Dimension Reduction Problems for Multiscale Materials in Nonlinear Elasticity.

Part 1: thin plates

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Overview

- Part 1: dimension reduction problem for homogeneous nonlinearly elastic plates.
  - Motivation and setting of the problem.
  - The membrane regime.
  - The bending regime.
  - ...and what about the other scenarios?
- Part 2/tutorial: static Γ-convergence, and the notion of 2-scale convergence.
- Part 3: simultaneous homogenization and dimension reduction.

Mathematical modeling of thin structures

Thin structures are **three-dimensional bodies** whose thickness in one direction is much smaller than the other dimensions (**membranes**, **plates**, or **shells**), or whose cross-section is much smaller than the length (**strings** or **rods**).

Applications: fuselages of aeroplanes, boat hulls and roof structures in some buildings, aero-spatial engineering, biology...

Dimension reduction for thin elastic plates

Consider a nonlinearly elastic thin plate whose reference configuration is described by the set

$$\Omega_h := \omega \times \left(-\tfrac{h}{2}, \tfrac{h}{2} \right),$$

$$\begin{cases} \omega \subset \mathbb{R}^2 \text{ bounded, open, connected,} \\ \partial\omega \text{ Lipschitz,} \\ h > 0. \end{cases}$$

Aim: for h small, to find a 2D approximate model in a rigorous way.

The variational approach

$$v : \Omega_h \to \mathbb{R}^3 \text{ deformation, } v \in W^{1,2}(\Omega_h; \mathbb{R}^3),$$

$$f^h : \Omega_h \to \mathbb{R}^3 \text{ body forces, } f^h \in L^2(\Omega_h; \mathbb{R}^3).$$

Total energy associated to the deformation v:

$$E_h(v) := \underbrace{\int_{\Omega_h} W(\nabla v(x))\,dx}_{\text{stored elastic energy}} - \underbrace{\int_{\Omega_h} f^h(x) \cdot v(x)\,dx}_{\text{work done by applied forces}}$$

The equilibrium configuration is given by a solution to

$$\min \left\{ E_h(v) : \; v \in W^{1,2}(\Omega_h; \mathbb{R}^3) \right\}.$$

Question: what is the behavior of these solutions as $h \to 0$? Γ-convergence.

A crash intro to Γ-convergence (E. De Giorgi, early '70s)

- a variational convergence;
- in dimension reduction theory it allows to replace a sequence of energies with a "limit energy" associated to a reduced model;
- yields convergence of minimizers of the sequence of energies to minimizers of the limit model (the same property a priori **does not hold** for non minimizing stationary points).

General strategy:

1. Establish compactness results for sequences of deformations with equibounded energies.
2. Identify possible candidates for being the "Γ-limit" by looking for a sharp lower bound for the relaxation of the sequence of energies along sequentially compact sequences of admissible fields.
3. Validate the optimality of the lower bound by constructing a sequence of fields such that the energies evaluated along them asymptotically converge to the Γ-limit.

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$$W(RF) = W(F) \quad \text{for every } F \in \mathbb{M}^{3\times 3}, \text{ and } R \in SO(3),$$

$$SO(3) := \{R \in \mathbb{M}^{3\times 3} : \ R^T R = RR^T = Id, \ \mathrm{det R} = 1\}$$

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Different from isotropy:

$$W(FR) = W(F) \text{ for every } F \in \mathbb{M}^{3\times 3}, \text{ and } R \in SO(3)$$

(the material has the same properties in every direction).

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  $$W(F) \geq C \mathrm{dist}^2(F; SO(3)) \quad \text{for every } F \in \mathbb{M}^{3\times 3},$$

- Non-interpenetrability:

  $$W(F) = +\infty \quad \text{if } \det F \leq 0, \qquad W(F) \to +\infty \text{ as } \det F \to 0^+.$$

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By contradiction: $W \equiv 0$ on $SO(3)$. W convex $\Rightarrow$ $W \equiv 0$ on $\mathrm{co}\, SO(3)$.

$$\frac{1}{2} Id + \frac{1}{2}\mathrm{diag}\,(-1,-1,1) = \mathrm{diag}\,(0,0,1), \quad \text{but } W(\mathrm{diag}\,(0,0,1)) = +\infty.$$

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Change of variables

We want to identify $\Gamma\text{-limit}_{h \to 0} E_h$.

