Overview

One equation $\dot u = \mathcal{V}(u)$ may have different gradient structures:

- Gradient structure $\dot u = -\mathbb{K}(u)\mathcal{E}(u)$ is additional physical information.
- Different physical problems may have the same PDE but different GS.

  heat equation $\dot\theta = \Delta\theta \qquad \neq \qquad \dot u = \Delta u$ diffusion equation
- In a multiscale problem only certain GS may have a pE-limit

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  heat equation $\dot\theta = \Delta\theta \qquad \neq \qquad \dot u = \Delta u$ diffusion equation
- In a multiscale problem only certain GS may have a pE-limit
- Even more dramatic: **Different gradient structures may lead to different effective equations!**

Tartar 1990: Nonlocal homogenization of hyperbolic equations:

$\Omega =]0,\ell[$, $u^\varepsilon(t,x) \in \mathbb{R}$

$\dot u^\varepsilon(t,x) = -a(x/\varepsilon)u^\varepsilon(t,x)$ soln. $u^\varepsilon(t,x) = u^\varepsilon(0,x)\exp\big(-ta(x/\varepsilon)\big)$

Problem $u^\varepsilon(0,\cdot) \rightharpoonup u_0^0 \;\not\Rightarrow\; u^\varepsilon(t,\cdot) = u_0^0 \exp(-t\, a_{\text{eff}})$

4. Energy-dissipation formulations

Philosophy: GS of $\dot u^\varepsilon(t,x) = -a(x/\varepsilon)u^\varepsilon(t,x)$ **is important!**

$(\boldsymbol{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ with $\boldsymbol{X} = \mathrm{L}^2(\Omega)$

(A) $\mathcal{E}_\varepsilon(u) = \int_\Omega \frac{a(x/\varepsilon)}{2} u(x)^2 \,\mathrm{d}x$ and $\mathcal{R}_\varepsilon(\dot u) = \mathcal{R}(\dot u) = \int_\Omega \frac12 \dot u(x)^2 \,\mathrm{d}x$

$\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_{\mathsf{harm}} : u \mapsto \int_\Omega \frac{a_{\mathsf{harm}}}{2} u^2 \,\mathrm{d}x$ $\qquad \mathcal{R}_\varepsilon = \mathcal{R}$

Guess (A) for limit $\dot u = -a_{\mathsf{harm}} u$ (cf. Braides 2013)

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$\overline{\mathcal{E}}_\varepsilon = \overline{\mathcal{E}}$ $\qquad \overline{\mathcal{R}}_\varepsilon(\dot u) \xrightarrow{\Gamma} \overline{\mathcal{R}}_0(\dot u) = \int_\Omega \frac{1}{2a_{\mathsf{arith}}} \dot u^2\,\mathrm{d}x$

Guess (B) for limit $\dot u = -a_{\mathsf{arith}} u$

Is (A) or (B) correct? Or both? or None?

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Guess (B) for limit $\dot u = -a_{\mathsf{arith}} u$

Is (A) or (B) correct? Or both? or None?

Neither $(\mathrm{L}^2(\Omega), \mathcal{E}_\varepsilon, \mathcal{R})$ **nor** $(\mathrm{L}^2(\Omega), \overline{\mathcal{E}}, \overline{\mathcal{R}}_\varepsilon)$ **do** pE**-converge!**

Two other gradient structures inspired by different physics
(namely by transport theory and growth or death of species)

$\boldsymbol{X}_{\rm M} := {\rm M}_{\geq 0}(\overline{\Omega})$ non-negative Radon measures

(C) $\widetilde{\mathcal{E}}_\varepsilon(u) = \int_\Omega a(\frac{x}{\varepsilon})u(x)\,{\rm d}x$ and $\widetilde{\mathcal{R}}_\varepsilon(u,\dot u) = \int_\Omega \frac{\dot u(x)^2}{2u(x)}\,{\rm d}x$

$${\rm D}_{\dot u}\widetilde{\mathcal{R}}_\varepsilon(u,\dot u) = \frac{\dot u}{u} = -a(\tfrac{x}{\varepsilon}) = -{\rm D}\widetilde{\mathcal{E}}_\varepsilon(u)$$ PDE is OK

(D) $\widehat{\mathcal{E}}_\varepsilon(u) = \int_\Omega \frac{1}{a(x/\varepsilon)}u(x)\,{\rm d}x$ and $\widehat{\mathcal{R}}_\varepsilon(u,\dot u) = \int_\Omega \frac{\dot u(x)^2}{2a(x/\varepsilon)^2u(x)}\,{\rm d}x$

$${\rm D}_{\dot u}\widehat{\mathcal{R}}_\varepsilon(u,\dot u) = \frac{\dot u}{a(x/\varepsilon)^2u} = -\frac{1}{a(x/\varepsilon)} = -{\rm D}\widehat{\mathcal{E}}_\varepsilon(u)$$ PDE is OK

4. Energy-dissipation formulations

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Theorem [Survey'16] (C) $(\boldsymbol{X}_{\mathrm{M}}, \widetilde{\mathcal{E}}_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon) \xrightarrow{\text{evol}} (w^*)\ (\boldsymbol{X}_{\mathrm{M}}, \widetilde{\mathcal{E}}_{\text{min}}, \widetilde{\mathcal{R}}_{\mathrm{H}})$ and
(D) $(\boldsymbol{X}_{\mathrm{M}}, \widehat{\mathcal{E}}_\varepsilon, \widehat{\mathcal{R}}_\varepsilon) \xrightarrow{\text{evol}} (w^*)\ (\boldsymbol{X}_{\mathrm{M}}, \widehat{\mathcal{E}}_{\text{max}}, \widehat{\mathcal{R}}_{\text{max}})$

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Different effective equations depending on choice of GS!

