_{m}\right|_{F_{m}}\left(Q_{j}\right)=\left(Q_{j}-Q_{k}\right) \cdot X_{j}, \\
\left(Q_{j}-c_{m}\right) \cdot \nabla f_{T}\left(Q_{j}\right) & =3\left(f_{T}\left(Q_{j}\right)-c_{T}\left(2 e_{3}+e_{2}\right)\right)=3\left(x_{j}-c_{T}\left(2 e_{3}+e_{2}\right)\right) \\
& =\left(Q_{j}-c_{m}\right) \cdot \nabla f_{m} \mid F_{m}\left(Q_{j}\right)=\left(Q_{j}-c_{m}\right) \cdot X_{j} .
\end{aligned}
$$

The third direction $Q_{j}-c_{K}$ is not parallel to the plane $\left\langle F_{m}\right\rangle$. Example A.3.c and A.2.a imply

$$
\left(Q_{j}-c_{K}\right) \cdot \nabla f_{T}\left(Q_{j}\right)=3\left(f_{T}\left(Q_{j}\right)-c_{T}\left(2 e_{3}+e_{1}\right)\right)=3\left(x_{j}-g\left(Q_{j}\right)\right)
$$

The choice of the nodal value $g\left(Q_{j}\right) \equiv x_{j}-\frac{1}{3}\left(Q_{j}-c_{K}\right) \cdot X_{j}$ in Algorithm 1.b implies

$$
\left(Q_{j}-c_{K}\right) \cdot \nabla f_{T}\left(Q_{j}\right)=\left(Q_{j}-c_{K}\right) \cdot X_{j} .
$$

This concludes the proof of $\nabla f_{T}\left(Q_{j}\right)=X_{j}=\left(x_{4+j}, x_{8+j}, x_{12+j}\right)^{\top}$. The proof of $\nabla f_{T}\left(Q_{k}\right)=$ $\left(x_{4+k}, x_{8+k}, x_{12+k}\right)^{\top}$ follows by symmetry of $j$ and $k$. Hence the gradient of the output function $f \in P_{3}(\mathcal{T})$ is single valued and continuous at every vertex $Q_{j} \in \mathcal{V}$ of $K$ for $j=1, \ldots, 4$ with $\nabla f\left(Q_{j}\right)=X_{j}$.

Proof of (A.20) for $\mu=20-2^{4-j}+2 k+1,20-2^{4-j}+2 k+2$. The remaining degrees of freedom $L_{20-2^{4-j}+2 k+1}$ and $L_{20-2^{4-j}+2 k+2}$ concern the gradient $\nabla f_{T}\left(Q_{j k}\right)$ in the edge midpoint $Q_{j k}$. Remark A. 2 explains how $x_{1}, \ldots, x_{28}$ determine a unique value $X_{j k}$ in Algorithm 1.a that represents $\nabla f\left(Q_{j k}\right)$ in Algorithm 1.b-c. Recall $f\left(Q_{j k}\right)=x_{j k}$ from (A.21) above. We want to verify that $\nabla f_{T}\left(Q_{j k}\right)=X_{j k}$ for $T \in \hat{\mathcal{T}}(T(m))$ identified with ( $c_{K}, c_{m}, Q_{j}, Q_{k}$ ). The gradient $\nabla f_{T}\left(Q_{j k}\right)$ of the output of Algorithm 1 is uniquely determined by its scalar product with $c_{m}-Q_{j k}, Q_{j}-Q_{j k}$, and $c_{K}-Q_{j k}$. In the directions $c_{m}-Q_{j k}$ and $Q_{j}-Q_{j k}$, parallel to the plane $\left\langle F_{m}\right\rangle$, the gradient is determined by $f_{m}$ in Algorithm 1.c. Example A.3.d-e lead to

$$
\begin{aligned}
\left(c_{m}-Q_{j k}\right) \cdot \nabla f_{T}\left(Q_{j k}\right) & =\frac{3}{4}\left(c_{T}(0,1,2,0)+2 c_{T}(0,1,1,1)+c_{T}(0,1,0,2)\right)-3 x_{j k} \\
& =\left.\left(c_{m}-Q_{j k}\right) \cdot \nabla f_{m}\right|_{F_{m}}\left(Q_{j k}\right)=\left(c_{m}-Q_{j k}\right) \cdot X_{j k} . \\
\left.\left(Q_{j}-Q_{j k}\right) \cdot \nabla f\left(Q_{j k}\right)\right|_{T} & =\frac{3}{8}\left(x_{j}+c_{T}\left(2 e_{\gamma}+e_{\kappa}\right)-c_{T}\left(2 e_{\kappa}+e_{\gamma}\right)-x_{k}\right) \\
& =\left.\left(Q_{j}-Q_{j k}\right) \cdot \nabla f_{m}\right|_{F_{m}}\left(Q_{j k}\right)=\left(Q_{j}-Q_{j k}\right) \cdot X_{j k} .
\end{aligned}
$$

For the remaining direction $c_{K}-Q_{j k}$, Example A.3.d and A.2.b show

$$
\begin{aligned}
\left(c_{K}-Q_{j k}\right) \cdot \nabla f_{T}\left(Q_{j k}\right) & =\frac{3}{4}\left(c_{T}\left(e_{1}+2 e_{3}\right)+2 c_{T}\left(e_{1}+e_{3}+e_{4}\right)+c_{T}\left(e_{1}+2 e_{4}\right)\right)-3 f_{T}\left(Q_{j k}\right) \\
& =3 g\left(Q_{j k}\right)-3 x_{j k} .
\end{aligned}
$$

Since the quadratic polynomial $g \in P_{2}(K)$ in Algorithm 1.b assumes the value $g\left(Q_{j k}\right) \equiv$ $\frac{1}{3}\left(c_{K}-Q_{j k}\right) \cdot X_{j k}+x_{j k}$ at $Q_{j k}$, it follows $\left(c_{K}-Q_{j k}\right) \cdot \nabla f_{T}\left(Q_{j k}\right)=\left(c_{K}-Q_{j k}\right) \cdot X_{j k}$. The last three displayed identities ensure $\nabla f_{T}\left(Q_{j k}\right)=X_{j k}$ for any $T \in \hat{\mathcal{T}}\left(Q_{j k}\right)$. Hence the gradient of the output function $f \in P_{3}(\hat{\mathcal{T}})$ is indeed single valued and continuous at every edge midpoint $Q_{j k}=\left(Q_{j}+Q_{k}\right) / 2$ of $K$ for $1 \leqslant j<k \leqslant 4$ with $\nabla f\left(Q_{j k}\right)=X_{j k}$. In particular, $L_{20-2^{4-j}+2 k+1}\left(f_{T}\right)=\nabla f_{T}\left(Q_{j k}\right) \cdot \nu_{j k, 1}=x_{20-2^{4-j}+2 k+1}$ and $L_{20-2^{4-j}+2 k+2}\left(f_{T}\right)=$ $\nabla f_{T}\left(Q_{j k}\right) \cdot \nu_{j k, 2}=x_{20-2^{4-j}+2 k+2}$. This concludes the proof of (A.20).

Step $2\left(f \in C^{1}(K)\right)$. It suffices to show $\left.f\right|_{T_{1} \cup T_{2}} \in C^{1}\left(T_{1} \cup T_{2}\right)$ for any two tetrahedra $T_{1}, T_{2} \in \hat{\mathcal{T}}$ with common face $F:=\partial T_{1} \cap \partial T_{2}$. There are two types of common faces in $\hat{\mathcal{T}}$.
Case 1. Assume $T_{1}, T_{2} \in \hat{\mathcal{T}}\left(c_{m}\right)$ for $m=1, \ldots, 4$. In other words, let $F=\operatorname{conv}\left\{c_{K}, c_{m}, Q_{j}\right\}$ and identify $T_{1}$ with $\left(c_{K}, c_{m}, Q_{j}, Q_{k}\right)$ and $T_{2}$ with $\left(c_{K}, c_{m}, Q_{j}, Q_{\ell}\right)$ for $\{j, k, \ell, m\}=\{1,2,3,4\}$. The definition of $g \in P_{2}(K) \subset C^{1}(K)$ in Algorithm 1.b ensures $\left.g\right|_{T_{1} \cup T_{2}}=\left.f^{\prime}\right|_{T_{1} \cup T_{2}}$ for $\xi=1$ in (A.13). Hence by Lemma A.2, $\left.f\right|_{T_{1} \cup T_{2}} \in C^{1}\left(T_{1} \cup T_{2}\right)$ is equivalent to the continuity of $f$ and $\nu_{F} \cdot \nabla f$ along $E:=\operatorname{conv}\left\{c_{m}, Q_{j}\right\}$. It remains to verify (i) continuity of $f$ along $E$ and (ii) continuity of $\nu_{F} \cdot \nabla f$ along $E$.

Proof of (i). Abbreviate $c_{\mu}(\alpha):=c_{T_{\mu}}(\alpha)$ for $\mu=1,2$. For $\alpha \in A_{3}$ with $\alpha_{1}=0=\alpha_{4}$, the continuity of $f_{m}$ from Algorithm 1.c shows $c_{1}(\alpha)=c_{2}(\alpha)$. For $\alpha_{1} \neq 0=\alpha_{4}$, the continuity of $g$ from Algorithm 1.b shows $c_{1}(\alpha)=c_{2}(\alpha)$. In particular, (A.10) implies that $\left.f\right|_{E}$ is continuous at $E$.
Proof of (ii). Note, since the ordinates $c_{1}\left(e_{2}+\alpha\right)=c_{1}\left(e_{2}+\alpha\right)$ are equal for all $\alpha \in A_{2}$ with $\alpha_{4}=0$, Lemma A. 1 proves the continuity of the piecewise quadratic function $g^{\prime \prime} \in P_{2}\left(\left\{T_{1}, T_{2}\right\}\right)$ defined by

$$
\begin{equation*}
\left.g^{\prime \prime}\right|_{T}:=2 \sum_{\beta \in A_{2}} c_{T}\left(e_{2}+\beta\right) \lambda^{\beta} / \beta!\quad \text { in } T \in\left\{T_{1}, T_{2}\right\} ; \tag{A.22}
\end{equation*}
$$

$g^{\prime \prime} \in C\left(T_{1} \cup T_{2}\right)$. In order to prove the continuity of $\nu_{F} \cdot \nabla f$ along $E$, recall that $\left.f\right|_{F_{m}}=$ $f_{m} \in C^{1}\left(F_{m}\right)$ on the face $F_{m}:=\operatorname{conv}\left\{Q_{\ell}, Q_{j}, Q_{k}\right\} \supset E$ of $K$ from Algorithm 1.c. Hence the tangential derivative $\nu_{F_{m}} \times \nabla f$ is continuous along $F_{m}$ and in particular along $E$. It remains to prove the continuity for one fixed direction $\tau:=c_{m}-c_{K} \in \mathbb{R}^{3} \backslash\{0\}$ oblique to $\left\langle F_{m}\right\rangle$. In $T \in\left\{T_{1}, T_{2}\right\}$, Remark A. 1 shows

$$
\left.\tau \cdot \nabla f\right|_{T}=6 \sum_{\beta \in A_{2}} \sum_{\kappa=1}^{4} c_{T}\left(\beta+e_{\kappa}\right) \frac{\lambda^{\beta}}{\beta!}\left(\tau \cdot \nabla \lambda_{\kappa}\right) .
$$

Since $\tau \cdot \nabla \lambda_{\kappa}=\lambda_{\kappa}\left(c_{m}\right)-\lambda_{\kappa}\left(c_{K}\right)=\delta_{2 \kappa}-\delta_{1 \kappa}$, this is

$$
\left.\tau \cdot \nabla f\right|_{T}=6 \sum_{\beta \in A_{2}}\left(c_{T}\left(\beta+e_{2}\right)-c_{T}\left(\beta+e_{1}\right)\right) \frac{\lambda^{\beta}}{\beta!}=\left.3 g^{\prime \prime}\right|_{T}-\left.3 g\right|_{T}
$$

with $g^{\prime \prime}$ from (A.22) and $g$ from Algorithm 1.b . Since $g^{\prime \prime}$ and $g$ are continuous in $T_{1} \cup T_{2}$, $\left.\nabla f\right|_{F_{m}}$ is continuous in particular along $E$.
Case 2. Assume $T_{1}, T_{2} \in \hat{\mathcal{T}}$ share a face $F=\operatorname{conv}\left\{c_{K}, Q_{j}, Q_{k}\right\}$ and identify $T_{1}$ with $\left(c_{K}\right.$, $\left.Q_{j}, Q_{k}, c_{m}\right)$ and $T_{2}$ with ( $\left.c_{K}, Q_{j}, Q_{k}, c_{\ell}\right)$ for $\{j, k, \ell, m\}=\{1,2,3,4\}$. The definition of $g \in$ $P_{2}(K) \subset C^{1}(K)$ in Algorithm 1.b ensures $\left.g\right|_{T_{1} \cup T_{2}}=\left.f^{\prime}\right|_{T_{1} \cup T_{2}}$ for $\xi=1$ in (A.13). Hence by

Lemma A.2, $\left.f\right|_{T_{1} \cup T_{2}} \in C^{1}\left(T_{1} \cup T_{2}\right)$ is equivalent to the continuity of $f$ and $\nu_{F} \cdot \nabla f$ along $E:=\operatorname{conv}\left\{Q_{j}, Q_{k}\right\}$. It remains to verify (i) continuity of $f$ along $E$ and (ii) continuity of $\nu_{F} \cdot \nabla f$ along $E$.
Proof of ( $i$. Since $\left.f\right|_{E}$ is uniquely determined in Algorithm 1.a, the equality of $f_{m}(x)=f_{\ell}(x)$ at $x=\lambda_{2} Q_{j}+\lambda_{3} Q_{k} \in E$ proves $c_{1}(\alpha)=c_{2}(\alpha)$ for $\alpha \in A_{3}$ with $\alpha_{1}=0=\alpha_{4}$. The equality of the ordinates shows the continuity of $f$ along $E$.
Proof of (ii). Since (each component of) $\left.\nabla f\right|_{T_{1}}$ and $\left.\nabla f\right|_{T_{2}}$ is a quadratic polynomial along $E$, which coincides at the three points $Q_{j}, Q_{j k}, Q_{k}$, it follows $\left.\nabla f\right|_{T_{1}}=\left.\nabla f\right|_{T_{2}}$ on $E$. In particular $\nu_{F} \cdot \nabla f$ is continuous along along $E$.

Proof of (b). Suppose two functions $f_{1}, f_{2} \in P_{3}(\hat{\mathcal{T}}) \cap C^{1}(K)$ satisfy $L_{\mu}\left(f_{j}\right)=x_{\mu}$ for $\mu=$ $1, \ldots, 28$ and $j=1,2$. Their difference $f:=f_{2}-f_{1}$ satisfies $L_{\mu}(f)=0$ for all $\mu=1, \ldots, 28$. Identify $T:=\operatorname{conv}\left\{c_{K}, c_{m}, Q_{j}, Q_{k}\right\} \in \hat{\mathcal{T}}$ with $\left(c_{K}, c_{m}, Q_{j}, Q_{k}\right)$ for $1 \leqslant j<k \leqslant 4$ and $m \in$ $\{1,2,3,4\} \backslash\{j, k\}$. Let $\left(c_{T}(\alpha) \mid \alpha \in A_{3}, T \in \hat{\mathcal{T}}\right)$ denote the coefficients of $f \in C^{1}(K)$ in

$$
\left.f\right|_{T}=6 \sum_{\alpha \in A_{3}} c_{T}(\alpha) \lambda^{\alpha} / \alpha!\quad \text { at } x=\lambda_{1} c_{K}+\lambda_{2} c_{m}+\lambda_{3} Q_{j}+\lambda_{4} Q_{k} \in T .
$$

Since $L_{1}(f)=\cdots=L_{28}(f)=0$, Remark A. 3 implies $f\left(Q_{\ell}\right)=f\left(Q_{j k}\right)=0$ and $\nabla f\left(Q_{\ell}\right)=$ $\nabla f\left(Q_{j k}\right)=0$ for all $\ell=1, \ldots, 4$ and $1 \leqslant j<k \leqslant 4$. The restriction $\left.f\right|_{F}$ of $f$ to a face $F$ of $K$ is a 2D HCT finite element function with vanishing associated degrees of freedom. Hence $\left.f\right|_{F} \equiv 0$. Consequently,

$$
0=\left.f_{T}\right|_{\partial K}=\sum_{\substack{\alpha \in A_{3} \\ \alpha_{1}=0}} c_{T}(\alpha) \frac{\lambda^{\alpha}}{\alpha!} \quad \text { at } x=\lambda_{2} c_{m}+\lambda_{3} Q_{j}+\lambda_{4} Q_{k} \in T \cap \partial K
$$

vanishes. This implies $c_{T}(\alpha)=0$ for all $\alpha \in A_{3}$ with $\alpha_{1}=0$. For $\alpha_{1} \geqslant 0, c_{T}(\alpha)$ is related to $f^{\prime} \in P_{2}(\widehat{\mathcal{T}})$ with

$$
\begin{equation*}
\left.f^{\prime}\right|_{T}:=2 \sum_{\beta \in A_{2}} c_{T}\left(e_{1}+\beta\right) \lambda^{\beta} / \beta!\quad \text { at } x=\lambda_{1} c_{K}+\lambda_{2} c_{m}+\lambda_{3} Q_{j}+\lambda_{4} Q_{k} \in T \text {. } \tag{A.23}
\end{equation*}
$$

Lemma A. 2 applies to any two $T_{+}, T_{-} \in \hat{\mathcal{T}}$ that share a face $T_{+} \cap T_{-}=F \in \hat{\mathcal{F}}$. Hence (A.23) defines $f^{\prime} \in C^{1}(K) \cap P_{2}(\widehat{\mathcal{T}})$ and Lemma A. 4 shows $f^{\prime} \in P_{2}(K)$. For any $T \in \mathcal{T}$ with vertex $Q_{\ell} \in \mathcal{V}(T)$ and $\left(\lambda_{1}, \ldots, \lambda_{4}\right)=e_{\kappa}$ for $\kappa \in\{3,4\}$, (A.7) in Example A. 3 reads

$$
0=\nabla f\left(Q_{\ell}\right)=3 \sum_{\mu=1}^{4} c_{T}\left(2 e_{\kappa}+e_{\mu}\right) \nabla \lambda_{\mu}=3 c_{T}\left(2 e_{\kappa}+e_{1}\right) \nabla \lambda_{1}
$$

with $c_{T}(\alpha)=0$ for $\alpha_{1}=0$ in the last equality. It follows with Example A.2.a that $0=$ $c_{T}\left(e_{1}+2 e_{\kappa}\right)=f^{\prime}\left(Q_{\ell}\right)$ for any $1 \leqslant \ell \leqslant 4$. Since for any $T \in \hat{\mathcal{T}}$ with vertices $Q_{j}$ and $Q_{k}$ the edge midpoint $Q_{j k}$ has barycentric coordinates $\left(\lambda_{1}, \ldots, \lambda_{4}\right)=e_{3} / 2+e_{4} / 2$, (A.8) in Example A. 3 implies

$$
0=\nabla f\left(Q_{j k}\right)=\frac{3}{4} \sum_{\mu=1}^{4}\left(c_{T}\left(2 e_{3}+e_{\mu}\right)+2 c_{T}\left(e_{3}+e_{4}+e_{\mu}\right)+c_{T}\left(2 e_{4}+e_{\mu}\right)\right) \nabla \lambda_{\mu}
$$

$$
=\frac{3}{4}\left(c_{T}\left(2 e_{3}+e_{1}\right)+2 c_{T}\left(e_{3}+e_{4}+e_{1}\right)+c_{T}\left(2 e_{4}+e_{1}\right)\right) \nabla \lambda_{1}
$$

with $c_{T}(\alpha)=0$ for $\alpha_{1}=0$ in the last equality. From Example A.2.b follows

$$
0=c_{T}\left(2 e_{3}+e_{1}\right)+2 c_{T}\left(e_{3}+e_{4}+e_{1}\right)+c_{T}\left(2 e_{4}+e_{1}\right)=\left.4 f^{\prime}\left(Q_{j k}\right)\right|_{T}
$$

for any $T \in \hat{\mathcal{T}}$ with vertices $Q_{j}$ and $Q_{k}$ for $1 \leqslant j<k \leqslant 4$. This shows that $f^{\prime} \in P_{2}(K)$ vanishes at all Lagrange points for a quadratic polynomial in $K$, whence $f^{\prime} \equiv 0$. It follows $c_{T}(\alpha)=0$ for all $T \in \widehat{\mathcal{T}}$ and $\alpha \in A_{3}$; in other words $f \equiv 0$.

Proof of (c). This follows directly from (a), which proves that Algorithm 1 allows the construction of $f \in P_{3}(\widehat{\mathcal{T}}) \cap C^{1}(K)$ for all input data $x_{1}, \ldots, x_{28}$, and (b), which proves the uniqueness of the function with the values $x_{1}, \ldots, x_{28}$ in the degrees of freedom $L_{1}, \ldots, L_{28}$ in Table A.1.

