Evolutionary Γ-convergence

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Analysis of Multi-Scale Systems Driven by Functionals

CENTRAL Workshop page: $\to$ Materials

Materials for lecture of Alexander Mielke

Survey Article

A. Mielke (2016):
On evolutionary Γ-convergence for gradient systems.

Chapter 3 (pages 187–249) in the Proceedings of Summer School 2012.

Muntean, Rademacher, Zagaris: *Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity.*
Lecture Notes in Applied Math. & Mechanics Vol. 3, Springer 2016.

Overview

Aim of these lectures:

- Evolutionary systems (time-dependent O/PDEs) with multiple scales
  $0 < \varepsilon = 1/n \ll 1$ small parameter
- Describe mathematical methods for limit passage $\varepsilon \to 0$
  ($\varepsilon = h$ contains the case of numerical convergence!)

Restriction:

- only generalized gradient systems
- only very simple applications
- proofs only for the simplest results

General evolutionary equations

Multiscale limit corresponds to interchanging to limits, namely

"$\lim_{\varepsilon\to 0}$" and "$u^\varepsilon(t) = u_0^\varepsilon + \int_0^t \ldots \mathrm{d}s$"

	microsc. system		macrosc. system
	$\dot u_\varepsilon = \boldsymbol{G}_\varepsilon(u_\varepsilon)$		$\dot y = \boldsymbol{G}_0(y)$
initial state	u_ε^0	$\xrightarrow[\boldsymbol{M}_\varepsilon]{\text{upscaling}}$	y^0
time evolution	$\downarrow$		$\downarrow$
	$u_\varepsilon(t) = \boldsymbol{S}_\varepsilon(t, u_\varepsilon^0)$	$\xrightarrow[\boldsymbol{M}_\varepsilon]{\text{upscaling}}$	$y(t) = \boldsymbol{S}_0(t, y^0)$

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Mathematical task: Prove $\lim_{\varepsilon\to 0} \boldsymbol{M}_\varepsilon \circ \boldsymbol{S}_\varepsilon(t,\cdot) = \boldsymbol{S}_0(t, \lim_{\varepsilon\to 0} \boldsymbol{M}_\varepsilon(\cdot))$

We say that **the PDEs $\dot u = G_\varepsilon(u)$ *evolutionary converge to* $\dot u = G_0(u)$.**

Γ-convergence is a purely static concept

At first sight there is no relation to evolution.

- $\mathcal{J}_0,\ \mathcal{J}_\varepsilon : \boldsymbol{X} \to \mathbb{R}$ are a functionals, $\boldsymbol{X}$ sep./refl. Banach space
- If $\mathcal{J}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{J}_0$, then *solutions of minimization problems converge*:

  $\forall\, \ell \in \boldsymbol{X}^*$ we have

$$\left.\begin{array}{c} u_\varepsilon \in \underset{w\in \boldsymbol{X}}{\mathrm{ArgMin}}\Big(\mathcal{J}_\varepsilon(w) - \langle \ell, w\rangle\Big) \\ \text{and } u_\varepsilon \rightharpoonup u \end{array}\right\} \Longrightarrow u \in \underset{w\in \boldsymbol{X}}{\mathrm{ArgMin}}\Big(\mathcal{J}_0(w) - \langle \ell, w\rangle\Big)$$

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Evolutionary Γ-convergence means "evolutionary convergence" for time-dependent O/PDEs that are given in terms of functionals.

$\rightsquigarrow$ evolution/dynamics driven by functionals

1. Introduction

An example for **evolution driven by functionals**

- the damped wave equation

$$\rho(x)\ddot{u}(t,x) + \delta(x)\dot{u}(t,x) = \mathrm{div}(A(x)\nabla u(t,x)) + f(t,x) \text{ in } \Omega + \text{Dir.B.C.}$$

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What are the relevant functionals?

