Traces and Duality Lemma

Recall the duality lemma with $H^{1/2}(\partial\Omega) := \gamma_0(H^1(\Omega))$ defined as the trace space of $H^1(\Omega)$ endowed with minimal extension norm; i.e., for $w \in H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$,

$$\|w\|_{H^{1/2}(\partial\Omega)} = \min\{\|\widehat{w}\|_{H^1(\Omega)} | \widehat{w} \in H^1(\Omega), \gamma_0\widehat{w} = w\},$$

$$\begin{aligned} H^{-1/2}(\partial\Omega) &:= \text{ dual to } H^{1/2}(\partial\Omega) =: H^{1/2}(\partial\Omega)^* \\ &\overset{!}{=} \gamma_\nu(H(\mathrm{div}, \Omega)). \end{aligned}$$

Any $q \in H(\mathrm{div}, \Omega)$ (i.e. $q \in L^2(\Omega, \mathbb{R}^2)$, $\mathrm{div}\, q \in L^2(\Omega)$) defines $\gamma_\nu q \in H^{-1/2}(\partial\Omega)$ by

$$(\gamma_\nu q)(w) =: \langle q \cdot \nu, w\rangle_{\partial\Omega} = \int_\Omega (q \cdot \nabla\hat{w} + \hat{w}\,\mathrm{div}\, q)\mathrm{d}x$$

for $w \in H^{1/2}(\partial\Omega)$ and $\widehat{w} \in H^1(\Omega)$ with $\gamma_0\widehat{w} = w$.
(Side note:

$$\begin{aligned} \langle q \cdot \nu, w\rangle_{\partial\Omega} &\leq \|q\| \,\|\!|\, \widehat{w} \,|\!\|\, + \|\,\mathrm{div}\, q\| \|\widehat{w}\| \\ &\leq \|q\|_{H(\mathrm{div},\Omega)} \|\widehat{w}\|_{H^1(\Omega)} \end{aligned}$$

implies $\|\gamma_\nu q\|_{H^{-1/2}(\partial\Omega)} \leq \|q\|_{H(\mathrm{div},\Omega)}$.)

Duality Lemma. *(a) There exists exactly one*

$$\gamma_\nu \in L(H(\mathrm{div}, \Omega); H^{-1/2}(\partial\Omega))$$

such that for all $q \in H^1(\Omega; \mathbb{R}^n)$

$$\gamma_\nu q = (\gamma_0 q) \cdot \nu \;\; a.e. \text{ on } \partial\Omega.$$

(b) Let $\langle \bullet, \bullet \rangle_{\partial\Omega}$ *denote the duality brackets of* $H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$*. All* $q \in H(\mathrm{div}, \Omega)$ *and* $v \in H^1(\Omega)$ *satisfy the formula*

$$\langle \gamma_\nu q, \gamma_0 v\rangle_{\partial\Omega} = \int_\Omega \Big(v\,\mathrm{div}\, q + q \cdot \nabla v\Big)\mathrm{d}x.$$

(c) The operator γ_ν is surjective and

$$\ker \gamma_\nu = H_0(\operatorname{div}, \Omega) := \overline{\mathcal{D}(\Omega; \mathbb{R}^n)}^{\| \bullet \|_{H(\operatorname{div})}}.$$

(d) (Duality lemma) For all $t \in H^{-1/2}(\partial\Omega)$ with $t(w) =: \langle t, w \rangle_{\partial\Omega}$ for $w \in H^{1/2}(\partial\Omega)$, it holds

$$\begin{aligned}
\|t\|_{H^{-1/2}(\partial\Omega)} &= \sup_{\substack{w \in H^{1/2}(\partial\Omega), \\ \|w\|_{H^{1/2}}=1}} \langle t, w \rangle_{\partial\Omega} \\
&= \sup_{\substack{v \in H^1(\Omega), \\ \gamma_0 v \neq 0}} \inf_{\varphi \in H^1(\Omega)} \frac{\langle t, \gamma_0 v \rangle_{\partial\Omega}}{\|v - \varphi\|_{H^1(\Omega)}} \\
&= \inf_{\substack{q \in H(\operatorname{div},\Omega), \\ \gamma_\nu q = t}} \|q\|_{H(\operatorname{div},\Omega)}.
\end{aligned}$$

Proof. Proof of (a). Let $q \in H(\operatorname{div}, \Omega)$. For all $v \in H^{1/2}(\partial\Omega)$ $\hat{v} \in H^1(\Omega)$ denotes the unique weak solution of

$$\begin{aligned}
-\Delta \hat{v} + \hat{v} &= 0 \text{ in } \Omega, \\
\gamma_0 \hat{v} &= v \text{ on } \partial\Omega.
\end{aligned}$$

Then $\|v\|_{H^{1/2}(\partial\Omega)} = \|\hat{v}\|_{H^1(\Omega)}$. Define

$$X_q(v) := \int_\Omega (\hat{v} \operatorname{div} q + q \cdot \nabla \hat{v}) \mathrm{d}x$$

The repeated application of the Cauchy Schwarz inequality shows

$$\begin{aligned}
X_q(v) &\leq \|\hat{v}\|_{L^2(\Omega)} \|\operatorname{div} q\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)} \|\nabla \hat{v}\|_{L^2(\Omega)} \\
&\leq \|q\|_{H(\operatorname{div})} \|\hat{v}\|_{H^1(\Omega)} = \|q\|_{H(\operatorname{div})} \|v\|_{H^{-1/2}(\partial\Omega)}.
\end{aligned}$$

