PLASTICITY AND DAMAGE
— PART I —
basic scenario:
rate-independent plasticity with rate-independent damage

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with computational contribution by
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Menagerie of options in plasticity/damage models:

Plasticity can influence damage:
1) indirectly through influencing the stress and strain
2) directly through influencing activation threshold for damage.

Damage can influence: 1) elasticity (through decaying elastic moduli)
2) plasticity (through decaying plastic yield stress)
3) both.

Damage evolution can be: 1) unidirectional,
2) with healing.

Plasticity/damage can be considered: 1) rate-independent
2) rate-dependent (visco-plasticity, viscous damage)
(4 options altogether, or more in damage/healing)

Plasticity can be: 1) with hardening,
2) without hardening (so-called perfect plasticity).

Length scale (gradients) in plasticity or/and damage, small vs large strains,..

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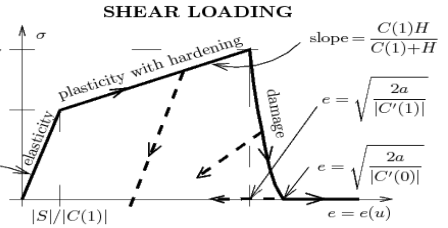
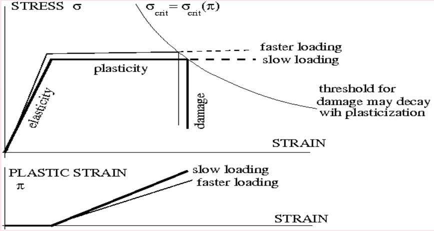
Plasticity with damage: two basic scenarios of the response under increasing mechanical load:

1) first plasticity, then damage:
  a) damage-activation threshold constant, reached by increasing stress after enough hardening
  b) damage-activation threshold decreasing, depending on plastification





...and a combination of a) and b) possible, too.

2) first damage, then plasticity:
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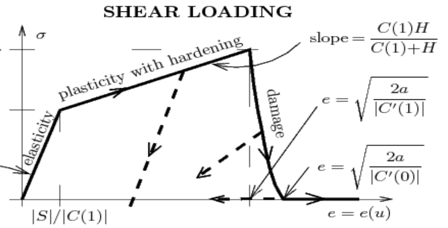

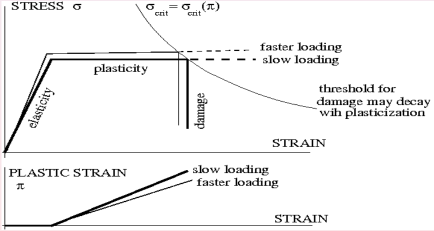

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The plot:

Part I: basic scenario: rate-independent plasticity + rate-independent damage

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Part II: perfect plasticity with rate dependent damage with a possible healing

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Part III: rate-independent unidirectional damage with visco-plasticity, thermodynamics, etc.

Part IV: tutorial – further outlooks
(combination with other processes, large strains, etc.)

1. Rate-independent plasticity, hardening, damage
   - Linearized plasticity and gradient damage
   - Weak solutions and various refinements
   - Dilemma: Global or local, energy or force?

2. Discretisation in time and convergence analysis outlined
   - Approximate max-diss principle for the semi-implicit scheme
   - Implicit discretisation – energetic solution

3. Stress-driven scenario, gradient plasticity and gradient damage
   - A fractional-step semi-implicit discretisation
   - Convergence towards local solutions
   - Numerical simulations - approximate maximum-dissipation principle

General scheme of mathematical modelling procedure:

a real-world phenomenon $\longrightarrow$ various mathematical models $\longrightarrow$ various concepts of solution

A solution concept may be a vital part of the model itself!

Equivalently, evolution governed formally by Biot-type equations (inclusions):

$$\partial_u\mathcal{E}(t,u,z)\ni 0 \quad \text{and} \quad \partial\mathcal{R}\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)+\partial_z\mathcal{E}(t,u,z)\ni 0\ ,$$

where the symbol "∂" refers to a (partial) (sub)differential, relying on that $\mathcal{E}(t,\cdot,z)$, $\mathcal{E}(t,u,\cdot)$, and $\mathcal{R}(\cdot)$ are convex functionals.

The main focuse in today's talk:

$\mathcal{E}(t,\cdot,\cdot)$ nonconvex, but at least $\mathcal{E}(t,\cdot,z)$ convex,
or possibly also $\mathcal{E}(t,u,\cdot)$ convex,

$\mathcal{R}\geq 0$ convex, positively homogenous of degree 1 (called 1-homogenous).
(such $\mathcal{R}$ is called a gauge)

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Evolution governed formally by a generalized gradient-flow equations (inclusions):

$$\partial_u \mathcal{E}(t,u,z) \ni 0 \quad \text{and} \quad \frac{\mathrm{d}z}{\mathrm{d}t} \in \partial\mathcal{R}^*(\xi) \quad \text{with} \quad \xi \in -\partial_z \mathcal{E}(t,u,z),$$

-admissible driving force; available driving force

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Linearized plasticity with hardening of Prager/Ziegler's type at small strains:

$\Omega \subset \mathbb{R}^d$ a bounded domain,
$u =$ displacement,
$z = (\pi, \eta) =$ the plastic strain and the isotropic-hardening parameter,

$$-\operatorname{div}\big(\mathbb{C}(e(u) - \pi)\big) = f, \qquad \text{(momentum equilibrium)}$$

$$\partial R \begin{pmatrix} \frac{\partial \pi}{\partial t} \\ \frac{\partial \eta}{\partial t} \end{pmatrix} + \begin{pmatrix} \mathbb{C}\pi + \mathbb{H}\pi \\ b\eta \end{pmatrix} \ni \begin{pmatrix} \mathbb{C}e(u) \\ 0 \end{pmatrix}, \qquad \text{(Biot inclusion)}$$

with $e(u) = \frac{1}{2}(\nabla u)^\top + \frac{1}{2}\nabla u$ small-strain tensor,
$b > 0$ isotropic-hardening coefficient,
$\mathbb{H} \geq 0$ kinematic-hardening coefficient (a $d \times d \times d \times d$-tensor),
$\mathbb{H}\pi$ is a *back stress* to the elastic stress σ.

δ_S is its indicator function, and δ_S^* the conjugate functional to δ_S.

Then $\partial R^* = \partial \delta_S^{**} = \partial \delta_S = N_S =$ the normal cone to S.

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$$\frac{\partial}{\partial t}\begin{pmatrix} \pi \\ \eta \end{pmatrix} \in \partial R^* \begin{pmatrix} \sigma - \mathbb{H}\pi \\ -b\eta \end{pmatrix}, \qquad \text{(flow rule)}$$

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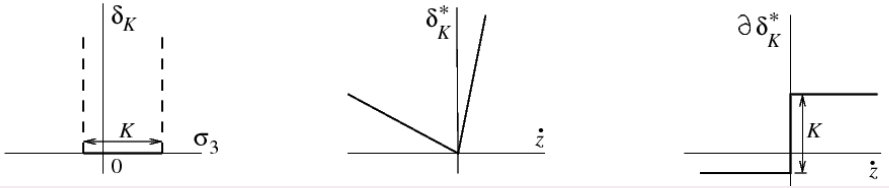
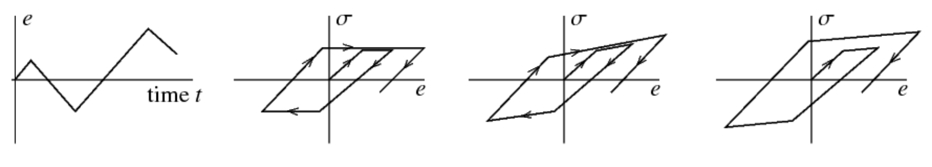


An illustration of an indicator function of K acting on a driving force σ,
its convex conjugate (1-homogeneous), and
its subdifferential (maximally responsive) =inverse to the normal cone to K
used here for $K = S$, later e.g. for $K = [-a_1, \infty)$ or $K = [-a, b]$.



A schematic response on cycling loading (left) of plastic material
without hardening, i.e. perfect (also called Prandtl-Reuss) plasticity,
with kinematic hardening, and
with isotropic hardening.

The concept of internal variables

P. DUHEM (1903), C. ECKART (1940), P.W. BRIDGMAN (1943)

G.A. Maugin: The saga of internal variables of state in continuum thermo-mechanics (1893-2013), *Mech. Res. Communic.*, 69 (2015), 79-86.

here now $z = (\pi, \eta)$

The state of the system: $q = (u, z) = (u, \pi, \eta)$.

Energy $E(t, u, z) = \frac{1}{2}\mathbb{C}(e(u)-\pi) : (e(u)-\pi) + \frac{1}{2}\mathbb{H}\pi : \pi + \frac{1}{2}b\eta^2 - f(t) \cdot u$.

The driving force $\xi = -\partial_{(u,z)}E(t, u, z)$

Biot equation:

$$\begin{pmatrix} 0 \\ \partial R(\frac{\partial z}{\partial t}) \end{pmatrix} + \begin{pmatrix} \partial_u E(t, u, z) \\ \partial_z E(t, u, z) \end{pmatrix} \ni \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note: $E(t, \cdot, \cdot)$ convex quadratic

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$\Rightarrow$ The Maximum-dissipation principle (R.Hill for convex problems, 1948):

maximal monotonicity of ∂R $\Rightarrow$

$$\xi \in \partial R\Big(\frac{\partial z}{\partial t}\Big) \quad \Leftrightarrow$$

$$\forall v\, \forall f \in \partial R(v) : \Big\langle f - \xi, v - \frac{\partial z}{\partial t}\Big\rangle \geq 0 \text{ with the driving force } \xi \in -\partial_z E(t,u,z).$$

— in particular for $v = 0$:

$$\Big\langle \frac{\partial z}{\partial t}, \xi \Big\rangle = \max_{f \in S} \Big\langle \frac{\partial z}{\partial t}, f \Big\rangle \qquad \text{for} \quad \xi \in -\partial_z E(t,u,z)$$

— also known as an orthogonality principle (H.Ziegler, 1958)

— or the isothermal variant of the maximal entropy production principle (K.R.Rajagopal, A.Srinivasa, 2004)

— An important message from the max.-diss. principle:

$\xi \in \mathrm{int}\, S \Rightarrow \frac{\partial z}{\partial t} = 0$ (a force-driven evolution)

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— in particular for $v = 0$: (if R only convex but not 1-homogeneous, then "$\geq$")

$$\Big\langle \frac{\partial z}{\partial t}, \xi \Big\rangle = \max_{f \in S} \Big\langle \frac{\partial z}{\partial t}, f \Big\rangle \qquad \text{for} \quad \xi \in -\partial_z E(t, u, z)$$

— also known as an orthogonality principle (H.ZIEGLER, 1958)

— or the isothermal variant of the maximal entropy production principle (K.R.RAJAGOPAL, A.SRINIVASA, 2004)

— An important message from the max.-diss. principle:

$$\xi \in -\mathrm{int}\, S \;\Rightarrow\; \frac{\partial z}{\partial t} = 0 \qquad \text{(a force-driven evolution)}$$

— Yet max.-diss. principle itself does not say much
$\Rightarrow$ must be combined with a (local) stability.

By the definition of the subdifferential:
$\partial R(\frac{\partial z}{\partial t}) \ni -\xi \ \Leftrightarrow\ \forall v: \quad R(v) \geq \langle \xi, v - \frac{\partial z}{\partial t}\rangle + R(\frac{\partial z}{\partial t})$
for $v \to v + \frac{\partial z}{\partial t}$: $\ \langle \xi, v\rangle \leq R(v + \frac{\partial z}{\partial t}) - R(\frac{\partial z}{\partial t}) \leq R(v)$

(triangle inequality $\Leftarrow$ 1-homogeneity of R)

$\Rightarrow$ Local stability: $\forall v$: $\ R(v) \geq \langle \xi, v\rangle$.
If combined with the maximum-dissipation principle: $S = \partial R(0) \Rightarrow$
$\langle \frac{\partial z}{\partial t}, \xi\rangle = \max_{f\in S} \langle \frac{\partial z}{\partial t}, f\rangle = R(\frac{\partial z}{\partial t})$ (again 1-homogeneity of R used)
$\Rightarrow\ \partial R(\frac{\partial z}{\partial t}) \ni -\xi$.