- kinetic energy $\mathcal{K}(\dot{u}) = \int_\Omega \frac{\rho}{2}\dot{u}^2 \,\mathrm{d}x$
- potential energy $\mathcal{E}(t,u) = \int_\Omega \big(\frac{1}{2}\nabla u \cdot A\nabla u - uf(t)\big)\,\mathrm{d}x$

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- dissipation potential $\mathcal{R}(\dot{u}) = \int_\Omega \frac{\delta}{2}\dot{u}^2\,\mathrm{d}x$

The PDE is given in terms of the three functionals $\mathcal{K}$, $\mathcal{E}$, and $\mathcal{R}$ via the force balance (cf. lectures by T. Roubíček or E. Davoli)

$$0 = \underbrace{\frac{\partial}{\partial t}\Big(\mathrm{D}_{\dot{u}}\mathcal{K}(\dot{u})\Big)}_{\text{inertial terms}} + \underbrace{\mathrm{D}_{\dot{u}}\mathcal{R}(\dot{u})}_{\text{dissipation}} + \underbrace{\mathrm{D}_u\mathcal{E}(t,u)}_{\text{potential force}}$$

Slightly more general O/PDE driven by functionals

$$\text{(DE)}_\varepsilon \quad 0 = \frac{\partial}{\partial t}\Big(\mathrm{D}_{\dot u}\mathcal{K}_\varepsilon(u,\dot u)\Big) + \mathrm{D}_{\dot u}\mathcal{R}_\varepsilon(u,\dot u) + \mathrm{D}_u\mathcal{E}_\varepsilon(t,u)$$

Naïve hope of evolutionary Γ convergence

$$\left.\begin{array}{c} \mathcal{K}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{K}_0 \\ \mathcal{R}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{R}_0 \\ \mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}_0 \end{array}\right\} \quad \Longrightarrow \quad \text{(DE)}_\varepsilon \xrightarrow{\text{evol}} \text{(DE)}_0$$

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- In general, this is wrong since the convergences need to be "compatible". (In numerics: discretizations of different parts need to be compatible.)
- True goal: Find sufficient compatibility conditions for the convergences.

Overview

1. Introduction

2. **Gradient systems**

3. Motivating examples

4. Energy-dissipation formulations

5. Evolutionary variational inequality (EVI)

6. Rate-independent systems (RIS)

Overview

Gradient flows = evolution driven by **gradient systems** $(\boldsymbol{X}, \mathcal{E}, \mathbb{G})$

- $u \in \boldsymbol{X}$ = state space (closed convex subset of a reflexive Banach space)
- $\mathcal{E} : [0,T] \times \boldsymbol{X} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ energy functional
- $\mathbb{G}(u) : \mathrm{T}_u\boldsymbol{X} = \boldsymbol{X} \to \mathrm{T}^*_u\boldsymbol{X} = \boldsymbol{X}^*$ metric structure (Riemannian tensor with $\mathbb{G}(u) = \mathbb{G}(u)^* \geq 0$)

A gradient system induces a DE via (the force balance)

$$0 = \underbrace{\mathbb{G}(u)\dot{u}}_{\text{visc.force}} + \underbrace{\mathrm{D}_u\mathcal{E}(t,u)}_{\text{rest.force}} \in \mathrm{T}^*_u\boldsymbol{X} = \boldsymbol{X}^*$$

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This gives the equivalent formulation (gradient flow)

$$\dot{u} = -\nabla_{\mathbb{G}}\mathcal{E}(t, u) = -\mathbb{G}(u)^{-1}\mathrm{D}_u\mathcal{E}(t, u) \in \mathrm{T}_u\boldsymbol{X} = \boldsymbol{X}$$

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$$0 = \mathbb{G}(u)\dot{u} + \mathrm{D}_u\mathcal{E}(t,u) \ \in \boldsymbol{X}^* \qquad \Longleftrightarrow \qquad \dot{u} = -\mathbb{K}(u)\mathrm{D}_u\mathcal{E}(t,u) \ \in \boldsymbol{X}$$

The metric tensor $\mathbb{G}$ is uniquely characterized by a quadratic form, namely the **(primal) dissipation potential**

$$\mathcal{R}(u,\dot{u}) := \tfrac{1}{2}\langle \underbrace{\mathbb{G}(u)\dot{u}}_{\in \boldsymbol{X}^*}, \underbrace{\dot{u}}_{\in \boldsymbol{X}} \rangle \qquad \Longrightarrow \qquad \mathrm{D}_{\dot{u}}\mathcal{R}(u,\dot{u}) = \mathbb{G}(u)\dot{u} \in \boldsymbol{X}^*$$