Hence $X_q : H^{1/2}(\partial\Omega) \to \mathbb{R}$ is linear and bounded. Thus for any $q \in H(\operatorname{div}, \Omega)$ there exists $g(q) \in H^{-1/2}(\partial\Omega)$ with $X_q = g(q)$. Define $\gamma_\nu : q \mapsto g(q)$. This operator is linear. The last inequality shows that the operator is also bounded, more precisely

$$\|\gamma_\nu\|_{L(H(\operatorname{div},\Omega); H^{-1/2}(\partial\Omega))} \leq 1.$$

Moreover, for all functions $v \in H^{1/2}(\partial\Omega)$ and $q \in H^1(\Omega, \mathbb{R}^n)$, an integration by parts leads to

$$\langle (\gamma_0 q) \cdot \nu, v \rangle_{\partial\Omega} = \langle (\gamma_0 q) \cdot \nu, \gamma_0 \hat{v} \rangle_{\partial\Omega} = \int_\Omega (\hat{v} \operatorname{div} q + q \cdot \nabla \hat{v}) \mathrm{d}x,$$

Thus $\langle(\gamma_0 q)\cdot\nu, v\rangle_{\partial\Omega} = \langle\gamma_\nu q, v\rangle_{\partial\Omega}$. Hence, for all $v \in H^{1/2}(\partial\Omega)$ it holds

$$\langle(\gamma_0 q)\cdot\nu - \gamma_\nu q, v\rangle_{\partial\Omega} = 0,$$

i.e., $(\gamma_0 q)\cdot\nu = \gamma_\nu q \in H^{-1/2}(\partial\Omega)$. This implies (a). Moreover, $\langle\bullet,\bullet\rangle_{\partial\Omega}$ extends the scalar product $(\bullet,\bullet)_{\partial\Omega}$ in $L^2(\partial\Omega)$ for smooth functions. □

Proof of (b). For all $v \in H^1(\Omega)$ and $q \in H(\mathrm{div},\Omega)$ it holds

$$\langle\gamma_\nu q, \gamma_0 v\rangle_{\partial\Omega} = \int_\Omega \Big(q\cdot\nabla\hat{v} + \hat{v}\,\mathrm{div}\,q\Big)\mathrm{d}x,$$

where $\hat{v} \in H^1(\Omega)$ is such that $\Delta\hat{v} + \hat{v} = 0$ and $\gamma_0 v = \gamma_0\hat{v}$ in the weak sense. Since $\ker\gamma_0 = H^1_0(\Omega)$ and $v - \hat{v} \in H^1_0(\Omega)$, this equals

$$\int_{\partial\Omega} \gamma_\nu q\cdot\gamma_0 v\,\mathrm{d}s = \int_\Omega \Big(q\cdot\nabla v + v\,\mathrm{div}\,q\Big)\mathrm{d}x$$

and implies (b). □

Proof of (c). For all $q \in \mathcal{D}(\Omega;\mathbb{R}^n)$, $\gamma_\nu q = (\gamma_0 q)\cdot\nu = 0$ a.e. by (a). Hence,

$$H_0(\mathrm{div},\Omega) = \overline{\mathcal{D}(\Omega;\mathbb{R}^n)}^{\|\bullet\|_{H(\mathrm{div})}} \subseteq \ker\gamma_\nu.$$

The proof of $\ker\gamma_\nu \subseteq H_0(\mathrm{div},\Omega)$ is more technical and can be found in the literature, i.e., in [Girault, V. and Raviart, P. A., *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, Berlin, Heidelberg, New York (1986)]. It remains to show the surjectivity of γ_ν. Given any $t \in H^{-1/2}(\partial\Omega)$, the functional

$$T : H^1(\Omega) \to \mathbb{R}, v \mapsto \langle t, \gamma_0 v\rangle_{\partial\Omega}$$

is linear and bounded, written $T \in H^1(\Omega)^*$. The Riesz representation $z \in H^1(\Omega)$ of T in the Hilbert space $H^1(\Omega)$ satisfies $\langle z,\bullet\rangle_{H^1(\Omega)} = T(\bullet)$. For $\varphi \in \mathcal{D}(\Omega)$, it follows for $\gamma_0\varphi = 0$ that

$$\langle z,\varphi\rangle_{H^1(\Omega)} = T(\varphi) = \langle t,\gamma_0\varphi\rangle = 0.$$

This proves $-\Delta z + z = 0$ in the weak sense. In particular, $q := \nabla z \in L^2(\Omega;\mathbb{R}^n)$ and $\mathrm{div}\,\nabla z = \Delta z$ leads to $\mathrm{div}\,q = z \in L^2(\Omega)$. Hence, $q \in H(\mathrm{div},\Omega)$ and

$$\begin{aligned}\|q\|_{H(\mathrm{div})} &= (\|\mathrm{div}\,q\|^2 + \|q\|^2)^{1/2} = (\|z\|^2 + \|\nabla z\|^2)^{1/2}\\ &= \|z\|_{H^1(\Omega)} = \|T\|_{(H^1(\Omega))^*}.\end{aligned}$$

For any $v \in H^1(\Omega)$, it follows

$$\begin{aligned}\langle \gamma_\nu q, \gamma_0 v\rangle_{\partial\Omega} &= \int_\Omega \Big(q\cdot\nabla v + v \operatorname{div} q\Big)\mathrm{d}x = \int_\Omega \Big(\nabla z\cdot\nabla v + vz\Big)\mathrm{d}x \\ &= \langle z, v\rangle_{H^1(\Omega)} = T(v) = \langle t, \gamma_0 v\rangle_{\partial\Omega}.\end{aligned}$$

This implies $\langle \gamma_\nu q - t, \gamma_0 v\rangle = 0$ for all $v \in H^1(\Omega)$, which is $\gamma_\nu q - t = 0$ in $H^{-1/2}(\partial\Omega)$. Consequently, $t = \gamma_\nu q \in \mathcal{R}(\gamma_\nu)$. □