When integrated $\mathcal{R}(z) = \int_\Omega R(z)\,\mathrm{d}x$ and $\mathcal{E}(t,u,z) = \int_\Omega E(t,u,z)\,\mathrm{d}x$:

$$\Big\langle \frac{\mathrm{d}z}{\mathrm{d}t}, \xi\Big\rangle_{\mathcal{Z}\times\mathcal{Z}^*} = \max_{f\in\partial\mathcal{R}(0)} \Big\langle \frac{\mathrm{d}z}{\mathrm{d}t}, f\Big\rangle_{\mathcal{Z}\times\mathcal{Z}^*} \qquad \text{for} \quad \xi \in -\partial_z\mathcal{E}(t,u,z).$$

— analytically only very formal because $\frac{\mathrm{d}z}{\mathrm{d}t}$ is typically not valued in $\mathcal{Z}$
and, as a function for time, is a measure but ξ jumps (+ is set-valued).

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A combination with damage: a scalar parameter ζ valued in $[0,1]$.
(the concept of L.M. KACHANOV 1958)

Now the internal variables are $z = (\pi, \eta, \zeta)$.

Stored energy

$$E(t,u,\pi,\eta,\zeta) = \frac{1}{2}\mathbb{C}(\zeta)(e(u){-}\pi){:}(e(u){-}\pi) + \frac{1}{2}\mathbb{H}\pi{:}\pi + \frac{1}{2}b\eta^2 + a_0(\zeta) + \frac{1}{2}\kappa|\nabla\zeta|^r - g(t){\cdot}u.$$

$\mathbb{C}(\cdot)$ elastic moduli subjected to damage

$a_0(\cdot)$ energy of damage (microscopically interpreted as
an energy of microcracks/microvoids).

Typically: $C(\cdot)$ and $a_0(\cdot)$ monotone (in Löwner ordering),
$C(0) = 0$ complete damage, but we will assume $C(0) > 0$ uncomplete damage.

Dissipation potential:

$$R(\dot\pi,\dot\eta,\dot\zeta) = \begin{cases} \delta_S^*(\dot\pi,\dot\eta) + a_1|\dot\zeta| & \text{if } \dot\zeta \le 0 \\ \infty & \text{if otherwise} \end{cases}$$

$a_1 > 0$ an activation energy for damage.

Note: $E(t,\cdot,\cdot)$ nonconvex, possibly only separately convex and quadratic, unidirectional damage (no healing allowed).

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The classical formulation of the Biot equation/inclusions $\partial\mathcal{R}(\frac{\mathrm{d}q}{\mathrm{d}t})+\partial_q\mathcal{E}(t,q)\ni 0$:

$$\mathrm{div}\big(\mathbb{C}(\zeta)e_{\rm el}\big)+g=0 \qquad \text{with} \quad e_{\rm el}=e(u)-\pi, \qquad \text{(momentum equilibrium)}$$

$$\partial\delta_S^*\begin{pmatrix}\frac{\partial\pi}{\partial t}\\ \frac{\partial\eta}{\partial t}\end{pmatrix}+\begin{pmatrix}\mathbb{H}\pi\\ b\eta\end{pmatrix}\ni\begin{pmatrix}\mathrm{dev}(\mathbb{C}(\zeta)e_{\rm el})\\ 0\end{pmatrix}, \qquad \text{(plastic flow rule)}$$

$$\partial\delta^*_{[-a_1,\infty)}\Big(\frac{\partial\zeta}{\partial t}\Big)+\frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}:e_{\rm el}$$
$$-\kappa\,\mathrm{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big)+N_{[0,1]}(\zeta)\ni a_0'(\zeta), \qquad \text{(damage flow rule)}$$

Boundary conditions: $u=u_{\rm Dir}(t)$ on $\Gamma_{\rm Dir}\subset\partial\Omega$,
$\nabla\zeta\cdot\vec{n}=0$ on $\Gamma:=\partial\Omega$.

A transformation to time-constant boundary condition: $u=0$ on $\Gamma_{\rm Dir}\subset\partial\Omega$ by a shift $u\mapsto u+u_{\rm Dir}(t)$ (with $u_{\rm Dir}(t)$ defined on Ω).

Weak solution:

$\partial\mathcal{R}(\frac{\mathrm{d}z}{\mathrm{d}t}) + \partial_z\mathcal{E}(t,u,z) \ni 0$ which, assuming $\mathcal{E}$ smooth for a moment, means

$$\forall v \in \mathcal{Z}: \qquad \mathcal{R}\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big) \le \Big\langle \mathcal{E}_z'(t,u,z), v - \frac{\mathrm{d}z}{\mathrm{d}t}\Big\rangle + \mathcal{R}(v).$$

substitute the troublesome term $\langle\mathcal{E}_z'(t,u,z),\frac{\mathrm{d}z}{\mathrm{d}t}\rangle$ by integration over time interval $[t_1,t_2]$ and using the chain rule

$$\mathcal{E}(t_2,u(t_2),z(t_2)) = \int_{t_1}^{t_2} \Big\langle \mathcal{E}_z'(t,u(t),z(t)),\frac{\mathrm{d}z}{\mathrm{d}t}\Big\rangle + \Big\langle \mathcal{E}_u'(t,u(t),z(t)),\frac{\mathrm{d}u}{\mathrm{d}t}\Big\rangle + \mathcal{E}_t'(t,u(t),z(t))\,\mathrm{d}t + \mathcal{E}(t_1,u(t_1),z(t_1)),$$

and, using $\mathcal{E}_u'(t,u(t),z(t)) = 0$, it eventually yields

$$\sim \int_{t_1}^{t_2}\mathcal{R}(\tfrac{\mathrm{d}z}{\mathrm{d}t})\mathrm{d}t$$

$$\forall v \in \mathcal{Z}\ \forall_{\text{a.a.}} 0 \le t_1 < t_2 \le T: \quad \mathcal{E}(t_2,u(t_2),z(t_2)) + \mathrm{Var}_{\mathcal{R}}(z;[t_1,t_2]) \le \mathcal{E}(t_1,u(t_1),z(t_1)) + \int_{t_1}^{t_2}\Big(\mathcal{E}_t'(t,u(t),z(t)) - \langle\xi,v\rangle + \mathcal{R}(v)\Big)\,\mathrm{d}t.$$

with the available driving force for evolution of z: $\xi = -\mathcal{E}_z'(t,u(t),z(t))$.

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A special case: $\mathcal{R}$ 1-homogeneous, $\mathcal{E}(t,u,\cdot)$ convex:

$\forall v\colon\ \partial\mathcal{R}(v)\subset\partial\mathcal{R}(0) \qquad \Rightarrow$

$\forall_{\rm a.a.}t:\quad \partial\mathcal{R}(0)\ni\xi(t)$ with (some) driving force $\xi(t)\in-\partial_z\mathcal{E}(t,u(t),z(t))$.

by convexity of $\mathcal{R}$ & $\mathcal{R}(0)=0$, this is equivalent to

$$\forall v\in\mathcal{Z}:\qquad \mathcal{R}(v)-\langle\xi(t),v\rangle\ge\mathcal{R}(0)=0.$$

Substituting $v=\tilde z-z(t)$ & convexity of $\mathcal{E}(t,u,\cdot)$ $\Rightarrow$

$$0\le\mathcal{R}(\tilde z-z(t))-\langle\xi(t),\tilde z-z(t)\rangle\le\mathcal{E}(t,u(t),\tilde z)+\mathcal{R}(\tilde z-z(t))-\mathcal{E}(t,u(t),z(t))$$

$\Rightarrow$ semi-stability:

$$\forall_{\rm a.a.}t\ \forall\tilde z\in\mathcal{Z}:\qquad \mathcal{E}(t,u(t),z(t))\le\mathcal{E}(t,u(t),\tilde z)+\mathcal{R}(\tilde z-z(t)).$$

Recall the property of the weak solution: $\partial_u\mathcal{E}(t,u,z)\ni 0$ for a.a. t and

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For $v=0$, it defines the a.e.-local solution (to use even for $\mathcal{E}(t,u,\cdot)$ nonconvex).
(a'la R.Toader & C.Zanini (2009) for crack problem, U.Stefanelli (2009), A.Mielke (2011))

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Substituting $v = \widetilde{z} - z(t)$ & convexity of $\mathcal{E}(t,u,\cdot)$ $\Rightarrow$

$0 \leq \mathcal{R}(\widetilde{z}-z(t)) - \langle \xi(t), \widetilde{z}-z(t)\rangle \leq \mathcal{E}(t,u(t),\widetilde{z}) + \mathcal{R}(\widetilde{z}-z(t)) - \mathcal{E}(t,u(t),z(t))$

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Recall the property of the weak solution: $\partial_u\mathcal{E}(t,u,z) \ni 0$ for a.a. t and

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$$\leq \mathcal{E}(t_1,u(t_1),z(t_1)) + \int_{t_1}^{t_2} \Big(\mathcal{E}_t'(t,u(t),z(t)) - \big\langle \xi, v\big\rangle + \mathcal{R}(v)\Big)\,\mathrm{d}t.$$

with $\mathrm{Var}_{\mathcal{R}}(z;[t_1,t_2]) := \sup_{\text{partitions } t_1 \leq t^0 < t^1 < \ldots < t^N \leq t_2} \sum_{k=1}^N \mathcal{R}(z(t^k) - z(t^{k-1}))$.

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$$\le \mathcal{E}(t_1,u(t_1),z(t_1)) + \int_{t_1}^{t_2} \Big(\mathcal{E}_t'(t,u(t),z(t))\,\mathrm{d}t$$

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For $v=0$, it defines the a.e.-local solution (to use even for $\mathcal{E}(t,u,\cdot)$ nonconvex).

(a'la R.Toader & C.Zanini (2009) for crack problem, U.Stefanelli (2009), A.Mielke (2011).)

A special case: $\mathcal{R}$ 1-homogeneous, $\mathcal{E}(t,u,\cdot)$ convex:

$\forall v$: $\partial\mathcal{R}(v) \subset \partial\mathcal{R}(0) \quad \Rightarrow$

$$\forall_{\mathrm{a.a.}} t: \quad \partial\mathcal{R}(0) \ni \xi(t) \quad \text{with (some) driving force} \quad \xi(t) \in -\partial_z\mathcal{E}(t,u(t),z(t)).$$

by convexity of $\mathcal{R}$ & $\mathcal{R}(0)=0$, this is equivalent to

$$\forall v \in \mathcal{Z}: \qquad \mathcal{R}(v) - \langle \xi(t), v\rangle \geq \mathcal{R}(0) = 0.$$

Substituting $v = \tilde{z} - z(t)$ & convexity of $\mathcal{E}(t,u,\cdot)$ $\Rightarrow$

$$0 \leq \mathcal{R}(\tilde{z}-z(t)) - \langle \xi(t), \tilde{z}-z(t)\rangle \leq \mathcal{E}(t,u(t),\tilde{z}) + \mathcal{R}(\tilde{z}-z(t)) - \mathcal{E}(t,u(t),z(t))$$

$\Rightarrow$ semi-stability:

$$\forall_{\mathrm{a.a.}} t \ \forall \tilde{z} \in \mathcal{Z}: \qquad \mathcal{E}(t,u(t),z(t)) \leq \mathcal{E}(t,u(t),\tilde{z}) + \mathcal{R}(\tilde{z}-z(t)).$$

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If $\mathrm{dom}\,\mathcal{R} = \mathcal{Z}$
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Proof. 1) a.e.-local solutions $\Rightarrow$ weak solutions proved (essentially) above

2) weak solutions $\Rightarrow$ a.e.-local solutions:

2a) put $v = 0$: energy inequality proved above.
2b) put $v = k\tilde{z}$ and use 1-homogeneity of $\mathcal{R}$:

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Functional setting: after transformation $u \mapsto u+u_{\rm Dir}$.