Sketch of proof for case (C) [(D) is analogous, cf. Survey'16]:

- $\widetilde{\mathcal{E}}_\varepsilon(u) = \int_0^\ell a(x/\varepsilon)\,\mathrm{d}u(x)$ is a linear energy functional in $\boldsymbol{X}_\mathrm{M}$
- $\widetilde{\mathcal{R}}_\varepsilon(u,\dot u) = \mathcal{R}_\mathrm{H}(u,\dot u) = \int_\Omega \dot u^2/(2u)\,\mathrm{d}x$ is a state-dependent dissipation potential that induces Hellinger distance $d_\mathrm{H}(u_0,u_1) = 2\|\sqrt{u_1}-\sqrt{u_0}\|_{\mathrm{L}^2}$

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$$\int_0^T \big(\widetilde{\mathcal{R}}_\varepsilon(u_\varepsilon,\dot u_\varepsilon) + \widetilde{\mathcal{R}}^*_\varepsilon(u_\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon))\big)\,\mathrm{d}t = \int_0^T\int_0^\ell \Big(\frac{\dot u_\varepsilon^2}{2u_\varepsilon} + \frac{u_\varepsilon}{2}a_\varepsilon^2\Big)\,\mathrm{d}x\,\mathrm{d}t$$

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(1)–(3) show that u is a solution of (EDE) for $(\boldsymbol{X}_\mathrm{M},\mathcal{E}_{\mathsf{min}},\mathcal{R}_\mathrm{H})$. □

Overview

(EDE) $\mathcal{E}_\varepsilon(u^\varepsilon(t)) + \int_0^T \mathcal{R}_\varepsilon(u^\varepsilon, \dot u^\varepsilon) + \mathcal{R}^*_\varepsilon(u^\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u^\varepsilon))\,\mathrm{d}t \leq \mathcal{E}_\varepsilon(u^\varepsilon(0))$

EDE is quite flexible
- general $\mathcal{R}_\varepsilon(u,\cdot)$
- λ_c-conv. of $\mathcal{E}_\varepsilon$ not needed
- convergence of individual terms not needed

It suffices to find $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$ and $\mathcal{M}$ such that

- $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$
- Chain rule holds for $(\boldsymbol{X}, \mathcal{E}_0, \mathcal{R}_0)$

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- $\int_0^T \mathcal{M}(u,\dot u)\,\mathrm{d}t \leq \liminf_\varepsilon \int_0^T \big(\mathcal{R}_\varepsilon(u^\varepsilon, \dot u^\varepsilon) + \mathcal{R}^*_\varepsilon(u^\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u^\varepsilon))\big)\,\mathrm{d}t$
  (a) $\mathcal{M}(u,v) \geq -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle$ and
  (b) $\mathcal{M}(u,v) = -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle \implies$
  $$\mathcal{R}_0(u,v) + \mathcal{R}^*_0(u, -\mathrm{D}\mathcal{E}_0(u)) = -\langle \mathrm{D}\mathcal{E}_0(u), v\rangle$$

Remark:
$\mathcal{M}(u,v) \geq \mathcal{R}_0(u,v) + \mathcal{R}^*_0(u, -\mathrm{D}\mathcal{E}_0(u))$ is suffic. for (a,b) but not necessary!

Even, passage from quadratic $\mathcal{R}_\varepsilon(v) = r_\varepsilon \|v\|^{\mathbf{2}}_H$

to 1-homogeneous $\mathcal{R}_0(v) = r_0 \|v\|^{\mathbf{1}}_{\boldsymbol{X}}$ is possible!

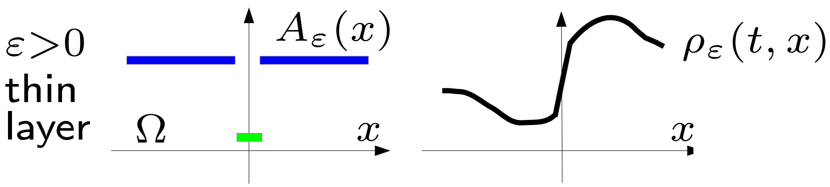
4. Energy-dissipation formulations

From diffusion to transmission (a case of dimension reduction)

(Liero'12 PhD thesis, Liero-M-Peletier-Renger'2015 WIAS preprint 2148)