## A.5. The WF on adjacent tetrahedra

This section is devoted to the compatibility of $W F$ for two tetrahedra with a common face. The point of departure is a first Lemma on one tetrahedron. Let $\widehat{\mathcal{T}}$ denote the WF partition of a simplex $K$ with vertices $Q_{1}, \ldots, Q_{4}$ and opposite faces $F_{1}, \ldots, F_{4}$. Recall, $c_{m} \in \operatorname{relint}\left(F_{m}\right)$ and $c_{K} \in \operatorname{relint}(K)$ from the $W F$ partition in Definition A.1, $T(m)=\operatorname{conv}\left(c_{K}, F_{m}\right)$, and abbreviate $\tau:=c_{m}-c_{K} \in \mathbb{R}^{3} \backslash\{0\}$ as well as $\widehat{\mathcal{T}}\left(c_{m}\right):=\widehat{\mathcal{T}}(T(m)$ ) for one fixed $m \in\{1, \ldots, 4\}$.

Lemma A.5. Let $f \in P_{3}\left(\hat{\mathcal{T}}\left(c_{m}\right)\right)$ and $g:=\partial f / \partial \tau \in P_{2}\left(\hat{\mathcal{T}}\left(c_{m}\right)\right)$. Then $f \in C^{1}(T(m))$ if and only if $\left.f\right|_{F_{m}} \in C^{1}\left(F_{m}\right)$ and $g \in P_{2}(T(m))$.

Proof. Without loss of generality let $m=4, \tau:=c_{4}-c_{K}$, and $T(4)=\operatorname{conv}\left(c_{K}, F_{4}\right)$.
$\Longrightarrow \Rightarrow$ " The continuity of $f \in C^{1}(T(4))$ and its derivatives imply $\left.f\right|_{F_{4}} \in C^{1}\left(F_{4}\right)$. Lemma A. 3 leads to $P_{2}\left(\widehat{\mathcal{T}}\left(c_{4}\right)\right) \cap C^{1}(T(4))=P_{2}(T(4))$. Since $g \in P_{2}\left(\widehat{\mathcal{T}}\left(c_{4}\right)\right) \cap C^{0}(T(4))$, it suffices to show $g \in C^{1}(T(4))$. Fix the common side $F=\operatorname{conv}\left\{c_{K}, c_{4}, Q_{j}\right\}=\partial T_{1} \cap \partial T_{2}$ of two neighbouring tetrahedra $T_{1}, T_{2} \in \hat{\mathcal{T}}\left(c_{4}\right)$ identified with $T_{1}=\left(c_{K}, c_{4}, Q_{j}, Q_{k}\right)$ and $T_{2}=\left(c_{K}, c_{4}, Q_{j}, Q_{\ell}\right)$ for $\{j, k, \ell\}=\{1,2,3\}$. Since $\left.f\right|_{T_{j}}=6 \sum_{\alpha \in A_{3}} c_{j}(\alpha) \lambda^{\alpha} / \alpha$ ! for $j=1,2$ defines $f \in C^{1}(T(m))$ in $T_{j}$, Lemma A. 2 shows that $f_{\xi}^{\prime} T_{j}:=2 \sum_{\beta \in A_{k-1}} c_{j}\left(e_{\xi}+\beta\right) \lambda^{\beta} / \beta$ ! for $j=1,2$ in (A.13) forms $f_{\xi}^{\prime} \in C^{1}\left(T_{1} \cup T_{2}\right)$ for any $\xi \in\{1,2,3\}$. Notice that (A.4) implies $\left.\tau \cdot \nabla f\right|_{T_{j}}=6 \sum_{\beta \in A_{2}} \sum_{\mu=1}^{4} c_{j}(\beta+$ $\left.e_{\mu}\right)\left(\tau \cdot \nabla \lambda_{\mu}\right) \lambda^{\beta} / \beta$ ! with $\tau \cdot \nabla \lambda_{\mu}=\lambda_{\mu}\left(c_{4}\right)-\lambda_{\mu}\left(c_{K}\right)=\delta_{2 \mu}-\delta_{1 \mu}$. Hence a.e. in $T_{j}$ it holds

$$
\left.g\right|_{T_{j}}=\left.\tau \cdot \nabla f\right|_{T_{j}}=6 \sum_{\beta \in A_{2}}\left(c_{j}\left(\beta+e_{2}\right)-c_{j}\left(\beta+e_{1}\right)\right) \lambda^{\beta} / \beta!=\left.3 f_{2}^{\prime}\right|_{T_{j}}-3 f_{1}^{\prime} \mid T_{j} \text { for } j=1,2
$$

This and $f_{1}^{\prime}, f_{2}^{\prime} \in C^{1}\left(T_{1} \cup T_{2}\right)$ show $g \in C^{1}\left(T_{1} \cup T_{2}\right)$. Since $\hat{\mathcal{T}}\left(c_{4}\right)$ is side-connected, it follows successively $g \in C^{1}(T(4)) \cap P_{2}\left(\widehat{\mathcal{T}}\left(c_{4}\right)\right)$. Lemma A. 3 implies $g \in P_{2}(T(4))$.
$„ \Leftarrow$ " Suppose $g \in P_{2}(T(4))$ and $\left.f\right|_{F_{4}} \in C^{1}\left(F_{4}\right)$. A one-dimensional integration shows for all $x \in F_{m}$ and $t \in \mathbb{R}$ with $x+t \tau \in T(4)$ that

$$
\begin{equation*}
f(x+t \tau)=f(x)+\int_{0}^{t} g(x+s \tau) \mathrm{d} s \tag{А.24}
\end{equation*}
$$

(Notice that $0<s<t$ and $x \in F_{4}$ with $x+t \tau \in T$ for some tetrahedron $T \in \hat{\mathcal{T}}$ imply hat $x+s \tau$ belongs to the same tetrahedron by definition of $\tau$.) The assumptions $g \in P_{2}(T(4))$ and $\left.f\right|_{F_{4}} \in C^{1}\left(F_{4}\right)$ show in combination with (A.24) that $f \in C(T(4))$ and that $\sigma \cdot \nabla f$ is continuous for all $\sigma \in \operatorname{span}\left\{\tau, Q_{2}-Q_{1}, Q_{3}-Q_{1}\right\} \in \mathbb{R}^{3}$ (in any direction parallel to the plane $\left\langle F_{4}\right\rangle$ and in the oblique direction $\left.\tau \notin \operatorname{span}\left\{Q_{2}-Q_{1}, Q_{3}-Q_{1}\right\} \|\left\langle F_{4}\right\rangle\right)$. This implies $\nabla f$ in $C^{0}\left(T(4) ; \mathbb{R}^{3}\right)$ and concludes the proof.

The main result of this section requires some notation on two tetrahedra with a common side. Suppose that $K_{+}, K_{-} \in \mathcal{T}$ are two neighbouring tetrahedra in a regular triangulation $\mathcal{T}$ that share a common face $F=\partial K_{+} \cap \partial K_{-}$. Suppose that the center points $c_{F}, c_{K_{+}}$, and $c_{K_{-}}$in the $W F$ partition $\widehat{\mathcal{T}}\left(K_{ \pm}\right)$of Definition A. 1 belong to one straight line. In other words $c_{K_{+}}-c_{F} \| c_{K_{-}}-c_{F}$. Suppose the degrees of freedom for some piecewise $W F$ (finite element) function $f \in W F\left(\left\{K_{+}, K_{-}\right\}\right)$with $f_{ \pm}:=\left.f\right|_{K_{ \pm}} \in W F\left(K_{ \pm}\right):=C^{1}\left(K_{ \pm}\right) \cap P_{3}\left(\widehat{\mathcal{T}}\left(K_{ \pm}\right)\right)$coincide at $F=\operatorname{conv}\left\{Q_{1}, Q_{2}, Q_{3}\right\}$. The latter means that

$$
\begin{equation*}
f_{+}\left(Q_{\ell}\right)=f_{-}\left(Q_{\ell}\right), \nabla f_{+}\left(Q_{\ell}\right)=\nabla f_{-}\left(Q_{\ell}\right), \text { and } \nabla f_{+}\left(Q_{j k}\right) \times \tau_{j k}=\nabla f_{-}\left(Q_{j k}\right) \times \tau_{j k} \tag{A.25}
\end{equation*}
$$

for all $1 \leqslant \ell \leqslant 3,1 \leqslant j<k \leqslant 3$. (Recall $Q_{j k}:=\left(Q_{j}+Q_{k}\right) / 2$ is the midpoint of the edge $E_{j k}:=\operatorname{conv}\left\{Q_{j}, Q_{k}\right\}$ with unit tangent vector $\tau_{j k}$ of fixed orientation, e.g., $\tau_{j k}=$ $\left(Q_{k}-Q_{j}\right) /\left|E_{j k}\right|$, in the triangle $F$.)

Theorem A.2. Under the present notation of 18 coinciding degrees of freedom (A.25), $f_{ \pm}$ form a $C^{1}$ function $f \in C^{1}\left(K_{+} \cup K_{-}\right)$on the patch $\omega(F):=\operatorname{int}\left(K_{+} \cup K_{-}\right)$of $F$.

Proof. Recall that the $2 D H C T$ function $\left.f_{m}\right|_{F_{m}} \in H C T\left(F_{m}\right)$ in Algorithm 1.c is computed from twelve degrees of freedom at $F_{m}$. For $f_{+}$and $f_{-}$these twelve degrees of freedom coincide along $F_{m} \equiv F$ owing to the assumption (A.25). Hence the interpolating $2 \mathrm{D} H C T$ function $f_{m \pm}$ in Algorithm 1.c is the same for each of the two tetrahedra $K_{ \pm}$, i.e., $\left.f_{m+}\right|_{K_{+}}=\left.f_{m-}\right|_{K_{-}}=$: $\left.f_{m}\right|_{F} \in H C T(F)$. Algorithm 1.c guarantees that $\left.f_{m}\right|_{F}$ defines the $W F$ function $\left.f_{ \pm}\right|_{F}=\left.f\right|_{F_{m}}$ at $F$. It follows $\left.f_{+}\right|_{F}=\left.f_{-}\right|_{F}=\left.f_{m}\right|_{F}$ and so $f \in C^{0}(\omega(F))$. This and $f_{ \pm} \in C^{1}\left(K_{ \pm}\right)$also imply that the tangential derivatives $\nu_{F} \times \nabla f_{+}(x)=\nu_{F} \times \nabla f_{-}(x)$ coincide at any $x \in F$ (owing to the Hadamard jump condition with unit normal vector $\nu_{F}$ of $F$ ). To conclude the proof of $f \in C^{1}(\omega(F))$, it remains to verify $\tau \cdot \nabla f_{+}(x)=\tau \cdot \nabla f_{-}(x)$ for the direction

$$
\begin{equation*}
\tau:=\frac{c_{K_{+}}-c_{F}}{\left|c_{K_{+}}-c_{F}\right|}=-\frac{c_{K_{-}}-c_{F}}{\left|c_{K_{-}}-c_{F}\right|} \in \mathbb{R}^{3} \backslash\{0\} \tag{A.26}
\end{equation*}
$$

oblique to $\langle F\rangle$. The equality in (A.26) results from the assumption $c_{K_{+}}-c_{F} \| c_{K_{-}}-c_{F}$. Since $f_{ \pm} \in C^{1}\left(K_{ \pm}\right)$, Lemma A. 5 implies that $g_{+}:=\tau \cdot \nabla f_{+} \in P_{2}\left(K_{+}(m)\right)$ and $g_{-}:=\tau$. $\nabla f_{-} \in P_{2}\left(K_{-}(m)\right)$ are quadratic polynomials on $K_{ \pm}(m):=\operatorname{conv}\left(F, c_{ \pm}\right)$, in particular $\left.g_{ \pm}\right|_{F}=$ $\left.\tau \cdot \nabla f_{ \pm}\right|_{F} \in P_{2}(F)$. The continuity of $f_{ \pm}$at $F$ and the equality (A.25) of the 18 degrees of freedom at $F$ imply that the quadratic polynomials $g_{+} \in P_{2}\left(K_{+}(m)\right)$ and $g_{-} \in P_{2}\left(K_{-}(m)\right)$ coincide at $Q_{1}, Q_{2}, Q_{3}, Q_{12}, Q_{13}$, and $Q_{23} \in F$ (see Remark A. 2 for the edge midpoints). The latter are the six Lagrange points of $P_{2}(F)$ in the triangle $F$ and uniquely determine $\left.g_{ \pm}\right|_{F} \in P_{2}(F)$. Consequently, $\left.g_{+}\right|_{F}=\left.g_{-}\right|_{F} \in P_{2}(F)$. This shows that $\tau \cdot \nabla f$ is continuous at $F$ and concludes the proof.


Figure A.4.: Notation in the profile of the intersection $\left\langle F_{1}\right\rangle \cap\left\langle F_{2}\right\rangle$ of two planes through the faces $F_{1}, F_{2} \in \mathcal{F}(T)$ with common edge $E$ indicated by $\tau_{E}$ as $\odot$ perpendicular to the plane depicted in (a) and of $F^{\prime}$ from (A.28) (in red) in (b).

## A.6. The shape-regularity of the WF partition

Let $\mathbb{T}$ denote the set of uniformly shape-regular triangulation $\mathcal{T}$ of a bounded polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^{3}$ into tetrahedra: There exists a global constant $C_{\mathrm{sr}}>0$ with $|T|^{1 / 3} \leqslant h_{T} \leqslant C_{\text {sr }}|T|^{1 / 3}$ for any tetrahedra $T \in \mathcal{T} \in \mathbb{T}$ with diameter $h_{T}$ and volume $|T|$. Throughout this section, let $\mathcal{V}$ denote set of vertices, $\mathcal{F}$ the set of faces, and $\mathcal{E}$ the set of edges in the triangulation $\mathcal{T} \in \mathbb{T}$. For any $E \in \mathcal{E}$ fix one unit tangent vector $\tau_{E} \| E$ and $\nu_{E, 1}, \nu_{E, 2} \in \mathbb{R}^{3}$ such that ( $\tau_{E}, \nu_{E, 1}, \nu_{E, 2}$ ) form an orthonormal basis of $\mathbb{R}^{3}$.

Definition A. 2 (global WF partition). For any $T \in \mathcal{T}$, let $c_{T}$ denote the midpoint of the incircle; for any interior face $F:=\partial T_{+} \cap \partial T_{-} \in \mathcal{F}(\Omega)$, let $c_{F}$ be the intersection of $F$ with the straight line through $c_{T_{+}}$and $c_{T_{-}}$of the neighbouring tetrahedra $T_{ \pm} \in \mathcal{T}$; for any boundary face $F \in \mathcal{F}(\partial \Omega)$, let $c_{F}:=\operatorname{mid}(F)$ denote the center of gravity. Let $\hat{\mathcal{T}}$ denote the subtriangulation of $\mathcal{T}$, where any tetrahedron $T \in \mathcal{T}$ is partitioned in $\widehat{\mathcal{T}}(T)=W F 3 D(T)$ according to Definition A.1.

The center points $c_{T}$ and $c_{F}$ from Definition A. 2 satisfy the conditions (a)-(b) in Section A. 3 according to the following Theorem A.3. In particular, the global WF partition in Definition A. 2 is well-defined.

Theorem A. 3 (shape regularity of $\hat{\mathcal{T}}$ ). There exists a constant $\varepsilon>0$ (that exclusively depends on $\mathbb{T}$ ), such that the points selected in Definition A. 2 satisfy $\operatorname{dist}\left(c_{T}, \partial T\right) \geqslant \varepsilon h_{T}$ for all $T \in \mathcal{T} \in \mathbb{T}$ and $\operatorname{dist}\left(c_{F}, \partial F\right) \geqslant \varepsilon h_{F}$ for all $F \in \mathcal{F}$. In particular, the global WF partition $\widehat{\mathcal{T}}$ in Definition A. 2 is uniformly shape-regular with a shape-regularity constant that is bounded in terms of $C_{\mathrm{sr}}$ in the definition of $\mathbb{T}$.

Proof. Step 1 - Preparation. First consider one tetrahedron $T \in \mathcal{T}$ with largest ball $B(m, \varrho) \subset T$ contained in $T$ with midpoint $m$ and radius $\varrho$ and a common edge $E=$ $\partial F_{1} \cap \partial F_{2} \in \mathcal{E}(T)$ of two of its faces $F_{1}, F_{2} \in \mathcal{F}(T), F_{1} \neq F_{2}$.
Fig. A.4.a illustrates the following notation. Let $\alpha(E)$ denote the dihedral angle of the distinct planes $\left\langle F_{1}\right\rangle$ and $\left\langle F_{2}\right\rangle$ through the faces $F_{1}, F_{2} \in \mathcal{F}(T)$ with common edge $E$. The two intersecting planes touch the ball $\bar{B}(m, \varrho)$ at the two points $m_{j}=\left\langle F_{j}\right\rangle \cap \partial B(m, \varrho) \in \operatorname{int}\left(F_{j}\right)$ for $j=1,2$. The plane $\left\langle m, m_{1}, m_{2}\right\rangle$ through the points $m, m_{1}$, and $m_{2}$ is perpendicular to the unit tangent vector $\tau_{E}$ of the edge $E$. The distance $d(E)$ of $m_{j}$ to the straight line $\langle E\rangle$ through $E$ coincides for $j=1$ and $j=2, d(E):=\operatorname{dist}\left(m_{1},\langle E\rangle\right)=\operatorname{dist}\left(m_{2},\langle E\rangle\right.$ ), according to elementary geometry in a deltoid. The shape regularity of a tetrahedron allows many equivalent characterizations e.g. in terms of angles as discussed in [BKK08]. The lower and upper bounds of $\alpha(E)$ from [BKK08, Eq.(4)] lead in $\varrho=d(E) \tan (\alpha(E) / 2)$ to the equivalence

$$
\begin{equation*}
d(E) \approx \varrho \approx h_{T} \approx h_{F} \quad \text { for all } E \in \mathcal{E}(T) \tag{A.27}
\end{equation*}
$$

(Recall that the equivalence constants in the notation $\approx$ depend exclusively on $C_{\mathrm{sr}}$.) This means that the point $m_{F} \in \partial B(m, \varrho) \cap F$ for the interior side $F$ belongs to the subtriangle

$$
\begin{equation*}
F^{\prime}:=\{x \in F \mid \operatorname{dist}(E, x) \geqslant d(E) \text { for all } E \in \mathcal{E}(F)\} \ni m_{F} \tag{A.28}
\end{equation*}
$$

depicted in Fig. A.4.b. This and (A.27) will be employed in the sequel.
Step 2 - Distance conditions. Suppose that $F=\partial T_{1} \cap \partial T_{2}$ is a common side of two distinct neighbouring tetrahedra $T_{1}, T_{2} \in \mathcal{T}$. Then each $T_{j}$ has a maximal incircle $B\left(m_{j}, \varrho_{j}\right) \subset T_{j}$ with midpoint $m_{j}$ and radius $\varrho_{j}$ and the touching point $m_{F, j} \in F_{j}^{\prime}$ from (A.28). The shape regularity [BKK08] means $h_{T_{1}} \approx \varrho_{1} \approx h_{F} \approx \varrho_{2} \approx h_{T_{2}}$. Hence the choice $m_{j}=c_{T_{j}}$ in Definition A. 2 guarantees $\varepsilon h_{T_{j}} \leqslant \operatorname{dist}\left(c_{T_{j}}, \partial T\right)=\varrho_{j} \approx h_{T_{j}}$ for $j=1,2$ with a constant $\varepsilon \approx 1$.
Let $d_{j}(E)$ replaces the distance $d(E)$ for $T$ replaced by $T_{j}$ for $j=1,2$ in (A.28) to define $F_{j}^{\prime}$. Then $F_{1}^{\prime} \subseteq F_{2}^{\prime}$ or $F_{2}^{\prime} \subseteq F_{1}^{\prime}$ holds. Let

$$
F^{\prime}:=F_{1}^{\prime} \cup F_{2}^{\prime}=\left\{x \in F \mid \operatorname{dist}(E, x) \geqslant \min \left\{d_{1}(E), d_{2}(E)\right\} \text { for all } E \in \mathcal{E}(F)\right\}
$$

denote the bigger set. The face center $c_{F}$ in Definition A. 2 lies in the convex hull of the two touching points $m_{F, 1}=\left(\partial B\left(m_{1}, \varrho_{1}\right) \cap F\right) \in F_{1}^{\prime} \subseteq F^{\prime}$ and $m_{F, 2}=\left(\partial B\left(m_{2}, \varrho_{2}\right) \cap F\right) \in$ $F_{2}^{\prime} \subseteq F^{\prime}$. Since the triangle $F^{\prime}$ depicted in Fig. A.4.b is convex, $c_{F} \in F^{\prime}$. This and (A.27) (with $h_{F} \approx|F|^{1 / 2} \approx \varrho_{1} \approx \varrho_{2}$ from shape regularity [BKK08]) prove that $\operatorname{dist}\left(c_{F}, \partial F\right) \geqslant$ $\min _{E \in \mathcal{E}(F)} \min _{j=1,2} d_{j}(E) \approx h_{F}$. This leads to $\varepsilon \approx 1$ and $\varepsilon h_{F} \leqslant \operatorname{dist}\left(c_{F}, \partial F\right)$.