New boundary conditions: $u = 0$ on $\Gamma_{\rm Dir} \subset \partial\Omega$:

The Banach state spaces:
$\mathcal{U} = \{W^{1,2}(\Omega;\mathbb{R}^d);\ u|_{\Gamma_{\rm Dir}} = 0\}$,
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$(u,\pi) \mapsto \mathcal{E}(t,u,\pi,\zeta)$ is smooth

$\Rightarrow$ $\partial\mathcal{R}(\frac{{\rm d}q}{{\rm d}t}) + \partial_q\mathcal{E}(t,q) \ni 0$ is more specifically as the system:

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A more selective concept uses a so-called stability condition:

by 1-homogeneity and positivity of $\delta_S^*(\cdot)$

$$-\mathcal{E}'_{(u,z)}(t,u,z) \in \Big(0, \partial\mathcal{R}\big(\frac{\mathrm{d}z}{\mathrm{d}t}\big)\Big) \quad \subset \quad \Big(0, \partial\mathcal{R}(0)\Big)$$

$\Longrightarrow$

$$0 = \mathcal{R}(0) \le \mathcal{R}(\tilde z) - \big\langle \mathcal{E}'_u(t,u,z), \tilde u\big\rangle - \big\langle \mathcal{E}'_z(t,u,z), \tilde z\big\rangle \qquad \forall(\tilde u, \tilde z)$$

write $\tilde u - u(t)$ instead of u
and $\tilde z - z(t)$ instead of z

$$0 \le \mathcal{R}(\tilde z - z(t)) - \big\langle \mathcal{E}'_z(t,u,z), \tilde z - z(t)\big\rangle - \big\langle \mathcal{E}'_u(t,u,z), \tilde u - u(t)\big\rangle \qquad \forall(\tilde u, \tilde z)$$

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We call $q=(u,z):[0,T]\to\mathcal{Q}=\mathcal{U}\times\mathcal{Z}$ an energetic solution to

$$\mathcal{E}_u'(t,u,z)=0,\quad \partial\mathcal{R}\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)+\mathcal{E}_z'(t,u,z)\ni 0,\quad u(0)=u_0,\ z(0)=z_0,$$

if

- the energy equality holds, i.e.

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Advantage: no $\frac{\mathrm{d}z}{\mathrm{d}t}$ and $\mathcal{E}_u'$ and $\mathcal{E}_z'$ explicitly involved.

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$$\mathcal{E}\big(t,u(t),z(t)\big)\le\mathcal{E}\big(t,\tilde{u},\tilde{z}\big)+\mathcal{R}\big(\tilde{z}-z(t)\big)$$

- the initial conditions $u(0)=u_0$ and $z(0)=z_0$ are satisfied.

Remark: it works even without convexity (our case here if damage is considered)

Remark: energetic solutions are (very special type of) local solutions.

A physically-justified attempt to get back the energy conservation:

a small ("vanishing" in the limit) viscosity in u or z:

$$\varepsilon_1 \partial\mathcal{R}_1\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big) + \partial_u\mathcal{E}(t,u,z) \ni 0 \qquad \text{and} \qquad \varepsilon_2 \partial\mathcal{R}_2\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big) + \partial\mathcal{R}\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big) + \partial_z\mathcal{E}(t,u,z) \ni 0$$

with $\mathcal{R}_1 \geq 0$ and $\mathcal{R}_2 \geq 0$ convex quadratic.

Again, semi-implicit time discretisation works efficiently. In the limit $\tau \to 0$:

The energy conservation (if $\mathcal{R}_1 > 0$ or $\mathcal{R}_2 > 0$) for $(u_\varepsilon, z_\varepsilon)$ with $\varepsilon := (\varepsilon_1, \varepsilon_2)$:

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In the vanishing-viscosity limit for $\varepsilon \to 0$ (as subsequences) $\Rightarrow$ "defect measure" μ

$$2\varepsilon_1\mathcal{R}_1\Big(\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}t}\Big) + 2\varepsilon_2\mathcal{R}_2\Big(\frac{\mathrm{d}z_\varepsilon}{\mathrm{d}t}\Big) \to \mu \geq 0 \qquad \text{weakly* as a measure on } [0,T].$$

The "semi-energetic solution" (u, z, μ) satisfies the energy equality

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Sometimes nonconvexity of $\mathcal{E}(t,\cdot,\cdot)$ & global minimization $\Rightarrow$ too early jumps.

General dilemma: energy vs force (global vs local),

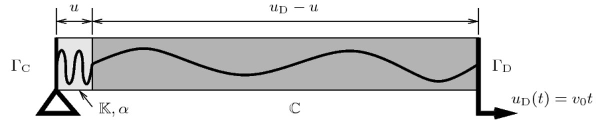
well recognized in mechanics, e.g. in
D.Leguillon, Strength or toughness? (Europ.J.Mech. A) 2002:

"...the incremental form of the energy criterion gives a lower bound of admissible crack lengths. On the contrary, the stress criterion leads to an upper bound."

and in math too – a comparison e.g. in
D.Knees, A.Mielke, C.Zanini 2008,
M.Negri, C.Ortner 2008,
U. Stefanelli, 2009, etc.

A concept of force-driven local solutions amenable by rigorous analysis and allowing for efficient computational schemes is desirable.

A 0-dimensional example: two elastic springs gradually stretched, one damageable (healing formally allowed).



$$\mathcal{E}(t,u,z)=\begin{cases}\frac12 z\mathbb{K}u^2+\frac12\mathbb{C}|u{-}v_0t|^2 & \text{if } 1\ge z\ge 0,\\ +\infty & \text{otherwise,}\end{cases}\qquad \mathcal{R}\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)=\alpha\Big|\frac{\mathrm{d}z}{\mathrm{d}t}\Big|.$$

Local solutions:

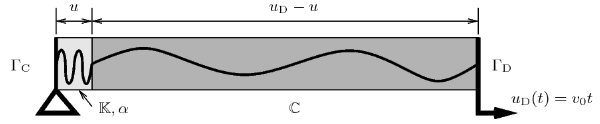
1) semi-stability $(\forall\widetilde z\in[0,1]:\ \frac12\mathbb{K}(\widetilde z{-}z)u^2+\alpha|z{-}\widetilde z|\ge0)\ \Rightarrow$ the rupture time of the local solution $(=\ t_{\rm LS})$ will be at most the time $(=\ t_{\rm MD})$ when the elastic energy of the undamaged spring reaches the activation threshold α, i.e. $\frac12\mathbb{K}u^2=\alpha$ (i.e. also $\frac12\mathbb{K}(v_0\mathbb{C}t_{\rm MD}/(\mathbb{K}{+}\mathbb{C}))^2=\alpha$)

2) $t_{\rm LS}$ cannot be earlier than when energetic solution ruptures $(=\ t_{\rm ES})$ because then the energy balance would be violated.

$$\Rightarrow\qquad t_{\rm ES}\ \le\ \underbrace{t_{\rm LS}}_{}\!\!{}^{\text{arbitrary}}\ \le\ t_{\rm MD}\qquad\Big(\text{and also}\quad t_{\rm ES}<t_{\rm MD}=t_{\rm VV}\Big).$$

$t_{\rm VV}$ = time when vanishing-viscosity solutions rupture.

A 0-dimensional example: two elastic springs gradually stretched, one damageable (healing formally allowed).



$$\mathcal{E}(t,u,z) = \begin{cases} \frac{1}{2} z \mathbb{K} u^2 + \frac{1}{2}\mathbb{C}|u - v_0 t|^2 & \text{if } 1 \ge z \ge 0, \\ +\infty & \text{otherwise,} \end{cases} \qquad \mathcal{R}\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big) = \alpha\Big|\frac{\mathrm{d}z}{\mathrm{d}t}\Big|.$$

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2) t_{LS} cannot be earlier than when energetic solution ruptures ($= t_{\mathrm{ES}}$) because then the energy balance would be violated.

$$\Rightarrow \qquad t_{\mathrm{ES}} \le \underset{\text{arbitrary}}{t_{\mathrm{LS}}} \le t_{\mathrm{MD}} \qquad \Big(\text{and also} \quad t_{\mathrm{ES}} < t_{\mathrm{MD}} = t_{\mathrm{VV}}\Big).$$

---

t_{VV} = time when vanishing-viscosity solutions rupture.

Energetic solution: time of break $t_{\rm ES}$

Analysing the incremental problem: min of $\mathcal{J}(t,z)-\alpha z$ subj. to $z\in[0,1]$ with the reduced functional $\mathcal{J}(t,z)=\mathcal{E}(t,\mathfrak{u}(t,z),z)$ with $\mathfrak{u}(t,z):=v_0\mathbb{C}t/(z\mathbb{K}+\mathbb{C})$, i.e.
$$\mathcal{J}(t,z)=\frac{\mathbb{K}\mathbb{C}^2zt^2+\mathbb{C}\mathbb{K}^2z^2t^2}{2(\mathbb{K}z+\mathbb{C})^2}.$$

$$\Rightarrow\qquad t_{\rm ES}=\sqrt{\frac{2\alpha\mathbb{K}+2\alpha\mathbb{C}}{v_0^2\mathbb{K}\mathbb{C}}}.$$

The upper bound for rupture of local solutions $t_{\rm MD}$:

Analysing the semi-stability: $\frac{1}{2}\mathbb{K}u^2=\alpha$ with $u=\mathfrak{u}(t,z)$ $\Rightarrow$ $t_{\rm MD}=\frac{\mathbb{K}+\mathbb{C}}{v_0\mathbb{C}}\sqrt{\frac{2\alpha}{\mathbb{K}}}$.

$$\text{the work of external loading}=\int_0^{t_{\rm LS}}\frac{\mathbb{K}v_0^2\mathbb{C}t}{\kappa+\mathbb{C}}\,\mathrm{d}t=\frac{v_0^2\mathbb{K}\mathbb{C}}{2\mathbb{K}+2\mathbb{C}}t_{\rm LS}^2.$$

Rupture at $t_{\rm ES}$: minimal dissipation
(all the work is dissipated into damaging)

Rupture at $t_{\rm MD}$: maximal dissipation
(the extra energy is due to neglected mechanisms like viscosity)

Vanishing-viscosity solution: time of break $t_{\rm VV}$ when $\partial_z\mathcal{J}(t_{\rm VV},1)=\alpha$: $t_{\rm VV}=t$

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Analysing the incremental problem: min of $\mathcal{J}(t,z)-\alpha z$ subj. to $z\in[0,1]$ with the reduced functional $\mathcal{J}(t,z)=\mathcal{E}(t,\mathfrak{u}(t,z),z)$ with $\mathfrak{u}(t,z):=v_0\mathbb{C}t/(z\mathbb{K}+\mathbb{C})$, i.e. $\mathcal{J}(t,z)=\dfrac{\mathbb{K}\mathbb{C}^2zt^2+\mathbb{C}\mathbb{K}^2z^2t^2}{2(\mathbb{K}z+\mathbb{C})^2}$.

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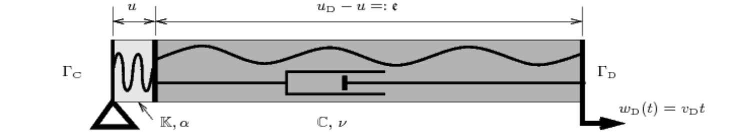
The vanishing-viscosity in the zero-dimensional example:



The energies $\mathcal{E} : \bar{I}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ and $\mathcal{R}, \mathcal{R}_1 : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ as:

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A combination with time-discretisation very difficult:

note $\lim_{\tau\to 0}\lim_{\nu\to 0}\mathcal{R}_1(\frac{{\rm d}u}{{\rm d}t}) = 0 \ne \mu$ in general!

Explicit solutions are known for the viscous variant.