Proof of (d). For any $t \in H^{-1/2}(\partial\Omega)$ let z and q be as above in the proof of (c). Then

$$\|t\|_{H^{-1/2}(\partial\Omega)} = \sup_{\substack{\hat{v}\in H^1(\Omega)\\ \|\gamma_0\hat{v}\|_{H^{1/2}(\partial\Omega)}=1}} \langle t, \gamma_0 \hat{v}\rangle$$

with $\langle t, \gamma_0\hat{v}\rangle = \langle \gamma_\nu \nabla z, \gamma_0 \hat{v}\rangle = \langle z, \hat{v}\rangle_{H^1(\Omega)} \le \|z\|_{H^1(\Omega)}\|\hat{v}\|_{H^1(\Omega)}$. Since $\|\hat{v}\|_{H^1(\Omega)} = 1$, this implies $\|t\|_{H^{-1/2}(\partial\Omega)} \le \|z\|_{H^1(\Omega)}$. Conversely, $\langle t, \gamma_0 z\rangle = \langle z, z\rangle_{H^1(\Omega)} = \|z\|^2_{H^1(\Omega)}$ proves $\|t\|_{H^{-1/2}(\partial\Omega)} \ge \|z\|_{H^1(\Omega)}$. □

This concludes the proof and characterizes $H^{-1/2}(\partial\Omega)$ completely. □

Primal PMP with test functions in $H^1(\Omega)$ without (BC) leads to

$$b(u,t;v) = a(u,v) - \langle t, v\rangle_{\partial\Omega} \overset{!}{=} F(v) \text{ for all } v \in H^1(\Omega). \tag{P}$$

Theorem. *u solves (PMP) $\iff$ $(u, \gamma_\nu\nabla u)$ solves (P).*

Proof. "$\Rightarrow$" $v \in H^1_0(\Omega)$ implies $\langle t, v\rangle_{\partial\Omega} = 0$. Hence u solves (PMP). □
"$\Leftarrow$" Let $u \in H^1_0(\Omega)$ solve (PMP), then $p := \nabla u \in H(\operatorname{div}, \Omega)$ leads to $t := \gamma_\nu p \in H^{-1/2}(\partial\Omega)$ so that, for all $v \in H^1(\Omega)$, it follows it follows

$$\begin{aligned}\langle t, v\rangle_{\partial\Omega} &= \langle p\cdot\nu, \gamma_0(v)\rangle_{\partial\Omega} \\ &= \int_\Omega (\underbrace{p}_{=\nabla u}\cdot\nabla v + v \underbrace{\operatorname{div} p}_{=-f})\mathrm{d}x = a(u,v) - F(v).\end{aligned}$$ □

Define $H_0(\operatorname{div}, \Omega) := \{q \in H(\operatorname{div}, \Omega) | \gamma_\nu q = 0\}$.

Interface trace spaces

For a shape-regular triangulation $\mathcal{T}$ of $\Omega \subset \mathbb{R}^n$ into simplices define

$$H^{-1/2}(\partial\mathcal{T}) := \{(t_K)_{K\in\mathcal{T}} \in \prod_{K\in\mathcal{T}} H^{-1/2}(\partial K) | \exists q \in H(\operatorname{div},\Omega) \forall K \in \mathcal{T}, \\ \gamma_\nu(q|_K) = t_K \in H^{-1/2}(\partial K)\}$$

endowed with the norm

$$\|(t_K)_{K\in\mathcal{T}}\|_{H^{-1/2}(\partial\mathcal{T})} := \min\{\|q\|_{H(\operatorname{div},\Omega)} | \forall K \in \mathcal{T}, \gamma_\nu(q|_K) = t_K\}$$

and

$$\begin{aligned} H^1(\mathcal{T}) &:= \{v \in L^2(\Omega) | \forall K \in \mathcal{T}, v|_K \in H^1(K)\} \\ &= \prod_{K\in\mathcal{T}} H^1(K) \end{aligned}$$

with

$$\|(v_K)_{K\in\mathcal{T}}\|_{H^1(\mathcal{T})} := \sqrt{\sum_{K\in\mathcal{T}} \|v|_K\|^2_{H^1(K)}}.$$

Given $t \equiv (t_K)_{K\in\mathcal{T}} \in H^{-1/2}(\partial\mathcal{T})$ and $v \in H^1(\mathcal{T})$ define

$$\langle t, v\rangle_{\partial\mathcal{T}} := \sum_{K\in\mathcal{T}} \langle t_K, v|_K\rangle_{\partial K}.$$

There exists $q \in H(\operatorname{div},\Omega)$ such that

$$t_K = \gamma_\nu(q|_K) \in H^{-1/2}(\partial K) \qquad \text{for all } K \in \mathcal{T}$$

and

$$\begin{aligned} \langle t, v\rangle_{\partial\mathcal{T}} &= \sum_{K\in\mathcal{T}} \int_K (q\cdot\nabla v + v \operatorname{div} q)\mathrm{d}x = \int_\Omega (q\cdot\nabla_{NC} v + v\cdot\operatorname{div} q) dtx \\ &\leq \|q\|_{H(\operatorname{div},\Omega)} \|v\|_{H^1(\mathcal{T})} \overset{!}{=} \|t\|_{H^{-1/2}(\partial\mathcal{T})} \|v\|_{H^1(\Omega)} \end{aligned}$$

Define

$$\begin{cases} b\colon (H^1_0(\Omega) \times H^{-1/2}(\partial\mathcal{T})) \times H^1(\mathcal{T}) \to \mathbb{R} \\ b(u,t;v) := ((u,t),v) \mapsto a_{NC}(u,v) - \langle t, v\rangle_{\partial\mathcal{T}} \end{cases}$$

Theorem. *u solves (PMP) and $t = (t_K)_{K\in\mathcal{T}} = (\gamma_\nu(\nabla u|_K))_{K\in\mathcal{T}}$ if and only if $(u,t) \in H^1_0(\Omega) \times H^{-1/2}(\partial\mathcal{T})$ solves*

$$b(u,t;v) = F(v)$$

for all $v \in H^1(\mathcal{T})$.