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$$v \in W^{1,2}(\Omega_h;\mathbb{R}^3) \mapsto y \in W^{1,2}(\Omega;\mathbb{R}^3), \quad y(x) := v(x', hx_3).$$

Thus,

$$E_h(v) = h\int_\Omega W(\nabla_h y(x))\,dx - h\int_\Omega \tilde{f}^h(x)\cdot y(x)\,dx =: hG_h(y),$$

where

$$\nabla_h y(x) := \left(\nabla' y \Big| \frac{\partial_3 y}{h}\right)\left(z', \frac{z_3}{h}\right) = \nabla v(z),$$

$$\tilde{f}^h(x) = f^h(x', hx_3).$$

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The problem becomes: to identify the $\Gamma\text{-limit}_{h\to 0} G_h$.

Scaling of minimal energy

Assume that

$$\|\tilde{f}^h\|_{L^2(\Omega;\mathbb{R}^3)} \leq Ch^\alpha,\ \alpha \geq 0, \text{ and } \int_\Omega \tilde{f}^h(x)\,dx = 0 \quad \text{for every } h$$

(this last condition prevents $\inf\ G_h = -\infty$, otherwise take $\bar{y}$ with finite energy, and $y_c = \bar{y} + c$).

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$$G_h(y^h) \leq G_h(x', hx_3) = -\int_\Omega \tilde{f}^h(x) \cdot (x', hx_3)\,dx \leq C\|\tilde{f}^h\|_{L^2(\Omega;\mathbb{R}^3)}.$$

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In particular,

$$\int_\Omega W(\nabla_h y^h(x))\,dx \leq C\|\tilde{f}^h\|_{L^2(\Omega;\mathbb{R}^3)} + \int_\Omega \tilde{f}^h(x) \cdot \Big(y^h(x) - \fint_\Omega y^h(x)\,dx\Big)\,dx$$
$$\leq Ch^\alpha(1 + \|\nabla y^h\|_{L^2(\Omega;\mathbb{M}^{3\times 3})}).$$

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Finally, by Poincaré inequality,

$$\Big|\int_\Omega \tilde{f}^h(x) \cdot y^h(x)\, dx\Big| \leq Ch^\alpha,$$

thus

$$|G_h(y^h)| \leq Ch^\alpha.$$

The problem becomes: to identify the Γ-limit$_{h\to 0}$ $\frac{1}{h^\alpha} G_h$.

The case $\alpha = 0$ (bounded forces)

[H. Le Dret - A. Raoult (1995)]

- Assume $\tilde{f}^h \rightharpoonup f$ weakly in $L^2(\Omega;\mathbb{R}^3)$, with $\int_\Omega \tilde{f}^h(x)\,dx = 0$ for every $h > 0$.

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Define

$$\tilde{G}_h(y) := \begin{cases} G_h(y) & \text{if } y \in W^{1,2}(\Omega;\mathbb{R}^3), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3). \end{cases}$$

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Theorem

$$\tilde{G}_h \overset{\Gamma(s-L^2)}{\underset{h\to 0}{\longrightarrow}} G,$$

where

$$G(y) = \begin{cases} \int_\omega QW_0(\nabla' y(x'))\,dx' - \int_\omega \bar{f}(x')\cdot y(x')\,dx' & \text{if } y \in W^{1,2}(\Omega;\mathbb{R}^3), \\ & \partial_3 y = 0, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3), \end{cases}$$

$$\nabla' y = (\partial_1 y|\partial_2 y), \quad \bar{f}(x') = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x', x_3)\,dx_3,$$

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$$W_0(F) = \min_{a\in\mathbb{R}^3} W(F|a) \quad \text{for every } F \in \mathbb{M}^{3\times 2},$$

QW_0 is the quasiconvex envelope of W_0.