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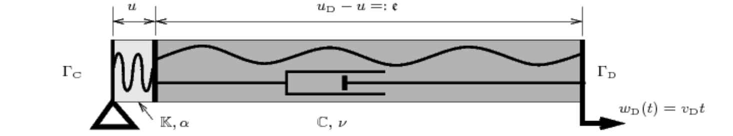

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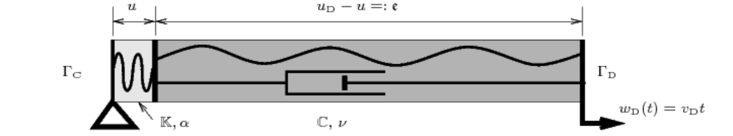

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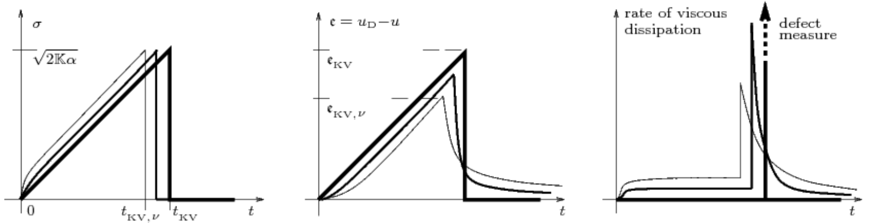


Fig. 2. A schematic response of the stress σ, the strain in the bulk $\mathfrak{e}$, and the rate of viscous dissipation $\nu\mathbb{C}\dot{\mathfrak{e}}^2$ depending on gradually decreasing $\nu > 0$ (two values of ν are depicted by gradually increasing thickness) and the limit for $\nu \to 0$ (depicted by the thickest line); the last picture shows schematically the defect measure (as a Dirac supported at $t = t_{\rm KV}$).

In this inviscid limit, the energetical picture during rupture is now clear:

$$\frac{1}{2}\mathbb{C}\mathfrak{e}_{\rm KV}^2 = \alpha\frac{\mathbb{K}}{\mathbb{C}} \leftarrow \int_{t_{\rm KV,\nu}}^{+\infty} \nu\mathbb{C}\Big|\frac{{\rm d}\mathfrak{e}_\nu}{{\rm d}t}\Big|^2 {\rm d}t \qquad \text{and} \qquad \frac{1}{2}\mathbb{K}u_{\rm KV}^2 = \alpha,$$

⇒ all energy stored in the bulk goes to the defect measure during the rupture

⇒ all energy stored in the damageable spring is dissipated by the delamination.

⇒ stress-driven delamination rather than the energy-driven one.
This is perfectly in accord with conventional engineering handling of fracture mechanics (which, however, typically ignores any energy balance).

Computational simulation: (made by C.G. PANAGIOTOPOULOS)

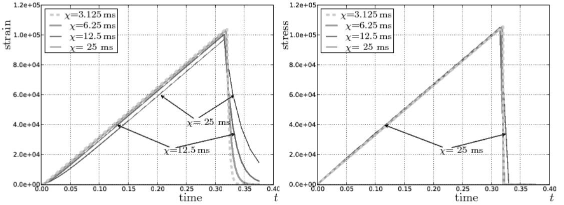


Fig. 4: The strain (left) and stress (right) response; due to the symmetry, these tensors have only one nonzero component.

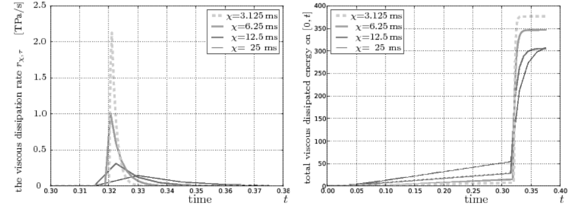


Fig. 5: Left. Convergence of the viscous dissipation rate $r_{\chi,\tau} = \chi\mathbb{C}e(\dot{u}_{\chi,\tau}){:}e(\dot{u}_{\chi,\tau})$ towards the defect measure μ from (4.8), i.e. here the Dirac at $t_{\text{BREAK}} = 0.322$ s for $\chi = 0.025 \times 2^{-k}$ with $k = 0, 1, 2, 3$ and decreasing τ chosen according the strategy from Table 1, zoomed in and depicted on a selected time subinterval $[0.3\,,\,0.375]$.

Right. Energy dissipated by viscosity over $[0,t]$, i.e. $\int_0^t \chi\mathbb{C}e(\dot{u}_{\chi,\tau}){:}e(\dot{u}_{\chi,\tau})\,\mathrm{d}t$, converging to the jump at $t_{\text{BREAK}} = 0.322$ s of the magnitude $\mathscr{E}_{\text{BREAK}} = 803.75$ J Also the convergence $t_{\text{BREAK},\chi} \nearrow t_{\text{BREAK}}$ from (4.6) is well documented.

The one-dimensional example discretised:

(all simulations made by C.G. Panagiotopoulos)

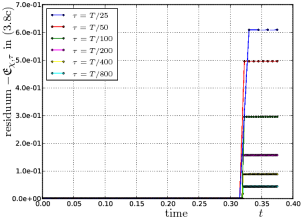


Fig. 2. Illustration of the time-dependent residuum $-\mathfrak{E}_{\chi,\tau}(\cdot)$ in the energy balance (3.8c) for $\chi = 0.00625$ s fixed and τ gradually decreasing as depicted. The numerical error occurs especially around suden rupture but is shown to converge to 0 for $\tau \to 0$, as also proved in (3.10).

χ=viscosity coefficient,
τ=time step of discretisation,
$\mathfrak{E}_{\chi,\tau}$ residuum in energy balance

a very fine time discretisation is is needed if viscosity $\to 0$

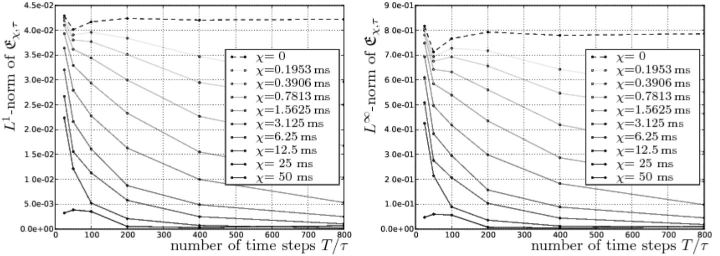
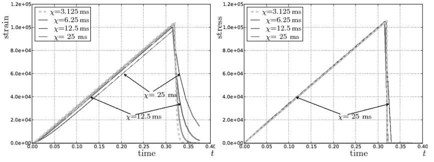


Fig. 3: Left. the convergence of L^1-norm of $\mathfrak{E}_{\chi,\tau}$ parameterized by χ, documenting the

A comparison of the maximally-dissipative local sln with the vanishing-viscosity sln:

Recall the figure:



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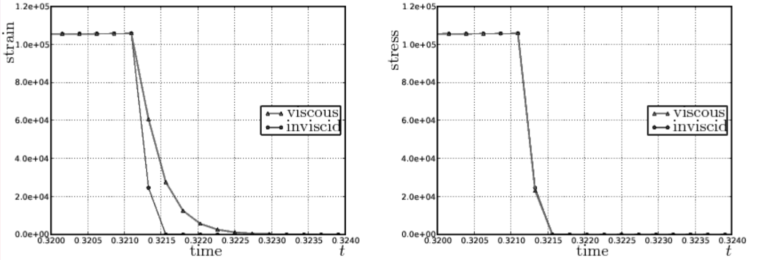

Fig. 6. A comparison of the strain (left) and stress (right) response of a energetically justified small-viscosity solution with an unphysical result without any viscosity obtained by a semi-implicit formula; strongly zoomed in and depicted on a selected short time subinterval around rupture [0.320 , 0.324]: a suprisingly good match is achieved although energy does not match at all (since $\mu \equiv 0$ without viscosity), cf. also Fig. 3 for $\chi = 0$.

The 0-dimensional example - the maximally dissipative local solution:

$\exists$ a continuous selection $\xi(t) \in -\partial_z \mathcal{E}(t, u(t), z(t))$

$$\left(\text{e.g. } \xi(t) = \begin{cases} -\partial_z \mathcal{E}(t, u(t), 1) & \text{for } t \leq t_{\rm MD} \\ = \alpha & \text{for } t > t_{\rm MD} \end{cases}\right)$$

such that the maximum dissipation principle

$$\left\langle \frac{\mathrm{d}z}{\mathrm{d}t}(t), \xi(t) \right\rangle = \max_{f \in \partial\mathcal{R}(0)} \left\langle \frac{\mathrm{d}z}{\mathrm{d}t}(t), f \right\rangle_{\mathcal{Z}\times\mathcal{Z}^*} = \mathcal{R}\left(\frac{\mathrm{d}z}{\mathrm{d}t}(t)\right)$$

holds in the sense of distributions (namely the Dirac $\alpha\delta_{t_{\rm MD}}$).

But for other local solutions the violation of the maximum principle is not obvious
- e.g. for energetic solution, a driving force of magnitude α may occur already immediately after this break time.

$\Rightarrow$ 1) only left-continuous local solutions (reflecting also causality)
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The lower Riemann-Stieltjes integral for ξ and z scalar-valued, z monotone, is

$$\int_r^s \xi(t)\,\mathrm{d}z(t) := \sup_{\substack{N\in\mathbb{N}\\ r=t_0<t_1<\ldots<t_{N-1}<t_N=s}} \underbrace{\sum_{j=1}^N \inf_{t\in[t_{j-1},t_j]} \big\langle \xi(t), z(t_j)-z(t_{j-1})\big\rangle}_{\text{lower Darboux sum}} .$$

- Sub-additivity in ξ and z, and additivity in the integration domain, too.
- The sum depends monotonically on the partition:
  finer partition $\Rightarrow$ bigger (or equal) sum.
- $\frac{\mathrm{d}z}{\mathrm{d}t} \in \mathrm{AC}([r,s];Z)$ & $\xi \in \mathrm{C}([r,s];Z^*)$
  $$\Rightarrow \int_r^s \xi(t)\,\mathrm{d}z(t) = \int_r^s \Big\langle \xi(t), \frac{\mathrm{d}z}{\mathrm{d}t}(t)\Big\rangle \mathrm{d}t \quad \text{(the Lebesgue integral).}$$

(but we will use $\int_r^s$ also for ξ discontinuous and $\frac{\mathrm{d}z}{\mathrm{d}t}$ a measure not valued in Z)

The maximum dissipation principle $\langle\frac{\mathrm{d}z}{\mathrm{d}t}(t),\xi(t)\rangle = \mathcal{R}(\frac{\mathrm{d}z}{\mathrm{d}t}(t))$
integrated over any $[t_1,t_2]\subset[0,T]$: $\exists$ selection $\xi(t)\in\mathcal{R}(0)$ $\forall\, t_1<t_2$:

$$\int_{t_1}^{t_2} \xi(t)\,\mathrm{d}z(t) = \mathrm{Var}_{\mathcal{R}}(z;[t_1,t_2]) \quad \& \quad \xi(t)\in -\partial_z\mathcal{E}(t,u(t),z(t)).$$

The lower Riemann-Stieltjes integral generalized a'la H.E. Moore & S. Pollard is

$$\underline{\int_r^s} \xi(t)\,\mathrm{d}z(t) := \limsup_{\substack{N\in\mathbb{N}\\ r=t_0<t_1<\ldots<t_{N-1}<t_N=s}} \underbrace{\sum_{j=1}^{N} \inf_{t\in[t_{j-1},t_j]} \big\langle \xi(t), z(t_j)-z(t_{j-1})\big\rangle}_{\text{lower Darboux sum}} .$$

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Illustration of selectivity of the integrated maximum-dissipation principle (IMDP):

left-continuous local solution which makes a complete rupture at time $t_{\rm LS}$, i.e.

$$u(t)=\begin{cases}\frac{\mathbb{C}}{\mathbb{C}+\mathbb{K}}v_0t,\\ v_0t,\end{cases}\qquad z(t)=\begin{cases}1,\\ 0\end{cases}\qquad \xi(t)\begin{cases}=-\frac12\mathbb{K}u(t)^2 & \text{for } t\le t_{\rm LS},\\ \in[-\alpha,\alpha] \text{ arbitrary} & \text{for } t>t_{\rm LS}.\end{cases}$$

$$\begin{aligned}\int_0^T\xi(t)\mathrm{d}z(t)&=\int_0^{t_{\rm LS}}\xi(t)\mathrm{d}z(t)+\int_{t_{\rm LS}}^T\xi(t)\mathrm{d}z(t)\\ &=0\ +\sup_{0<\varepsilon\le T-t_{\rm LS}}\ \inf_{t\in[t_{\rm LS},t_{\rm LS}+\varepsilon]}\xi(t)\big(z(t_{\rm LS}+\varepsilon)-z(t_{\rm LS})\big)\\ &=0\ +\sup_{0<\varepsilon\le T-t_{\rm LS}}\ \min\Big(-\xi(t_{\rm LS}),\inf_{t\in(t_{\rm LS},t_{\rm LS}+\varepsilon]}-\xi(t)\Big)\ \le-\xi(t_{\rm LS}).\end{aligned}$$

$t_{\rm LS}<t_{\rm MD}\ \Rightarrow\ -\xi(t_{\rm LS})<\alpha=\mathrm{Var}_{\mathcal{R}}(z;[0,T])\ \Rightarrow$ (IMDP) not satisfied.