Consider diffusion in $]-l, l[$ with much lower mobility in thin layer $]-\varepsilon, \varepsilon[$:

$$\boxed{\dot{u} = \mathrm{div}(A_\varepsilon(x)\nabla u) + \text{Neum.BC}} \quad \text{with } A_\varepsilon(x) = \begin{cases} a & \text{for } \varepsilon < |x| < l, \\ \varepsilon b & \text{for } |x| \leq \varepsilon \end{cases}$$



$\mathcal{E}_\varepsilon(u) = \int_\Omega \lambda_\mathrm{B}(u(x))\,\mathrm{d}x$ with $\lambda_\mathrm{B}(z) = z\log z - z + 1 \geq 0$

$\mathcal{R}^*_\varepsilon(u, \xi) = \frac{1}{2}\int_\Omega A_\varepsilon(x)u(x)\xi'(x)^2\,\mathrm{d}x$ quadratic Wasserstein diffusion

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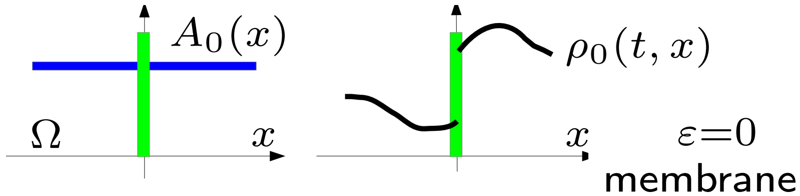

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Theorem (LMPR'15) $(\mathrm{L}^1_{\ge}(\Omega),\mathcal{E},\mathcal{R}^*_\varepsilon) \overset{\text{evol}}{\rightharpoonup} (\mathrm{L}^1_{\ge}(\Omega),\mathcal{E},\mathcal{R}^*_0)$

with $\mathcal{R}^*_0(u,\xi) = \frac{a}{2}\int_{]-l,0[} u|\xi'|^2\,\mathrm{d}x + \frac{a}{2}\int_{]0,l[} u|\xi'|^2\,\mathrm{d}x$ Wasserstein diffusion

$+\, b\sqrt{u(0^-)u(0^+)}\Big(\cosh\big(\tfrac{1}{2}(\xi(0^+)-\xi(0^-))\big)-1\Big)$ non-quadratic

Limit gradient system $(\mathrm{L}^1_{\geq}(\Omega), \mathcal{E}, \mathcal{R}^*_0)$ with $\mathcal{E}(u) = \int_{-l}^{l} \lambda_{\mathrm{B}}(u(x))\,\mathrm{d}x$ and

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Chemical potential $\xi(x) = \mathrm{D}\mathcal{E}(u)(x) = \log u(x)$

Transmission cond. arises from $\dot u = \mathrm{D}_\xi \mathcal{R}^*_0(u, -\mathrm{D}\mathcal{E}(u))$ via integr.by parts:

$$x = 0^+ : \quad a u(0^+)\xi'(0^+) = -b\sqrt{u(0^-)u(0^+)}\,\tfrac{1}{2}\sinh\big(\tfrac{1}{2}(\xi(0^+){-}\xi(0^-))\big)$$

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$$au'(0^+) = -b\big(u(0^+){-}u(0^-)\big)$$

$$x = 0^- : \quad au'(0^-) = +b\big(u(0^+){-}u(0^-)\big)$$

⊖ Linear transmission conditions arise in nontrivial nonlinear way.

⊕ Obtain Marcelin-de Donder kinetics (as used in physics) for membrane.

Since $\mathcal{E}_\varepsilon = \mathcal{E}$ the evol. Γ-convergence follows easily using the next result.

Proposition. Define the time-space functional

$$\mathcal{J}_\varepsilon(u) = \int_0^T \big(\mathcal{R}_\varepsilon(u,\dot u) + \mathcal{R}^*_\varepsilon(u,\text{-}\log u)\big)\,\mathrm{d}x = \int_0^T\!\!\int_{-l}^{l} \Big(\frac{(\int_x^1 \dot u\,\mathrm{d}y)^2}{2A_\varepsilon(x)u} + \frac{A_\varepsilon(x)(u')^2}{2u} \Big)\,\mathrm{d}x\,\mathrm{d}t,$$

then $\mathcal{J}_\varepsilon \xrightarrow{\Gamma} \mathcal{J}_0$ in $\mathrm{L}^1([0,T]\times\Omega)$ with $\mathcal{J}_0(u) = \int_0^T \big(\mathcal{R}_0(u,\dot u) + \mathcal{R}^*_0(u,\text{-}\log u)\big)\mathrm{d}x.$

- The Sandier-Serfaty approach does not work:
  For general u (not solutions $u_\varepsilon \to u$) we have separate Γ-limits
  - $u \mapsto \int_0^T \mathcal{R}_\varepsilon(u,\dot u)\,\mathrm{d}t \xrightarrow{\Gamma} \mathcal{J}_{\text{veloc}} \lneqq \int_0^T \mathcal{R}_0\,\mathrm{d}t$
  - $u \mapsto \int_0^T \mathcal{R}^*_\varepsilon(u,\text{-}\log u)\,\mathrm{d}t \xrightarrow{\Gamma} \mathcal{J}_{\text{slope}} \lneqq \int_0^T \mathcal{R}^*_0(\cdot,\text{-}\log\cdot)\,\mathrm{d}t$
- There is a non-trivial interplay between the two terms,
  recovery sequences for $\mathcal{J}_{\text{veloc}}$ and $\mathcal{J}_{\text{slope}}$ are different: $\mathcal{J}_0 \gneqq \mathcal{J}_{\text{veloc}} + \mathcal{J}_{\text{slope}}$