Step 3 - Shape regularity of $\hat{\mathcal{T}}$. Any tetrahedron $\widehat{T}=\operatorname{conv}\left\{c_{T}, c_{F}, Q_{j}, Q_{k}\right\} \in \hat{\mathcal{T}}$ is contained in one $T \in \mathcal{T}$ with center point $c_{T}$ of the maximal incircle $B\left(c_{T}, \varrho\right) \subset T$ in $T$ and face $F \in \mathcal{F}(T)$ such that $c_{F} \in \operatorname{relint}(F)$ is the point $c_{F} \in F^{\prime}$ from Definition A. 2 and $E=\operatorname{conv}\left\{Q_{j}, Q_{k}\right\} \in$ $\mathcal{E}(F)$ is an edge of $F$. Then

$$
h_{\hat{T}}:=\operatorname{diam}(\widehat{T})=\max \left\{\left|c_{T}-c_{F}\right|,\left|c_{T}-Q_{j}\right|,\left|c_{T}-Q_{k}\right|,\left|c_{F}-Q_{j}\right|,\left|c_{F}-Q_{k}\right|,\left|Q_{j}-Q_{k}\right|\right\} \leqslant h_{T} .
$$

By construction and shape regularity of $\mathcal{T}$ with (A.27)-(A.28) it follows $h_{T} \approx \varrho \leqslant \min \left\{\mid c_{T}-\right.$ $c_{F}\left|,\left|c_{T}-Q_{j}\right|,\left|c_{T}-Q_{k}\right|\right\}, h_{T} \approx d(E) \leqslant \min \left\{\left|c_{F}-Q_{j}\right|,\left|c_{F}-Q_{k}\right|\right\}$, and $h_{T} \approx h_{E}=\left|Q_{j}-Q_{k}\right|$.

In other words $h_{\hat{T}} \approx h_{T}$. The equivalence $h_{\hat{T}} \approx h_{T} \approx \varrho \approx|F|^{1 / 2}$ and the observation $\hat{T}=\operatorname{conv}\left(F, c_{T}\right)$ imply

$$
|\widehat{T}|=\frac{|F| \varrho}{3} \approx h_{T}^{3} \approx h_{\hat{T}}^{3} .
$$

This guarantees the shape regularity of $\hat{\mathcal{T}}$ [BKK08, Eq.(1)] and the shape regularity constant solely depends on $\mathbb{T}$.

## A.7. The scaling of the WF basis functions

Throughout this section, $\hat{\mathcal{T}}$ is the $W F$ partition of a triangulation $\mathcal{T} \in \mathbb{T}$ as in Definition A. 2 with set of vertices $\mathcal{V}$, set of faces $\mathcal{F}$, and set of edges $\mathcal{E}$. For any edge $E \in \mathcal{E}$ fix the orientation of a unit tangent vector $\tau_{E}$ along $E$ as well as $\nu_{E, 1}, \nu_{E, 2} \in \mathbb{R}^{3}$ such that ( $\tau_{E}, \nu_{E, 1}, \nu_{E, 2}$ ) form an orthonormal basis of $\mathbb{R}^{3}$. Set

$$
\begin{equation*}
W F(\mathcal{T}):=P_{3}(\widehat{\mathcal{T}}) \cap H_{0}^{2}(\Omega) \tag{A.29}
\end{equation*}
$$

Definition A. 3 (global degrees of freedom). The global $M=4|\mathcal{V}(\Omega)|+2|\mathcal{E}(\Omega)|$ degrees of freedom are $\left(L_{z, \mu} \mid z \in \mathcal{V}(\Omega), \mu=1, \ldots, 4\right)$ and ( $\left.L_{E, \mu} \mid E \in \mathcal{E}(\Omega), \mu=1,2\right)$ defined, for all $f \in C^{1}(\bar{\Omega})$, by

$$
\begin{array}{ll}
L_{z, 1} f:=f(z), L_{z, \kappa+1}:=\frac{\partial f}{\partial x_{\kappa}}(z) & \text { for } \kappa=1,2,3 \\
L_{E, \mu} f:=\frac{\partial f}{\partial \nu_{E, \mu}}(\operatorname{mid}(E)) & \text { for } \mu=1,2 \tag{A.30}
\end{array}
$$

These $M$ degrees of freedom are also enumerated as $L_{1}, \ldots, L_{M}$. Note, that each degree of freedom $L_{\ell}$ for $\ell=1, \ldots, M$ is a directional derivative of order $\mu_{\ell}=0$ or $\mu_{\ell}=1$ at an interior node $z_{\ell} \in \mathcal{N}(\Omega):=\mathcal{V}(\Omega) \cup\{\operatorname{mid}(E): E \in \mathcal{E}(\Omega)\}$.
For each node $z_{\ell} \in \mathcal{N}(\Omega)$ let $\mathcal{T}\left(z_{\ell}\right):=\left\{T \in \mathcal{T} \mid z_{\ell} \in \partial T\right\}$ denote the set of adjacent simplices and let $\omega_{\ell}:=\operatorname{int}\left(\bigcup \mathcal{T}\left(z_{\ell}\right)\right)$ denote the nodal patch with volume $h_{\ell}^{3}:=\left|\omega_{\ell}\right|$.
Theorem A. 4 below extends the analysis for a quasi-affine family of finite elements in [Cia78, $\S 6.1]$ to 3D and implies the correct scaling of the dual basis functions for $W F(\mathcal{T})$. The point is that Theorem A. 3 lead to uniform bounds in (b). Recall that $a \lesssim b$ abbreviates $a \leqslant C b$ with a generic constant $C$ that only depends the shape-regularity constant $C_{\text {sr }}$ of $\mathcal{T} \in \mathbb{T}$. (The constants below depend on $\varepsilon>0$ from Theorem A. 3 which depends solely on $\mathbb{T}$.)

Theorem A. 4 (scaling of WF basis functions). (a) There exists a unique nodal basis $\left(\varphi_{1}, \ldots, \varphi_{M}\right)$ of $W F(\mathcal{T})$ which satisfies $L_{k}\left(\varphi_{\ell}\right)=\delta_{k \ell}$ for $k, \ell=1, \ldots, M$.

Notation. If $L_{\ell}=L_{z, \mu}$ for $z \in \mathcal{V}, \mu=1, \ldots, 4$, relabel $\varphi_{z, \mu}:=\varphi_{\ell}$. If $L_{\ell}=L_{E, \mu}$ for $E \in \mathcal{E}, \mu=$ 1,2 , relabel $\varphi_{E, \mu}:=\varphi_{\ell}$.
(b) Suppose the degree of freedom $L_{\ell}$ for $\ell=1, \ldots, M$ is a directional derivative of order $\mu_{\ell} \in\{0,1\}$ at the node $z_{\ell} \in \mathcal{N}(\Omega)$ with nodal patch $\omega_{\ell}$ of volume $h_{\ell}^{3}:=\left|\omega_{\ell}\right|$. Then

$$
\left\|\varphi_{\ell}\right\|_{H^{s}(\Omega)} \lesssim h_{\ell}^{3 / 2+\mu_{\ell}-s} \quad \text { for all } s=0,1,2
$$

Proof of (a). Consider one global degree of freedom $L_{\ell}$ with $\ell=1, \ldots, M$ from Definition A. 3 associated with the node $z_{\ell} \in \mathcal{N}(\Omega)$. For any $K \in \mathcal{T}\left(z_{\ell}\right)$ the duality translates in input data $\left(x_{1}, \ldots, x_{28}\right) \in \mathbb{R}^{28}$ for Algorithm 1 as follows: Suppose $L_{\ell}=L_{E, \mu}$ for some edge $E=\operatorname{conv}\left\{Q_{j}, Q_{k}\right\} \in \mathcal{E}(K)$ in $K$ and $\mu \in\{1,2\}$, i.e., $z_{\ell}=\operatorname{mid}(E)$. Recall $\left(\tau_{E}, \nu_{E, 1}, \nu_{E, 2}\right)$ is an orthonormal basis of $\mathbb{R}^{3}$ and $\left(\tau_{j k}, \nu_{j k, 1}, \nu_{j k, 2}\right)$ for $E=E_{j k}$ denotes another basis of $\mathbb{R}^{3}$ in Subsection A.4. Since $\tau_{j k} \| \tau_{E}$, it holds $\nu_{E, \mu}=\sum_{m=1}^{2} \alpha_{m} \nu_{j k, m}$ for some $\alpha_{1}, \alpha_{2} \in \mathbb{R}$. Let $x_{20-2^{4-j}+2 k+m}=\alpha_{m}$ for $m=1,2$ (for the enumeration of Table A.1). Algorithm 1 with input data $\left(0, \ldots, 0, x_{20-2^{4-j}+2 k+1}, x_{20-2^{4-j}+2 k+2}, 0, \ldots, 0\right) \in \mathbb{R}^{28}$ (and $\left.\widehat{\mathcal{T}}(K)\right)$ computes a function $\varphi_{\ell, K} \in W F(K):=P_{3}(\hat{\mathcal{T}}(K)) \cap C^{1}(K)$ with

$$
\begin{aligned}
L_{\ell}\left(\varphi_{\ell, K}\right) & =\nu_{E, \mu} \cdot \nabla \varphi_{\ell, K}(\operatorname{mid}(E))=\alpha_{1} \nu_{j k, 1} \cdot \nabla \varphi_{\ell, K}\left(Q_{j k}\right)+\alpha_{2} \nu_{j k, 2} \cdot \nabla \varphi_{\ell, K}\left(Q_{j k}\right) \\
& =\alpha_{1} x_{20-2^{4-j}+2 k+1}+\alpha_{2} x_{20-2^{4-j}+2 k+2}=\alpha_{1}^{2}+\alpha_{2}^{2}=\left|\nu_{E, \mu}\right|^{2}=1
\end{aligned}
$$

which vanishes $L_{k}\left(\varphi_{\ell, k}\right)=0$ in all other degrees of freedom $k=1, \ldots, M$ with $k \neq \ell$. (For the second normal vector $\nu_{E, \kappa}=\sum_{m=1,2} \beta_{m} \nu_{j k, m}$ for some $\beta_{1}, \beta_{2} \in \mathbb{R}, \kappa \in\{1,2\} \backslash\{\mu\}$ holds $0=\nu_{E, \kappa} \cdot \nu_{E, \mu}=\alpha_{1} \beta_{1}+\alpha_{2} \beta_{2}$ and whence $L_{E, \kappa}\left(\varphi_{\ell, K}\right)=\sum_{m=1}^{2} \beta_{1} x_{20-2^{4-j}+2 k+m}=0$.) Similarly suppose $L_{\ell}=L_{z, \mu}$ for some vertex $z=Q_{j} \in \mathcal{V}(K)$ in $K$ and $\mu \in\{1, \ldots, 4\}$. Set $x_{4(\mu-1)+j}=1$. Algorithm 1 with input data $\left(0, \ldots, 0, x_{4(\mu-1)+j}, 0, \ldots, 0\right) \in \mathbb{R}^{28}($ and $\hat{\mathcal{T}}(K))$ computes a function $\varphi_{\ell, K} \in W F(K):=P_{3}(\widehat{\mathcal{T}}(K)) \cap C^{1}(K)$ with

$$
\begin{aligned}
& L_{\ell}\left(\varphi_{\ell, K}\right)=\varphi_{\ell, K}\left(Q_{j}\right)=x_{j}=1 \quad \text { for } \mu=1, \\
& L_{\ell}\left(\varphi_{\ell, K}\right)=e_{\mu-1} \cdot \varphi_{\ell, K}\left(Q_{j}\right)=x_{4(\mu-1)+j}=1 \quad \text { for } \mu=2,3,4,
\end{aligned}
$$

which vanishes $L_{k}\left(\varphi_{\ell, k}\right)=0$ in all other degrees of freedom $k=1, \ldots, M$ with $k \neq \ell$. Let $\varphi_{\ell} \in P_{3}(\hat{\mathcal{T}})$ denote the global function locally defined as $\left.\varphi_{\ell}\right|_{K}:=\varphi_{\ell, K}$ for any $K \in \mathcal{T}\left(z_{\ell}\right)$ (and $\left.\varphi_{\ell}\right|_{K}=0$ otherwise). In summary the functions $\varphi_{1}, \ldots, \varphi_{M} \in P_{3}(\hat{\mathcal{T}})$ satisfy the duality property in (a). For any two tetrahedra $K_{ \pm}$with common side $F=\partial K_{+} \cap \partial K_{-} \in \mathcal{F}(\Omega)$ the respective center points $c_{K_{+}} \in \operatorname{int}\left(K_{+}\right), c_{K_{-}} \in \operatorname{int}\left(K_{-}\right)$, and $c_{F} \in \operatorname{relint}(F)$ in Definition A. 2 lie on a straight line and the values in the degrees of freedom in (A.25) along $F$ coincide. Hence Theorem A. 2 guarantees $C^{1}$ conformity of $\left.\varphi_{\ell}\right|_{K_{+} \cup K_{-}}$on the side patch $\bar{\omega}(F)=K_{+} \cup K_{-}$. Successive application of this argument leads to $\varphi_{\ell} \in C^{1}(\Omega)$ with $\operatorname{support} \operatorname{supp}\left(\varphi_{\ell}\right)=\overline{\omega_{\ell}}$ in the nodal patch of $z_{\ell} \in \mathcal{N}(\Omega)$. In particular this implies vanishing boundary conditions. Hence $\varphi_{\ell} \in P_{3}(\hat{\mathcal{T}}) \cap H_{0}^{2}(\Omega)=W F(\mathcal{T})$ is a $W F$ function. For any $f \in W F(\mathcal{T})$ holds $\left.f\right|_{K} \in W F(K)$ and along any interior face the values in the degrees of freedom (cf. (A.25)) coincide. Hence the definition of $\left.\varphi_{\ell}\right|_{K}=\varphi_{\ell, K}$ for any $K \in \mathcal{T}$ and Theorem A. 1 imply $f \in \operatorname{span}\left\{\varphi_{\ell}: \ell=1, \ldots, M\right\}$. This concludes the proof of (a).

The proof of Theorem A.4.b follows the methodology of [Cia78, Thm. 6.1.3] with an affine equivalent alternative finite element. Consider a tetrahedron $K=\operatorname{conv}\left\{Q_{1}, Q_{2}, Q_{3}, Q_{4}\right\}$ with vertices $Q_{1}, \ldots, Q_{4} \in \mathcal{V}(K)$ and edges $E_{j k}:=\operatorname{conv}\left\{Q_{j}, Q_{k}\right\} \in \mathcal{E}(K)$ with midpoints $Q_{j k}:=$ $\left(Q_{j}+Q_{k}\right) / 2$ for $1 \leqslant j<k \leqslant 4$.
Definition A. 4 (alternative set of linear functionals). Define $28=16+12$ linear functionals $\mathcal{L}(K):=\left(L_{j k} \mid j, k=1, \ldots, 4\right) \cup\left(L_{a b c} \mid\{a, b, c, d\}=\{1,2,3,4\}, a<b\right)$ on $K=$ $\operatorname{conv}\left\{Q_{1}, Q_{2}, Q_{3}, Q_{4}\right\}$, for $j, k=1, \ldots, 4$ and all pairwise distinct $a, b, c \in\{1,2,3,4\}$ with $a<b$, by

$$
L_{j k}:=\left\{\begin{array}{ll}
\delta_{Q_{j}} & \text { if } j=k  \tag{A.31}\\
\left(Q_{k}-Q_{j}\right) \cdot \delta_{Q_{j}} \nabla & \text { if } j \neq k
\end{array} \quad \text { and } \quad L_{a b c}:=\left(Q_{c}-Q_{a b}\right) \cdot \delta_{Q_{a b}} \nabla .\right.
$$

(Recall the point evaluation $\delta_{P}(f)=f(P)$ and $\delta_{P} \nabla(f)=\nabla f(P)$ for any $P \in \bar{\Omega}$ and $f \in$ $\left.C^{1}(\bar{\Omega}).\right)$
Lemma A. 6 (local basis). There exits a constant $C_{a}>0$ (that exclusively depends on $\mathbb{T}$ ), such that for any tetrahedron $K \in \mathcal{T} \in \mathbb{T}$ with WF partition $\widehat{\mathcal{T}}(K)$, there exists a basis $\left(\varphi_{j k}: j, k=1, \ldots, 4\right) \cup\left(\varphi_{a b c}:\{a, b, c, d\}=\{1,2,3,4\}, a<b\right)$ of $W F(K)$ that satisfies the following. The 28 functions are dual to the linear functionals in $\mathcal{L}(K)$ from (A.31), in that

$$
\begin{equation*}
L_{j k}\left(\varphi_{\ell m}\right)=\delta_{j \ell} \delta_{\ell m}, \quad L_{a b c}\left(\varphi_{d e f}\right)=\delta_{a d} \delta_{b e} \delta_{c f}, \quad \text { and } \quad L_{j k}\left(\varphi_{a b c}\right)=0=L_{a b c}\left(\varphi_{j k}\right) \tag{A.32}
\end{equation*}
$$

for all $j, k, \ell, m=1, \ldots, 4$, all pairwise distinct $a, b, c \in\{1,2,3,4\}$ with $a<b$, and all pairwise distinct $d, e, f \in\{1,2,3,4\}$ with $d<e$, and satisfy, for any $s=0,1,2$, that

$$
\begin{equation*}
\sum_{j, k=1}^{4}\left\|\varphi_{j k}\right\|_{H^{s}(K)}+\sum_{\substack{\{a, b, c, d\}=\{1,2,3,4\} \\ a<b}}\left\|\varphi_{a b c}\right\|_{H^{s}(K)} \leqslant C_{a} h_{K}^{3 / 2-s} . \tag{A.33}
\end{equation*}
$$

Proof. Step 1 - linear independence. For each $K:=\operatorname{conv}\left\{Q_{1}, Q_{2}, Q_{3}, Q_{4}\right\} \in \mathcal{T} \in \mathbb{T}$ the linear functionals in $\mathcal{L}(K)$ from Definition A. 4 are linearly independent. The proof concerns $f \in W F(K):=P_{3}(\hat{\mathcal{T}}(K)) \cap C^{1}(K)$ with $L_{j k} f=0=L_{a b c} f$ for all indices $j, k$ and $a, b, c$. Hence (A.31) implies that $f$ vanishes at the vertices $Q_{1}, \ldots, Q_{4} \in \mathcal{V}(K)$ of $K$. Since for fixed $j \in\{1,2,3,4\}$ the three vectors $Q_{j}-Q_{k} \in \mathbb{R}^{3}$ for $k \in\{1,2,3,4\} \backslash\{j\}$ are linearly independent, (A.31) implies that $\nabla f$ vanishes at the vertices $Q_{1}, \ldots, Q_{4} \in \mathcal{V}(K)$ of $K$ as well. Moreover the tangential derivative $\tau_{a b} \cdot \nabla f$ vanishes along each edge $E_{a b}:=\operatorname{conv}\left\{Q_{a}, Q_{b}\right\} \in \mathcal{E}(K)$ of $K$ with unit tangent vector $\tau_{a b}$ for $1 \leqslant a<b \leqslant 4$. Since $\operatorname{span}\left\{\tau_{a b}, Q_{c}-Q_{a b}, Q_{d}-Q_{a b}\right\}=\mathbb{R}^{3}$ for all $\{a, b, c, d\}=\{1,2,3,4\}$, (A.31) implies that $\nabla f\left(Q_{a b}\right)$ vanishes in all midpoints $Q_{a b}:=$ $\left(Q_{a}+Q_{b}\right) / 2$ of edges $E_{a b} \in \mathcal{E}(K)$ of $K$ as well. We conclude that $L_{1} f=\cdots=L_{28} f=0$ for the $W F$ degrees of freedom from Table A. 1 and so $f \equiv 0$ by Theorem A.1. Hence $(K, W F(K), \mathcal{L}(K))$ is a finite element in the sense of Ciarlet. Consequently, there exist nodal basis functions $\varphi_{j k} \in W F(K)$ and $\varphi_{a b c} \in W F(K)$ with the required duality properties (A.33).