Proof. The Proof is left as an exercise. □

Remark. $\langle t, v\rangle_{\partial\mathcal{T}} = 0$ for $v \in H^1_0(\Omega)$.

inf-sup Condition

This section is devoted to some immediate estimation for $\beta > 0$. Recall $X := X_1 \times X_2 := H^1_0(\Omega) \times H^{-1/2}(\partial\mathcal{T})$, $Y := H^1(\mathcal{T})$ and the bounded bilinear form $b : X \times Y \to \mathbb{R}$ with

$$b(u,t;v) = b((u,t),v) = a_{\mathrm{NC}}(u,v) - \langle t, v\rangle_{\partial\mathcal{T}} \quad \forall\,(u,t) \in X, v \in Y.$$

For any $(u,t) \in S(X)$ and $v \in S(Y)$ the Cauchy-Schwarz inequality leads to

$$\begin{aligned} b(u,t;v) &\le \vert\!\vert\!\vert u \vert\!\vert\!\vert \, \vert\!\vert\!\vert v \vert\!\vert\!\vert_{\mathrm{NC}} + \|t\|_{H^{-1/2}(\partial\mathcal{T})}\|v\|_{H^1(\mathcal{T})} \\ &\le \sqrt{\vert\!\vert\!\vert u \vert\!\vert\!\vert^2 + \|t\|^2_{H^{-1/2}(\partial\mathcal{T})}}\sqrt{\vert\!\vert\!\vert v \vert\!\vert\!\vert^2_{\mathrm{NC}} + \|v\|^2_{H^1(\mathcal{T})}}. \end{aligned}$$

With $\vert\!\vert\!\vert v \vert\!\vert\!\vert_{\mathrm{NC}} \le \|v\|_{H^1(\mathcal{T})}$ the choice of (u,t) and v finally shows, that $b(u,t;v) \le \sqrt{2}$. Given $(u,t) \in S(X)$ set $M := \|b(u,t;\bullet)\|_{H^1(\mathcal{T})^*}$. For $u \ne 0$ choose $v := u/\|u\|_{H^1(\mathcal{T})}$ to obtain

$$\langle t, u\rangle_{\partial\mathcal{T}} = \sum_{K\in\mathcal{T}} \langle t, u\rangle_{\partial K} = \int_\Omega \Big(q \cdot \nabla u + u \operatorname{div} q\Big)\mathrm{d}x = \int_{\partial\Omega} u\, q \cdot \nu \mathrm{d}s = 0.$$

Hence,

$$b(u,t;v) = \frac{a_{\mathrm{NC}}(u,u)}{\|u\|_{H^1(\mathcal{T})}} = \frac{\vert\!\vert\!\vert u \vert\!\vert\!\vert^2}{\sqrt{\|u\|^2 + \vert\!\vert\!\vert u \vert\!\vert\!\vert^2}}.$$

The Friedrichs inequality implies $\|u\| \le C_F(\Omega)\,\vert\!\vert\!\vert u \vert\!\vert\!\vert$ with $C_F \le \mathrm{width}(\Omega)/\pi$. This leads to

$$b(u,t;v) \le \frac{\vert\!\vert\!\vert u \vert\!\vert\!\vert}{\sqrt{1 + C_F^2(\Omega)}} \le M.$$

Hence

$$\|\!| u \|\!| \le M\sqrt{1+C_F^2(\Omega)}. \tag{1}$$

Given t let $q \in H(\mathrm{div},\Omega)$ have minimal extension norm in $H(\mathrm{div},\Omega)$ with $q\cdot\nu = t$ on ∂K for all $K\in\mathcal{T}$. The duality lemma leads to some $v\in H^1(\mathcal{T})$ with $\|v\|_{H^1(\mathcal{T})} = 1$ and $\|t\|_{H^{-1/2}(\partial\mathcal{T})} = \langle t, v\rangle_{\partial\mathcal{T}}$ (i.e. v is the normed Riesz representation of $\langle t, \bullet\rangle_{\partial\mathcal{T}}$ in $H^1(\mathcal{T})$). This implies

$$-\|\!| u \|\!| \, \|\!| v \|\!|_{\mathrm{NC}} + \|t\|_{H^{-1/2}} = a_{\mathrm{NC}}(u,v) + \|t\|_{H^{-1/2}} = b(u,t;v) \le M,$$

whence

$$\|t\|_{H^{-1/2}(\partial\mathcal{T})} - \|\!| u \|\!| \le M. \tag{2}$$

The inequalities (1) and (2) show that

$$1 = \|\!| u \|\!|^2 + \|t\|^2_{H^{-1/2}(\partial\mathcal{T})} \le M^2\Big(1+\sqrt{1+C_F^2(\Omega)}\Big)^2 + M^2(1+C_F^2(\Omega)).$$

This leads to

$$\frac{1}{(1+\sqrt{C_F^2(\Omega)})^2 + 1 + C_F^2(\Omega)} \le M^2 = \|b(u,t;\bullet)\|^2_{H^1(\mathcal{T})^*}.$$

Since this holds for all $(u,t)\in S(X)$, it implies

$$0 < \frac{1}{\sqrt{3+2C_F(\Omega)^2 + 2\sqrt{1+C_F^2(\Omega)}}} \le \beta = \inf_{x\in S(X)} \sup_{v\in S(H^1(\mathcal{T}))} b(u,t;v).$$

Splitting Lemmas

Splitting Lemma I. *Given real Hilbert spaces* $X_1, X_2, X := X_1\times X_2, \{0\}\neq Y_1 \subseteq Y$ *and bounded bilinear forms* $b_j : X_j \to Y$ *for* $j = 1,2$, *let* $b : X\times Y \to \mathbb{R}$, $(x_1,x_2;y)\mapsto b_1(x_1,y)+b_2(x_2,y)$. *Suppose*

(A1) $0 < \beta_1 := \inf_{x_1\in S(X_1)} \sup_{y_1\in S(Y_1)} b_1(x_1,y_1)$,

(A2) $0 < \beta_2 := \inf_{x_2\in S(X_2)} \sup_{y\in S(Y)} b_2(x_2,y)$,

(A3) $b_2|_{X_2\times Y_1} = 0$.