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$$\begin{aligned} \underline{\int_0^T} \xi(t)\mathrm{d}z(t) &= \underline{\int_0^{t_{\rm LS}}} \xi(t)\mathrm{d}z(t) + \underline{\int_{t_{\rm LS}}^T} \xi(t)\mathrm{d}z(t) \\ &= 0 \ + \sup_{0<\varepsilon\le T-t_{\rm LS}} \ \inf_{t\in[t_{\rm LS},t_{\rm LS}+\varepsilon]} \xi(t)\big(z(t_{\rm LS}+\varepsilon) - z(t_{\rm LS})\big) \\ &= 0 \ + \sup_{0<\varepsilon\le T-t_{\rm LS}} \ \min\Big(-\xi(t_{\rm LS}), \inf_{t\in(t_{\rm LS},t_{\rm LS}+\varepsilon]} -\xi(t)\Big) \ \le -\xi(t_{\rm LS}). \end{aligned}$$

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The 0-dimensional example modified:

both springs damageable, two internal parameters z_1 and z_2,
fully symmetric ($\mathbb{C} = \mathbb{K}$, $z_1(0) = 1 = z_2(0)$):
Left-continuous local solutions:

Number of energetic solutions: 2
both breaks at $t = t_{\rm ES}$, either z_1 or z_2 jumps to 0.
No energetic solution is symmetric.

Number of maximally-dissipative solutions: ∞
all breaks at $t = t_{\rm MD}$ when z_1 or z_2 (meaning that possibly both) jump to 0,
but either z_1 or z_2 may possibly not jump completely up to 0.

$2t_{\rm ES}^2 \leq t_{\rm MD}^2 \Rightarrow$ One of these solutions is symmetric (both springs completely damaged and dissipate maximal energy during the break (U.Stefanelli's principle)

Although all these solutions rupture at $t = t_{\rm MD}$ and dissipate maximal work of external load, the contribution to $\mathrm{Var}_{\mathcal{R}}(z; 0, T)$ varies from α to 2α for the symmetric maximal-dissipative local solutions.

The later one is also the vanishing-viscosity solution (with symmetric viscosity) and "generically" obtained (for $\tau \to 0$) by semi-implicit discretisation.

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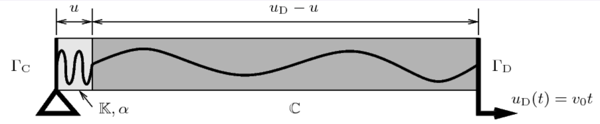
The 0-dimensional damage example - the maximally dissipative local solution



$$\mathcal{E}(t,u,z) = \begin{cases} \frac{1}{2} z\mathbb{K}u^2 + \frac{1}{2}\mathbb{C}|u - v_0 t|^2 & \text{if } 1 \geq z \geq 0, \\ +\infty & \text{otherwise,} \end{cases} \qquad \mathcal{R}\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big) = \alpha\Big|\frac{\mathrm{d}z}{\mathrm{d}t}\Big|.$$

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$$\left(\text{e.g. } \xi(t) \begin{cases} = -\partial_z \mathcal{E}(t, u(t), 1) & \text{for } t < t_{\rm MD} \\ \in -\partial_z \mathcal{E}(t, u(t), 1) & \text{for } t = t_{\rm MD} \\ = \alpha & \text{for } t > t_{\rm MD} \end{cases}\right)$$

such that the maximum dissipation principle

$$\Big\langle \frac{\mathrm{d}z}{\mathrm{d}t}(t), \xi(t) \Big\rangle = \max_{f \in \partial\mathcal{R}(0)} \Big\langle \frac{\mathrm{d}z}{\mathrm{d}t}(t), f \Big\rangle_{\mathcal{Z}\times\mathcal{Z}^*} = \mathcal{R}\Big(\frac{\mathrm{d}z}{\mathrm{d}t}(t)\Big)$$

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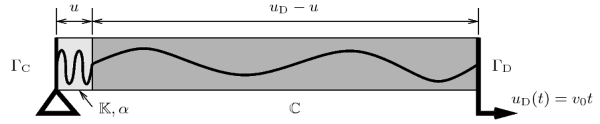

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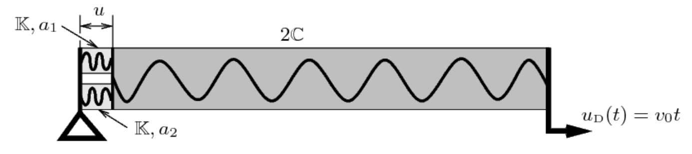
two parallel damageable springs of the same stiffness $\mathbb{K}$ but different fracture toughness a_1 and a_2 coupled by an elastic spring of the stiffness $2\mathbb{C}$:



$$\frac{a_2}{a_1} > 2\frac{(\mathbb{K}+\mathbb{C})^2}{(\mathbb{K}+2\mathbb{C})^2} \qquad \Rightarrow \qquad \text{one max.-diss. left-continuous local solution}$$

$$a_1 < a_2 \leq \frac{2a_1(\mathbb{K}+\mathbb{C})^2}{(\mathbb{K}+2\mathbb{C})^2} \qquad \Rightarrow \qquad \text{no max.-diss. left-continuous local solution}$$

the 2nd-spring breaks immediately when the 1st-spring breaks,

the jump of $z = (z_1, z_2)$ from $(1,1)$ to $(0,0)$ is not orthogonal to the elastic domain $\partial\mathcal{R}(0,0) = [-a_1, \infty) \times [a_2, \infty)$.

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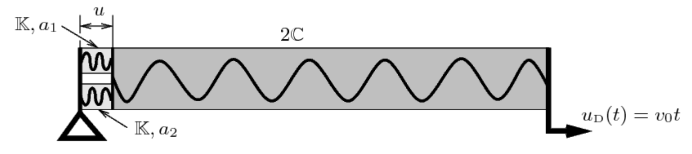

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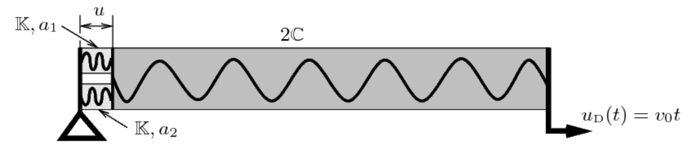

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Maximum-dissipation principle for approximate solutions
$\Rightarrow$ Approximate maximum-dissipation principle (AMDP):

$$\underline{\int_0^T} \bar{\xi}_\tau(t)\mathrm{d}\bar{z}_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal{R}}(\bar{z}_\tau;[0,T]) \quad \text{with} \quad \bar{\xi}_\tau(t) \in -\partial_z\bar{\mathcal{E}}_\tau(t,\bar{u}_\tau(t),\bar{z}_\tau(t)).$$

We can explicitly evaluate the left-hand side as

$$\underline{\int_0^T} \bar{\xi}_\tau(t)\mathrm{d}\bar{z}_\tau(t) = \sum_{k=1}^{T/\tau}\langle \xi_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}\rangle \quad \text{with} \quad \xi_\tau^{k-1} \in -\partial_z\mathcal{E}((k-1)\tau, u_\tau^{k-1}, z_\tau^{k-1}).$$

(the supremum in $\underline{\int_0^T}$ attained just on the partition $\{k\tau;\ k=0,\dots,T/\tau\}$)

Unfortunately: one cannot expect equality even in the limit –
the (even left-continuously modified) limit not maximally-dissipative in general.

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Discretization in time by a fully implicit formula

$$\partial_u \mathcal{E}_\tau^k(u_\tau^k, z_\tau^k) = 0,$$
$$\partial \mathcal{R}\Big(\frac{z_\tau^k - z_\tau^{k-1}}{\tau}\Big) + \partial_z \mathcal{E}_\tau^k(u_\tau^k, z_\tau^k) \ni 0$$

where $\mathcal{E}_\tau^k(u,z) := \mathcal{E}_\tau(k\tau, u, z)$ with $\mathcal{E}_\tau(t,u,z) := \frac{1}{\tau}\int_{-\tau}^0 \mathcal{E}(t+\xi, u, z)\mathrm{d}\xi$, for $k = 1, ..., T/\tau$ and using, for $k=1$,

$$z_\tau^0 = z_0,$$

The existence of the discrete solution (u_τ^k, z_τ^k):

the direct method: (u_τ^k, z_τ^k) can be taken as a solution to:

$$\left.\begin{array}{ll} \text{minimize} & \tau\mathcal{R}\Big(\frac{z-z_\tau^{k-1}}{\tau}\Big) + \mathcal{E}_\tau^k(u,z) \\ \text{subject to} & (u,z) \in \mathcal{Q} = \mathcal{U} \times \mathcal{Z}. \end{array}\right\} \qquad (P_\tau^k)$$

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Discretization in time by a fully implicit formula and in space by P_0/P_1-FEM

$$\partial_u \mathcal{E}^k_{\tau h}(u^k_{\tau h}, z^k_{\tau h}) = 0,$$

$$\partial \mathcal{R}\Big(\frac{z^k_{\tau h} - z^{k-1}_{\tau h}}{\tau}\Big) + \partial_z \mathcal{E}^k_{\tau h}(u^k_{\tau h}, z^k_{\tau h}) \ni 0$$

where $\mathcal{E}^k_{\tau h}(u,z) := \mathcal{E}_\tau(k\tau, u, z) + \delta_{\mathcal{Q}_h(u,z)}$,
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$$\left.\begin{array}{ll} \text{minimize} & \mathcal{R}\Big(z - z^{k-1}_{\tau h}\Big) + \mathcal{E}^k_\tau(u,z) \\ \text{subject to} & (u,z) \in \mathcal{Q}_h = \mathcal{U}_h \times \mathcal{Z}_h. \end{array}\right\} \qquad (P^k_{\tau h})$$

Properties of the discrete solution:

- Comparing values $(P_{\tau h}^k)$ at the level k with those in a general $(\tilde{u}, \tilde{z})$ and using degree-1 homogeneity of $\mathcal{R}$, we obtain the discrete stability:

$$\mathcal{E}_\tau^k(u_{\tau h}^k, z_{\tau h}^k) \le \mathcal{E}_\tau^k(\tilde{u}, \tilde{z}) + \mathcal{R}(\tilde{z} - z_{\tau h}^{k-1}) - \mathcal{R}(z_{\tau h}^k - z_{\tau h}^{k-1})$$
$$\le \mathcal{E}_\tau^k(\tilde{u}, \tilde{z}) + \mathcal{R}(\tilde{z} - z_{\tau h}^k);$$

we thus get the stability for the discrete solution, i.e.:

$$\bar{\mathcal{E}}_\tau\big(t, \bar{u}_{\tau h}(t), \bar{z}_{\tau h}(t)\big) \le \bar{\mathcal{E}}_\tau\big(t, \tilde{u}, \tilde{z}\big) + \mathcal{R}\big(\tilde{z} - \bar{z}_{\tau h}(t)\big)$$

holds for all $\tilde{u} \in \mathcal{U}$, $\tilde{z} \in \mathcal{Z}$, and $t \in [0, T]$.