Idea of the proof of proposition: $\mathfrak{J}_\varepsilon(u) = \int_{-l}^{l} \left(\frac{\big(\int_{-1}^{x} \dot u \,\mathrm{d}y\big)^2}{2A_\varepsilon(x)u} + \frac{A_\varepsilon(x)(u')^2}{2u} \right) \mathrm{d}x$

Blow up of membrane to size 1: $x = X_\varepsilon(\hat x) = \begin{cases} \hat x & \text{for } \hat x \in [-l,-\varepsilon], \\ \frac{\varepsilon(2\hat x-1)}{1+2\hat\varepsilon} & \text{for } \hat x \in [-\varepsilon, 1+\varepsilon], \\ \hat x - 1 & \text{for } \hat x \in [1+\varepsilon, l+1]. \end{cases}$

Setting $\hat u(\hat x) = u(X_\varepsilon(\hat x))$ and $\widehat a_\varepsilon(\hat x) := \frac{A_\varepsilon(X_\varepsilon(\hat x))}{X'_\varepsilon(\hat x)} \in \{a,b\}$ yields transformed fnctl

$$\widehat{\mathfrak{J}}_\varepsilon(\hat u) = \int_{-l}^{l+1} \left(\frac{\big(\int_{-1}^{\hat x} \dot{\hat u}\, X'_\varepsilon(\hat y)\,\mathrm{d}\hat y\big)^2}{2\widehat a_\varepsilon(\hat x)\hat u} + \frac{\widehat a_\varepsilon(\hat x)(\hat u')^2}{2\hat u} \right) \mathrm{d}\hat x \xrightarrow{\Gamma} \widehat{\mathfrak{J}}_0 := \widehat{\mathfrak{J}}_{[-1,0]} + \widehat{\mathfrak{J}}_{\mathsf{memb}} + \widehat{\mathfrak{J}}_{[1,l+1]}$$

4. Energy-dissipation formulations

Idea of the proof of proposition: $\mathfrak{J}_\varepsilon(u) = \int_{-l}^{l} \left(\frac{\left(\int_{-1}^{x} \dot u \,\mathrm{d}y\right)^2}{2A_\varepsilon(x)u} + \frac{A_\varepsilon(x)(u')^2}{2u} \right) \mathrm{d}x$

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where $\widehat{\mathfrak{J}}_{\text{memb}}(\hat u) = \int_0^1 \left(\frac{\alpha^2}{2b\widehat u} + \frac{b(\hat u')^2}{2\hat u} \right) \mathrm{d}\hat x$ with $\alpha = \int_{-l}^{0} \dot{\hat u}(\hat y)\,\mathrm{d}\hat y = \text{const.}$

Now we use $\min \left\{ \int_0^1 \frac{\beta^2 + (\hat u')^2}{2\widehat u} \mathrm{d}\hat x \;\middle|\; \begin{array}{l} \widehat u(0) = u(0^-) \\ \widehat u(1) = u(0^+) \end{array} \right\} = \ldots$

$= \sqrt{u(0^-)u(0^+)} \Big(\mathfrak{S}\big(\frac{\beta}{\sqrt{u(0^-)u(0^+)}}\big) + \mathfrak{S}^*\big(\log \frac{u(0^+)}{u(0^-)}\big)\Big)$ with $\mathfrak{S}^*(\xi) = 4\cosh(\frac12 \xi) - 4$

Overview

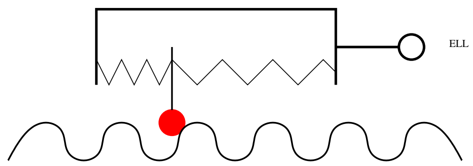
Aim: Derive dry friction as evol. Γ-limit of viscous friction

$(\mathbb{R}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}, \mathcal{E}_0, \Psi_0)$

where $\Psi_\varepsilon(v) = \frac{\varepsilon^\alpha}{2} v^2$ (quadratic)
and $\Psi_0(v) = \rho|v|$ (one-homogeneous)

Here $\mathcal{E}_\varepsilon(t, \cdot)$ is a **wiggly energy landscape**

James '96, Puglisi&Truskinovsky '02,'05



Prandtl Gedankenmodell 1928

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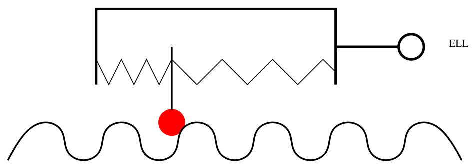