Remark

1. The limit problem is 2D. The map $y : \omega \to \mathbb{R}^3$ represents the deformation of the mid-plane ω into a surface in $\mathbb{R}^3$. The energy that we get is called a **membrane model**: it depends only on the first derivatives of the deformation, there are **no bending effects**.

The case $\alpha = 2$

[G. Friesecke - R.D. James - S. Müller (2002)]

- $W : \mathbb{M}^{3\times 3} \to [0, +\infty]$ is continuous,
- Frame-indifference: $W(RF) = W(F)$ for every $F \in \mathbb{M}^{3\times 3}$, and $R \in SO(3)$,
- W is of class C^2 in a neighborhood of $SO(3)$,
- $W(Id) = \min W = 0$,
- Growth conditions from below: $W(F) \geq C\mathrm{dist}^2\,(F; SO(3))$ for every $F \in \mathbb{M}^{3\times 3}$.

No growth conditions from above!

The case $\alpha = 2$: heuristic argument in 2D

[G. Friesecke - R.D. James - S. Müller (2002)]

$$\Omega_h = (0, L) \times (-\tfrac{h}{2}, -\tfrac{h}{2}),$$

$$v(z_1, z_2) := \gamma(z_1) + z_2 b(z_1), \quad \gamma, b : (0, L) \to \mathbb{R}^2,$$

$$\nabla v(z_1, z_2) = (\gamma'(z_1)|b(z_1)) + z_2(b'(z_1)|0).$$

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We observe that

$$\frac{1}{h}\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} W(\nabla v(z_1, z_2))\, dz_2\, dz_1 \to 0$$

$$\iff \ (\gamma'(z_1)|b(z_1)) \in SO(2) \quad \text{for every } z_1 \in (0, L)$$

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$$\frac{1}{h}\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} W(\nabla v(z_1, z_2))\, dz_2\, dz_1 = \mathrm{o}(1)$$
$$\iff (\gamma'(z_1)|b(z_1)) \in SO(2) \quad \text{for every } z_1 \in (0, L)$$
$$\iff |\gamma'(z_1)| = 1 \quad \text{for every } z_1 \in (0, L)$$
$$\iff \gamma \text{ is an isometry, and } b = \nu \text{ (normal vector to } \gamma).$$

The case $\alpha = 2$: heuristic argument in 2D

[G. Friesecke - R.D. James - S. Müller (2002)]

$$W(\nabla v(z_1, z_2)) \underset{\substack{\uparrow \\ \text{Frame indifference}}}{=} W((\gamma'(z_1)|b(z_1))^T \nabla v(z_1, z_2))$$

$$= W(Id + z_2(\gamma'(z_1)|b(z_1))^T (b'(z_1)|0))$$

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$$\underset{\substack{\uparrow \\ \text{Taylor expansion}}}{\cong} \frac{z_2^2}{2} D^2 W(Id) \begin{pmatrix} \gamma'(z_1) \cdot b'(z_1) & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \gamma'(z_1) \cdot b'(z_1) & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \frac{z_2^2}{2} \frac{\partial^2 W(Id)}{\partial F_{11}^2} (\gamma'(z_1) \cdot b'(z_1))^2 =$$

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$$= \frac{z_2^2}{2} \frac{\partial^2 W(Id)}{\partial F_{11}^2} (\gamma'(z_1) \cdot b'(z_1))^2 = \frac{z_2^2}{2} a \, \underset{\substack{\uparrow \\ \text{Curvature of } \gamma}}{k^2(z_1)}.$$

The case $\alpha = 2$: heuristic argument in 2D

[G. Friesecke - R.D. James - S. Müller (2002)]

$$\frac{1}{h}\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} W(\nabla v(z_1, z_2))\, dz_2\, dz_1 \cong \frac{1}{h}\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{z_2^2}{2} a k^2(z_1)\, dz_2\, dz_1$$

$$= \frac{ah^2}{24}\int_0^L k^2(z_1)\, dz_1.$$

$\Rightarrow$ We expect the limit energy to take into account **bending**.