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- Comparing values of $(P^k_{\tau h})$ at the level k with those in $(u^{k-1}_{\tau h}, z^{k-1}_{\tau h})$ gives an upper estimate of the energy balance:

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- Eventually, written the stability at the level $k-1$ and test it by $(\tilde{u}, \tilde{z}) = (u^k_{\tau h}, z^k_{\tau h})$ gives a lower estimate of the energy balance:

$$\begin{aligned}&\mathcal{E}^k_\tau(u^k_{\tau h}, z^k_{\tau h}) + \mathcal{R}(z^k_{\tau h} - z^{k-1}_{\tau h}) - \mathcal{E}^{k-1}_\tau(u^{k-1}_{\tau h}, z^{k-1}_{\tau h})\\ &= \mathcal{E}^{k-1}_\tau(u^k_{\tau h}, z^k_{\tau h}) + \int_{(k-1)\tau}^{k\tau} \frac{\partial}{\partial t}\mathcal{E}(t, u^k_{\tau h}, z^k_{\tau h})\mathrm{d}t + \mathcal{R}(z^k_{\tau h} - z^{k-1}_{\tau h}) - \mathcal{E}^{k-1}_\tau(u^{k-1}_{\tau h}, z^k_\tau\\ &\ge \int_{(k-1)\tau}^{k\tau} \frac{\partial}{\partial t}\mathcal{E}(t, u^k_{\tau h}, z^k_{\tau h})\mathrm{d}t.\end{aligned}$$

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&\geq \int_{(k-1)\tau}^{k\tau} \frac{\partial}{\partial t}\mathcal{E}(t, u^k_{\tau h}, z^k_{\tau h})\mathrm{d}t.
\end{aligned}$$

Summing it for $k = 1, ..., s/\tau \in \mathbb{N}$, we get the two-sided approximate energy balance:

$$\mathcal{E}(0, u_0, z_0) + \int_0^s \partial_t \mathcal{E}_\tau\big(t, \overline{u}_{\tau h}(t), \overline{z}_{\tau h}(t)\big)\mathrm{d}t$$
$$\leq \mathcal{E}\big(s, u_{\tau h}(s), z_{\tau h}(s)\big) + \mathrm{Var}_{\mathcal{R}}\big(z_{\tau h}; 0, s\big)$$
$$\leq \mathcal{E}(0, u_0, z_0) + \int_0^s \partial_t \mathcal{E}_\tau\big(t, \underline{u}_{\tau h}(t), \underline{z}_{\tau h}(t)\big)\mathrm{d}t,$$

where

$u_{\tau h} :=$ piecewise affine interpolation of $\{u_{\tau h}^k\}_{k=0}^{T/\tau}$,
$\overline{u}_{\tau h} :=$ "forward" piecewise constant interpolation of $\{u_{\tau h}^k\}_{k=0}^{T/\tau}$,
$\underline{u}_{\tau h} :=$ "backward" piecewise constant interpolation of $\{u_{\tau h}^k\}_{k=0}^{T/\tau}$,

and similarly for $z_{\tau h}$, $\overline{z}_{\tau h}$, and $\underline{z}_{\tau h}$.

Possibility of certain a-posteriori information about the discretisation error.

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Possibility of certain a-posteriori information about the discretisation error.

Convergence analysis outlined

Step 1: a-priori estimates: from the approximate energy balance by Gronwall inequality:

$$\left\|u_{\tau h}\right\|_{L^\infty([0,T];\mathcal{U}))} \leq C_1,$$

$$\max_{t\in[0,T]} \bar{\mathcal{E}}_\tau(t, \bar{u}_{\tau h}(t), \bar{z}_{\tau h}(t)) \leq C_2,$$

$$\left\|z_{\tau h}\right\|_{L^\infty([0,T];\mathcal{Z})} \leq C_3,$$

$$\mathrm{Var}_{\mathcal{R}}(\bar{z}_{\tau h}; 0, T) \leq C_4.$$

Step 2: selection of subsequences

weakly* converging (Banach's selection principle) to some u and z,

pointwise converging (Helly's selection principle):

$$z_{\tau h}(t) \to z(t) \text{ weakly in } \mathcal{Z} \text{ for all } t.$$

the uniform monotonicity of $\partial_u \mathcal{E}(t, \cdot, z)$ also

$$u_{\tau h} \to u \text{ strongly in } L^2([0,T];\mathcal{U}).$$

Step 3: limit passage in the stability:

An essential assumption:
Mutual recovery sequence (MRS) exists, (MIELKE, R., STEFANELLI, 2008):

$$\forall (t_\ell, u_\ell, z_\ell) \to (t, u, z) \;\; \forall (\widetilde{u}, \widetilde{z}) \in \mathcal{U} \times \mathcal{Z} \quad \exists (\widetilde{u}_\ell, \widetilde{z}_\ell)_{\ell \in \mathbb{N}} :$$

$$\limsup_{\ell \to \infty} \big(\mathcal{E}(t_\ell, \widetilde{u}_\ell, \widetilde{z}_\ell) + \mathcal{R}(\widetilde{z}_\ell - z_\ell) - \mathcal{E}(t_\ell, u_\ell, z_\ell)\big) \\ \leq \mathcal{E}(t, \widetilde{u}, \widetilde{z}) + \mathcal{R}(\widetilde{z} - z) - \mathcal{E}(t, u, z).$$

For plasticity only:

MRS by the "binomial trick" ($\mathbb{H}=0$ and no t-dependence for simplicity):

$$\begin{aligned}
&\limsup_{\ell\to\infty}\Big(\mathcal{E}(t_\ell,\widetilde{u}_\ell,\widetilde{z}_\ell)+\mathcal{R}(\widetilde{z}_\ell-z_\ell)-\mathcal{E}(t_\ell,u_\ell,z_\ell)\Big)\\
&\quad=\limsup_{\ell\to\infty}\bigg(\int_\Omega\frac{1}{2}\mathbb{C}(e(\widetilde{u}_\ell{+}u_\ell)-\pi_\ell{-}\widetilde{\pi}_\ell):(e(\widetilde{u}_\ell{-}u_\ell)+\pi_\ell{-}\widetilde{\pi}_\ell)\\
&\qquad\qquad+\frac{1}{2}b(\widetilde{\eta}_\ell{+}\eta_\ell)(\widetilde{\eta}_\ell{-}\eta_\ell)\,\mathrm{d}x+\mathcal{R}(\widetilde{\pi}_\ell{-}\pi_\ell,\widetilde{\eta}_\ell{-}\eta_\ell)\bigg)\\
&\quad=\int_\Omega\frac{1}{2}\mathbb{C}(e(\widetilde{u}{+}u)-\pi{-}\widetilde{\pi}):(e(\widetilde{u}{-}u)+\pi{-}\widetilde{\pi})\\
&\qquad\qquad+\frac{1}{2}b(\eta{+}\widetilde{\eta})(\eta{-}\widetilde{\eta})\,\mathrm{d}x+\mathcal{R}(\widetilde{\pi}{-}\pi,\widetilde{\eta}{-}\eta)\\
&\quad=\mathcal{E}(t,\widetilde{u},\widetilde{z})+\mathcal{R}(\widetilde{z}-z)-\mathcal{E}(t,u,z),
\end{aligned}$$

if we choose $\widetilde{u}_\ell:=\widetilde{u}-u+u_\ell$, $\widetilde{\pi}_\ell:=\widetilde{\pi}-\pi+\pi_\ell$ and $\widetilde{\eta}_\ell:=\widetilde{\eta}-\eta+\eta_\ell$.

We use it for $\bar{u}_\tau(t)\to u(t)$ weakly in $H^1(\Omega;\mathbb{R}^d)$
and $\bar{\pi}_\tau(t)\to\pi(t)$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$
and $\bar{\eta}_\tau(t)\to\eta(t)$ weakly in $L^2(\Omega)$!

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$$\limsup_{\ell\to\infty}\Big(\mathcal{E}(t_\ell,\widetilde{u}_\ell,\widetilde{z}_\ell)+\mathcal{R}(\widetilde{z}_\ell-z_\ell)-\mathcal{E}(t_\ell,u_\ell,z_\ell)\Big)$$
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\limsup_{\ell\to\infty}\Big(&\mathcal{E}(t_\ell,\widetilde{u}_\ell,\widetilde{z}_\ell)+\mathcal{R}(\widetilde{z}_\ell-z_\ell)-\mathcal{E}(t_\ell,u_\ell,z_\ell)\Big)\\
&=\lim_{\ell\to\infty}\bigg(\int_\Omega\frac{1}{2}\mathbb{C}(e(\widetilde{u}_\ell+u_\ell)-\pi_\ell-\widetilde{\pi}_\ell):(e(\widetilde{u}_\ell-u_\ell)+\pi_\ell-\widetilde{\pi}_\ell)\\
&\qquad\qquad+\frac{1}{2}b(\widetilde{\eta}_\ell+\eta_\ell)(\widetilde{\eta}_\ell-\eta_\ell)\,\mathrm{d}x+\mathcal{R}(\widetilde{\pi}_\ell-\pi_\ell,\widetilde{\eta}_\ell-\eta_\ell)\bigg)\\
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MRS by the "binomial trick" ($\mathbb{H}=0$ and no t-dependence for simplicity):

$$\begin{aligned}
&\limsup_{\ell\to\infty}\Big(\mathcal{E}(t_\ell,\widetilde u_\ell,\widetilde z_\ell)+\mathcal{R}(\widetilde z_\ell-z_\ell)-\mathcal{E}(t_\ell,u_\ell,z_\ell)\Big)\\
&\quad=\lim_{\ell\to\infty}\bigg(\int_\Omega\frac12\mathbb{C}(e(\widetilde u_\ell+u_\ell)-\pi_\ell-\widetilde\pi_\ell):(e(\widetilde u_\ell-u_\ell)+\pi_\ell-\widetilde\pi_\ell)\\
&\qquad\qquad+\frac12 b(\widetilde\eta_\ell+\eta_\ell)(\widetilde\eta_\ell-\eta_\ell)\,\mathrm{d}x+\mathcal{R}(\widetilde\pi_\ell-\pi_\ell,\widetilde\eta_\ell-\eta_\ell)\bigg)\\
&\quad=\int_\Omega\frac12\mathbb{C}(e(\widetilde u+u)-\pi-\widetilde\pi):(e(\widetilde u-u)+\pi-\widetilde\pi)\\
&\qquad\qquad+\frac12 b(\eta+\widetilde\eta)(\eta-\widetilde\eta)\,\mathrm{d}x+\mathcal{R}(\widetilde\pi-\pi,\widetilde\eta-\eta)\\
&\quad=\mathcal{E}(t,\widetilde u,\widetilde z)+\mathcal{R}(\widetilde z-z)-\mathcal{E}(t,u,z),
\end{aligned}$$

if we choose $\widetilde u_\ell:=\widetilde u-u+u_\ell$, $\widetilde\pi_\ell:=\widetilde\pi-\pi+\pi_\ell$ and $\widetilde\eta_\ell:=\widetilde\eta-\eta+\eta_\ell$.

We use it for $\bar u_\tau(t)\to u(t)$ weakly in $H^1(\Omega;\mathbb{R}^d)$
and $\bar\pi_\tau(t)\to\pi(t)$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$
and $\bar\eta_\tau(t)\to\eta(t)$ weakly in $L^2(\Omega)$!