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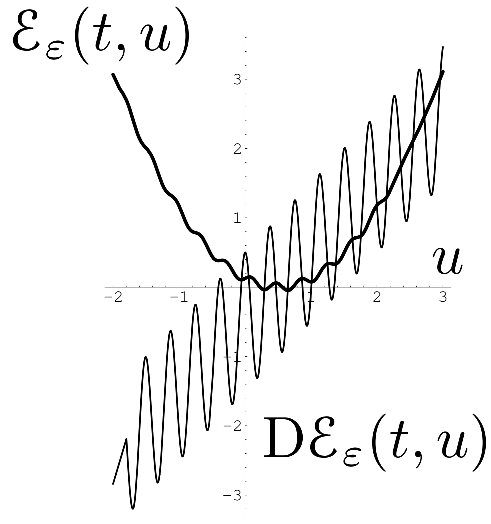
Driven gradient system $(\mathbb{R}, \mathcal{E}_\varepsilon, \Psi_\varepsilon)$

$$\mathcal{E}_\varepsilon(t,u) = \underbrace{\tfrac{1}{2}u^2 - \ell(t)u}_{\text{macroscopic part}} + \underbrace{\varepsilon\rho\cos(u/\varepsilon)}_{\text{wiggly part}}$$



$$\varepsilon^\alpha \dot{u} = -\mathrm{D}_u \mathcal{E}_\varepsilon(t,u) = -\big(u - \ell(t)\big) + \rho\sin(u/\varepsilon)$$

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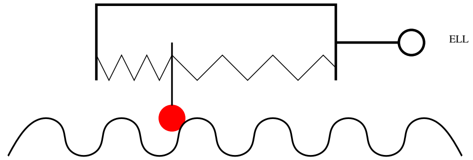

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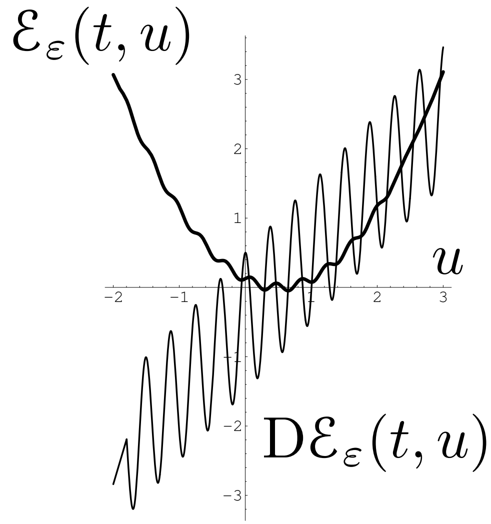


$$\varepsilon^\alpha \dot{u} = -\mathrm{D}_u \mathcal{E}_\varepsilon(t, u) = -\big(u - \ell(t)\big) + \rho \sin(u/\varepsilon)$$

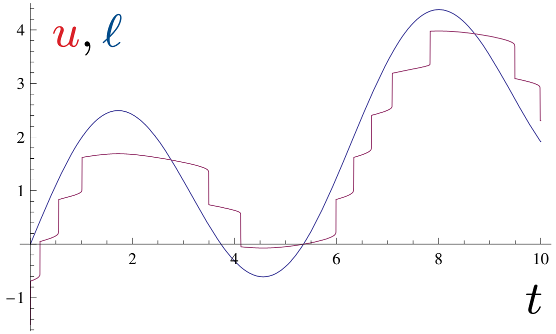
$\mathcal{E}_\varepsilon(t, u) \xrightarrow{\text{pw}} \mathcal{E}_0(t, u) = \frac{1}{2} u^2 - \ell(t) u + 0$ and $\Psi_\varepsilon \to \Psi_0 \equiv 0$

However, $u = \lim u^\varepsilon$ **does not solve** $0 = -\mathrm{D}_u \mathcal{E}_0(t, u(t))$ **!!**

Simulation: $\mathcal{E}_\varepsilon(t,u) = \frac{1}{2}u^2 - \ell(t)u - \varepsilon\cos(u/\varepsilon),$

$\ell(t) = 2\sin t + 0.3\,t, \quad q(0) = -1.0, \quad \varepsilon^\alpha = 10^{-3}$

$\varepsilon = 0.125$



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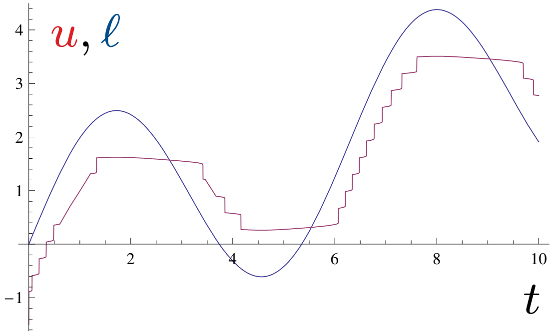

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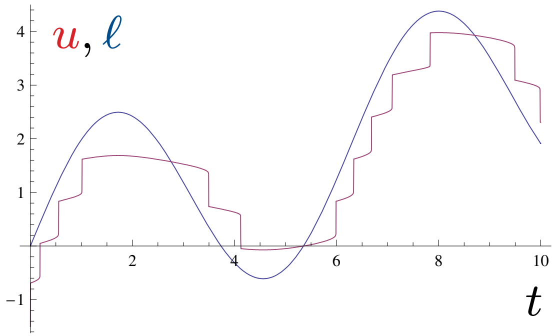