The case $\alpha = 2$: the rigidity estimate

[G. Friesecke - R.D. James - S. Müller (2002)]

Theorem

Let $U \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, $N \geq 2$. Then there exists a constant $C(U) > 0$ such that for every $v \in W^{1,2}(U;\mathbb{R}^N)$ there exists $R \in SO(N)$ such that

$$\int_U |\nabla v(x) - R|^2\,dx \leq C(U) \int_U \operatorname{dist}^2\left(\nabla v(x); SO(N)\right) dx.$$

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Remark

The constant is invariant by translations and dilations of U, and is uniform for families of sets which are uniform bi-Lipschitz images of a cube.

The case $\alpha = 2$: the rigidity estimate

[G. Friesecke - R.D. James - S. Müller (2002)]

Remark

1. *Quantitative version of the following.*

Theorem (Liouville's theorem)

Let $v \in C^\infty(U;\mathbb{R}^N)$, be such that $\nabla v(x) \in SO(N)$ for every $x \in U$. Then v is affine.

The case $\alpha = 2$: the rigidity estimate

[G. Friesecke - R.D. James - S. Müller (2002)]

Remark

1. *Quantitative version of the following.*

Theorem (Liouville's theorem)

Let $v \in C^\infty(U;\mathbb{R}^N)$, be such that $\nabla v(x) \in SO(N)$ for every $x \in U$. Then v is affine.

2. *Nonlinear version of Korn's inequality.*

Theorem (Korn's inequality)

Let $U \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, $N \geq 2$. Then there exists a constant $C(U) > 0$ such that for every $w \in W^{1,2}(U;\mathbb{R}^N)$ there exists $A \in \mathbb{M}^{3\times 3}_{\rm skew}$ such that

$$\int_U |\nabla w(x) - A|^2\,dx \leq C(U)\int_U |\mathrm{sym}\,\nabla w(x)|^2\,dx.$$

Consequences of the rigidity estimate

Rigidity estimate on cubes of size h

$$\Downarrow$$

A sequence of piecewise constant maps $R^h : \omega \to SO(3)$, such that

$$\int_{\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)} |\nabla_h y^h(x) - R^h(x')|^2\,dx \leq Ch^2, \tag{1}$$

$$\int_S |R^h(x'+\xi) - R^h(x')|^2\,dx' \leq C(|\xi| + h)^2 \quad \text{for every } S \subset\subset \omega. \tag{2}$$

Consequences of the rigidity estimate

$$\nabla_h y^h \to (\nabla' y|b) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}),$$

with $(\nabla' y(x)|b(x)) \in SO(3)$ for a.e. $x \in \Omega$.

$$\Downarrow$$

$$y^h - \fint_\Omega y^h(x)\,dx \to y \quad \text{strongly in } W^{1,2}(\Omega;\mathbb{R}^3),$$

with $y \in \mathscr{A}_{\rm iso}$, where

$$\mathscr{A}_{\rm iso} := \Big\{ y \in W^{2,2}(\omega;\mathbb{R}^3) : \, |\partial_1 y| = |\partial_2 y| = 1 \text{ and } \partial_1 y \cdot \partial_2 y = 0 \Big\}.$$

In particular,

$$b = \partial_1 y \wedge \partial_2 y.$$

The case $\alpha = 2$

[G. Friesecke - R.D. James - S. Müller (2002)]

Theorem

$$\frac{1}{h^2} G_h \overset{\Gamma(s-W^{1,2})}{\underset{h\to 0}{\longrightarrow}} G_{Kir},$$

where

$$G_{Kir}(y) := \begin{cases} \frac{1}{24}\int_\omega Q_2(\Pi(x'))\,dx' - \int_\omega \bar{f}(x')\cdot y(x')\,dx' & y \in \mathscr{A}_{\rm iso} \\ +\infty & \textit{otherwise in } W^{1,2}(\Omega;\mathbb{R}^3), \end{cases}$$

with

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$$\bar{f}(x') = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x', x_3)\,dx_3, \quad \Pi(x') := \underbrace{(\nabla' y(x'))^T \nabla' (\underbrace{\partial_1 y(x') \wedge \partial_2 y(x')}_{\textit{Normal vector to the deformed surface}})}_{\textit{Curvature tensor (2nd fundamental form of } y(\omega))}$$