Step 2 - nodal basis in the reference tetrahedron $\widehat{T}$. Let $\Phi: \widehat{T} \rightarrow K$ denote an affine diffeomorphism from the reference tetrahedron $\widehat{T}:=\operatorname{conv}\left\{0, e_{1}, e_{2}, e_{3}\right\}$ onto $K$. The center points in the $W F$ partition $\hat{\mathcal{T}}$ in Definition A. 2 are input parameter in Algorithm 1. For each $K \in \mathcal{T}$ the center points $c_{K} \in \operatorname{int}(K)$ and $c_{m} \in \operatorname{relint}\left(F_{m}\right)$ for $m=1, \ldots, 4$ of the faces $F_{1}, \ldots, F_{4} \in \mathcal{F}(K)$ of $K$ are mapped to five points in $\widehat{T}$. Namely the point $\widehat{c}_{0}:=\Phi^{-1}\left(c_{K}\right)$ and $\widehat{c}_{m}:=\Phi^{-1}\left(c_{m}\right) \in \widehat{F}_{m}=\Phi^{-1}\left(F_{m}\right)$ for $m=1, \ldots, 4$. For each set of points $S:=\left(\hat{c}_{0}, \hat{c}_{1}, \ldots, \hat{c}_{4}\right) \in$ $\mathbb{R}^{15}$ suppressed in the notation, there exists a set of nodal basis functions $\hat{\varphi}_{j k}$ and $\hat{\varphi}_{a b c}$ for $j, k=1, \ldots, 4$ and for all $\{a, b, c, d\}=\{1,2,3,4\}$ with $a<b$ associated with $\widehat{\mathcal{L}}:=\mathcal{L}(\widehat{T})$ in (A.31). Theorem A. 3 shows that the parameters $S:=\left(\hat{c}_{0}, \widehat{c}_{1}, \ldots, \widehat{c}_{4}\right) \in \mathbb{R}^{15}$ belong to a compact set $\mathcal{C}$ of points in $\mathbb{R}^{15}$ such that the distance to the respective relative boundaries is positive. Note that for each fixed parameter $S \in \mathcal{C}$ the algorithm Algorithm 1 (with adapted preprocessing in step (a)) maps input data $x_{j k}$ and $x_{a b c}$ onto a function $f \in W F(\mathcal{T})$ with $L_{j k}(f)=x_{j k}$ and $L_{a b c}(f)=x_{a b c}$ from $\hat{\mathcal{L}}:=\left(L_{j k} \mid j, k=1, \ldots, 4\right) \cup\left(L_{a b c} \mid\{a, b, c, d\}=\right.$ $\{1,2,3,4\}, a<b)$. Remark A. 4 points out that the evaluation of this interpolation operator depends continuously on the parameters in the compact set $\mathcal{C}$. Hence the norms of the nodal basis functions $\hat{\varphi}_{j k}$ and $\hat{\varphi}_{a b c}$ for any $j, k=1, \ldots, 4$ and for all $\{a, b, c, d\}=\{1,2,3,4\}$ with
$a<b$ for the resulting $W F$ partition of the reference tetrahedron $\widehat{T}$ remain bounded by a universal constant $C_{1}>0$ as in [Cia78, Eq. (6.1.30)] for 2D,

$$
\begin{equation*}
\sum_{j, k=1}^{4}\left\|\widehat{\varphi}_{j k}\right\|_{H^{s}(\hat{T})}+\sum_{\substack{\{a, b, c, d\}=\{1,2,3,4\} \\ a<b}}\left\|\widehat{\varphi}_{a b c}\right\|_{H^{s}(\widehat{T})} \leqslant C_{1} . \tag{A.34}
\end{equation*}
$$

Note that the constant $C_{1}$ depends exclusively on $\mathcal{C}$ and so on $\varepsilon>0$ from Theorem A.3.
Step 3 - affine equivalence. Recall the affine diffeomorphism $\Phi: \widehat{T} \rightarrow K$ from the reference tetrahedron $\widehat{T}:=\operatorname{conv}\left\{0, e_{1}, e_{2}, e_{3}\right\}$ onto $K$ from Step 2. The point is that the alternative finite element $(K, W F(K), \mathcal{L}(K))$ is affine equivalent to $(\widehat{T}, W F(\mathcal{T}), \widehat{\mathcal{L}})$ in that $\widehat{\mathcal{L}}=\mathcal{L}(K) \circ \Phi$; cf. [Cia78, BS08] for the concept of affine equivalence. In particular the local nodal basis $\varphi_{j k} \in W F(K)$ and $\varphi_{a b c} \in W F(K)$ associated with (A.31) and the local nodal basis functions $\widehat{\varphi}_{j k} \in W F(\mathcal{T})$ and $\widehat{\varphi}_{a b c} \in W F(\mathcal{T})$ on the reference tetrahedron $\widehat{T}$ satisfy

$$
\varphi_{j k}=\hat{\varphi}_{j k} \circ \Phi^{-1} \quad \text { and } \quad \varphi_{a b c}=\hat{\varphi}_{a b c} \circ \Phi^{-1}
$$

for any $j, k=1, \ldots, 4$ and all pairwise distinct $a, b, c \in\{1,2,3,4\}$ with $a<b$. Standard scaling properties in [Cia78, Thm. 3.1.2-3.1.3] lead to a universal constant $C_{2}$ (that depends only on the reference tetrahedron $\widehat{T}$ ) such that for any $s=0,1,2$

$$
\left\|\varphi_{a b c}\right\|_{H^{s}(K)} \leqslant C_{2}|K|^{1 / 2} \varrho_{K}^{-s}\left\|\widehat{\varphi}_{a b c}\right\|_{H^{s}(\hat{T})} \quad \text { and } \quad\left\|\varphi_{j k}\right\|_{H^{s}(K)} \leqslant C_{2}|K|^{1 / 2} \varrho_{K}^{-s}\left\|\hat{\varphi}_{j k}\right\|_{H^{s}(\hat{T})}
$$

with the radius $\varrho_{K}$ of the biggest incircle in $K$. The shape regularity of $K \in \mathcal{T} \in \mathbb{T}$ guarantees $\varrho_{K} \approx h_{K} \approx|K|^{1 / 3}$ and so

$$
|K|^{1 / 2} \varrho_{K}^{-s} \leqslant C_{3} h_{K}^{3 / 2-s} .
$$

The combination of the last two displayed estimates with (A.34) concludes the proof of (A.33) with $C_{a}=C_{1} C_{2} C_{3}$.

The point in the proof of Theorem A.4.b below is that each (global) nodal basis function $\varphi_{\ell}$ for $\ell=1, \ldots, M$ restricted to one tetrahedron $K \subset \operatorname{supp}\left(\varphi_{\ell}\right)$ is identical with its interpolation in the local basis from Lemma A.6, i.e.,

$$
\begin{equation*}
\left.\varphi_{\ell}\right|_{K}=\sum_{j, k=1}^{4} L_{j k}\left(\varphi_{\ell}\right) \varphi_{j k}+\sum_{\substack{\{a, b, c, d\}=\{1,2,3,4\} \\ a<b}} L_{a b c}\left(\varphi_{\ell}\right) \varphi_{a b c} . \tag{A.35}
\end{equation*}
$$

The interpolation (A.35) and a triangle inequality show

$$
\begin{equation*}
\left\|\varphi_{\ell}\right\|_{L^{2}(K)} \leqslant \sum_{j, k=1}^{4}\left|L_{j k}\left(\varphi_{\ell}\right)\left\|\varphi_{j k}\right\|_{L^{2}(K)}+\sum_{\substack{\{a, b, c, d\}=\{1,2,3,4\} \\ a<b}}\right| L_{a b c}\left(\varphi_{\ell}\right)\left\|\varphi_{a b c}\right\|_{L^{2}(K)} . \tag{A.36}
\end{equation*}
$$

A careful analysis of the contributions $\left|L_{j k}\left(\varphi_{\ell}\right)\right|$ and $\left|L_{a b c}\left(\varphi_{\ell}\right)\right|$ on the right-hand side of (A.36) for each of the three types of global nodal basis functions from Definition A. 3 and the application of the scaling in (A.33) conclude the proof.

Proof of Theorem A.4.b for $\varphi_{\ell}=\varphi_{E, \mu}$ for $E \in \mathcal{E}(\Omega)$. We discuss the scaling first for the basis function related to the global degree of freedom

$$
L_{E, \mu}:=\nu_{E, \mu} \cdot \delta_{\operatorname{mid}(E)} \nabla \quad \text { for } \mu=1,2 \text { and } E \in \mathcal{E}
$$

Consider one tetrahedron $K:=\operatorname{conv}\left\{Q_{1}, \ldots, Q_{4}\right\} \in \mathcal{T}(E)$ with $W F$ partition $\widehat{\mathcal{T}}(K)$. Suppose, without loss of generality, $E=\operatorname{conv}\left\{Q_{1}, Q_{2}\right\}=E_{12}$. Note $\left.\varphi_{\ell}\right|_{K} \in W F(K)$ satisfies $L_{E, \mu} \varphi_{\ell}=1$ and all the other degrees of freedom of $\varphi_{\ell}$ in Definition A. 3 vanish. In particular, $\varphi_{\ell}$ and $\nabla \varphi_{\ell}$ vanish at $Q_{1}, \ldots, Q_{4}$. This implies in (A.36), that

$$
L_{j k}\left(\varphi_{\ell}\right)=0 \quad \text { for all } j, k=1, \ldots, 4
$$

Along any edge $E_{j k} \in \mathcal{E}(K)$ with unit tangent vector $\tau_{j k}$ for $1 \leqslant j<k \leqslant 4$ we consider $\left.\varphi_{\ell}\right|_{E_{j k}} \in$ $P_{3}\left(E_{j k}\right)$ as a one-dimensional cubic polynomial with four vanishing degrees of freedom. Hence the one-dimensional derivative $\varphi_{\ell}^{\prime}=\tau_{j k} \cdot \nabla \varphi_{\ell}=0$ vanishes. For any edge $E_{j k} \in \mathcal{E}(K) \backslash\left\{E_{12}\right\}$ in $K$ besides $E_{12}$ the gradient $\nabla \varphi_{\ell}$ in the midpoint $Q_{j k}$ vanishes in the direction $\nu_{j k, 1}$ and $\nu_{j k, 2}$. In other words $\nabla \varphi_{\ell}\left(Q_{j k}\right)=0$ for all $(j, k) \neq(1,2)$ with $1 \leqslant j<k \leqslant 4$. This implies

$$
L_{a b c}\left(\varphi_{\ell}\right)=0 \quad \text { for all }(a, b, c) \neq(1,2,3) \text { or }(a, b, c) \neq(1,2,4)
$$

Consequently, (A.35) reads
$\varphi_{\ell}=L_{123}\left(\varphi_{\ell}\right) \varphi_{123}+L_{124}\left(\varphi_{\ell}\right) \varphi_{124}=\left(Q_{3}-Q_{12}\right) \cdot \nabla \varphi_{\ell}\left(Q_{12}\right) \varphi_{123}+\left(Q_{4}-Q_{12}\right) \cdot \nabla \varphi_{\ell}\left(Q_{12}\right) \varphi_{124}$.
Recall $\tau_{12} \cdot \nabla \varphi_{\ell}\left(Q_{12}\right) \equiv 0$ for the unit tangent vector $\tau_{12} \equiv \tau_{E}$ of the edge $E_{12} \equiv E$ and $L_{E, \kappa}\left(\varphi_{\ell}\right)=\nu_{E, \kappa} \cdot \nabla \varphi_{\ell}\left(Q_{12}\right)=0$ for the normal $\nu_{E, \kappa}$ with $\kappa \in\{1,2\} \backslash\{\mu\}$. Since $L_{\ell}\left(\varphi_{\ell}\right)=$ $\nu_{E, \mu} \cdot \nabla \varphi_{\ell}\left(Q_{12}\right)=1$, this implies for $k=3,4$

$$
\begin{aligned}
\left|L_{12 k}\left(\varphi_{\ell}\right)\right| & =\left|\left(Q_{k}-Q_{12}\right) \cdot\left(\left(\nu_{E, \mu} \cdot \nabla \varphi_{\ell}\left(Q_{12}\right)\right) \nu_{E, \mu}+\left(\nu_{E, \kappa} \cdot \nabla \varphi_{\ell}\left(Q_{12}\right)\right) \nu_{E, \kappa}+\left(\tau_{E} \cdot \nabla \varphi_{\ell}\left(Q_{12}\right)\right) \tau_{E}\right)\right| \\
& =\left|\left(Q_{k}-Q_{12}\right) \cdot \nu_{E, \mu}\right| \leqslant h_{K}
\end{aligned}
$$

For (A.36) this implies that

$$
\left\|\varphi_{\ell}\right\|_{H^{s}(K)}=\left\|\varphi_{E, \kappa}\right\|_{H^{s}(K)} \leqslant h_{K} \sum_{k=3,4}\left\|\varphi_{12 \ell}\right\|_{H^{s}(K)} \leqslant C_{a} h_{K}^{5 / 2-s}
$$

with the upper bound from Lemma A. 6 in the last step. The shape regularity of $\mathcal{T} \in \mathbb{T}$ implies $|K| \approx h_{K}^{3}$ and the boundedness of the number of tetrahedra $|\mathcal{T}(E)|$ in the support of $\varphi_{E, \kappa}$. This concludes the proof.

Proof of Theorem A.4.b for $\varphi_{\ell}=\varphi_{z, 1}$ for $z \in \mathcal{V}(\Omega)$. Consider the basis function associated with

$$
L_{z, 1}:=\delta_{z} \quad \text { for } z \in \mathcal{V}
$$

and $K:=\operatorname{conv}\left\{Q_{1}, \ldots, Q_{4}\right\} \in \mathcal{T}(z)$ with $W F$ partition $\hat{\mathcal{T}}(K)$. Suppose, without of loss of generality, $z=Q_{1}$. Note $\left.\varphi_{\ell}\right|_{K} \in W F(K)$ satisfies $L_{z, 1}\left(\varphi_{\ell}\right)=\varphi_{\ell}\left(Q_{1}\right)=1=L_{11}\left(\varphi_{\ell}\right)$ and all the other degrees of freedom in Definition A. 3 of $\varphi_{\ell}$ vanish. Since $\varphi_{\ell}$ vanishes at $Q_{2}, Q_{3}, Q_{4}$ and $\nabla \varphi_{\ell}$ vanish at $Q_{1}, Q_{2}, Q_{3}, Q_{4},($ A.31) implies

$$
L_{k k}\left(\varphi_{\ell}\right)=0 \quad \text { and } \quad L_{j k}\left(\varphi_{\ell}\right)=0 \quad \text { for } 1 \leqslant j<k \leqslant 4
$$

For any edge $E \in \mathcal{E}$ with midpoint $\operatorname{mid}(E)$ holds $L_{E, 1}\left(\varphi_{\ell}\right)=\nu_{E, 1} \cdot \nabla f(\operatorname{mid}(E))=0=$ $\nu_{E, 2} \cdot \nabla \varphi_{\ell}(\operatorname{mid}(E))=L_{E, 2}\left(\varphi_{\ell}\right)$. This implies for all pairwise distinct $a, b, c \in\{1,2,3,4\}$ with $a<b$ and edges $E_{a b} \in \mathcal{E}(K)$ with unit tangent vector $\tau_{a b}$ and midpoint $Q_{a b}$, that

$$
L_{a b c}\left(\varphi_{\ell}\right)=\left(Q_{c}-Q_{a b}\right) \cdot \nabla \varphi_{\ell}\left(Q_{a b}\right)=\left(Q_{c}-Q_{a b}\right) \cdot\left(\tau_{a b} \cdot \nabla \varphi_{\ell}\left(Q_{a b}\right)\right) \tau_{a b}
$$

Along any $E_{j k} \in \mathcal{E}(K)$ for $2 \leqslant j<k \leqslant 4$ in $K$ without the vertex $Q_{1}$ with unit tangent vector $\tau_{j k}$, consider $\varphi_{\ell} \mid E_{j k} \in P_{3}\left(E_{j k}\right)$ as a one-dimensional cubic polynomial with vanishing degrees of freedom. Hence the one-dimensional derivative $\varphi_{\ell}^{\prime}=\tau_{j k} \cdot \nabla \varphi_{\ell}=0$ vanishes. In other words,

$$
L_{a b c}\left(\varphi_{\ell}\right)=\left(Q_{c}-Q_{a b}\right) \cdot\left(\tau_{a b} \cdot \nabla \varphi_{\ell}\left(Q_{a b}\right)\right) \tau_{a b}=0 \quad \text { for } 2 \leqslant a<b \leqslant 4
$$

and (A.35) reads

$$
\varphi_{\ell}=L_{11}(f) \varphi_{11}+\sum_{\substack{\leqslant \leqslant b, c \leqslant 4 \\ b c c}} L_{1 b c} \varphi_{1 b c}=\varphi_{11}+\sum_{\substack{2 \leqslant b, c \leqslant 4 \\ b \neq c}}\left(Q_{c}-Q_{1 b}\right) \cdot\left(\left(\tau_{1 b} \cdot \nabla \varphi_{\ell}\left(Q_{1 b}\right)\right) \tau_{1 b}\right) \varphi_{1 b c} .
$$

Along any edge $E=E_{1 b} \in \mathcal{E}(K)$ for $b=2,3,4$ with unit tangent vector $\tau_{E} \equiv \tau_{1 b}$ consider $\left.\varphi_{\ell}\right|_{E} \in P_{3}(E)$ as one-dimensional function uniquely determined by the given values of $\varphi_{\ell}$ and the tangential derivative $\tau_{E} \cdot \nabla \varphi_{\ell}$ in the endpoints $Q_{1}$ and $Q_{b}$ of $E$. The parametrisation $f_{\tau}(s)$ for $s \in(0, h)$ of $\left.\varphi_{\ell}\right|_{E}$ with $f_{\tau}(0)=\varphi_{\ell}\left(Q_{1}\right)=1, f_{\tau}(h)=\varphi_{\ell}\left(Q_{b}\right)=0$, and $f_{\tau}^{\prime}=\tau_{E} \cdot \nabla \varphi_{\ell}$ vanishing in both endpoints, reads

$$
f_{\tau}(s)=h^{-3}(s-h)^{2}(2 s+h) .
$$

Hence, $\left|\tau_{E} \cdot \nabla \varphi_{\ell}\left(Q_{1 b}\right)\right|=\left|f_{\tau}^{\prime}(h / 2)\right|=1.5 h^{-1} \leqslant C_{4} h_{K}^{-1}$. In other words, for all $b=2,3,4$,

$$
\left|L_{1 b c}\left(\varphi_{\ell}\right)\right|=\left|\left(Q_{c}-Q_{1 b}\right) \cdot\left(\tau_{1 b} \cdot \nabla \varphi_{\ell}\left(Q_{1 b}\right)\right) \tau_{1 b}\right| \leqslant\left|\left(Q_{c}-Q_{1 b}\right) \cdot \tau_{1 b}\right| C_{4} h_{K}^{-1} \leqslant C_{4} .
$$

For (A.36) this implies that

$$
\left\|\varphi_{\ell}\right\|_{H^{s}(K)}=\left\|\varphi_{z, 1}\right\|_{H^{s}(K)}=\left\|\varphi_{11}\right\|_{H^{s}(K)}+C_{4} \sum_{\substack{2 \leqslant b, c \leqslant 4 \\ b \neq c}}\left\|\varphi_{1 b c}\right\|_{H^{s}(K)} \leqslant \max \left\{1, C_{4}\right\} C_{a} h_{K}^{3 / 2-s}
$$

with the upper bound from Lemma A. 6 in the last step. The shape regularity of $\mathcal{T} \in \mathbb{T}$ implies $|K| \approx h_{K}^{3}$ and the boundedness of the number of tetrahedra $|\mathcal{T}(z)|$ in the support of $\varphi_{z, 1}$. This concludes concludes the proof.