Then b satisfies an inf-sup condition with $\beta > 0$ and

$$0 < \frac{\beta_1 \beta_2}{\sqrt{(\beta_1 + \|b_1\|)^2 + \beta_2^2}} \le \beta \le \beta_2.$$

Example (Application to primal dPG for PMP)**.** Let $X_1 := H_0^1(\Omega)$, $X_2 := H^{-1/2}(\partial\mathcal{T})$ and $Y_1 := H_0^1(\Omega) \subseteq H^1(\mathcal{T}) =: Y$.

Ad (A1). Show that

$$\beta_1 = \inf_{u \in H_0^1(\Omega), \||\, u \,\|| = 1} \; \sup_{v \in H_0^1(\Omega), \||\, v \,\||^2 + \|v\|^2 = 1} a(u,v) = \frac{1}{\sqrt{1 + C_F^2(\Omega)}}.$$

Proof of "$\le$" is as above. For "$\ge$" utilize the first Dirichlet eigenpair (λ_1, Φ_1) with $\|\Phi_1\| = 1$ and $\||\, \Phi_1 \,\|| = \lambda_1^{1/2}$ so that $C_F(\Omega) = \lambda_1^{-1/2}$ and $\|\Phi_1\|_{H^1(\mathcal{T})} = \sqrt{1 + \lambda_1}$. Consequently

$$\beta_1 \le \sup_{v \in H_0^1(\Omega), \|v\|_{H^1(\Omega)} = 1} \frac{a(\Phi_1, v)}{\||\, \Phi_1 \,\||}.$$

Since the eigenvectors $(\Phi_j)_{j\in\mathbb{N}}$ form an L^2-orthonormal and a-orthogonal basis of $H_0^1(\Omega)$ the supremum is attained by $v := \Phi_1 / \|\Phi_1\|_{H^1()\Omega}$. This leads to

$$\beta_1 \le \frac{\||\, \Phi_1 \,\||^2}{\||\, \Phi_1 \,\|| \, \|\Phi_1\|_{H^1(\Omega)}} = \frac{\||\, \Phi_1 \,\||}{\|\Phi_1\|_{H^1(\Omega)}} = \frac{\sqrt{\lambda_1}}{\sqrt{1+\lambda_1}} = \frac{1}{\sqrt{1 + \lambda^{-1}}} = \frac{1}{\sqrt{1 + C_F^2(\Omega)}}.$$

Ad (A2). By duality lemma it holds

$$\beta_2 := \inf_{t \in S(H^{-1/2}(\partial\mathcal{T}))} \; \sup_{v \in S(H^1(\mathcal{T}))} -\langle t, v \rangle_{\partial\mathcal{T}} = \inf_{t \in S(H^{-1/2}(\partial\mathcal{T}))} \|t\|_{H^{-1/2}(\partial\mathcal{T})} = 1.$$

Ad (A3). The Cauchy-Schwarz inequality implies

$$\|b_1\| = \sup_{u \in S(H_0^1(\Omega))} \; \sup_{v \in S(H^1(\mathcal{T}))} a_{\mathrm{NC}}(u,v) \le \sup_{u \in S(H_0^1(\Omega))} \; \sup_{v \in S(H^1(\mathcal{T}))} \||\, u \,\|| \, \||\, v \,\|| \le 1.$$

The proof of $\|b_1\| \ge 1$ is left as an exercise. This leads to the inf-sup estimate

$$\frac{1}{\sqrt{1 + C_F^2(\Omega) + (\sqrt{1 + C_F^2(\Omega)} + 1)^2}} = \frac{1}{\sqrt{3 + 2C_F^2(\Omega) + 2\sqrt{1 + C_F^2(\Omega)}}} \le \beta.$$

Proof of the first splitting lemma. Given $(x_1, x_2) \in S(X_1 \times X_2)$ let $s := \|x_1\|_{X_1}$ and $\|x_2\|_{X_2} = \sqrt{1-s^2}$ for $0 \le s \le 1$. Then (A1) and (A3) imply

$$\beta_1 s \le \|b_1(x_1, \bullet)\|_{Y_1^*} = \|b(x_1, x_2; \bullet)\|_{Y_1^*} =: M.$$

Moreover, (A2), the definition of b and triangle inequality show that

$$\beta_2 \sqrt{1-s^2} \le \|b_2(x_2, \bullet)\|_{Y_*} \le \|b(x_1, x_2; \bullet)\|_{Y_*} + \|b_1(x_1, \bullet)\|_{Y_*} \le M + \|b_1\| s.$$