The case $\alpha = 2$: main result

[G. Friesecke - R.D. James - S. Müller (2002)]

Theorem

$$\frac{1}{h^2} G_h \overset{\Gamma(s-W^{1,2})}{\underset{h\to 0}{\longrightarrow}} G_{\mathrm{Kir}},$$

where

$$G_{\mathrm{Kir}}(y) := \begin{cases} \frac{1}{24}\int_\omega Q_2(\Pi(x'))\,dx' - \int_\omega \bar{f}(x')\cdot y(x')\,dx' & y \in \mathscr{A}_{\mathrm{iso}} \\ +\infty & \textit{otherwise in } W^{1,2}(\Omega;\mathbb{R}^3), \end{cases}$$

with

$$Q_3(F) := D^2W(Id)F : F \quad \textit{for every } F \in \mathbb{M}^{3\times 3},$$

$$Q_2(G) := \min_{a\in\mathbb{R}^2,\, b\in\mathbb{R}} Q_3\left(\begin{array}{c|c} G & a \\ \hline a^T & b \end{array}\right) \quad \textit{for every } G \in \mathbb{M}_{2\times 2}.$$

Some remarks on the general case

$$I_h(y) = \int_\Omega W(\nabla_h y(x))\,dx, \qquad \Omega = \omega \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$$

Question: how do we characterize the Γ-limit of $\frac{1}{h^\alpha} I^h$?

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- $\alpha = 0$: membrane model,
- $\alpha = 2$: Kirchhoff's plate theory.

Both cases are borderline.

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Question: how do we characterize the Γ-limit of $\frac{1}{h^\alpha} I^h$?

- $\alpha = 0$: membrane model,
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Both cases are borderline.
Under the assumptions of [G. Friesecke - R.D. James - S. Müller (2002)],

$$I_h(y^h) \leq C h^\alpha.$$

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Rigidity estimate on cubes of size h

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$$\int_{\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)} |\nabla_h y^h(x) - R^h(x')|^2 \, dx \leq C h^{\alpha}, \tag{3}$$

$$\int_S |R^h(x' + \xi) - R^h(x')|^2 \, dx' \leq C h^{\alpha - 2} (|\xi| + h)^2 \quad \text{for every } S \subset\subset \omega. \tag{4}$$

Some remarks on the general case

- For $\alpha = 2$, by (3) and (4) there holds $R^h \to R$ strongly in $L^2_{\rm loc}(\omega; \mathbb{M}^{3\times 3})$, with $R \in W^{1,2}(\omega; \mathbb{M}^{3\times 3})$, and $y^h \to y$ strongly in $W^{1,2}(\Omega; \mathbb{R}^3)$.

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$$\nabla_h y^h \rightharpoonup (\nabla' y | b) \in \mathrm{co}(SO(3)) \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times 3}),$$

with $(\nabla' y)^T \nabla' y - Id \leq 0$.

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$$\Downarrow$$

The Γ-limit will be finite only on those maps.

Some remarks on the general case

[G. Friesecke - R.D. James - S. Müller (2006)]

- For $\alpha > 2$, by (4),

$$y^h \to y \quad \text{strongly in } W^{1,2},$$
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$$y(x') = R \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + c \quad \text{(Rigid motion)}.$$

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$$y(x') = R \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + c \quad \text{(Rigid motion)}.$$
$$\Downarrow$$
We expect linearization effects in the Γ-limit around this rigid motion.
WLOG: $R = Id$.