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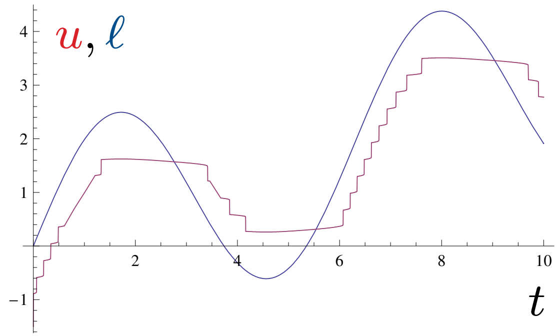

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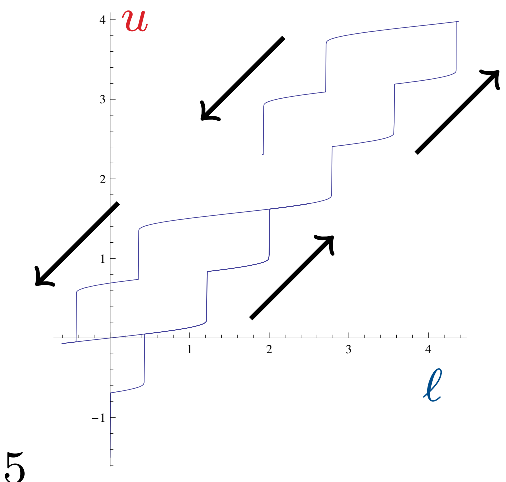

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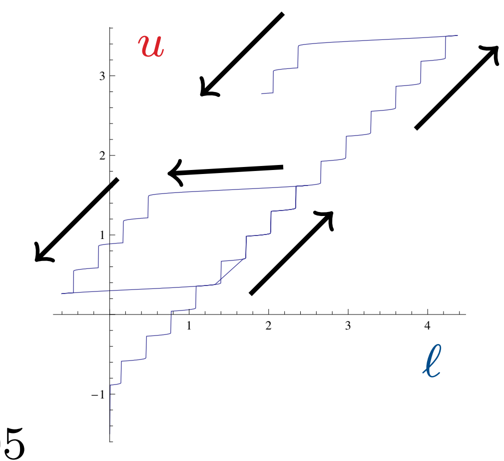

$\varepsilon = 0.05$

For $\varepsilon \to 0$ (vanishing oscillations and vanishing viscosity):
Convergence to a rate-independent hysteresis operator

$$\mathcal{E}_\varepsilon(t,u) = \frac{1}{2}u^2 - \ell(t)u + \varepsilon\rho\cos(u/\varepsilon), \qquad \Psi_\varepsilon(v) = \frac{\varepsilon^\alpha}{2}v^2, \qquad \Psi^*_\varepsilon(\xi) = \frac{1}{2\varepsilon^\alpha}\xi^2$$

Theorem (M'11 Cont. Mech. Thermodyn. / Puglisi-Truskinovsky'05)

$$(\mathbb{R}, \mathcal{E}_\varepsilon, \Psi_\varepsilon) \xrightarrow{\text{evol}} (\mathbb{R}, \mathcal{E}_0, \Psi_0)$$

where $\mathcal{E}_0(u) = \frac{1}{2}u^2 - \ell(t)u$ *and* $\Psi_0(v) = \rho|v|$

Use (EDE) $\mathcal{E}_\varepsilon(T, u_\varepsilon(T)) + \mathfrak{J}_\varepsilon(u_\varepsilon) = \mathcal{E}_\varepsilon(u_\varepsilon(0))$ with

$$\mathfrak{J}_\varepsilon(u) = \int_0^T \Psi_\varepsilon(\dot u) + \Psi^*_\varepsilon(-\mathrm{D}\mathcal{E}_\varepsilon(t,u))\,\mathrm{d}t \geq \int_0^T (1-\varepsilon^{\frac{\alpha}{2}})|\dot u||\mathrm{D}\mathcal{E}_\varepsilon(t,u)| + \frac{1/2}{\varepsilon^{\alpha/2}}\mathrm{D}\mathcal{E}_\varepsilon(t,u)^2\,\mathrm{d}t$$

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Proposition: $u^\varepsilon \rightsquigarrow u^0 \implies \liminf_{\varepsilon\to 0} \mathbb{J}_\varepsilon(u^\varepsilon) \geq \int_0^T \mathcal{M}(u^0, \dot u^0, t)\,\mathrm{d}t$ with

$\mathcal{M}(u,v,t) = |v|K(\ell(t)-u) + \chi_{[\text{-}\rho,\rho]}(\ell(t)-u)$ and $K(\xi) = \frac{1}{2\pi}\int_0^{2\pi}|\xi+\rho\cos y|\,\mathrm{d}y$