Proof of Theorem A.4.b for $\varphi_{\ell}=\varphi_{z, \kappa+1}$ for $z \in \mathcal{V}(\Omega)$. Finally, regard the basis function associated with

$$
L_{z, \kappa+1}:=e_{\kappa} \cdot \delta_{z} \nabla \quad \text { for } \kappa=1,2,3, \text { and } z \in \mathcal{V}
$$

and consider $K:=\operatorname{conv}\left\{Q_{1}, \ldots, Q_{4}\right\} \in \mathcal{T}(z)$ with $W F$ partition $\widehat{\mathcal{T}}(K)$. Suppose, without of loss of generality, $z=Q_{1}$. Note $\left.\varphi_{\ell}\right|_{K} \in W F(K)$ satisfies $L_{z, \kappa+1} \varphi_{\ell}=\partial \varphi_{\ell} / \partial x_{\kappa}=1$ and all the other degrees of freedom of $\varphi_{\ell}$ vanish. Since $\varphi_{\ell}$ vanishes at $Q_{1}, Q_{2}, Q_{3}, Q_{4}$ and $\nabla \varphi_{\ell}$ vanish at $Q_{2}, Q_{3}, Q_{4}$, the definition in (A.31) shows

$$
L_{j j}\left(\varphi_{\ell}\right)=0 \quad \text { for } j=1, \ldots, 4 \quad \text { and } \quad L_{j k}\left(\varphi_{\ell}\right)=0 \quad \text { for } 2 \leqslant j<k \leqslant 4
$$

For any edge $E \in \mathcal{E}$ with midpoint $\operatorname{mid}(E)$ holds $L_{E, 1}\left(\varphi_{\ell}\right)=\nu_{E, 1} \cdot \nabla \varphi_{\ell}(\operatorname{mid}(E))=0=$ $\nu_{E, 2} \cdot \nabla \varphi_{\ell}(\operatorname{mid}(E))=L_{E, 2}\left(\varphi_{\ell}\right)$. This implies all pairwise distinct $a, b, c \in\{1,2,3,4\}$ with $a<b$ and edges $E_{a b} \in \mathcal{E}(K)$ with unit tangent vector $\tau_{a b}$ and midpoint $Q_{a b}$, that

$$
L_{a b c}(f)=\left(Q_{c}-Q_{a b}\right) \cdot \nabla \varphi_{\ell}\left(Q_{a b}\right)=\left(Q_{c}-Q_{a b}\right) \cdot\left(\tau_{a b} \cdot \nabla \varphi_{\ell}\left(Q_{a b}\right)\right) \tau_{a b} .
$$

Along any $E_{j k} \in \mathcal{E}(K)$ for $2 \leqslant j<k \leqslant 4$ in $K$ without the vertex $Q_{1}$ with unit tangent vector $\tau_{j k}$, consider $\varphi_{\ell} \mid E_{j k} \in P_{3}\left(E_{j k}\right)$ as a one-dimensional cubic polynomial with vanishing degrees of freedom. Hence the one-dimensional derivative $\varphi_{\ell}^{\prime}=\tau_{j k} \cdot \nabla \varphi_{\ell}=0$ vanishes. In other words,

$$
L_{a b c}\left(\varphi_{\ell}\right)=\left(Q_{c}-Q_{a b}\right) \cdot\left(\tau_{a b} \cdot \nabla \varphi_{\ell}\left(Q_{a b}\right)\right) \tau_{a b}=0 \quad \text { for } 2 \leqslant a<b \leqslant 4
$$

and (A.35) reads

$$
\begin{aligned}
\varphi_{\ell} & =\sum_{k=2}^{4} L_{1 k}\left(\varphi_{\ell}\right) \varphi_{1 k}+\sum_{\substack{2 \leqslant b, c \leqslant 4 \\
b \neq c}} L_{1 b c}\left(\varphi_{\ell}\right) \varphi_{1 b c} \\
& =\sum_{k=2}^{4}\left(Q_{k}-Q_{1}\right) \cdot \nabla \varphi_{\ell}\left(Q_{1}\right) \varphi_{1 k}+\sum_{\substack{2 \leqslant b, c \leqslant 4 \\
b \neq c}}\left(Q_{c}-Q_{1 b}\right) \cdot\left(\left(\tau_{1 b} \cdot \nabla \varphi_{\ell}\left(Q_{1 b}\right)\right) \tau_{1 b}\right) \varphi_{1 b c} .
\end{aligned}
$$

Since $\partial \varphi_{\ell} / \partial x_{\ell}\left(Q_{1}\right)=0$ for $\ell \in\{1,2,3\} \backslash\{\kappa\}$ and $\partial \varphi_{\ell} / \partial x_{\kappa}\left(Q_{1}\right)=L_{z, \kappa+1}\left(\varphi_{\ell}\right)=1$, it holds for $k=2,3,4$

$$
\left|L_{1 k}\left(\varphi_{\ell}\right)\right|=\left|\left(Q_{k}-Q_{1}\right) \cdot \nabla \varphi_{\ell}\left(Q_{1}\right)\right|=\left|\left(Q_{k}-Q_{1}\right) \cdot e_{\kappa} \partial \varphi_{\ell} / \partial x_{\kappa}\left(Q_{1}\right)\right|=\left|\left(Q_{k}-Q_{1}\right) \cdot e_{\kappa}\right| \leqslant h_{K} .
$$

Along any edge $E=E_{1 b} \in \mathcal{E}(K)$ for $b=2,3,4$ with unit tangent vector $\tau_{E}=\tau_{1 b}$ consider $\left.\varphi_{\ell}\right|_{E} \in P_{3}(E)$ as one-dimensional function uniquely determined by the given values of $\varphi_{\ell}$ and the tangential derivative $\tau_{E} \cdot \nabla \varphi_{\ell}$ in the endpoints $Q_{1}$ and $Q_{b}$ of $E$. Let $c:=e_{\kappa} \cdot \tau_{E} \leqslant 1$. The parametrisation $f_{\tau}(s)$ for $s \in(0, h)$ of $\left.f\right|_{E}$ with $f_{\tau}(0)=\varphi_{\ell}\left(Q_{1}\right)=0, f_{\tau}(h)=\varphi_{\ell}\left(Q_{b}\right)=0$, and $f_{\tau}^{\prime}=\tau \cdot \nabla \varphi_{\ell}$ with $f^{\prime}(0)=c$ and $f^{\prime}(h)=0$, reads

$$
f_{\tau}(s)=c(s-h)^{2} s / h^{2} .
$$

Hence $\left|\tau_{1 b} \cdot \nabla \varphi_{\ell}\left(Q_{1 b}\right)\right|=\left|f_{\tau}^{\prime}(h / 2)\right|=c / 4 \leqslant 1$ and

$$
\left|L_{1 b c}(f)\right|=\left|\left(Q_{c}-Q_{1 b}\right) \cdot\left(\tau_{1 b} \cdot \nabla \varphi_{\ell}\left(Q_{1 b}\right)\right) \tau_{1 b}\right| \leqslant\left|\left(Q_{c}-Q_{1 b}\right) \cdot \tau_{1 b}\right| \leqslant h_{K} .
$$

For (A.36) this implies that

$$
\left\|\varphi_{\ell}\right\|_{H^{s}(K)}=\left\|\varphi_{z, \kappa+1}\right\|_{H^{s}(K)} \leqslant h_{K} \sum_{k=2}^{4}\left\|\varphi_{1 k}\right\|_{H^{s}(K)}+h_{K} \sum_{\substack{2 \leqslant b, c \leqslant 4 \\ b \neq c}}\left\|\varphi_{1 b c}\right\|_{H^{s}(K)} \leqslant C_{a} h_{K}^{5 / 2-s}
$$

with the upper bound from Lemma A. 6 in the last step. The shape regularity of $\mathcal{T} \in \mathbb{T}$ implies $|K| \approx h_{K}^{3}$ and the boundedness of the number of tetrahedra $|\mathcal{T}(z)|$ in the support of $\varphi_{z, \kappa+1}$. This concludes the proof.

## B. Conforming companion

## B.1. Overview

Given a regular triangulation $\mathcal{T} \in \mathbb{T}$ and the Morley finite element space $M(\mathcal{T})$, this sections contains the details of the definition of the conforming companion operator $J_{M}: M(\mathcal{T}) \rightarrow$ $V:=H_{0}^{2}(\Omega)$ in 3D and the proof of Theorem 3.1.

Theorem 3.1 (properties of $\boldsymbol{J}_{\mathbf{M}}$ ). There exists a constant $M_{2} \approx 1$ (that exclusively depends on $\mathbb{T}$ ) and a conforming companion $J_{M} v_{M} \in V:=H_{0}^{2}(\Omega)$ for any $v_{M} \in M(\mathcal{T})$ with
(a) $J_{M}$ is a right inverse to the interpolation $I_{M}$ in that $I_{M} \circ J_{M}=\mathrm{id}$ in $M(\mathcal{T})$,
(b) $\left\|h_{\mathcal{T}}^{-2}\left(1-J_{M}\right) v_{M}\right\|_{L^{2}(\Omega)}+\left\|\left(1-J_{M}\right) v_{M}\right\|_{\mathrm{pw}} \leqslant M_{2} \min _{v \in V}\left\|v_{M}-v\right\|_{\mathrm{pw}}$,
(c) the orthogonality $\left(1-J_{M}\right)(M(\mathcal{T})) \perp P_{2}(\mathcal{T})$ holds in $L^{2}(\Omega)$.

For the triangulation $\mathcal{T} \in \mathbb{T}$ with set edges $\mathcal{E}$ fix one unit tangent vector $\tau_{E} \| E$ and choose two unit vectors $\nu_{E, 1}, \nu_{E, 2} \in \mathbb{R}^{3}$ for any edge $E \in \mathcal{E}$ such that ( $\tau_{E}, \nu_{E, 1}, \nu_{E, 2}$ ) form an orthonormal basis of $\mathbb{R}^{3}$, and define the $W F$ partition $\hat{\mathcal{T}}$ as in Definition A.2. Consider the conforming subspace with homogeneous boundary conditions as in (A.29)

$$
W F(\mathcal{T}):=P_{3}(\widehat{\mathcal{T}}) \cap H_{0}^{2}(\Omega) .
$$

The design in Sections B.2-B. 5 below contains four steps. Given any Morley function $v_{M} \in$ $M(\mathcal{T})$, first the enrichment $J_{1} v_{M} \in W F(\mathcal{T})$ is computed via averaging of nodes. The global degrees of freedom for the Morley finite element read

$$
\begin{equation*}
L_{E}(f):=\oint_{E} f \mathrm{~d} s \text { and } L_{F}(f):=\oint_{F} \nabla f \cdot \nu_{F} \mathrm{~d} \sigma \text { for any } E \in \mathcal{E}, F \in \mathcal{F}, \text { and } f \in H^{2}(\Omega) \tag{B.1}
\end{equation*}
$$

Note, for a Morley function $v_{M} \in M(\mathcal{T})$ the integral means in (B.1) are single valued, in that for any edge $E \in \mathcal{E}$ and any adjacent tetrahedron $T \in \mathcal{T}(E)$ holds $\left.f_{E} v_{M}\right|_{T} \mathrm{~d} s=L_{E}\left(v_{M}\right)$ and for any face $F \in \mathcal{F}$ and any adjacent tetrahedron $T \in \mathcal{T}(F)$ holds $\left.f_{F} \nabla v_{M}\right|_{T} \cdot \nu_{F} \mathrm{~d} \sigma=L_{F}\left(v_{M}\right)$. Two separate corrections are necessary to guarantee $L_{E}\left(v_{M}\right)=f_{E} v_{M} \mathrm{~d} s=L_{E}\left(J_{M} v_{M}\right)$ for any $E \in \mathcal{E}$ and $L_{F}\left(v_{M}\right)=f_{F} \nabla v_{M} \cdot \nu_{F} \mathrm{~d} \sigma=L_{F}\left(J_{M} v_{M}\right)$ for any $F \in \mathcal{F}$ in 3D. The integral means over the edges are corrected in $J_{2}$. The integral means of the normal derivatives along the faces are corrected in $J_{3}$. This guarantees that $J_{3}$ is a right inverse of $I_{M}$ in that $I_{M} J_{3} v_{M}=v_{M}$. Finally the correction $J_{4} v_{M}$ is designed such that its $L^{2}$ projection in $P_{2}(\mathcal{T})$ equals $v_{M} \in M(\mathcal{T}), \Pi_{2} J_{M} v_{M}=v_{M}$. Note that in an abstract point of view Subsection B. 2 concerns nodal values for vertices $z \in \mathcal{V}$ and midpoints $\operatorname{mid}(E)$ of edges $E \in \mathcal{E}$ ( 0 -simplices), Subsection B. 3 concerns the integral mean for each edge $E \in \mathcal{E}$ (1-simplex), Subsection B. 4 concerns one integral mean for each face $F \in \mathcal{F}$ (2-simplex), and finally Subsection B. 5 concerns the volume contribution for each tetrahedron $T \in \mathcal{T}$ (3-simplex). The corrections in the last three steps are independent of the choice of the conforming space $W F(\mathcal{T})$ and possible in any space dimension; cf. [Gal15, VZ19] for $n=2$ and Theorem 3.1.a-b.

## B.2. Design of $J_{1}$

Determine the values for the degrees of freedom of $W F(\mathcal{T})$ by averaging the respective values of the Morley function on all adjacent simplices $T \in \mathcal{T}$ and assure homogeneous boundary conditions to define $J_{1}: M(\mathcal{T}) \rightarrow W F(\mathcal{T}) \subset V$ as follows.

Definition B. 1 (enrichment $\boldsymbol{J}_{1}$ ). For any vertex $z \in \mathcal{V}$ with nodal patch $\omega(z)=\operatorname{int}(\bigcup \mathcal{T}(z))$, $\mathcal{T}(z):=\{T \in \mathcal{T} \mid z \in \mathcal{V}(T)\}$, of $|\mathcal{T}(z)|$ many tetrahedra with the vertex $z$ and for any multiindex $\alpha \in \mathbb{N}_{0}^{3}$ with $0 \leqslant|\alpha| \leqslant 1$, set

$$
\begin{equation*}
\left(D^{\alpha} J_{1} v_{M}\right)(z):=\frac{1}{|\mathcal{T}(z)|} \sum_{T \in \mathcal{T}(z)}\left(\left.D^{\alpha} v_{M}\right|_{T}\right)(z) \quad \text { for } z \in \mathcal{V}(\Omega) \tag{B.2}
\end{equation*}
$$

(and $=0$ for $z \in \mathcal{V}(\partial \Omega)$ owing to the homogeneous boundary conditions in $V$ ). For any edge $E \in \mathcal{E}$ with midpoint $\operatorname{mid}(E)$, edge patch $\omega(E)=\operatorname{int}(\bigcup \mathcal{T}(E)), \mathcal{T}(E):=\{T \in \mathcal{T} \mid E \in \mathcal{E}(T)\}$, of $|\mathcal{T}(E)|$ many tetrahedra with common edge $E$, and unit tangential vector $\tau_{E}$ with a fixed orientation, set

$$
\begin{equation*}
\tau_{E} \times \nabla J_{1} v_{M}(\operatorname{mid}(E)):=\frac{1}{|\mathcal{T}(E)|} \sum_{T \in \mathcal{T}(E)} \tau_{E} \times\left(\left.\nabla v_{M}\right|_{T}\right)(\operatorname{mid}(E)) \text { for } E \in \mathcal{E}(\Omega) \tag{B.3}
\end{equation*}
$$

(and $=0$ for $E \in \mathcal{E}(\partial \Omega)$ owing to the homogeneous boundary conditions in $V$ ). That means equality in the directions $\left\{\nu_{E, 1}, \nu_{E, 2}\right\}$ perpendicular to the tangential unit vector $\tau_{E}$.

The enrichment operator $J_{1}$ satisfies the approximation condition as shown in Lemma B. 2 below. We aim to provide explicit constants whenever possible. To this end we need in the proof of Lemma B. 2 the following Lemma B. 1 as addition to the related observations in [CP20, Lem. B-C], which may be of independent interest. Similar results are contained in [CH17, Lem 4.2].
Lemma B. 1 (inequality). Any $x=\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}^{m}, m \in \mathbb{N}$ satisfies

$$
\begin{equation*}
2\left(1-\cos \left(\frac{\pi}{2 m+1}\right)\right)|x|^{2} \leqslant \sum_{j=1}^{m-1}\left(x_{j+1}-x_{j}\right)^{2}+\min \left\{x_{1}^{2}, \ldots, x_{m}^{2}\right\} \tag{B.4}
\end{equation*}
$$

The factor $2\left(1-\cos \left(\frac{\pi}{2 m+1}\right)\right)$ in (B.4) is sharp in that there exists some $x \in \mathbb{R}^{m} \backslash\{0\}$ with equality in (B.4).
Proof. Abbreviate $\varrho(x)^{2}:=\sum_{j=1}^{m-1}\left(x_{j+1}-x_{j}\right)^{2}+\min \left\{x_{1}^{2}, \ldots, x_{m}^{2}\right\}$ for any $x=\left(x_{1}, \ldots, x_{m}\right) \in$ $\mathbb{R}^{m} \backslash\{0\}$, let $S:=\left\{x \in \mathbb{R}^{m}| | x \mid=1\right\}$ denote the unit sphere in $\mathbb{R}^{m}$, and $S_{+}:=\{x \in S \mid 0 \leqslant$ $x_{j}$ for all $\left.j=1, \ldots, m\right\}$ its positive half. Then $\varrho\left(\left(\left|x_{1}\right|, \ldots,\left|x_{m}\right|\right)\right) \leqslant \varrho(x)$ follows from $\left(\left|x_{j+1}\right|-\right.$ $\left.\left|x_{j}\right|\right)^{2} \leqslant\left|x_{j+1}-x_{j}\right|^{2}$ for $j=1, \ldots, m-1$. This implies the equality after the definition of

$$
\Lambda:=\min _{x \in S} \varrho(x)^{2}=\min _{x \in S_{+}} \varrho(x)^{2}
$$

Lemma B in the appendix of [CP20] shows that a minimum of the sum $\sum_{j=1}^{m-1}\left(x_{j+1}-x_{j}\right)^{2}$ is attained for ordered coefficients in the vector $x \in S_{+}$, hence

$$
\Lambda=\min _{\substack{x \in S_{+} \\ 0 \leqslant x_{1} \leqslant x_{2} \leqslant \cdots \leqslant x_{m}}} \varrho(x)^{2}=x \cdot B x
$$

with the tridiagonal matrix

$$
B:=\left(\begin{array}{ccccc}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & -1 & 2 & -1 \\
0 & \cdots & 0 & -1 & 1
\end{array}\right)
$$

The positive eigenvalues $0<\lambda_{k}=2(1-\cos ((2 k-1) \pi /(2 m+1)))$ for $k=1, \ldots, m$ of the $B$ are computed in [YSC08, Thm.3.2.viii], hence from the Rayleigh quotient follows $\lambda_{1}|x|^{2} \leqslant$ $x \cdot B x$ for any $x \in \mathbb{R}^{m}$ and $\Lambda=\lambda_{1}=2(1-\cos (\pi /(2 m+1)))$. The eigenspace $E\left(\lambda_{1}\right)$ is spanned by $x=(\sin (\pi /(2 m+1)), \sin (2 \pi /(2 m+1)), \ldots, \sin (m \pi /(2 m+1))) \in \mathbb{R}^{m} \backslash\{0\}$. Since $0 \leqslant \sin (k \pi /(2 m+1)) \leqslant \sin ((k+1) \pi /(2 m+1))$ for any $k=1, \ldots, m-1$, the normed eigenvector $v:=x /|x| \in E\left(\lambda_{1}\right) \cap S_{+}$has ordered coefficients and leads to the equality $\Lambda=\varrho(v)^{2}$; in other words $\Lambda=\lambda_{1}$ is the optimal constant.

Lemma B. 2 (properties of $\boldsymbol{J}_{\mathbf{1}}$ ). There exists a constant $C_{1} \approx 1$ (that exclusively depends on $\mathbb{T}$ ) such that the enrichment operator $J_{1} v_{M} \in W F(\mathcal{T})$ for any $v_{M} \in M(\mathcal{T}), \mathcal{T} \in \mathbb{T}$, satisfies

$$
\left\|h_{\mathcal{T}}^{-2}\left(1-J_{1}\right) v_{M}\right\|_{L^{2}(\Omega)}+\left\|\left(1-J_{1}\right) v_{M}\right\|_{\mathrm{pw}} \leqslant C_{1} \min _{v \in V}\left\|v_{M}-v\right\|_{\mathrm{pw}} .
$$

Proof. This proof consist of three steps. For Step 1-Step 2 fix one tetrahedron $K \in \mathcal{T}$.
Step 1. Proof of $h_{K}^{-4}\left\|v_{M}-J_{1} v_{M}\right\|_{L^{2}(K)}^{2} \lesssim \sum_{z \in \mathcal{V}(K)} \sum_{F \in \mathcal{F}(z)} h_{F}\left\|\left[D^{2} v_{M}\right]_{F} \times \nu_{F}\right\|_{L^{2}(F)}^{2}$. The WF interpolation on $K$ preserves the enrichment $J_{1} v_{M} \in W F(K)$ and quadratic polynomials like the Morley function $\left.v_{M}\right|_{K} \in P_{2}(K) \subset W F(K)$. Recall the nodal basis functions $\left.\varphi_{z, 1}\right|_{K}$, $\left.\varphi_{z, \kappa+1}\right|_{K}$, and $\left.\varphi_{E, \mu}\right|_{K}$ for any $z \in \mathcal{V}(K), E \in \mathcal{E}(K), \kappa=1,2,3, \mu=1,2$, and the degrees of freedom of $W F(K)$ from Definition A.3. Then the $W F$ interpolation reads

$$
\begin{aligned}
\left.\left(v_{M}-J_{1} v_{M}\right)\right|_{K}= & \sum_{z \in \mathcal{V}(K)}\left(\left(v_{M}-J_{1} v_{M}\right)(z) \varphi_{z, 1}+\sum_{\kappa=1}^{3} e_{\kappa} \cdot \nabla\left(v_{M}-J v_{M}\right)(z) \varphi_{z, \kappa+1}\right) \\
& +\sum_{E \in \mathcal{E}(K)} \sum_{\mu=1}^{2} \nu_{E, \mu} \cdot \nabla\left(v_{M}-J_{1} v_{M}\right)(\operatorname{mid}(E)) \varphi_{E, \mu} .
\end{aligned}
$$

The triangle inequality and the scaling of the basis functions $\left\|\varphi_{z, 1}\right\|_{L^{2}(K)} \lesssim h_{K}^{3 / 2},\left\|\varphi_{z, \kappa+1}\right\|_{L^{2}(K)} \lesssim$ $h_{K}^{5 / 2}$, and $\left\|\varphi_{E, \mu}\right\|_{L^{2}(K)} \lesssim h_{K}^{5 / 2}$ in Theorem A. 4 reveal for a constant $c_{1} \lesssim 1$

$$
\begin{align*}
c_{1}^{-1}\left\|v_{M}-J_{1} v_{M}\right\|_{L^{2}(K)}^{2} \leqslant & h_{K}^{3} \sum_{z \in \mathcal{V}(K)}\left(\left|\left(\left.v_{M}\right|_{K}-J_{1} v_{M}\right)(z)\right|^{2}+h_{K}^{2} \sum_{\kappa=1}^{3}\left|e_{\kappa} \cdot \nabla\left(\left.v_{M}\right|_{K}-J_{1} v_{M}\right)(z)\right|^{2}\right) \\
& +h_{K}^{5} \sum_{E \in \mathcal{E}(K)} \sum_{\mu=1}^{2}\left|\nu_{E, \mu} \cdot \nabla\left(\left.v_{M}\right|_{K}-J_{1} v_{M}\right)(\operatorname{mid}(E))\right|^{2} . \tag{B.5}
\end{align*}
$$

In Definition B. 1 the given local values $L_{j}\left(\left.v_{M}\right|_{T}\right)$ of $v_{M}$ for every $W F$ degree of freedom $L_{j}$ for $j=1, \ldots, 28$ and every tetrahedron $T \in \mathcal{T}$ are averaged (and homogeneous boundary conditions ensured). This in mind, suppose $z \in \mathcal{V}(K)$ and analyse the term $\left|v_{M}(z)-J_{1} v_{M}(z)\right|$ first. There are three cases of interest that lead to the constant $C_{z} \lesssim 1$ in (B.9) below. The vertex $z \in \mathcal{V}(K)$ could be
Case 1. a boundary vertex $z \in \mathcal{V}(\partial \Omega)$, but $K$ has no boundary side that contains $z$ in that $\mathcal{F}(K) \cap \mathcal{F}(z) \cap \mathcal{F}(\partial \Omega)=\varnothing$,
Case 2. a boundary vertex $z \in \mathcal{V}(\partial \Omega) \cap F$ that belongs to a boundary side $F \in \mathcal{F}(\partial \Omega) \cap \mathcal{F}(K)$ of $K$, or
Case 3. an interior vertex $z \in \mathcal{V}(\Omega)$.