The case $\alpha > 2$

[G. Friesecke - R.D. James - S. Müller (2006)]

$$\text{Tangential displacement: } u^h(x') := \frac{1}{h^\gamma}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\begin{pmatrix} y_1^h \\ y_2^h \end{pmatrix}(x', x_3) - x' \right) dx_3,$$

$$\text{Normal displacement: } v^h(x') := \frac{1}{h^\sigma}\int_{-\frac{1}{2}}^{\frac{1}{2}} y_3^h(x', x_3)\, dx_3,$$

where

$$\sigma = \frac{\alpha}{2} - 1, \quad \gamma = \begin{cases} \alpha - 2 & \text{if } 2 < \alpha < 4, \\ \frac{\alpha}{2} & \text{if } \alpha \geq 4. \end{cases}$$

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Compactness:

$$u^h \rightharpoonup u \quad \text{weakly in } W^{1,2}(\omega; \mathbb{R}^2),$$

$$v^h \rightharpoonup v \quad \text{weakly in } W^{1,2}(\omega), \quad v \in W^{2,2}(\omega).$$

$$I_h(y^h) \leq Ch^\alpha \Rightarrow y^h(x) \cong \begin{pmatrix} x_1 \\ x_2 \\ hx_3 \end{pmatrix} + \begin{pmatrix} h^\gamma u_1(x') \\ h^\gamma u_2(x') \\ h^\sigma v(x') \end{pmatrix} + \dots$$

The general picture

$$\frac{1}{h^\alpha} I_h \underset{h\to 0}{\overset{\Gamma}{\to}} I_\alpha,$$

- $\alpha > 4$: linearized Von Kàrmàn plate theory:

$$I_\alpha(v) = \frac{1}{24}\int_\omega Q_2((\nabla')^2 v(x'))\,dx',$$

The general picture

$$\frac{1}{h^\alpha} I_h \xrightarrow[h\to 0]{\Gamma} I_\alpha,$$

- $\alpha > 4$: linearized Von Kàrmàn plate theory:

$$I_\alpha(v) = \frac{1}{24}\int_\omega Q_2((\nabla')^2 v(x'))\, dx',$$

- $\alpha = 4$: Von Kàrmàn plate theory:

$$I_\alpha(u,v) = \frac{1}{24}\int_\omega Q_2((\nabla')^2 v(x'))\, dx' + \frac{1}{2}\int_\omega Q_2\Big(\mathrm{sym}\,\nabla' u(x') + \frac{1}{2}\nabla v(x') \otimes \nabla v(x')\Big)\, dx',$$

The general picture

$$\frac{1}{h^\alpha} I_h \underset{h\to 0}{\overset{\Gamma}{\to}} I_\alpha,$$

- $2 < \alpha < 4$: constrained Von Kàrmàn plate theory:

$$I_\alpha(u,v) = \frac{1}{24}\int_\omega Q_2((\nabla')^2 v(x'))\, dx'$$

if $\mathrm{sym}\, \nabla' u(x') + \frac{1}{2}\nabla v(x') \otimes \nabla v(x') = 0 \iff \det(\nabla')^2 v(x') = 0$ a.e. in ω.

The general picture

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- $0 < \alpha < \frac{5}{3}$: [S. Conti - F. Maggi (2008)]

$$I_\alpha(y) = 0$$

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$$I_\alpha(y) = 0$$

if $(\nabla' y)^T \nabla' y - Id \leq 0$,

- $\frac{5}{3} < \alpha < 2$: ???

References

- E. Acerbi, G. Buttazzo, D. Percivale. A variational definition for the strain energy of an elastic string. *J.Elasticity*, **25** (1991), 137–148.
- S. Conti, F. Maggi. Confining Thin Elastic Sheets and Folding Paper. *Arch. Rational Mech. Anal.* **187** (2008), 1–48.
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Dimension Reduction Problems for Multiscale Materials in Nonlinear Elasticity.

Part 2: tutorial-some references

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CENTRAL Summerschool
Humboldt-Universität zu Berlin, August 29th-September 2nd 2016.

Some references on static Γ-convergence

- A. Braides, Γ-convergence for beginners, Oxford University Press, Oxford (2002).
- G. Dal Maso, An introduction to Γ-convergence, Boston, Birkhäuser (1993).

A reference on two-scale convergence

- S. Neukamm's PhD thesis, which can be found here: https://mediatum.ub.tum.de/doc/976438/976438.pdf

The approach I used to introduce Γ-convergence in class was taken from the PhD thesis above, Section 2. A very thorough list of references can also be found there.