For mere damage:

$$\limsup_{\ell\to\infty}\Big(\mathcal{E}(t_\ell,\widetilde{u}_\ell,\widetilde{z}_\ell)+\mathcal{R}(\widetilde{z}_\ell{-}z_\ell)-\mathcal{E}(t_\ell,u_\ell,z_\ell)\Big)$$

$$=\limsup_{\ell\to\infty}\int_\Omega\frac{1}{2}\mathbb{C}(\widetilde{\zeta}_\ell)e(\widetilde{u}_\ell){:}e(\widetilde{u}_\ell)+\frac{\kappa}{r}|\nabla\widetilde{\zeta}_\ell|^r$$

$$-\frac{1}{2}\mathbb{C}(\zeta_\ell)e(u_\ell){:}e(u_\ell)-\frac{\kappa}{r}|\nabla\zeta_\ell|^r+a_1(\zeta_\ell{-}\widetilde{\zeta}_\ell)\,dx$$

$$\le\mathcal{E}(t,\widetilde{u},\widetilde{z})+\mathcal{R}(\widetilde{z}{-}z)-\mathcal{E}(t,u,z),$$

Now we choose $\widetilde{u}_\ell:=u_\ell$ (resp. $\widetilde{u}_\ell$ fixed) and also $\tilde{\zeta}_\ell=\big(\tilde{\zeta}-\|\zeta_\ell{-}\zeta\|_{C(\bar\Omega)}\big)^+$.

Note that $0\le\tilde{\zeta}_\ell\le\zeta_\ell$ if $\tilde{\zeta}\le\zeta$
and that $\tilde{\zeta}_\ell\to\tilde{\zeta}$ in $W^{1,r}(\Omega)$ if $\zeta_\ell\to\zeta$ weakly in $W^{1,r}(\Omega)$.

We use $$\nabla\tilde{\zeta}_\ell(x)=\begin{cases}\nabla\tilde{\zeta}(x) & \text{if } \tilde{\zeta}_\ell(x)>\|\zeta_\ell{-}\zeta\|_{C(\bar\Omega)},\\ 0 & \text{otherwise.}\end{cases}$$

Thus, as $\|\zeta_\ell{-}\zeta\|_{C(\bar\Omega)}\to0$, we have $\nabla\tilde{\zeta}_\ell\to\nabla\tilde{\zeta}$ a.e. on Ω and thus $\int_\Omega|\nabla\tilde{\zeta}_\ell-\nabla\tilde{\zeta}|\to0$ by Lebesgue theorem with the integrable majorant: $\big|\nabla\widetilde{z}_k{-}\nabla\widetilde{z}\big|^r\le2^{r-1}\big(|\nabla\widetilde{z}_k|^r{+}|\nabla\widetilde{z}|^r\big)\le2^r|\nabla\widetilde{z}|^r$.

We use it for $\bar{u}_\tau(t)\to u(t)$ strongly (resp. weakly) in $H^1(\Omega;\mathbb{R}^d)$ and $\bar{\zeta}_\tau(t)\to\zeta(t)$ weakly in $W^{1,r}(\Omega)$.

For mere damage:

$$\limsup_{\ell\to\infty}\Big(\mathcal{E}(t_\ell,\widetilde{u}_\ell,\widetilde{z}_\ell)+\mathcal{R}(\widetilde{z}_\ell{-}z_\ell)-\mathcal{E}(t_\ell,u_\ell,z_\ell)\Big)$$
$$=\limsup_{\ell\to\infty}\int_\Omega\frac12\mathbb{C}(\widetilde{\zeta}_\ell)e(\widetilde{u}_\ell){:}e(\widetilde{u}_\ell)+\frac{\kappa}{r}|\nabla\widetilde{\zeta}_\ell|^r$$
$$-\frac12\mathbb{C}(\zeta_\ell)e(u_\ell){:}e(u_\ell)-\frac{\kappa}{r}|\nabla\zeta_\ell|^r+a_1(\zeta_\ell{-}\widetilde{\zeta}_\ell)\,dx$$
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For mere damage alternatively if $\mathbb{C}$ monotonically dependent on ζ:

$$\begin{aligned}
&\limsup_{\ell\to\infty}\Big(\mathcal{E}(t_\ell,\widetilde{u}_\ell,\widetilde{z}_\ell)+\mathcal{R}(\widetilde{z}_\ell-z_\ell)-\mathcal{E}(t_\ell,u_\ell,z_\ell)\Big)\\
&\quad=\limsup_{\ell\to\infty}\int_\Omega\frac12\mathbb{C}(\widetilde{\zeta}_\ell)e(\widetilde{u}_\ell){:}e(\widetilde{u}_\ell)+\frac{\kappa}{r}|\nabla\widetilde{\zeta}_\ell|^r\\
&\qquad\qquad-\frac12\mathbb{C}(\zeta_\ell)e(u_\ell){:}e(u_\ell)-\frac{\kappa}{r}|\nabla\zeta_\ell|^r+a_1(\zeta_\ell-\widetilde{\zeta}_\ell)\,dx\\
&\quad\le\limsup_{\ell\to\infty}\int_\Omega\frac12\mathbb{C}(\zeta_\ell)e(\widetilde{u}_\ell){:}e(\widetilde{u}_\ell)+\frac{\kappa}{r}|\nabla\widetilde{\zeta}_\ell|^r\\
&\qquad\qquad-\frac12\mathbb{C}(\zeta_\ell)e(u_\ell){:}e(u_\ell)-\frac{\kappa}{r}|\nabla\zeta_\ell|^r+a_1(\zeta_\ell-\widetilde{\zeta}_\ell)\,dx\\
&\quad=\limsup_{\ell\to\infty}\int_\Omega\frac12\mathbb{C}(\zeta_\ell)e(\widetilde{u}_\ell{+}u_\ell){:}e(\widetilde{u}_\ell{-}u_\ell)+\frac{\kappa}{r}|\nabla\widetilde{\zeta}_\ell|^r\\
&\qquad\qquad-\frac{\kappa}{r}|\nabla\zeta_\ell|^r+a_1(\zeta_\ell-\widetilde{\zeta}_\ell)\,dx\\
&\quad\le\mathcal{E}(t,\widetilde{u},\widetilde{z})+\mathcal{R}(\widetilde{z}-z)-\mathcal{E}(t,u,z).
\end{aligned}$$

Now we choose $\widetilde{u}_\ell:=\widetilde{u}-u+u_\ell$ and $\tilde{\zeta}_\ell=(\tilde{\zeta}-\|\zeta_\ell{-}\zeta\|_{C(\bar\Omega)})^+$.

We use it for $\bar{u}_\tau(t)\to u(t)$ weakly in $H^1(\Omega;\mathbb{R}^d)$
and $\bar{\zeta}_\tau(t)\to\zeta(t)$ weakly in $W^{1,r}(\Omega)$.

And for plasticity with damage if $\mathbb{C}$ monotonically dependent on ζ:

$$\limsup_{\ell\to\infty}\Big(\mathcal{E}(t_\ell,\widetilde u_\ell,\widetilde z_\ell)+\mathcal{R}(\widetilde z_\ell-z_\ell)-\mathcal{E}(t_\ell,u_\ell,z_\ell)\Big)$$

$$=\limsup_{\ell\to\infty}\int_\Omega\frac12\mathbb{C}(\widetilde\zeta_\ell)(e(\widetilde u_\ell)-\widetilde\pi_\ell){:}(e(\widetilde u_\ell)-\widetilde\pi_\ell)+\frac{\kappa}{r}|\nabla\widetilde\zeta_\ell|^r$$
$$-\frac12\mathbb{C}(\zeta_\ell)(e(u_\ell)-\pi_\ell){:}(e(u_\ell)-\pi_\ell)-\frac{\kappa}{r}|\nabla\zeta_\ell|^r+a_1(\zeta_\ell-\widetilde\zeta_\ell)+\delta_S^*(\widetilde\pi_\ell-\pi_\ell)\,dx$$
$$\le\limsup_{\ell\to\infty}\int_\Omega\frac12\mathbb{C}(\zeta_\ell)(e(\widetilde u_\ell)-\widetilde\pi_\ell){:}(e(\widetilde u_\ell)-\widetilde\pi_\ell)+\frac{\kappa}{r}|\nabla\widetilde\zeta_\ell|^r$$
$$-\frac12\mathbb{C}(\zeta_\ell)(e(u_\ell)-\pi_\ell){:}(e(u_\ell)-\pi_\ell)-\frac{\kappa}{r}|\nabla\zeta_\ell|^r+a_1(\zeta_\ell-\widetilde\zeta_\ell)+\delta_S^*(\widetilde\pi_\ell-\pi_\ell)\,dx$$
$$=\limsup_{\ell\to\infty}\int_\Omega\frac12\mathbb{C}(\zeta_\ell)(e(\widetilde u_\ell+u_\ell)-\widetilde\pi_\ell-\pi_\ell){:}(e(\widetilde u_\ell-u_\ell)-\widetilde\pi_\ell+\pi_\ell)+\frac{\kappa}{r}|\nabla\widetilde\zeta_\ell|^r$$
$$-\frac{\kappa}{r}|\nabla\zeta_\ell|^r+a_1(\zeta_\ell-\widetilde\zeta_\ell)+\delta_S^*(\widetilde\pi_\ell-\pi_\ell)\,dx$$
$$\le\mathcal{E}(t,\widetilde u,\widetilde z)+\mathcal{R}(\widetilde z-z)-\mathcal{E}(t,u,z).$$

We choose $\widetilde u_\ell:=\widetilde u-u+u_\ell$, $\widetilde\pi_\ell:=\widetilde\pi-\pi+\pi_\ell$, and $\tilde\zeta_\ell=(\tilde\zeta-\|\zeta_\ell-\zeta\|_{C(\bar\Omega)})^+$.
(R. Toader, 3.2.2015, personal communication)

We use it for $\bar u_\tau(t)\rightharpoonup u(t)$ in $H^1(\Omega;\mathbb{R}^d)$, $\bar\pi_\tau(t)\rightharpoonup\pi(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$, and $\bar\zeta_\tau(t)\to\zeta(t)$ weakly in $W^{1,r}(\Omega)$. (For isotropic hardening works too.)

Step 4: limit passage in the upper energy inequality:

$$\mathcal{E}\big(T, u_{\tau h}(T), z_{\tau h}(T)\big) + \mathrm{Var}_{\mathcal{R}}\big(z_{\tau h}; 0, T\big) \\ \leq \mathcal{E}\big(0, u_{0,h}, z_{0,h}\big) + \int_0^T \partial_t \mathcal{E}_\tau\big(t, u_{\tau h}(t), z_{\tau h}(t)\big)\mathrm{d}t.$$

by lower semicontinuity in the l.h.s. and continuity in the r.h.s.

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Step 5: the lower energy inequality:

stability (suffices a.e.) allows

by Riemann-sum approximation of Lebesgue integral to show

the opposite inequality $\Rightarrow$ the energy equality!

Step 6: Improved convergence.

$$\forall\, t \in [0,T]:\ \mathrm{Var}_{\mathcal{R}}(z_{\tau h};[0,t]) \to \mathrm{Var}_{\mathcal{R}}(z;[0,t]);$$

$$\forall\, t \in [0,T]:\ \mathcal{E}(t,u_{\tau h}(t),z_{\tau h}(t)) \to \mathcal{E}(t,u(t),z(t));$$

$$\partial_t\mathcal{E}(\cdot,u_{\tau h}(\cdot),z_{\tau h}(\cdot)) \to \partial_t\mathcal{E}(\cdot,u(\cdot),z(\cdot)) \text{ in } L^1((0,T)).$$

Mere convergence (W.Han, B.D.Reddy, 1999, A.Mielke, T.R., 2009):

Rate of convergence (D.Knees, 2009):

$$\big\|u-\bar u_{\tau,h}\big\|_{L^\infty(I;H^1(\Omega;\mathbb{R}^d))}+\big\|z-\bar z_{\tau,h}\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}\times\mathbb{R}))}=\mathcal{O}\big(\sqrt{\tau}+\sqrt[4-\epsilon]{h}\big),\quad \epsilon>0.$$

for smooth Ω and time-regular loading, based on regularity

$$u\in L^\infty(I;W^{3/2-\epsilon}(\Omega;\mathbb{R}^d)),\qquad z\in L^\infty(I;W^{1/2-\epsilon}(\Omega;\mathbb{R}^{d\times d}\times\mathbb{R})),\qquad \epsilon>0,$$

with $\varepsilon>0$ the ellipticity constant of $\mathcal{E}(t,\cdot,\cdot)$.
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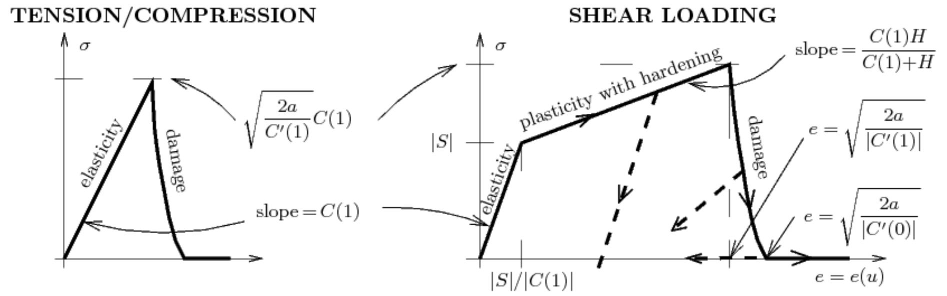
The goal: to realize the stress-driven scenario:



Schematic response of the mechanical stress σ on the total strain e during a "one-dimensional" tenson (left) or shear (right) loading experiment under a stress-driven scenario. The latter option combines plasticity with eventual (complete) damage. Dashed lines outline a response on unloading, $C = C(\zeta)$ refers to Young's modulus (left) or the shear modulus (right).