In case 1 holds $J_{1} v_{M}(z)=0$ due to the homogeneous boundary conditions in $V$. Then Lemma B. 1 applies and contains the optimal constant. Choose a face-connected subset $\left\{T_{1}, \ldots, T_{k}\right\} \subset \mathcal{T}(z)$ with $2 \leqslant k \leqslant|\mathcal{T}(z)|=: J_{z} \lesssim 1, T_{k}=K, z \in F_{j}:=T_{j} \cap T_{j+1} \in \mathcal{F}$ for any $j=1, \ldots, k-1$, and boundary simplex $T_{1}$ such that $z \in F_{0} \in \mathcal{F}\left(T_{1}\right) \cap \mathcal{F}(\partial \Omega)$. Set $x_{j}:=\left.v_{M}\right|_{T_{j}}(z)-J_{1} v_{M}(z)=\left.v_{M}\right|_{T_{j}}(z) \in \mathbb{R}$ for $j=1, \ldots, k$ and $x:=\left(x_{1}, \ldots, x_{k}\right) \in \mathbb{R}^{k}$. If $x \in \mathbb{R}^{k} \backslash\{0\}$, Lemma B. 1 shows

$$
2(1-\cos (\pi /(2 k+1)))|x|^{2} \leqslant \sum_{j=1}^{k}\left(x_{j+1}-x_{j}\right)^{2}+\min \left\{x_{1}^{2}, \ldots, x_{k}^{2}\right\} \leqslant \sum_{j=1}^{k}\left(x_{j+1}-x_{j}\right)^{2}+x_{1}^{2} .
$$

The jump $[v]_{F} \in L^{1}(F)$ of a piecewise Lipschitz continuous function $v$ across $F \in \mathcal{F}$ reads $[v]_{F}:=\left.\left.v\right|_{T_{+}-} v\right|_{T_{-}}$for an interior side $F=\partial T_{+} \cap \partial T_{-} \in \mathcal{F}(\Omega)$ that belongs to the neighbouring simplicies $T_{+}, T_{-} \in \mathcal{T}$, which are labelled such that the unit outward normal along the boundary of $T_{+}$(resp. $T_{-}$) satisfies $\left.\nu_{T_{+}}\right|_{F}=\nu_{F}$ (resp. $\left.\nu_{T_{-}}\right|_{F}=-\nu_{F}$ ), while $[v]_{F}:=v$ along $F \in \mathcal{F}(\partial \Omega)$. Hence the side-connectivity of $\left\{T_{1}, \ldots, T_{k}\right\} \subset \mathcal{T}(z)$ implies, for any $j=1, \ldots, k-1$, that

$$
\left.\left|x_{j+1}-x_{j}\right|=\left|v_{M}\right| T_{j}(z)-v_{M}\left|T_{j+1}(z)\right|=\mid\left[v_{M}\right]_{F_{j}}(z)\right] \mid \text { and }\left|x_{1}\right|=\left|v_{M}\right| T_{1}(z)\left|=\left|\left[v_{M}\right]_{F_{0}}(z)\right| .\right.
$$

We conclude (with the Euclidean norm $|\cdot|$ of $x \in \mathbb{R}^{k}$ ) in case 1,

$$
\begin{equation*}
\left|v_{M}\right|_{K}(z)-\left.J_{1} v_{M}(z)\right|^{2} \leqslant|x|^{2} \leqslant \frac{1}{2(1-\cos (\pi /(2 k+1))} \sum_{j=0}^{k}\left|\left[v_{M}\right]_{F_{j}}(z)\right|^{2} . \tag{B.6}
\end{equation*}
$$

In case 2 , for the boundary node $z \in \mathcal{V}(\partial \Omega) \cap F$ that belongs to a boundary side $F \in$ $\mathcal{F}(\partial \Omega) \cap \mathcal{F}(K)$ of $K$, holds $J_{1} v_{M}(z)=0$ and the jump definition with $[v]_{F}:=v$ along any boundary side, in particular along $F \in \mathcal{F}(\partial \Omega) \cap \mathcal{F}(K)$, shows

$$
\begin{equation*}
\left|v_{M}\right|_{K}(z)-J_{1} v_{M}(z)\left|=\left|v_{M}\right|_{K}(z)\right|=\left|\left[v_{M}\right]_{F}(z)\right| . \tag{B.7}
\end{equation*}
$$

In case 3, for the interior vertex $z \in \mathcal{V}(\Omega)$, holds $J_{1} v_{M}(z)=\left.J_{z}^{-1} \sum_{T \in \mathcal{T}(z)} v_{M}\right|_{T}(z)$ with the abbreviation $J_{z}:=|\mathcal{T}(z)| \lesssim 1$. This is the standard averaging and an analogue argumentation with explicit constant is provided Step 3 in the proof of Theorem 6.1 in [CP20]. For convenience we repeat the main arguments in the present notation. Choose an enumeration $\left\{T_{1}, \ldots, T_{J_{z}}\right\}$ of $\mathcal{T}(z)$ such that the values $x_{j}:=\left.v_{M}\right|_{T_{j}}(z)-J_{1} v_{M}(z) \in \mathbb{R}$ for $j=1, \ldots, J_{z}$ are ordered in the sense that $x_{1} \leqslant x_{2} \leqslant \cdots \leqslant x_{J_{z}}$. The sum $\sum_{j=1}^{J_{z}} x_{j}=0$ vanishes by definition of $J_{1} v_{M}(z)$. Since $\left|v_{M}\right|_{K}(z)-\left.J_{1} v_{M}(z)\left|\leqslant \max _{T \in \mathcal{T}(z)}\right| v_{M}\right|_{T}-J_{1} v_{M}(z)\left|=\max _{j=1, \ldots, J_{z}}\right| x_{j} \mid$, Lemma C in [CP20] implies (in the same abstract notation as Lemma B. 1 above) that

$$
\left|v_{M}\right|_{K}(z)-\left.J_{1} v_{M}(z)\right|^{2} \leqslant \frac{\left(J_{z}-1\right)\left(2 J_{z}-1\right)}{6 J_{z}} \sum_{j=1}^{J_{z}-1}\left|x_{j+1}-x_{j}\right|^{2}
$$

It remains a reordering to arrange for jump contributions on the right-hand side. To this end let $\mathcal{J}:=\left\{\{\alpha, \beta\}: T_{\alpha}, T_{\beta} \in \mathcal{T}(z)\right.$ and $\left.\partial T_{\alpha} \cap \partial T_{\beta} \in \mathcal{F}(z)\right\}$ denote unordered index pairs of all tetrahedra in $\mathcal{T}(z)$ which share a side in $\mathcal{F}(z):=\{F \in \mathcal{F} \mid z \in \mathcal{V}(z)\}$. Since $\mathcal{T}(z)$ is side-connected, $\mathcal{J}$ is connected, in the sense that for all $\alpha, \beta \in\left\{1, \ldots, J_{z}\right\}$ and $\alpha \neq \beta$ there
are $k \in \mathbb{N}$ pairs $\left\{\alpha_{1}, \alpha_{2}\right\},\left\{\alpha_{2}, \alpha_{3}\right\}, \ldots,\left\{\alpha_{k}, \alpha_{k+1}\right\} \in \mathcal{J}$ with $\alpha_{1}=\alpha$ and $\alpha_{k+1}=\beta$. Therefore some arguments in the language of graph theory in Lemma B in [CP20] imply that

$$
\sum_{j=1}^{J_{z}-1}\left|x_{j+1}-x_{j}\right|^{2} \leqslant \sum_{\{\alpha, \beta\} \in \mathcal{J}}\left|x_{\alpha}-x_{\beta}\right|^{2}=\sum_{\{\alpha, \beta\} \in \mathcal{J}}\left|\left(\left.v_{M}\right|_{T_{\alpha}}\right)(z)-\left(\left.v_{M}\right|_{T_{\beta}}\right)(z)\right|^{2} .
$$

The combination of the last two displayed estimates with the jump definition shows in case 3 that

$$
\begin{equation*}
\left|v_{M}\right|_{K}(z)-\left.J_{1} v_{M}(z)\left|\leqslant \frac{\left(J_{z}-1\right)\left(2 J_{z}-1\right)}{6 J_{z}} \sum_{F \in \mathcal{F}(z)}\right|\left[v_{M}\right]_{F}(z)\right|^{2} . \tag{B.8}
\end{equation*}
$$

Abbreviate

$$
C_{z}:=\max \left\{1, \frac{\left(J_{z}-1\right)\left(2 J_{z}-1\right)}{6 J_{z}}, \frac{1}{2\left(1-\cos \left(\frac{\pi}{2 J_{z}+1}\right)\right)}\right\} \lesssim 1 .
$$

The combination of (B.6)-(B.8) implies for any vertex $z \in \mathcal{V}(K)$ of $K \in \mathcal{T}$ that

$$
\begin{equation*}
\left|v_{M}(z)\right|_{K}-\left.J_{1} v_{M}(z)\left|\leqslant C_{z} \sum_{F \in \mathcal{F}(z)}\right|\left[v_{M}\right]_{F}(z)\right|^{2} . \tag{B.9}
\end{equation*}
$$

Since $J_{1}$ is defined by nodal averaging for all degrees of freedom $L_{1}, \ldots, L_{28}$ of $W F$ (with homogeneous boundary conditions), the argumentation for $\left|v_{M}\right|_{K}(z)-J_{1} v_{M}(z) \mid$ with case 1case 3 applies verbatim for $\left|e_{\kappa} \cdot \nabla\left(\left.v_{M}\right|_{K}-J_{1} v_{M}\right)(z)\right|$ and $\left|\nu_{E, \mu} \cdot \nabla\left(\left.v_{M}\right|_{K}-J_{1} v_{M}\right)(\operatorname{mid}(E))\right|$. Let $\mathcal{N}:=\mathcal{V} \cup\{\operatorname{mid}(E) \mid E \in \mathcal{E}\}$ denote the set of all vertices and edge midpoints. Let $\mathcal{T}(z):=\{T \in$ $\mathcal{T} \mid z \in \partial T\}$ denote the set of all tetrahedra with node $z \in \mathcal{N}$ and $\mathcal{F}(z):=\{F \in \mathcal{F} \mid z \in \partial F\}$ set of all faces with node $z \in \mathcal{N}$. In abstract notation let $J_{A}$ denote a general averaging operator; for any $k \in \mathbb{N}_{0}$, a piecewise polynomial $v \in P_{k}(\mathcal{T})$ of degree at most $k$, the tetrahedron $K \in \mathcal{T}$, and the node $z \in \mathcal{N}(K)$ set $J_{A} v(z):=\left.|\mathcal{T}(z)|^{-1} \sum_{T \in \mathcal{T}(z)} v\right|_{T}(z)$ for each interior node $z \in \mathcal{N}(\Omega)$ and $J_{A} v(z):=0$ else. Given this notation, $\left(D^{\alpha} J_{1} v_{M}\right)(z)=J_{A}\left(D^{\alpha} v_{M}\right)(z)$ holds for $z \in \mathcal{V}$, $\alpha \in \mathbb{N}_{0}^{3}$ with $0 \leqslant|\alpha| \leqslant 1$, as well as, $\tau_{E} \times \nabla J_{1} v_{M}(\operatorname{mid}(E))=J_{A}\left(\tau_{E} \times \nabla v_{M}\right)(\operatorname{mid}(E))$ for $E \in \mathcal{E}$. In the argumentation above we could replace $J_{1}$ by $J_{A}$ and $z \in \mathcal{V}$ by $z \in \mathcal{N}$ to obtain (B.9) in the general setting. Hence,

$$
\begin{aligned}
\left|e_{\kappa} \cdot \nabla\left(\left.v_{M}\right|_{K}-J_{1} v_{M}\right)(z)\right| & \leqslant C_{z} \sum_{F \in \mathcal{F}(z)}\left|\left[e_{\kappa} \cdot \nabla v_{M}\right]_{F}(z)\right|^{2}, \\
\left|\nu_{E, \mu} \cdot \nabla\left(\left.v_{M}\right|_{K}-J_{1} v_{M}\right)(\operatorname{mid}(E))\right| & \leqslant C_{E} \sum_{F \in \mathcal{F}(E)}\left|\left[\nu_{E, \mu} \cdot \nabla v_{M}\right]_{F}(\operatorname{mid}(E))\right|^{2}
\end{aligned}
$$

with

$$
C_{E}:=\max \left\{1, \frac{\left(J_{E}-1\right)\left(2 J_{E}-1\right)}{6 J_{E}}, \frac{1}{2\left(1-\cos \left(\frac{\pi}{2 J_{E}+1}\right)\right)}\right\} \lesssim 1
$$

for any edge $E \in \mathcal{E}$ with set of adjacent faces $\mathcal{F}(E):=\{F \in \mathcal{F}: E \in \mathcal{E}(F)\}=\mathcal{F}(\operatorname{mid}(E))$ of cardinality $J_{E}=|\mathcal{F}(E)| \lesssim 1$.

Hence (B.5) simplifies to

$$
c_{1}^{-1}\left\|v_{M}-J_{1} v_{M}\right\|_{L^{2}(K)}^{2} \leqslant h_{K}^{3} \sum_{z \in \mathcal{V}(K)} C_{z} \sum_{F \in \mathcal{F}(z)}\left(\left|\left[v_{M}\right]_{F}(z)\right|^{2}+h_{K}^{2}\left|\left[\nabla v_{M}\right]_{F}(z)\right|^{2}\right)
$$

$$
\begin{equation*}
+h_{K}^{5} \sum_{E \in \mathcal{E}(K)} C_{E} \sum_{F \in \mathcal{F}(E)}\left|\left[\nabla v_{M}\right]_{F}(\operatorname{mid}(E))\right|^{2} \tag{B.10}
\end{equation*}
$$

(The shape regularity of $\mathcal{T} \in \mathbb{T}$ guarantees that the number of tetrahedra $J_{z}$ and $J_{E}$ in the patches, i.e., $C_{z}$ and $C_{E}$, are uniformly bounded for any $z \in \mathcal{V}$ and $E \in \mathcal{E}$.) Let $S$ denote either the set of vertices $S=\mathcal{V}$ or the set of edge midpoints $S=\{\operatorname{mid}(E) \mid E \in \mathcal{E}\}=: \mathcal{M}$. For a fixed polynomial degree $k \in \mathbb{N}_{0}$ and any $z \in \mathcal{N}=\mathcal{V} \cup \mathcal{M}$, there exists a constant $C(k, S)>0$, such that the following inverse estimate holds

$$
\begin{equation*}
|F|\left|[v]_{F}(z)\right|^{2} \leqslant C(k, S)\left\|[v]_{F}\right\|_{L^{2}(F)}^{2} \quad \text { for any } v \in P_{k}(\mathcal{T}) \text { and } F \in \mathcal{F}(z) \tag{B.11}
\end{equation*}
$$

In the cases of interest the optimal constant $C(k, S)>0$ can be computed as in [CP20, Lem. D]. The inverse mass matrix for the basis functions dual to the Lagrange interpolation points and a Cauchy-Schwarz inequality prove $C(1, \mathcal{V})=n^{2}=9$ and $C(2, \mathcal{V})=36$ for any vertex $z \in \mathcal{V}$ as well as $C(1, \mathcal{M})=n(n-1) / 2=3$ and $C(2, \mathcal{M})=39 / 4$ for any edge $\operatorname{midpoint} z=\operatorname{mid}(E) \in \mathcal{M}$. Moreover, an inverse estimate for $\left[v_{M}\right]_{F} \in P_{2}(F)$ ensures $c_{\text {inv }}^{-1}\left\|\left[v_{M}\right]_{F}\right\|_{L^{2}(F)} \leqslant h_{F}\left\|\left[\nabla v_{M}\right]_{F}\right\|_{L^{2}(F)} \leqslant h_{K}\left\|\left[\nabla v_{M}\right]_{F}\right\|_{L^{2}(F)}$. In combination with (B.11) this recasts (B.10) as

$$
\begin{aligned}
c_{1}^{-1}\left\|v_{M}-J_{1} v_{M}\right\|_{L^{2}(K)}^{2} \leqslant & h_{K}^{5} \sum_{z \in \mathcal{V}(K)} C_{z} \sum_{F \in \mathcal{F}(z)}\left(c_{\mathrm{inv}}^{2} 36\left\|\left[\nabla v_{M}\right]_{F}\right\|_{L^{2}(F)}^{2} /|F|+9\left\|\left[\nabla v_{M}\right]_{F}\right\|_{L^{2}(F)}^{2} /|F|\right) \\
& +h_{K}^{5} \sum_{E \in \mathcal{E}(K)} C_{E} \sum_{F \in \mathcal{F}(E)} 3\left\|\left[\nabla v_{M}\right]_{F}\right\|_{L^{2}(F)}^{2} /|F|
\end{aligned}
$$

Since for any edge $E=\operatorname{conv}\left\{P_{j}, P_{k}\right\} \in \mathcal{E}$ with vertices $P_{j}, P_{k} \in \mathcal{V}(E)$, it holds $\mathcal{F}(E) \subset$ $\mathcal{F}\left(P_{j}\right) \cup \mathcal{F}\left(P_{k}\right)$, this simplifies to

$$
h_{K}^{-5}\left\|v_{M}-J_{1} v_{M}\right\|_{L^{2}(K)}^{2} \lesssim \sum_{z \in \mathcal{V}(K)} \sum_{F \in \mathcal{F}(z)}\left\|\left[\nabla v_{M}\right]_{F}\right\|_{L^{2}(F)}^{2} /|F|
$$

Recall that the integral mean of the gradient $f_{F}\left[\nabla v_{M}\right]_{F} \mathrm{~d} \sigma$ vanishes for any side $F \in \mathcal{F}$ from [MX06, Lem. 4] as utilized in [CP21, Thm. 2.1.b]. Hence the Poincaré inequality with Payne-Weinberger constant [PW60, Beb03] shows

$$
\left\|\left[\nabla v_{M}\right]_{F}\right\|_{L^{2}(F)} \leqslant h_{F} \pi^{-1}\left\|\left[D^{2} v_{M}\right]_{F} \times \nu_{F}\right\|_{L^{2}(F)}
$$

with the tangential components $\left[D^{2} v_{M}\right]_{F} \times \nu_{F}$ of the jump $\left[D^{2} v_{M}\right]_{F}$ along any edge $F \in \mathcal{F}$ and the (piecewise) Hessian $D^{2}$. This concludes the first step, since $h_{F} \approx h_{K} \approx|F|^{1 / 2}$ holds for any $K \in \mathcal{T}$ and $F \in \mathcal{F}(K)$.