$K(\xi) = |\xi|$ for $|\xi| \geq \rho$ and $K(\xi) \gneqq |\xi|$ for $|\xi| < \rho \implies$

$$\mathcal{M}(u,v,t) \geq |v|\,|\ell(t)-u| \geq -v\,\mathrm{D}\mathcal{E}_0(t,u) \implies \;\ldots\ldots\; \implies \Psi_0(v) = \rho|v|$$

Overview

Overview

Ambrosio-Gigli-Savaré'05, Daneri-Savaré'08'10

Gradient system $(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$ with **quadratic** $\mathcal{R}(u,v) = \frac{1}{2}\langle \mathbb{G}(u)v, v\rangle$

- **Geodesic distance** $d_{\mathcal{R}} : \boldsymbol{X} \times \boldsymbol{X} \to [0,\infty]$ defined via
$$d_{\mathcal{R}}(u_0,u_1)^2 = \inf\{ \int_0^1 2\mathcal{R}(\widetilde{u},\dot{\widetilde{u}})\,\mathrm{d}s \mid u_0 \overset{\widetilde{u}}{\rightsquigarrow} u_1 \}$$
- $\widetilde{u} : [s_0,s_1] \to \boldsymbol{X}$ is called a **geodesic curve** in $(\boldsymbol{X}, d_{\mathcal{R}})$
if $d_{\mathcal{R}}(\widetilde{u}(r),\widetilde{u}(t)) = |t-r| d_{\mathcal{R}}(\widetilde{u}(s_0),\widetilde{u}(s_1))$ for all $r,t \in [s_0,s_1]$

5. Evolutionary variational inequality (EVI)

Ambrosio-Gigli-Savaré'05, Daneri-Savaré'08'10

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- $\mathcal{E} : \boldsymbol{X} \to \mathbb{R}_\infty$ is called **geodesically λ-convex** on $(\boldsymbol{X}, d_\mathcal{R})$ if
$s \mapsto \mathcal{E}(\widetilde{u}(s)) - \lambda \frac{d_\mathcal{R}(\widetilde{u}(s_0),\widetilde{u}(s))^2}{2}$ is convex on $[s_0,s_1]$ for all geod. $\widetilde{u}$

Trivial but useful and important case: Hilbert spaces!!

$\mathbb{G}(u) = \mathbb{G}_\varepsilon$ =const. $\Longrightarrow$ $d_{\mathcal{R}_\varepsilon}(u_0,u_1) = \|u_1-u_0\|_{\mathbb{G}_\varepsilon}$ with $\|w\|^2_{\mathbb{G}_\varepsilon} = \langle \mathbb{G}_\varepsilon w, w\rangle$

Then, $\mathcal{E}$ geod. λ-convex on $(\boldsymbol{X}, d_{\mathbb{G}_\varepsilon})$ $\iff$ $\mathrm{D}^2\mathcal{E} \geq \lambda \mathbb{G}_\varepsilon$

Formulations used so far:

(i) $0 \in \mathbb{G}(u)\dot{u} + \mathrm{D}\mathcal{E}(u)$ (ii) $\dot{u} = -\nabla_{\mathbb{G}}\mathcal{E}(u) = -\mathbb{K}(u)\mathrm{D}\mathcal{E}(u)$ (iii) ....

(EDE) $\mathcal{E}(u(T)) + \int_0^T \mathcal{R}(u,\dot{u}) + \mathcal{R}^*(u,-\mathrm{D}\mathcal{E}(u))\,\mathrm{d}t \leq \mathcal{E}(u(0))$

Truely derivative-free reformulation for λ-convex gradient system

Theorem [AGS'05] (Benilan'72: Hilbert-space case $d = d_{\mathbb{G}_{\mathsf{const}}}$)

If $(\boldsymbol{X}, \mathcal{E}, \mathbb{G})$ is geodesically λ-convex, then

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (EDE) $\Leftrightarrow$ **(EVI)**$_\lambda$ $\Leftrightarrow$ **(EVI')**$_\lambda$

where

(EVI)$_\lambda$ $\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t} d_{\mathbb{G}}(u(t),w)^2 + \frac{\lambda}{2} d_{\mathbb{G}}(u(t),w)^2 + \mathcal{E}(u(t)) \leq \mathcal{E}(w)$
for $t > 0$, $w \in \boldsymbol{X}$

(EVI')$_\lambda$ $\frac{\mathrm{e}^{\lambda\tau}}{2} d_{\mathbb{G}}(u(t{+}\tau),w)^2 - \frac{1}{2} d_{\mathbb{G}}(u(t),w)^2$
$\leq \frac{\mathrm{e}^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}(w) - \mathcal{E}(u(t{+}\tau))\big)$ for $t, \tau > 0$, $w \in \boldsymbol{X}$

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$\leq \frac{\mathrm{e}^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}(w) - \mathcal{E}(u(t{+}\tau))\big)$ for $t, \tau > 0$, $w \in \boldsymbol{X}$

Exercise:

(a) Prove (EDE) $\Leftrightarrow$ (EVI)$_\lambda$ (b) Prove (EVI)$_\lambda$ $\Leftrightarrow$ (EVI'$_\lambda$