Consequently

$$f(s) := \max\{\beta_1 s, \beta_2\sqrt{1-s^2} - \|b_1\| s\} \le M.$$

It remains to compute $\min f := \min_{0 \le s \le 1} f(s) \le M$. Since $(x_1, x_2) \in S(X)$ is arbitrary, this lead to $\beta_0 \le \beta$. The monotony of $\beta_1 s$ and $\beta_2\sqrt{1-s^2} - \|b_1\| s$ shows that the minimizer s exists in $(0, 1)$ with

$$(\|b_1\| + \beta_1) s = \beta_2 \sqrt{1-s^2}.$$

Set $\kappa := \beta_2/(\beta_1 + \|b_1\|)$, so $s^2 = \kappa^2(1-s^2)$, whence $s = \kappa/\sqrt{1+\kappa}$. Consequently,

$$\beta_0 = \frac{\beta_1 \kappa}{\sqrt{1+\kappa^2}}$$

concludes the proof. □

Splitting Lemma II. *In addition to the notation of the first splitting lemma with (A1)-(A2), suppose*

$$Y_1 := \{y \in Y \,|\, b_2(\bullet, y) = 0 \text{ in } X_2\}$$

(then (A3) follows and characterizes maximal Y_1 in (A3)) and

$$N_1 := \{y_1 \in Y_1 |\, b_1(\bullet, y_1) = 0 \text{ in } X_1\} = \{0\}.$$

Then

$$N := \{y \in Y \,|\, b(\bullet, y) = 0 \text{ in } X\} = 0$$

and

$$\underline{\beta} := \frac{\sqrt{2}\beta_1\beta_2}{\sqrt{\beta_1^2 + \beta_2^2 + \|b_1\|^2 + \sqrt{(\beta_1^2 + \beta_2^2 + \|b_1\|^2)^2 - 4\beta_1^2\beta_2^2}}} \le \beta.$$

Example (Application to primal dPG for PMP)**.** Given $v \in Y_1$, then for any $q \in H(\mathrm{div}, \Omega)$ follows

$$0 = \int_\Omega (v \,\mathrm{div}\, q + q \cdot \nabla_{\mathrm{NC}} v) \,\mathrm{d}x.$$

Hence, any $\alpha = 1, 2$ and $\varphi \in H^1(\Omega)$ satisfy

$$0 = \int_\Omega (v \partial\varphi/\partial\alpha + \varphi\, e_\alpha \cdot \nabla_{\mathrm{NC}} v) \,\mathrm{d}x.$$

Hence, $\nabla_{\mathrm{NC}} v$ is the weak gradient of $v \in L^2(\Omega)$, i.e. $v \in H^1(\Omega)$. Consequently,

$$\int_{\partial\Omega} v\, q \cdot \nu \,\mathrm{d}s = 0 \quad \text{for all } q \in H(\mathrm{div}, \Omega).$$

This implies $v = 0$ on $\partial\Omega$, whence $v \in H^1_0(\Omega)$. Consequently,

$$Y_1 = \{v \in H^1(\mathcal{T}) | \forall\, t \in H^{-1/2}(\partial\mathcal{T}), \langle t, v\rangle_{\partial\mathcal{T}} = 0\} = H^1_0(\Omega).$$

Moreover,

$$N_1 := \{w \in H^1_0(\Omega) |\, a_{\mathrm{NC}}(\bullet, w) = 0 \text{ in } H^1_0(\Omega)\} = \{0\}.$$

Recall $1 = \beta_2 = \|b_1\|$ and $1/\sqrt{1 + C_F^2(\Omega)}$ and compute

$$\begin{aligned}\beta_0 \le \underline{\beta} &= \frac{\sqrt{2}}{\sqrt{2(1 + C_F^2(\Omega)) + 1 + \sqrt{(3 + 2C_F^2(\Omega))^2 - 4(1 + C_F^2(\Omega))}}} \\ &= \frac{\sqrt{2}}{\sqrt{3 + 2C_F^2(\Omega) + \sqrt{5 + 4C_F^2(\Omega) + 8C_F^2(\Omega)}}}.\end{aligned}$$

Proof of the second splitting lemma. Since Y_1 is a closed subspace of the Hilbert space Y, there is an orthogonal decomposition $Y = Y_1 \oplus Y_2$ with $Y_1^\perp = Y_2$. Then

$$0 < \inf_{x_2 \in S(X_2)} \sup_{y \in S(Y)} b_2(x_2, y) = \inf_{x_2 \in S(X_2)} \sup_{y_2 \in S(Y_2)} b_2(x_2, y_2) = \beta_2.$$

Any $y_2 \in Y_2$ with $b_2(\bullet, y_2) = 0$ in X_2 belongs to Y_1, whence $y_2 \in Y_1 \cap Y_2 = \{0\}$. Consequently, $b_2|_{X_2 \times Y_2}$ satisfies inf-sup condition with β_2 and is non-degenerate. General theory of bilinear forms shows

$$\beta_2 = \inf_{y_2 \in S(Y_2)} \sup_{x_2 \in S(X_2)} b_2(x_2, y_2) > 0.$$

Given any $(x_1, x_2) \in S(X_1 \times X_2)$ there exists a unique solution $y_2 \in Y_2$ to $b_2(\bullet, y_2) = \langle x_2, \bullet \rangle_{X_2}$. From Riesz isomorphism follows $\|b(\bullet, y_2)\|_{X_2^*} = \|x_2\|_{X_2}$. Then for any $y_2 \in Y_2$

$$\beta_2 \|y_2\|_Y \leq \|b_2(\bullet, y_2)\|_{X_2^*} = \|x_2\|_{X_2}.$$

Since $\beta_1 > 0$ and $N_1 = \{0\}$, $b_1|_{X_1 \times Y_1}$ satisfies inf-sup conditions and is non-degenerate, whence there exists a unique solution $y_1 \in Y_1$ to

$$b_1(\bullet, y_1) = \langle \bullet, x_1 \rangle_{X_1} - b_1(\bullet, y_2) \quad \text{in } X_1.$$

Consequently,

$$\beta_1 \|Y_1\|_Y \leq \|b_1(\bullet, y_1)\|_{X_1^*} \leq \|x_1\|_{X_1} + \|b_1\| \, \|y_2\|_Y.$$