Step 2. Proof of $h_{F}\left\|\left[D^{2} v_{M}\right]_{F} \times \nu_{F}\right\|_{L^{2}(F)}^{2} \lesssim \min _{v \in H^{2}(\omega(F))}\left\|D_{\mathrm{pw}}^{2}\left(v_{M}-v\right)\right\|_{L^{2}(\omega(F))}^{2}$. This step is comparable to the 2D analysis in [Gal15, Prop. 2.3] but demands the observations on the Curl operator in 3D from [CBJ02, Thm 3.2]. For $\boldsymbol{\Psi}=\left(\Psi_{1}, \Psi_{2}, \Psi_{2}\right)^{\top} \in H^{1}(\Omega)^{3 \times 3}$ the Curl is applied row-wise,

$$
\operatorname{Curl}(\boldsymbol{\Psi}):=\left(\nabla \times \Psi_{1}, \nabla \times \Psi_{2}, \nabla \times \Psi_{3}\right)^{\top}
$$

First, suppose $F \in \mathcal{F}(\Omega)=\partial T_{+} \cap \partial T_{-}$is an interior side and for any vertex $z \in \mathcal{V}(T)$ let $\phi_{z} \in S^{1}(\mathcal{F}):=P_{1}(\mathcal{T}) \cap C(\Omega)$ denote the hat-function with $\phi_{z}(P)=\delta_{z P}$ for any $P \in \mathcal{V}$.

Define the face bubble $b_{F}:=60 \prod_{z \in \mathcal{V}(F)} \phi_{z} \in H_{0}^{1}(\omega(F))$ that vanishes on the boundary of the face patch $\omega(F):=\operatorname{int}\left(T_{+} \cup T_{-}\right)$. Then $\int_{F} b_{F} \mathrm{~d} \sigma=|F|$ and $\left\|b_{F}\right\|_{L^{\infty}(\omega(E))}=20 / 3$. Given $\left[D^{2} v_{M}\right]_{F} \in \mathbb{R}^{3 \times 3}$, set $\Psi_{F}:=b_{F}\left(\left[D^{2} v_{M}\right]_{F} \times \nu_{F}\right) \in H_{0}^{1}\left(\omega(F) ; \mathbb{R}^{3 \times 3}\right)$. Since $\left[D^{2} v_{M}\right]_{F} \in \mathbb{R}^{3 \times 3}$ is constant,

$$
\left\|\left[D^{2} v_{M}\right]_{F} \times \nu_{F}\right\|_{L^{2}(F)}^{2}=\left\|b_{F}^{1 / 2}\left[D^{2} v_{M}\right]_{F} \times \nu_{F}\right\|_{L^{2}(F)}^{2}
$$

For any $v \in H^{2}(\omega(E))$ the tangential jump $\left[D^{2} v\right]_{F} \times \nu_{F}$ vanishes. Hence an integration by parts shows

$$
\begin{aligned}
\left\|b_{F}^{1 / 2}\left[D^{2} v_{M}\right]_{F} \times \nu_{F}\right\|_{L^{2}(F)}^{2} & =\int_{F}\left(\left[D^{2} v_{M}\right]_{F} \times \nu_{F}\right) \Psi \mathrm{d} \sigma=\sum_{T \in \omega(F)} \int_{\partial T}\left(\left(D^{2} v_{M}-v\right) \times \nu_{F}\right): \Psi \mathrm{d} \sigma \\
& =\left(\operatorname{Curl} \Psi_{F}, D_{\mathrm{pw}}^{2}\left(v_{M}-v\right)\right)_{L^{2}(\omega(F))} .
\end{aligned}
$$

The Cauchy-Schwarz inequality, an inverse estimate, and the boundedness of $\left\|b_{F}\right\|_{L^{\infty}(\omega(E))}$ prove that the last term is bounded by
$\left\|\operatorname{Curl} \Psi_{F}\right\|_{L^{2}(\omega(F))}\left\|D_{\mathrm{pw}}^{2}\left(v_{M}-v\right)\right\|_{L^{2}(\omega(F))} \lesssim h_{F}^{-1 / 2}\left\|\left[D^{2} v_{M}\right]_{F} \times \nu_{F}\right\|_{L^{2}(F)}\left\|D_{\mathrm{pw}}^{2}\left(v_{M}-v\right)\right\|_{L^{2}(\omega(F))}$.
This implies for any interior side $F \in \mathcal{F}(\Omega)$ that

$$
\begin{equation*}
h_{F}\left\|\left[D^{2} v_{M}\right]_{F} \times \nu_{F}\right\|_{L^{2}(F)}^{2} \lesssim \min _{v \in H^{2}(\omega(F))}\left\|D_{\mathrm{pw}}^{2}\left(v_{M}-v\right)\right\|_{L^{2}(\omega(F))}^{2} . \tag{B.12}
\end{equation*}
$$

Second, suppose $F \in \mathcal{F}(\partial \Omega)$ is a boundary face with face patch $\omega(F)=T_{+}$. The face bubble $b_{F}:=60 \prod_{z \in \mathcal{V}(F)} \phi_{z} \in H_{0}^{1}(\omega(F))$ vanishes on all faces $G \in \mathcal{F}\left(T_{+}\right) \backslash\{F\}$ of $T_{+}$besides $F$. Hence the argumentation above holds verbatim and shows (B.12) for $F \in \mathcal{F}(\partial \Omega)$.

Step 3. Conclusion of the proof. The combination of Step 1-Step 2 implies for any $K \in \mathcal{T}$

$$
h_{K}^{-4}\left\|v_{M}-J_{1} v_{M}\right\|_{L^{2}(K)}^{2} \lesssim \sum_{F \in \mathcal{F}(z)} \min _{v \in H^{2}(\omega(F))}\left\|D_{\mathrm{pw}}^{2}\left(v_{M}-v\right)\right\|_{L^{2}(\omega(F))}^{2}
$$

The sum over all tetrahedra $K \in \mathcal{T}$ with the finite overlap of the face patches $(\omega(F): F \in \mathcal{F})$ concludes the proof of the approximation property

$$
\left\|h_{\mathcal{T}}^{2}\left(1-J_{1}\right) v_{M}\right\|_{L^{2}(\Omega)} \lesssim \min _{v \in H_{0}^{2}(\Omega)}\left\|v-v_{M}\right\|_{\mathrm{pw}}
$$

The combination with the inverse estimate $\left\|\left(1-J_{1}\right) v_{M}\right\|_{\mathrm{pw}} \lesssim\left\|h_{\mathcal{T}}^{-2}\left(1-J_{1}\right) v_{M}\right\|_{L^{2}(\Omega)}$ for the piecewise polynomials in $W F(\mathcal{T}) \subset P_{3}(\widehat{\mathcal{T}})$ concludes the proof of Lemma B.2.

## B.3. Design of $J_{2}$

This step corrects the integral mean over each internal edge $E \in \mathcal{E}$ such that $f_{E} J_{2} v_{M} \mathrm{~d} s=$ $f_{E} v_{M} \mathrm{~d} s$. For any edge $E \in \mathcal{E}(\Omega)$ define the ball $B:=B\left(\operatorname{mid}(E), R_{E}\right) \subset \widehat{\omega}(E)$ of radius $R_{E}>0$ with midpoint $\operatorname{mid}(E)$. The radius $R_{E}$ is chosen maximal such that
(i) $B$ belongs to the edge patch $\widehat{\omega}(E):=\operatorname{int}\left(\bigcup_{T \in \hat{\mathcal{T}}(E)} T\right)$ of $E$ in the $W F$ triangulation $\hat{\mathcal{T}}$ and


Figure B.1.: Subtetrahedron $T(4)=\operatorname{conv}\left(c_{K}, F_{4}\right)$ of $K \in \mathcal{T}$ with face $F=F_{4}=\operatorname{conv}\left\{P_{1}, P_{2}, P_{3}\right\} \in \mathcal{F}(K)$ and edge $E=\operatorname{conv}\left\{P_{1}, P_{2}\right\} \in \mathcal{E}(K)$ and dashed $W F$ triangulation $\widehat{\mathcal{T}}(K)$ with $T=\operatorname{conv}\left(c_{K}, c_{F}, E\right) \in \widehat{\mathcal{T}}$ marked in red, the support $\operatorname{supp}\left(\xi_{E}\right)=B$ of $\xi_{E}$ from (B.13) in blue and the disjoint ball $\widehat{B} \subset T$ in green.
(ii) in any adjacent $W F$-subtetrahedra $T \in \hat{\mathcal{T}}(E)$ exists a point $\widehat{c}_{T} \in T$ such that $\widehat{B}:=$ $B\left(\hat{c}_{T}, R_{E}\right) \subset \operatorname{int}(T)$ is a disjunct ball $B \cap \widehat{B}=\varnothing$ of the same size as $B$ and lies in the interior of $T$.

Fig. B.1.b illustrates the disjoint balls $B$ and $\widehat{B}$. The shape regularity of $\mathcal{T} \in \mathbb{T}$ and $\hat{\mathcal{T}}$ in Theorem A. 3 guarantees $h_{T} \approx R_{E} \approx h_{E}$. Set for any $E \in \mathcal{E}(\Omega)$

$$
\xi_{E}(x):= \begin{cases}\frac{|E|}{R_{E} \mid}\left(1-3 \frac{|y|^{2}}{R_{E}^{2}}+2 \frac{|y|^{3}}{R_{E}^{3}}\right) & \text { for } x \in \bar{B} \text { and } y:=x-\operatorname{mid}(E),  \tag{B.13}\\ 0 & \text { for } x \notin \bar{B} .\end{cases}
$$

By construction holds $\xi_{E} \in C^{1}\left(\mathbb{R}^{3}\right) \cap H_{0}^{2}(B)$ with $f_{E} \xi_{D}$ ds $=\delta_{E D}$ for any $E, D \in \mathcal{E}$ and the $\operatorname{support} \operatorname{supp}\left(\xi_{E}\right)=B \subset \widehat{\omega}(E)$.

Definition B. $2\left(J_{2}\right)$. For any $v_{M} \in M(\mathcal{T})$ set

$$
\begin{equation*}
J_{2}\left(v_{M}\right):=J_{1} v_{M}+\sum_{E \in \mathcal{E}(\Omega)}\left(f_{E}\left(v_{M}-J_{1} v_{M}\right) \mathrm{d} s\right) \xi_{E} \in V . \tag{B.14}
\end{equation*}
$$

Lemma B. 3 (properties of $\boldsymbol{J}_{\mathbf{2}}$ ). There exists a constant $C_{2} \approx 1$ (that exclusively depends on $\mathbb{T}$ ) such that the companion $J_{2} v_{M} \in V$ for any $v_{M} \in M(\mathcal{T})$ satisfies
(a) $L_{E}\left(J_{2} v_{M}\right)=L_{E}\left(v_{M}\right)$ for any $E \in \mathcal{E}(\Omega)$,
(b) $\left\|h_{\mathcal{T}}^{-2}\left(1-J_{2}\right) v_{M}\right\|_{L^{2}(\Omega)}+\left\|\left(1-J_{2}\right) v_{M}\right\|_{\mathrm{pw}} \leqslant C_{2} \min _{v \in V}\left\|v_{M}-v\right\|_{\mathrm{pw}}$.

Proof of (a). This holds by construction, since for any $v_{M} \in M(\mathcal{T})$ and $E \in \mathcal{E}(\Omega)$

$$
\begin{aligned}
L_{E}\left(J_{2} v_{M}\right) & =f_{E} J_{2} v_{M} \mathrm{~d} s=f_{E} J_{1} v_{M} \mathrm{~d} s+\sum_{D \in \mathcal{E}(\Omega)}\left(f_{D}\left(v_{M}-J_{1} v_{M}\right) \mathrm{d} s\right) f_{D} \xi_{E} \mathrm{~d} s \\
& =f_{E} J_{1} v_{M} \mathrm{~d} s+f_{E}\left(v_{M}-J_{1} v_{M}\right) \mathrm{d} s=f_{E} v_{M} \mathrm{~d} s=L_{E}\left(v_{M}\right)
\end{aligned}
$$

Proof of (b). Recall the discrete trace inequality $\|v\|_{L^{2}(F)}^{2} \leqslant|F| /|T|\|v\|_{L^{2}(T)}\left(\|v\|_{L^{2}(T)}+\right.$ $\left.2 h_{T} / m\|\nabla v\|_{L^{2}(T)}\right)$ for any $v \in H^{1}(T)$ and any $m$-simplex $T$ with ( $m-1$ )-subsimplex $F$, $m \geqslant 2$. (This is a direct consequence from the discrete trace identity [CP21, Eq. (2.5)] (see [CGR12, CH17]).) For any $g \in W F(\mathcal{T})$ and any edge $E \in \mathcal{E}$ of a side $F \in \mathcal{F}(E)$, the trace inequality and a Cauchy-Schwarz inequality show

$$
\begin{aligned}
f_{E} g \mathrm{~d} s & \leqslant|E|^{-1 / 2}\|g\|_{L^{2}(E)} \leqslant|E|^{-1 / 2} \sqrt{\frac{|E|}{|F|}\|g\|_{L^{2}(F)}\left(\|g\|_{L^{2}(F)}+h_{F}\|\nabla g\|_{L^{2}(F)}\right)} \\
& \leqslant|F|^{-1 / 2} \sqrt{1+c_{\mathrm{inv}}}\|g\|_{L^{2}(F)}
\end{aligned}
$$

with an inverse estimate $\|\nabla g\|_{L^{2}(F)} \leqslant c_{\text {inv }} h_{F}^{-1}\|g\|_{L^{2}(F)}$ in the last step. The trace inequality for any side $F \in \mathcal{F}$ of a tetrahedron $K \in \mathcal{T}(F)$ reveals, for any $g \in W F(\mathcal{T})$, that

$$
\begin{equation*}
\frac{|K|}{|F|}\|g\|_{L^{2}(F)}^{2} \leqslant\|g\|_{L^{2}(K)}\left(\|v\|_{L^{2}(K)}+2 h_{K} / 3\|\nabla v\|_{L^{2}(K)}\right) \leqslant\left(1+3 C_{\mathrm{inv}} / 2\right)\|v\|_{L^{2}(K)} \tag{B.15}
\end{equation*}
$$

with an inverse estimate $\|\nabla g\|_{L^{2}(K)} \leqslant C_{\mathrm{inv}} h_{K}^{-1}\|g\|_{L^{2}(K)}$ in the last step. The combination of the two displayed inequalities shows, for $g=v_{M}-J_{1} v_{M} \in W F(\mathcal{T})$ and any edge $E \in \mathcal{E}$ of a tetrahedron $K \in \mathcal{T}(E)$, that

$$
f_{E}\left(v_{M}-J_{1} v_{M}\right) \mathrm{d} s \leqslant \sqrt{1+c_{\mathrm{inv}}} \sqrt{1+3 C_{\mathrm{inv}} / 2}|K|^{-1 / 2}\left\|v_{M}-J_{1} v_{M}\right\|_{L^{2}(K)}
$$

Since straightforward computations reveal $\left\|\xi_{E}\right\|_{L^{2}(K)}=|E| \sqrt{38 \pi R_{E} / 315} \approx h_{K}^{3 / 2}$ for any $E \in$ $\mathcal{E}(K)$ and $\xi_{E}$ in (B.13), the last displayed estimate and a triangle inequality prove

$$
\begin{aligned}
\| v_{M} & -J_{2} v_{M}\left\|_{L^{2}(K)}-\right\| v_{M}-J_{1} v_{M}\left\|_{L^{2}(K)} \leqslant \sum_{E \in \mathcal{E}(K) \cap \mathcal{E}(\Omega)} \mid f_{E}\left(v_{M}-J_{1} v_{M}\right) \mathrm{d} s\right\| \xi_{E} \|_{L^{2}(K)} \\
& \leqslant 6 \frac{|E| \sqrt{38 \pi R_{E}}}{\sqrt{315}} \frac{\sqrt{1+c_{\mathrm{inv}}} \sqrt{1+3 C_{\mathrm{inv}} / 2}}{|K|^{1 / 2}}\left\|v_{M}-J_{1} v_{M}\right\|_{L^{2}(K)} \lesssim\left\|v_{M}-J_{1} v_{M}\right\|_{L^{2}(K)}
\end{aligned}
$$

for any $K \in \mathcal{T}$ with the shape-regularity $h_{K} \approx|E| \approx R_{E} \approx|K|^{1 / 3}$ in the last step. The inverse estimate $\left|v_{M}-J_{2} v_{M}\right|_{H^{2}(K)} \leqslant\left\|h_{K}^{-2}\left(v_{M}-J_{2} v_{M}\right)\right\|_{L^{2}(K)}$, a summation over all $K \in \mathcal{T}$, and Lemma B. 2 conclude the proof.

The following Lemma B. 4 provides more insight into the inverse estimates for functions of the type $\xi_{E}$ in (B.13). In particular standard inverse estimates hold for the operator $J_{M} \equiv J_{4}$ in Definition B. 4 below.

Lemma B. 4 (inverse estimates). There exists a constant $C \approx 1$ (that exclusively depends on $\mathbb{T}$ ), such that for any tetrahedron $G \in \mathcal{T} \in \mathbb{T}$ or triangle $G \in \mathcal{F}$,

$$
\left|v_{M}-J_{M} v_{M}\right|_{H^{1}(G)} \leqslant C \operatorname{diam}(G)^{-1}\left\|v_{M}-J_{M} v_{M}\right\|_{L^{2}(G)}
$$

Notice that the $H^{1}$-seminorm concerns the tangential gradient $\nu_{F} \times \nabla v_{M}$ for a face $G=F \in \mathcal{F}$ with unit normal $\nu_{F}$ of fixed orientation and the full gradient $\nabla v_{M}$ for a tetrahedra $G=T \in \mathcal{T}$.

Proof. The difference $v_{M}-J_{M} v_{M}$ can be rewritten locally for any $G \in \mathcal{T}$ or $G \in \mathcal{F}$ as

$$
\left.\left(v_{M}-J_{M} v_{M}\right)\right|_{G}=p_{2}+g_{W F}+p_{7}+p_{10}+\sum_{E \in \mathcal{E}(G)} \alpha_{E} \xi_{E}
$$

with polynomials $p_{2} \in P_{2}(G), p_{7} \in P_{7}(G), p_{10} \in P_{10}(G)$, a $W F$ function $g_{W F} \in P_{3}(\hat{\mathcal{T}}), \alpha_{E} \in \mathbb{R}$, and $\xi_{E}$ in (B.13) with $\operatorname{supp}\left(\xi_{E}\right)=B\left(\operatorname{mid}(E), R_{E}\right)=: B$ for any edge $E \in \mathcal{E}(G)$.