5. Evolutionary variational inequality (EVI)

Formulations used so far:

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⊕ no derivatives of $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ appear ⇝ ideal for Γ-convergence

⊕ no time derivative $\dot{u}$ is involved

5. Evolutionary variational inequality (EVI)

$$(\text{EVI'})_\lambda \quad \frac{e^{\lambda\tau}}{2} d_\varepsilon(u(t{+}\tau),w)^2 - \frac{1}{2} d_\varepsilon(u(t),w)^2 \ \le \ \frac{e^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t{+}\tau))\big)$$

Theorem (Savaré'11 (personal communication))

If $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon)$ is geodesically λ-convex, $\mathcal{E}_\varepsilon$ $\boldsymbol{X}$-coercive (both unif. in ε), $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}$, and $d_\varepsilon \xrightarrow{\text{cont}} d$ in $\boldsymbol{X}$, then $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X}, \mathcal{E}, d)$.

(Convergence of the whole sequence u^ε to u, since solutions are unique.)

(EVI')$_\lambda$ $\frac{\mathrm{e}^{\lambda\tau}}{2} d_\varepsilon(u(t{+}\tau),w)^2 - \frac12 d_\varepsilon(u(t),w)^2 \ \leq\ \frac{\mathrm{e}^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t{+}\tau))\big)$

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If $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon)$ is geodesically λ-convex, $\mathcal{E}_\varepsilon$ $\boldsymbol{X}$-coercive (both unif. in ε), $\mathcal{E}_\varepsilon \stackrel{\Gamma}{\rightharpoonup} \mathcal{E}$, and $d_\varepsilon \stackrel{\text{cont}}{\rightharpoonup} d$ in $\boldsymbol{X}$, then $(\boldsymbol{X}, \mathcal{E}_\varepsilon, d_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X}, \mathcal{E}, d)$.

(Convergence of the whole sequence u^ε to u, since solutions are unique.)

The relatively strong assumption $d_\varepsilon \stackrel{\text{cont}}{\rightharpoonup} d$ in $\boldsymbol{X}$ means
$u_\varepsilon \rightharpoonup u$ & $w_\varepsilon \rightharpoonup w$ in $\boldsymbol{X}$ $\quad\Longrightarrow\quad d_\varepsilon(u_\varepsilon, w_\varepsilon) \to d(u,w)$

This can be weakened to
Gromov-Hausdorff convergence $(\boldsymbol{X}, d_\varepsilon) \xrightarrow{\text{GH}} (\boldsymbol{X}, d)$.

5. Evolutionary variational inequality (EVI)

(EVI')$_\lambda$ $\quad \frac{e^{\lambda\tau}}{2} d_\varepsilon(u(t{+}\tau),w)^2 - \frac{1}{2} d_\varepsilon(u(t),w)^2 \ \leq \ \frac{e^{\lambda\tau}-1}{\lambda}\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u(t{+}\tau))\big)$

Theorem (Savaré'11 (personal communication))

If $(\boldsymbol{X},\mathcal{E}_\varepsilon,d_\varepsilon)$ is geodesically λ-convex, $\mathcal{E}_\varepsilon$ $\boldsymbol{X}$-coercive (both unif. in ε), $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}$, and $d_\varepsilon \xrightarrow{\text{cont}} d$ in $\boldsymbol{X}$, then $(\boldsymbol{X},\mathcal{E}_\varepsilon,d_\varepsilon) \xrightarrow{\text{evol}} (\boldsymbol{X},\mathcal{E},d)$.

(Convergence of the whole sequence u^ε to u, since solutions are unique.)

Sketch of proof: $\quad u_\varepsilon$ solves (EVI')$_\lambda$ for $(\boldsymbol{X},\mathcal{E}_\varepsilon,d_\varepsilon)$

- ε-uniform bounds from (EVI')$_\lambda$ $\quad \Longrightarrow \quad u_{\varepsilon_k}(t) \rightharpoonup u(t)$ for all $t \in [0,T]$
- Pass to the limit in (EVI')$_\lambda$ using
  recovery sequence $w_\varepsilon \rightharpoonup w$ with $\mathcal{E}_\varepsilon(w_\varepsilon) \to \mathcal{E}(w)$
  $\Rightarrow \quad d_\varepsilon(u_\varepsilon(t{+}\tau),w_\varepsilon) \to d(u(t{+}\tau),w)$ and $d_\varepsilon(u_\varepsilon(t),w_\varepsilon) \to d(u(t),w)$
  $\Rightarrow \quad \mathcal{E}(u(t{+}\tau)) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(t{+}\tau))$ by Γ-liminf estimate
- Hence, $u : [0,T] \to \boldsymbol{X}$ satisfies (EVI')$_\lambda$ for $(\boldsymbol{X},\mathcal{E},d)$ QED

Overview

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3. Motivating examples

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5. Evolutionary variational inequality (EVI)

5.1. Abstract theory of $(\mathrm{EVI})_\lambda$

5.2. Application of $(\mathrm{EVI})_\lambda$ to homogenization

6. Rate-independent systems (RIS)