We introduce the short-hand $\mathbb{K}(u) := \mathbb{G}(u)^{-1}$ and define the **dual dissipation potential**

$$\mathcal{R}^*(u,\xi) := \tfrac{1}{2}\langle \underbrace{\xi}_{\in \boldsymbol{X}^*}, \underbrace{\mathbb{K}(u)\xi}_{\in \boldsymbol{X}} \rangle \qquad \Longrightarrow \qquad \mathrm{D}_{\xi}\mathcal{R}^*(u,\xi) = \mathbb{K}(u)\xi \in \boldsymbol{X}$$

2. Gradient systems

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- Gradient systems can also be denoted by

$$(\boldsymbol{X},\mathcal{E},\mathbb{G}) \ = \ (\boldsymbol{X},\mathcal{E},\mathcal{R}) \ = \ (\boldsymbol{X},\mathcal{E},\mathcal{R}^*) \ = \ (\boldsymbol{X},\mathcal{E},\mathbb{K})$$

- The induced equation can be written as

$$0 = \mathrm{D}_{\dot{u}}\mathcal{R}(u,\dot{u}) + \mathrm{D}_u\mathcal{E}(t,u) \in \boldsymbol{X}^* \qquad \Longleftrightarrow \qquad \dot{u} = \mathrm{D}_{\xi}\mathcal{R}^*\big(u, -\mathrm{D}_u\mathcal{E}(t,u)\big) \in \boldsymbol{X}$$

Generalized gradient systems $(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$

$\mathcal{R}(u, \dot{u})$ general dissipation potential, which means that

$$\mathcal{R}(u,\cdot){:}\boldsymbol{X} \to [0,\infty] \text{ is convex, lower semi-continuous, and } \mathcal{R}(u,0)=0.$$

The possible dissipative forces are given by the (set-valued) convex subdifferential $\partial_{\dot{u}}\mathcal{R}(u,\dot{u}) = \{\, \xi{\in}\boldsymbol{X}^* \mid \forall w{\in}\boldsymbol{X}{:}\ \mathcal{R}(u,w){\geq}\mathcal{R}(u,\dot{u}){+}\langle \xi, w{-}u\rangle \,\}$.

$$\text{(DE)} \quad 0 \in \partial_{\dot{u}}\mathcal{R}(u,\dot{u}) + \mathrm{D}_u\mathcal{E}(t,u)$$

Classical gradient system $\mathcal{R}(u,v) = \frac{1}{2}\langle \mathbb{G}(u)v, v\rangle$ (quadratic)

More general $\mathcal{R}(u,v) = \|\mathbb{A}(u)v\|_B + \frac{1}{2}\|\mathbb{V}(u)v\|_H^2 + \frac{1}{p}\|\mathbb{M}(u)\|_Z^p$

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In **multiscale modeling** on is interested in

Γ-convergence for families of gradient systems $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$

- homogenization
- dimension reductions (plates, interfaces, ...)
- singular perturbations
- numerical approximation $\varepsilon = h \to 0$

Overview

Our working definition for this course:

Definition (Γ-convergence of geneneralized gradient systems = evolutionary Γ-convergence)

We write $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$ if and only if

$$\left.\begin{array}{c} u^\varepsilon : [0,T] \to \boldsymbol{X} \\ \text{solves } (\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \\ u^\varepsilon(0) \rightharpoonup u_0, \\ \mathcal{E}_\varepsilon(u^\varepsilon(0)) \to \mathcal{E}_0(u_0) \end{array}\right\} \Longrightarrow \left\{\begin{array}{r} \exists\, u \text{ sln. of } (\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0) \text{ with } u(0) = u_0 \\ \text{and a subsequence } \varepsilon_k \to 0 : \\ \forall\, t \in [0,T] : \; u^{\varepsilon_k}(t) \rightharpoonup u(t) \\ \mathcal{E}_\varepsilon(u^{\varepsilon_k}(t)) \to \mathcal{E}_0(u(t)) \end{array}\right.$$

Aim: Find conditions of $(\mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \rightsquigarrow (\mathcal{E}_0, \mathcal{R}_0)$ to guarantee evolutionary Γ-convergence.

Overview

Overview

Why do we want to use gradient structures?