Case 1. Suppose $G=F \in \mathcal{F}(T)$ is a face and without loss of generality regard $F \subset \mathbb{R}^{2}$. The key argument is that the support $\operatorname{supp}\left(\xi_{E}\right) \subset \widehat{\omega}(E)$ of $\xi_{E}$ lies in the edge patch of $E \in \mathcal{E}$ in $\widehat{\mathcal{T}}$. In other words the $W F$ function $\left.g_{W F}\right|_{\bar{\omega}} \in P_{3}(\bar{\omega})$ is a cubic polynomial in $\bar{\omega}:=\operatorname{supp}\left(\xi_{E}\right) \cap G$ and the intersection $\omega=B \cap F \subset \mathbb{R}^{2}$ is the half ball in $\mathbb{R}^{2}$ illustrated in Fig. B.1.a. Hence, it remains to show, for all $p_{10} \in P_{10}(\omega)$ and $\alpha_{E} \in \mathbb{R}$, that

$$
\left|p_{10}+\alpha_{E} \xi_{E}\right|_{H^{1}(\omega)} \lesssim h_{E}^{-1}\left\|p_{10}+\alpha_{E} \xi_{E}\right\|_{L^{2}(\omega)} .
$$

Let $H_{+}:=\mathbb{R} \times(0, \infty)$ and consider the re-scaling $\varphi: B\left(0, R_{E}\right) \rightarrow B(0,1), x \mapsto \varphi(x)=$ $x / R_{E}:=\vartheta$ with $|\operatorname{det} D \varphi|=R_{E}^{-2}$ and abbreviate $\hat{f}(x):=f\left(x R_{E}\right)$ for $|x|<1$, such that $f(x)=\hat{f}\left(x / R_{E}\right)=\hat{f}(\vartheta)$ and $\nabla_{\vartheta} \hat{f}(\vartheta)=R_{E} \nabla_{x} f(x)$. A substitution shows

$$
\begin{aligned}
\| \hat{f}_{L^{2}\left(B(0,1) \cap H_{+}\right)}^{2} & =R_{E}^{-2}\|f\|_{L^{2}\left(B\left(0, R_{E}\right) \cap H_{+}\right)}^{2}, \\
|\hat{f}|_{H^{1}\left(B(0,1) \cap H_{+}\right)}^{2} & =\int_{B(0,1) \cap H_{+}}\left|\nabla_{y} \hat{f}(y)\right|^{2} \mathrm{~d} y=\int_{B\left(0, R_{E}\right) \cap H_{+}}\left|\nabla_{\vartheta} \hat{f}(\vartheta)\right|^{2}|\operatorname{det} D \varphi| \mathrm{d} x \\
& =\int_{B\left(0, R_{E}\right) \cap H_{+}} R_{E}^{2}\left|\nabla_{x} f(x)\right|^{2}|\operatorname{det} D \varphi| \mathrm{d} x=|f|_{\left.H^{1}\left(B\left(0, R_{E}\right)\right) \cap H_{+}\right)}^{2} .
\end{aligned}
$$

For radius one there exists a constant $C_{1}>0$, such that for all $p_{10} \in P_{10}$ and $\alpha_{E} \in \mathbb{R}$,

$$
\begin{equation*}
\left|p_{10}+\alpha_{E} \xi_{E}\right|_{H^{1}\left(B(0,1) \cap H_{+}\right)} \leqslant C_{1}\left\|p_{10}+\alpha_{E} \xi_{E}\right\|_{L^{2}\left(B(0,1) \cap H_{+}\right)}, \tag{B.16}
\end{equation*}
$$

since $\left\|p_{10}+\alpha_{E} \xi_{E}\right\|_{L^{2}\left(B(0,1) \cap H_{+}\right)}=0$ on the right-hand side of (B.16) implies on the left-hand side $\left|p_{10}+\alpha_{E} \xi_{E}\right|_{H^{1}\left(B(0,1) \cap H_{+}\right)}=0$. The substitution shows for $\omega=B\left(0, R_{E}\right) \cap H^{+}$,

$$
\left|p_{10}+\alpha_{E} \xi_{E}\right|_{H^{1}(\omega)} \leqslant C_{1} R_{E}^{-1}\left\|p_{10}+\alpha_{E} \xi_{E}\right\|_{L^{2}(\omega)}
$$

and $R_{E} \approx h_{E} \approx h_{F} \approx h_{T}$ concludes the proof since $C_{1}$ is independent of any mesh-size factor.
Case 2. For a tetrahedron, the shape regularity of the $W F$ triangulation $\hat{\mathcal{T}}$ as well as $\mathcal{T} \in \mathbb{T}$ allow the restriction to $G=T \in \hat{\mathcal{T}}$. Hence, it remains to show for all $p_{10} \in P_{10}(T)$ and $\alpha_{E} \in \mathbb{R}$,

$$
\left|p_{10}+\alpha_{E} \xi_{E}\right|_{H^{1}(T)} \lesssim h_{T}^{-1}\left\|p_{10}+\alpha_{E} \xi_{E}\right\|_{L^{2}(T)} .
$$

The key argument in this case is the existence of $\widehat{B}=B\left(\hat{c}_{T}, R_{E}\right)$ with $\hat{B} \cap B=\varnothing$. A triangle inequality, the inverse estimate for $p_{10} \in P_{10}(T)$ with constant $c_{10}>0$ and for $\xi_{E}$ in (B.13) with directly calculated constant show

$$
\left|p_{10}+\alpha_{E} \xi_{E}\right|_{H^{1}(T)} \leqslant \frac{6 \sqrt{6}}{\sqrt{19} R_{E}}\left\|\alpha_{E} \xi_{E}\right\|_{L^{2}(T)}+h_{T}^{-1} c_{k}\left\|p_{10}\right\|_{L^{2}(T)} .
$$

Since the shape regularity ensures $R_{E} \approx h_{T}$, a triangle inequality leads to

$$
\left|p_{10}+\alpha_{E} \xi_{E}\right|_{H^{1}(T)} \lesssim h_{T}^{-1}\left(\left\|\alpha_{E} \xi_{E}\right\|_{L^{2}(T)}+\left\|p_{10}\right\|_{L^{2}(T)}\right) \leqslant h_{T}^{-1}\left(\left\|\alpha_{E} \xi_{E}+p_{10}\right\|_{L^{2}(T)}+2\left\|p_{10}\right\|_{L^{2}(T)}\right) .
$$

The shape regularity guarantees further that there exists a ball $\widetilde{B}:=B\left(\widehat{c}_{T}, \widetilde{R}\right)$ with radius $\widetilde{R} \approx h_{T} \approx R_{E}$ that contains $T, \widehat{B} \subset T \subset \widetilde{B}$. Standard scaling arguments for polynomials show for the polynomial extension $p_{10} \in P_{10}(\widetilde{B})$ (not-relabelled), that

$$
\left\|p_{10}\right\|_{L^{2}(T)} \leqslant\left\|p_{10}\right\|_{L^{2}(\widetilde{B})} \lesssim\left\|p_{10}\right\|_{L^{2}(\widehat{B})} .
$$

The constant in the last step is bounded because the radius quotient $\widetilde{R} / R_{E} \lesssim 1$ is bounded. Since the support of $\xi_{E}$ satisfies $\operatorname{supp}\left(\xi_{E}\right) \cap \widehat{B}=\varnothing$, the estimate $\left\|p_{10}\right\|_{L^{2}(\hat{B})}=\| p_{10}+$ $\alpha_{E} \xi_{E}\left\|_{L^{2}(\hat{B})} \leqslant\right\| p_{10}+\alpha_{E} \xi_{E} \|_{L^{2}(T)}$ concludes the proof.

## B.4. Design of $J_{3}$

This step corrects the integral mean of the normal derivative over each internal face $F \in \mathcal{F}(\Omega)$ such that $f_{F} \nu_{F} \cdot \nabla J_{3} v_{M} \mathrm{~d} s=f_{F} \nu_{F} \cdot \nabla v_{M} \mathrm{~d} s$. Suppose $T=\operatorname{conv}\left\{z_{1}, \ldots, z_{4}\right\} \in \mathcal{T}$ denotes a tetrahedron with face $F=\operatorname{conv}\left\{z_{1}, z_{2}, z_{3}\right\} \in \mathcal{F}(T)$ opposite to the vertex $z_{4} \in \mathcal{V}(T)$. Recall the barycentric coordinate $\lambda_{k} \in P_{1}(T)$ in $T$ associated with $z_{k} \in \mathcal{V}(T)$ for $k=1, \ldots, 4$ and set

$$
\left.\zeta_{F}\right|_{T}:=\frac{7!}{2}\left(\nu_{T} \cdot \nu_{F}\right) \operatorname{dist}\left(z_{4}, F\right)\left(\lambda_{1} \lambda_{2} \lambda_{3}\right)^{2} \lambda_{4} \in P_{7}(T)
$$

for $T \in \mathcal{T}(F)$. This defines $\zeta_{F} \in P_{7}(\mathcal{T}(F)) \cap W_{0}^{2, \infty}(\omega(F))$ with $f_{G} \nabla \zeta_{F} \cdot \nu_{G} \mathrm{~d} s=\delta_{G F}$ for all sides $F, G \in \mathcal{F}$ in the face patch $\omega_{F}:=\operatorname{int}\left(\bigcup_{T \in \mathcal{T}(F)} T\right)$; cf. [Gal15, Prop.2.6] for 2D.
Definition B. $3\left(J_{3}\right)$. For any $v_{M} \in M(\mathcal{T})$ set

$$
\begin{equation*}
J_{3}\left(v_{M}\right):=J_{2} v_{M}+\sum_{F \in \mathcal{F}(\Omega)}\left(f_{F} \nabla\left(v_{M}-J_{2} v_{M}\right) \cdot \nu_{F} \mathrm{~d} s\right) \zeta_{F} \in V . \tag{B.17}
\end{equation*}
$$

Lemma B. 5 (properties of $\boldsymbol{J}_{3}$ ). There exists a constant $C_{3} \approx 1$ (that exclusively depends on $\mathbb{T}$ ) such that the companion $J_{3} v_{M} \in V$ for any $v_{M} \in M(\mathcal{T})$ satisfies
(a) $J_{3}$ is a right inverse to the interpolation $I_{M}$ in that $I_{M} \circ J_{3}=\operatorname{id}$ in $M(\mathcal{T})$,
(b) $\left\|h_{\mathcal{T}}^{-2}\left(1-J_{3}\right) v_{M}\right\|_{L^{2}(\Omega)}+\left\|\left(1-J_{3}\right) v_{M}\right\|_{\mathrm{pw}} \leqslant C_{3} \min _{v \in V}\left\|v_{M}-v\right\|_{\mathrm{pw}}$.

Proof of (a). Since $\left.\zeta_{F}\right|_{\partial T} \equiv 0$, it holds $L_{E}\left(J_{3} v_{M}\right)=L_{E}\left(J_{2} v_{M}\right)$ for any $E \in \mathcal{E}$. Hence Lemma B.3.a implies $L_{E}\left(J_{3} v_{M}\right)=L_{E}\left(v_{M}\right)$ and by construction holds $L_{F}\left(J_{3} v_{M}\right)=L_{F}\left(v_{M}\right)$ for any $F \in \mathcal{F}(\Omega)$. This proves that $J_{3}$ satisfies the right-inverse property (a), since

$$
I_{M}(v):=\sum_{F \in \mathcal{F}(\Omega)} f_{F} \nabla v \cdot \nu_{F} \mathrm{~d} \sigma \phi_{F}+\sum_{E \in \mathcal{E}(\Omega)} f_{E} v \mathrm{~d} s \phi_{E} \quad \text { for any } v \in V \text {. }
$$

Proof of (b). The inverse estimate in Lemma B. 4 guarantees in particular for $g:=v_{M}-J_{2} v_{M}$, $\|\nabla g\|_{L^{2}(F)} \leqslant h_{F}^{-1} \tilde{c}_{\text {inv }}\|g\|_{L^{2}(F)}$. Moreover the discrete trace inequality (B.15) holds with a constant $\tilde{C}_{\text {inv }}$ from Lemma B.4. For any $F \in \mathcal{F}, K \in \mathcal{T}(F)$, and $v_{M} \in M(\mathcal{T})$, this and a Cauchy-Schwarz inequality imply

$$
\begin{aligned}
f_{F} \nabla\left(v_{M}-J_{2} v_{M}\right) \cdot \nu_{F} \mathrm{~d} s & \leqslant|F|^{-1 / 2}\left\|\nabla\left(v_{M}-J_{2} v_{M}\right)\right\|_{L^{2}(F)} \leqslant|F|^{-1 / 2} h_{F}^{-1} \tilde{c}_{\mathrm{inv}}\left\|v_{M}-J_{2} v_{M}\right\|_{L^{2}(F)} \\
& \leqslant|K|^{-1 / 2} h_{F}^{-1} \tilde{c}_{\mathrm{inv}} \sqrt{1+2 \tilde{C}_{\mathrm{inv}} / 3}\left\|v_{M}-J_{2} v_{M}\right\|_{L^{2}(K)} .
\end{aligned}
$$

Straightforward computations reveal that $\left\|\zeta_{F}\right\|_{L^{2}(K)}=6 h_{K}|K|^{1 / 2} / \sqrt{12155} \approx h_{K}^{5 / 2}$ holds for any side $F \in \mathcal{F}$ of $K \in \mathcal{T}(F)$. In combination with the last displayed estimate and a triangle inequality, this proves

$$
\begin{aligned}
\| v_{M} & -J_{3} v_{M}\left\|_{L^{2}(K)}-\right\| v_{M}-J_{2} v_{M}\left\|_{L^{2}(K)} \leqslant \sum_{F \in \mathcal{F}(K) \cap \mathcal{F}(\Omega)}\left|f_{F}\left(v_{M}-J_{2} v_{M}\right) \mathrm{d} s\right|\right\| \zeta_{F} \|_{L^{2}(K)} \\
& \leqslant 4 \frac{6 h_{K}}{\sqrt{12155}} \frac{\tilde{c}_{\mathrm{inv}} \sqrt{1+2 \tilde{C}_{\mathrm{inv}} / 3}}{h_{F}}\left\|v_{M}-J_{2} v_{M}\right\|_{L^{2}(K)} \lesssim\left\|v_{M}-J_{2} v_{M}\right\|_{L^{2}(K)}
\end{aligned}
$$

for any $K \in \mathcal{T}$ with the shape-regularity $h_{K} \approx h_{F}$ in the last step. The inverse estimate $\left|v_{M}-J_{3} v_{M}\right|_{H^{2}(K)} \lesssim\left\|h_{K}^{2}\left(v_{M}-J_{3} v_{M}\right)\right\|_{L^{2}(K)}$ from Lemma B.4, a summation over all $K \in \mathcal{T}$, and Lemma B.3.b conclude the proof.

## B.5. Design of $J_{4}=: J_{M}$

In order to assure the $L^{2}$ orthogonality of $v_{M}-J_{M} v_{M}$ onto $P_{2}(\mathcal{T})$, we propose a local correction for each tetrahedron $T \in \mathcal{T}$ with the product of a squared volume bubble-function and a Riesz representation in $P_{2}(T)$. For any $T:=\operatorname{conv}\left\{z_{1}, z_{2}, z_{3}, z_{4}\right\} \in \mathcal{T}$, let $\lambda_{k} \in P_{1}(T)$ denote the barycentric coordinate in $T$ for $k=1, \ldots, 4$ associated with the vertex $z_{k} \in \mathcal{V}(T)$. Let

$$
\begin{equation*}
b_{T}:=4^{8} \prod_{k=1}^{4} \lambda_{k}^{2} \in P_{8}(T) \cap H_{0}^{2}(T) \subset V \tag{B.18}
\end{equation*}
$$

denote the squared volume bubble that satisfies $\left\|b_{T}\right\|_{L^{\infty}(T)}=1$. Let $v_{T} \in P_{2}(T)$ denote the Riesz representation of the linear functional

$$
\ell_{T}: P_{2}(T) \rightarrow \mathbb{R}, \quad w_{T} \mapsto \int_{T}\left(v_{M}-J_{3} v_{M}\right) w_{T} \mathrm{~d} x
$$

in the Hilbert space $P_{2}(T)$ endowed with the weighted $L^{2}$ scalar product $\left(b_{T} \bullet, \bullet\right)_{L^{2}(T)}$. The Riesz representation $v_{T} \in P_{2}(T)$ satisfies for any $v_{M} \in M(\mathcal{T})$

$$
\begin{equation*}
\left(v_{M}-J_{3} v_{M}, w_{T}\right)_{L^{2}(T)}=\left(b_{T} v_{T}, w_{T}\right)_{L^{2}(T)} \quad \text { for all } w_{T} \in P_{2}(T) \tag{B.19}
\end{equation*}
$$

It can be computed as $v_{T}=\sum_{\ell=1}^{10}\left(\int_{T}\left(v_{M}-J_{3} v_{M}\right) \Phi_{T, \ell} \mathrm{~d} x\right) \Phi_{T, \ell} b_{T}$ with a basis $\Phi_{T, 1}, \ldots, \Phi_{T, 10}$ of $P_{2}(T)$ that satisfies $\int_{T} \Phi_{T, \ell} \Phi_{T, k} b_{T} \mathrm{~d} x=\delta_{\ell, k}$ for any $\ell, k=1, \ldots, 10$.

Definition B. $4\left(J_{4} \equiv \boldsymbol{J}_{\boldsymbol{M}}\right)$. Define $J_{4} v_{M} \in V$ for any $v_{M} \in \mathcal{M}(T)$ by

$$
\begin{equation*}
J_{M} v_{M}:=J_{4} v_{M}:=J_{3} v_{M}+\sum_{T \in \mathcal{T}} v_{T} b_{T} . \tag{B.20}
\end{equation*}
$$

Lemma B. 6 (properties of $\boldsymbol{J}_{\mathbf{4}} \equiv \boldsymbol{J}_{\boldsymbol{M}}$ ). For any $v_{M} \in \mathcal{M}(T)$ the conforming companion $J_{4} v_{M}=J_{M} v_{M}$ satisfies Theorem 3.1.

Proof of Theorem 3.1.a. Since $\left.\left.\left(v_{T} b_{T}\right)\right|_{\partial T} \equiv 0 \equiv\left(D\left(v_{T} b_{T}\right)\right)\right|_{\partial T}$ vanishes along the boundary of any $T \in \mathcal{T}, L_{E}\left(J_{M} v_{M}\right)=L_{E}\left(J_{3} v_{M}\right)$ and $L_{F}\left(J_{M} v_{M}\right)=L_{F}\left(J_{3} v_{M}\right)$ holds for any $E \in \mathcal{E}$ and $F \in \mathcal{F}$. Hence $J_{M} \equiv J_{4}$ satisfies the right inverse property according to Lemma B.5.a.

Proof of Theorem 3.1.b. For $w_{T}=v_{T} \in P_{2}(T)$, (B.19) and a Cauchy-Schwarz inequality imply

$$
\begin{equation*}
\left(b_{T} v_{T}, v_{T}\right)_{L^{2}(T)}=\left(v_{M}-J_{3} v_{M}, v_{T}\right)_{L^{2}(T)} \leqslant\left\|v_{M}-J_{3} v_{M}\right\|_{L^{2}(T)}\left\|v_{T}\right\|_{L^{2}(T)} . \tag{B.21}
\end{equation*}
$$

The volume bubble $b_{T}$ in (B.18) generates an inverse estimate (cf. [Ver13, §3.6])

$$
\begin{equation*}
C_{\mathrm{inv}}\left\|w_{T}\right\|_{L^{2}(T)} \leqslant\left\|b_{T}^{1 / 2} w_{T}\right\|_{L^{2}(T)} \leqslant\left\|w_{T}\right\|_{L^{2}(T)} \quad \text { for all } w_{T} \in P_{2}(T) \tag{B.22}
\end{equation*}
$$

with a universal constant $C_{\text {inv }} \approx 1$ and the bound $\left\|b_{T}^{2}\right\|_{L^{\infty}(T)}=1$ in the last step. The combination of (B.21)-(B.22) proves

$$
C_{\mathrm{inv}}^{2}\left\|v_{T}\right\|_{L^{2}(T)}^{2} \leqslant\left\|b_{T}^{1 / 2} v_{T}\right\|_{L^{2}(T)} \leqslant\left\|v_{M}-J_{3} v_{M}\right\|_{L^{2}(T)}\left\|v_{T}\right\|_{L^{2}(T)} .
$$

Hence $C_{\text {inv }}^{2}\left\|v_{T}\right\|_{L^{2}(T)} \leqslant\left\|v_{M}-J_{3} v_{M}\right\|_{L^{2}(T)}$. The definition (B.20) and $\left\|b_{T}^{2}\right\|_{L^{\infty}(T)}=1$ imply

$$
\left\|J_{M} v_{M}-J_{3} v_{M}\right\|_{L^{2}(T)}=\left\|v_{T} b_{T}\right\|_{L^{2}(T)} \leqslant\left\|v_{T}\right\|_{L^{2}(T)} \leqslant C_{\text {inv }}^{-2}\left\|v_{M}-J_{3} v_{M}\right\|_{L^{2}(T)} .
$$

This holds for all $T \in \mathcal{T}$ and so a triangle inequality implies

$$
\begin{aligned}
\left\|h_{\mathcal{T}}^{-2}\left(v_{M}-J_{M} v_{M}\right)\right\|_{L^{2}(\Omega)} & =\left\|h_{\mathcal{\mathcal { T }}}^{-2}\left(v_{M}-J_{3} v_{M}\right)\right\|_{L^{2}(\Omega)}+\left\|h_{\mathcal{T}}^{-2}\left(J_{M} v_{M}-J_{3} v_{M}\right)\right\|_{L^{2}(\Omega)} \\
& \leqslant\left(1+C_{\mathrm{inv}}^{-2}\right)\left\|h_{\mathcal{T}}^{-2}\left(v_{M}-J_{3} v_{M}\right)\right\|_{L^{2}(\Omega)} .
\end{aligned}
$$

The combination with Lemma B.5.b concludes the proof of

$$
\left\|h_{\mathcal{T}}^{-2}\left(v_{M}-J_{M} v_{M}\right)\right\|_{L^{2}(\Omega)} \leqslant\left(1+C_{\mathrm{inv}}^{-2}\right) C_{3} \min _{v \in V}\left\|v_{M}-v\right\|_{\mathrm{pw}} .
$$

This and the inverse estimate $\left\|v_{M}-J_{M} v_{M}\right\|_{\mathrm{pw}} \leqslant\left\|h_{\mathcal{T}}^{-2}\left(v_{M}-J_{M} v_{M}\right)\right\|_{L^{2}(\Omega)}$ from Lemma B. 4 conclude the proof of (b).

Proof of Theorem 3.1.c. The definition (B.20) and the identity (B.19) imply
$\left(v_{M}-J_{M} v_{M}, w_{T}\right)_{L^{2}(T)}=\left(v_{M}-J_{3} v_{M}, w_{T}\right)_{L^{2}(T)}-\left(b_{T} v_{T}, w_{T}\right)_{L^{2}(T)}=0 \quad$ for all $w_{T} \in P_{2}(T)$.
This proves the claim.
Remark B. 1 (general boundary conditions). The construction in Subsections B.2-B. 5 works for more general boundary conditions of $M(\mathcal{T})$. In (B.2)-(B.3) the vanishing boundary values can be replaced by the same averaging formula as in the interior degrees of freedom. This simplifies the proof of Lemma B. 2 in that (B.8) holds directly for any $z \in \mathcal{V}$ etc. The corrections in (B.14)-(B.17) are possible for boundary edges and sides as well and guarantee the right inverse property. Subsection B. 5 is independent of boundary conditions.

## Supplement2DirectGLB4BiLaplace

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