Altogether

$$\begin{aligned} b(x, y_1 + y_2) &= b_1(x_1, y_1 + y_2) + b_2(x_2, y_1 + y_2) \\ &= \|x_1\|_{X_1}^2 + b_2(x_2, y_2) = \|x_1\|_{X_1}^2 + \|x_2\|_{Y_2}^2 = 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|y_1 + y_2\|_Y^2 &= \|y_1\|_Y^2 + \|y_2\|_Y^2 \\ &\leq \frac{1}{\beta_1^2} (\|x_1\|_{X_1} + \|b_1\| \, \|x_2\|_{X_2}/\beta_2)^2 + \|x_2\|_{X_2}^2/\beta_2^2 \\ &= (\|x_1\|_{X_1}, \|x_2\|_{X_2}) \begin{pmatrix} \beta_1^{-2} & \|b_1\| \beta_1^{-2} \beta_2^{-1} \\ \|b_1\| \beta_1^{-2} \beta_2^{-1} & \beta_2^{-2}(1 + \|b_1\|^2/\beta_1^2) \end{pmatrix} \begin{pmatrix} \|x_1\|_{X_1} \\ \|x_2\|_{X_2} \end{pmatrix} \end{aligned}$$

is bounded from above by the maximal eigenvalue Λ of the 2×2 matrix

$$\beta_1^{-2} \begin{pmatrix} 1 & \|b_1\|/\beta_2 \\ \|b_1\|/\beta_2 & \frac{\beta_1^2 + \|b_1\|}{\beta_2^2} \end{pmatrix}.$$

This implies

$$\Lambda^{-1/2} \leq \frac{b(x, y_1 + y_2)}{\|y_1 + y_2\|_Y} \leq \|b(x, \bullet)\|_{Y^*}.$$

Since $x \in S(X)$ is arbitrary, this proves $\beta \geq \Lambda^{-1/2}$. The formula follows from explicit calculations of the above 2×2 matrix. $\square$

Discretization

Define for $k \in \mathbb{N}_0$

$$
\begin{aligned}
&S_0^{k+1}(\mathcal{T}) \subset \mathrm{X}_1 = H_0^1(\Omega)\\
&P_k(\mathcal{E}) \subset \mathrm{X}_2 = H^{-1/2}(\partial\mathcal{T})\\
&\mathrm{X}_h := S_0^{k+1} \times P_k(\mathcal{E})\\
&\mathrm{Y}_h := P_{k+d}(\mathcal{T}) \subset \mathrm{Y} = H^1(\mathcal{T}).
\end{aligned}
$$

Suggest $d =$ dimension of domain and all $k \in \mathbb{N}_0$. This lecture studies $d = 1$ for $n = 2$ space dimensions and $k = 0$.

Remark (on $P_0(\mathcal{E}) \subset H^{-1/2}(\partial\mathcal{T})$). Given any $t_0 \in P_0(\mathcal{E})$. Let $\tau_{RT} \in RT_0(\mathcal{T}) \subset H(\mathrm{div}, \Omega)$ satisfy

$$\forall E \in \mathcal{E} : t_0 = \tau_{RT} \cdot \nu_E \text{ on } E.$$

Then

$$\langle t_0, v\rangle_{\partial\mathcal{T}} = \sum_{K\in\mathcal{T}} \int_{\partial K} (\tau_{RT}|_K \cdot \nu_K) v \, \mathrm{d}s$$

for all $v \in H^1(\mathcal{T})$.

Discrete duality lemma. *For any $t_0 \in P_0(\mathcal{E})$ there exists exactly one $p_{\mathrm{RT}} \in \mathrm{RT}_0(\mathcal{T}) \subseteq H(\mathrm{div}, \Omega)$ such that for all $K \in \mathcal{T}$ and $E \in \mathcal{E}(K)$*

$$(\nu_E \cdot \nu_K|_E) t_0 = (p_{\mathrm{RT}} \cdot \nu_K)|_E.$$

Then

$$\|t_0\|_{H^{-1/2}(\partial\mathcal{T})} \le \|p_{\mathrm{RT}}\|_{H(\mathrm{div},\Omega)} \le \sqrt{1 + \frac{h_{\max}^2}{\pi^2}} \|t_0\|_{H^{-1/2}(\partial\mathcal{T})}.$$

Proof. Recall that $\|t_0\|_{H^{-1/2}(\partial\mathcal{T})}$ is the minimum of all $\|q\|_{H(\mathrm{div},\Omega)}$ for any $q \in H(\mathrm{div}, \Omega)$ with

$$(\nu_E \cdot \nu_K|_E) t_0 = (q \cdot \nu_K)|_E \quad \text{for all } K \in \mathcal{T}, E \in \mathcal{E}(K). \tag{3}$$

This proves the first inequality. Given any $q \in H(\mathrm{div}, \Omega)$, $(p_{\mathrm{RT}} - q) \cdot \nu_K = 0$ on ∂K defined by the integration-by-parts formula. In particular

$$0 = \int_{\partial K} (p_{\mathrm{RT}} - q) \cdot \nu_K \, \mathrm{d}s = \int_K \mathrm{div}(p_{\mathrm{RT}} - q) \, \mathrm{d}x \quad \text{for all } K \in \mathcal{T}.$$

Consequently,

$$\operatorname{div} p_{\mathrm{RT}} = \Pi_0 \operatorname{div} q \quad \text{a.e. in } \Omega.$$

An integration by parts shows for any $v \in H^1(\mathcal{T})$ that

$$\begin{aligned} |\int_\Omega (p_{\mathrm{RT}} - q) \cdot \nabla_{\mathrm{NC}} v \, \mathrm{d}x| &= |\int_\Omega (v - \Pi_0 v) \operatorname{div}(q - p_{\mathrm{RT}}) \, \mathrm{d}x| \\ &\le h_{\max}/\pi \, \| v \|_{\mathrm{NC}} \, \|(1-\Pi_0) \operatorname{div} q\|_{L^2(\Omega)}. \end{aligned}$$