- They displays the physics behind:
  one DE may have several gradient structures

- The gradient structure defines function space
  energy space $u \in \boldsymbol{Z} \;\Leftrightarrow\; \mathcal{E}(t,u) < \infty$
  dynamic space $\dot{u} \in \boldsymbol{X} \;\Leftrightarrow\; \mathcal{R}(\dot{u}) < \infty$

- Using the gradient structure may simplify the proof of showing evolutionary convergence

  However: Evol. Γ-conv. for $(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathbb{G}_\varepsilon) \underset{\not\Leftarrow}{\Rightarrow}$ Evol. conv. for $\dot{u} = -\nabla_{\mathbb{G}_\varepsilon} \mathcal{E}_\varepsilon(u)$

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- **Most importantly:** We will see an example, where one equation has different gradient structures having evolutionary Γ-limits that do not coincide

$$\dot{u} = \boldsymbol{G}_\varepsilon(u) \begin{array}{l} \nearrow \dot{u} = -\mathbb{K}_\varepsilon(u)\mathrm{D}\mathcal{E}_\varepsilon(u) \xrightarrow{\text{evol.}\Gamma\text{-conv.}} \dot{u} = -\mathbb{K}_0(u)\mathrm{D}\mathcal{E}_0 = \boldsymbol{G}_0(u) \\ \searrow \dot{u} = -\widetilde{\mathbb{K}}_\varepsilon(u)\mathrm{D}\widetilde{\mathcal{E}}_\varepsilon(u) \xrightarrow{\text{evol.}\Gamma\text{-conv.}} \dot{u} = -\widetilde{\mathbb{K}}_0(u)\mathrm{D}\widetilde{\mathcal{E}}_0 = \widetilde{\boldsymbol{G}}_0(u) \end{array} \quad \text{different!!}$$

Heat equation $\dot{\theta} = \kappa\Delta\theta$ $\neq$ Diffusion equation $\dot{u} = m\Delta u$

(This will important for coupling reaction-diffusion and heat transfer in one thermodynamic framework. $\rightsquigarrow$ Tutorial)

Heat equation for temperature $\neq$ diffusion equation

Diffusion equation $\dot{v} = m\Delta v = -\mathbb{K}_{\mathsf{diff}}(v)\mathrm{D}\mathcal{E}_{\mathsf{diff}}(v)$ with

$\mathcal{E}_{\mathsf{diff}}(v) = \int_\Omega v\log v - v \,\mathrm{d}x$ and $\mathbb{K}_{\mathsf{diff}}(v)\xi = -m\,\mathrm{div}(v\nabla\xi)$ JKO/Wasserstein

Pure heat equation for temperature

total entropy $\mathcal{S}(\theta) := \int_\Omega S(\theta(x))\,\mathrm{d}x$

total energy $\mathcal{E}(\theta) := \int_\Omega E(\theta(x))\,\mathrm{d}x$ with Gibbs relation $0 < c(\theta) = E'(\theta) = \frac{1}{\theta}S'(\theta)$

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Physical gradient structure $\dot\theta = +\mathbb{K}_{\mathsf{heat}}(\theta)\mathrm{D}\mathcal{S}(\theta)$

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Only choice: $\mathbb{K}_{\text{heat}}(\theta)\xi = -\frac{1}{E'(\theta)}\,\mathrm{div}\,\Big(\mu(\theta)\nabla\big(\frac{\xi}{E'(\theta)}\big)\Big)$ (note $\mathbb{K}_{\text{heat}}(\theta)\mathrm{D}\mathcal{E}(\theta) \equiv 0$!)

Using $\frac{S'(\theta)}{E'(\theta)} = \frac{1}{\theta}$ we have to choose $\mu(\theta) = \theta^2\kappa(\theta)$

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Physical gradient structure $\dot\theta = +\mathbb{K}_{\mathsf{heat}}(\theta)\mathrm{D}\mathcal{S}(\theta)$

For coupling it is better to use the **internal energy** $u = E(\theta)$ as variable

$\widehat{\mathcal{E}}(u) = \int_\Omega u(x)\,\mathrm{d}x$ and $\widehat{\mathcal{S}}(u) = \int_\Omega \widehat{S}(u(x))\,\mathrm{d}x$

Equivalent gradient structure $\dot u = \widehat{\mathbb{K}}(u)\mathrm{D}\widehat{\mathcal{S}}(u)$ with $\widehat{\mathbb{K}}(u)\eta = -\,\mathrm{div}\,\big(\widehat{\mu}(u)\nabla\eta\big)$