Set $v(x) := (\Pi_0 p_{\mathrm{RT}}) \cdot (x - \mathrm{mid}(K)) + 1/4(\operatorname{div} p_{\mathrm{RT}})|x - \mathrm{mid}(K)|^2$ with

$$\nabla_{\mathrm{NC}} v = \Pi_0 p_{\mathrm{RT}} + 1/2(\operatorname{div} p_{\mathrm{RT}}) + 1/2 \operatorname{div} p_{\mathrm{RT}}(\bullet - \mathrm{mid}(\mathcal{T})) = p_{\mathrm{RT}}$$

in the previous estimate to deduce

$$\begin{aligned} \|p_{\mathrm{RT}}\|^2_{L^2(\Omega)} &= \int_\Omega q \cdot p_{\mathrm{RT}} \, \mathrm{d}x + \int_\Omega (p_{\mathrm{RT}} - q) \cdot \nabla_{\mathrm{NC}} v \, \mathrm{d}x \\ &\le \|q\|_{L^2(\Omega)} \|p_{\mathrm{RT}}\|_{L^2(\Omega)} + \frac{h_{\max}}{\pi} \|p_{\mathrm{RT}}\|_{L^2(\Omega)} \|(1-\Pi_0) \operatorname{div} q\|_{L^2(\Omega)}, \end{aligned} \tag{4}$$

whence

$$\|p_{\mathrm{RT}}\|_{L^2(\Omega)} \le \|q\|_{L^2(\Omega)} + \frac{h_{\max}}{\pi} \|(1-\Pi_0) \operatorname{div} q\|_{L^2(\Omega)}.$$

This and (4) imply with $\lambda = h_{\max}/\pi$

$$\begin{aligned} \|p_{\mathrm{RT}}\|^2_{H(\mathrm{div},\Omega)} &\le (\|q\|_{L^2(\Omega)} + \lambda \|(1-\Pi_0) \operatorname{div} q\|_{L^2(\Omega)})^2 + \|\Pi_0 \operatorname{div} q\|_{L^2(\Omega)} \\ &\le (1+\lambda^2) \|q\|^2_{L^2(\Omega)} + (1 + 1/\lambda^2)\lambda^2 \|(1-\Pi_0) \operatorname{div} q\|^2_{L^2(\Omega)} \\ &\le (1+\lambda^2)(\|q\|^2_{L^2(\Omega)} + \|(1-\Pi_0) \operatorname{div} q\|^2_{L^2(\Omega)} + \|\Pi_0 \operatorname{div} q\|). \end{aligned}$$

In other words, $\|p_{\mathrm{RT}}\|_{H(\mathrm{div},\Omega)}/\sqrt{1 + h^2_{\max}/\pi^2}$ is a lower bound of $\|q\|_{H(\mathrm{div},\Omega)}$ for all q with (3). By definition of $\|t_0\|_{H^{1/2}(\partial\mathcal{T})}$ as the minimum, this shows

$$\frac{\|p_{\mathrm{RT}}\|_{H(\mathrm{div},\Omega)}}{\sqrt{1 + h^2_{\max}/\pi^2}} \le \|t_0\|_{H^{-1/2}(\partial\mathcal{T})}. \qquad \square$$

Annulation property for $P := I^{\mathrm{loc}}_{\mathrm{NC}} : H^1(\mathcal{T}) \to H^1(\mathcal{T})$ projection onto $P_1(\mathcal{T})$ defined by

$$I^{\mathrm{loc}}_{\mathrm{NC}} v|_K := \sum_{E \in \mathcal{E}(K)} \fint_E (v|_K) \, \mathrm{d}s \Psi_E|_K \in P_1(K) \quad \text{for any } v \in H^1(\mathcal{T}), K \in \mathcal{T}.$$

Given any $v \in H^1(\mathcal{T})$,

$$\|(1-P)v\|_Y = \sqrt{\|v - I_{\mathrm{NC}}^{\mathrm{loc}} v\|^2 + ||| \, v - I_{\mathrm{NC}}^{\mathrm{loc}} v \, |||^2} \le \sqrt{1 + \kappa^2 h_{\max}^2} \, ||| \, v \, |||_{\mathrm{NC}} \, .$$

Consequently, the Kato lemma implies

$$\|P\| = \|1 - P\| \le \sqrt{1 + \kappa^2 h_{\max}^2}.$$

Mean value property of the gradients $\Pi_0 \nabla_{\mathrm{NC}} I_{\mathrm{NC}}^{\mathrm{loc}} v$ for all $v \in H^1(\mathcal{T})$ leads to the annulation property

$$\sum_{K \in \mathcal{T}} \int_{\partial K} t_0 (v|_K - I_{\mathrm{NC}}^{\mathrm{loc}} v|_K) \, \mathrm{d}s = \langle t_0, v - Pv \rangle_{\partial \mathcal{T}}.$$

Hence, for all $x_h = (u_c, t_0) \in X_h$ and $v \in H^1(\mathcal{T})$, it follows

$$b(x_h, v - Pv) = a_{\mathrm{NC}}(u_c, v - P\, v) - \langle t_0, v - P\, v \rangle_{\partial \mathcal{T}} = 0.$$

The abstract theory asserts discrete inf-sup condition with $\beta \le \|P\| \beta_h \le \|b\|$. This shows

$$\frac{\beta}{\sqrt{1 + \kappa^2 h_{\max}^2}} \le \beta_h.$$

The a posteriori analysis involves $\|F \circ (1 - P)\|_{Y^*} \le \kappa \|h_{\mathcal{T}} f\|_{L^2(T)}$, which is computable but not of higher order.