(The analysis will work only for incomplete damage, however!)

A requirement: to eliminate unphysically "too early" jumps and global minimization:

1) Physically motivated option: small viscosity:
Here there are 3 options: viscosity in $e_{\rm el}$ and η, or
viscosity in ζ, or
viscosity in both $e_{\rm el}$ and η and ζ.
Numerically difficult for very small viscosities (as shown above),
analytically difficult for limitting towards vanishing viscosity.

2) Suitable semi-implicit discretisation:
A general intuitive strategy to facilitate numerical handling:
fractional splitting of variables in accord to separate convexity of $\mathcal{E}(t,\cdot)$ and
in accord to additive splitting of $\mathcal{R}$.
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Global minimization is difficult if $\mathcal{E}(t,\cdot,\cdot)$ is not convex.

Various local minimization algorithms (typically alternating minimisation algorithm = AMA) with suitable choice of initial iteration (backtracking exploiting the double sided energy inequality).

An engineering approach: mere AMA (= a sequence of convex problems). At level k, z_τ^{k-1} is fixed during AMA iterations. If AMA converges, then it gives only critical points of P_τ^k and thus a solution to the Rothe formula

$$\partial_u\mathcal{E}_\tau^k(u_\tau^k,z_\tau^k)\ni 0 \quad\text{and}\quad \partial\mathcal{R}\Big(\frac{z_\tau^k-z_\tau^{k-1}}{\tau}\Big)+\partial_z\mathcal{E}_\tau^k(u_\tau^k,z_\tau^k)\ni 0.$$

But testing the inclusions by $\frac{u_\tau^k-u_\tau^{k-1}}{\tau}$ and $\frac{z_\tau^k-z_\tau^{k-1}}{\tau}$ respectively does not give any a-priori estimates unless $\mathcal{E}(t,\cdot,\cdot)$ is convex (or unless (u_τ^k,z_τ^k) is, in addition, a global minimizer).

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An idea: only 1 iteration of AMA:

The semi-implicit Rothe method ($\tau > 0$ a time step):

$$\partial_u \mathcal{E}_\tau^k(u_\tau^k, z_\tau^{k-1}) \ni 0 \quad \text{and} \quad \partial\mathcal{R}\Big(\frac{z_\tau^k - z_\tau^{k-1}}{\tau}\Big) + \partial_z \mathcal{E}_\tau^k(u_\tau^k, z_\tau^k) \ni 0.$$

It yields two convex decoupled problems:

$$\left.\begin{array}{ll} \text{minimize} & \mathcal{E}_\tau^k(u, z_\tau^{k-1}) \\ \text{subject to} & u \in \mathcal{U}, \end{array}\right\} \qquad ([P_1]_\tau^k)$$

$$\left.\begin{array}{ll} \text{minimize} & \mathcal{R}(z - z_\tau^{k-1}) + \mathcal{E}_\tau^k(u_\tau^k, z) \\ \text{subject to} & z \in \mathcal{Z}. \end{array}\right\} \qquad ([P_2]_\tau^k)$$

Fractional-step strategy: $q = (u, z)$, $\mathcal{R}_{\rm ext}(q) = \mathcal{R}(u)$, $\partial\mathcal{E}_\tau^k(q) := \sum_{i=1}^2 A_{\tau,i}^k(q)$,
$A_{\tau,1}^k(q) := (\partial_u \mathcal{E}_\tau^k(q), 0)$, $\quad A_{\tau,2}^k(q) := (0, \partial_z \mathcal{E}_\tau^k(q))$:

$$\mathcal{R}_{\rm ext}\Big(\frac{q_\tau^{k-1+i/2} - q_\tau^{k-3/2+i/2}}{\tau}\Big) + A_{\tau,i}^k(q_\tau^{k-1+i/2}) \ni 0, \qquad i = 1, 2.$$

Then $q_\tau^{k-1} = (u_\tau^{k-1}, z_\tau^{k-1})$, $q_\tau^{k-1/2} = (u_\tau^k, z_\tau^{k-1})$, and $q_\tau^k = (u_\tau^k, z_\tau^k)$.

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Numerical stability of this semi-implicit scheme:

test of $([P_1]^k_\tau)$ by $u^k_\tau - u^{k-1}_\tau$ and use convexity of $\mathcal{E}^k_\tau(\cdot, z^{k-1}_\tau)$:

$$\mathcal{E}^k_\tau(u^k_\tau, z^{k-1}_\tau) \le \mathcal{E}^k_\tau(u^{k-1}_\tau, z^{k-1}_\tau)$$
$$= \mathcal{E}^{k-1}_\tau(u^{k-1}_\tau, z^{k-1}_\tau) + \int_{(k-1)\tau}^{k\tau} \mathcal{E}'_t(t, u^{k-1}_\tau, z^{k-1}_\tau)\mathrm{d}t$$

and then compare the value of $([P_2]^k_\tau)$ at z^k_τ with the value at z^{k-1}_τ:

$$\mathcal{R}\Big(\frac{z^k_\tau - z^{k-1}_\tau}{\tau}\Big) + \mathcal{E}^k_\tau(u^k_\tau, z^k_\tau) \le \mathcal{E}^k_\tau(u^k_\tau, z^{k-1}_\tau).$$

Summing it up $\Rightarrow$ cancelation of $\pm\mathcal{E}^k_\tau(u^k_\tau, z^{k-1}_\tau)$ and the energy imbalance:

$$\mathcal{E}^k_\tau(u^k_\tau, z^k_\tau) + \tau\mathcal{R}\Big(\frac{z^k_\tau - z^{k-1}_\tau}{\tau}\Big) \le \mathcal{E}^{k-1}_\tau(u^{k-1}_\tau, z^{k-1}_\tau) + \int_{(k-1)\tau}^{k\tau} \mathcal{E}'_t(t, u^{k-1}_\tau, z^{k-1}_\tau)\mathrm{d}t$$

$\Rightarrow$ again a-priori estimates ($=$ numerical stability).

Yet, we do not have the two-sided inequality

(but we do not need it for controlling global minimization)

Numerical stability of this semi-implicit scheme:

test of $([P_1]_\tau^k)$ by $u_\tau^k - u_\tau^{k-1}$ and use convexity of $\mathcal{E}_\tau^k(\cdot, z_\tau^{k-1})$:

$$\mathcal{E}_\tau^k(u_\tau^k, z_\tau^{k-1}) \le \mathcal{E}_\tau^k(u_\tau^{k-1}, z_\tau^{k-1}) = \mathcal{E}_\tau^{k-1}(u_\tau^{k-1}, z_\tau^{k-1}) + \int_{(k-1)\tau}^{k\tau} \mathcal{E}_t'(t, u_\tau^{k-1}, z_\tau^{k-1})\mathrm{d}t$$

and then compare the value of $([P_2]_\tau^k)$ at z_τ^k with the value at z_τ^{k-1}:

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Yet, we do not have the two-sided inequality
(but we do not need it for controlling global minimization).

General strategy of convergence towards local solutions: difficult parts:

1) semi-stability: needs a mutual recovery sequences:

$$\forall \text{ semistable sequence } (t_k,u_k,z_k) \rightharpoonup (t,u,z) \ \ \forall \widetilde{z}\in\mathcal{Z} \ \ \exists (\widetilde{z}_k)_{k\in\mathbb{N}}:$$
$$\limsup_{k\to\infty}\big(\mathcal{E}(t_k,u_k,\widetilde{z}_k)+\mathcal{R}(\widetilde{z}_k-z_k)-\mathcal{E}(t_k,u_k,z_k)\big)\le\mathcal{E}(t,u,\widetilde{z})+\mathcal{R}(\widetilde{z}-z)-\mathcal{E}(t,u,z).$$

2) energy inequality:

$$\mathcal{E}(t_2,u(t_2),z(t_2))+\mathrm{Var}_{\mathcal{R}}(z;[t_1,t_2])\le\mathcal{E}(t_1,u(t_1),z(t_1))+\int_{t_1}^{t_2}\mathcal{E}_t'(t,u(t),z(t))\,\mathrm{d}t$$

needs typically the strong convergence for $u(t_1)$ and,
if $\mathcal{E}(t,u,\cdot)$ is not affine, also for $z(t_1)$.

Therefore, some "good convexity" of $\mathcal{E}(t,\cdot,z)$ is needed.

Contra-intuitively, maybe more difficult than convergence to energetic solutions,
although the local solutions form the widest reasonable concept.

Here it additionally needs 1) gradient plasticity,
2) to allow (at least formally) for healing
(still as rate independent)

The governing equation/inclusions read as (no isotropic hardening for simplicity):

$$\mathrm{div}\big(\mathbb{C}(\zeta)e_{\rm el}\big) + g = 0 \qquad \text{with} \quad e_{\rm el} = e(u) - \pi,$$

$$\partial\delta_S^*\Big(\frac{\partial\pi}{\partial t}\Big) \ni \mathrm{dev}\big(\mathbb{C}(\zeta)e_{\rm el}\big) - \mathbb{H}\pi + \kappa_1\Delta\pi,$$

$$\partial\delta_{[-a,b]}^*\Big(\frac{\partial\zeta}{\partial t}\Big) \ni -\frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el} : e_{\rm el} + \kappa_2\,\mathrm{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big) - N_{[0,1]}(\zeta),$$

with the boundary conditions:

$$u = w_{\rm Dir} \qquad \text{on } \Gamma_{\rm Dir},$$

$$\big(\mathbb{C}(\zeta)e_{\rm el}\big)\cdot\vec{n} = f \qquad \text{on } \Gamma_{\rm Neu},$$

$$\nabla\pi\vec{n} = 0 \quad \text{and} \quad \nabla\zeta\cdot\vec{n} = 0 \qquad \text{on } \Gamma.$$

Healing only rather formal: if $\mathbb{C}(\cdot)$ monotone, b large, $\kappa_2 > 0$ small,
$\Rightarrow$ not much chance for healing (except at most very small regions).

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The transformed problem with time-constant (homogeneous) Dirichlet condition:

$$e_{\rm el} = e(u){-}\pi \quad \text{replaces by} \quad e_{\rm el} = e(u{+}u_{\rm Dir}){-}\pi,$$
$$w_{\rm Dir} \quad \text{replaces by } 0.$$

The governing functionals:

$$\mathcal{E}(t,u,\pi,\zeta) := \int_\Omega \frac{1}{2}\mathbb{C}(\zeta)\big(e(u{+}u_{\rm Dir}(t)){-}\pi\big) : \big(e(u{+}u_{\rm Dir}(t)){-}\pi\big) + \frac{1}{2}\mathbb{H}\pi : \pi + \frac{\kappa_1}{2}|\nabla\pi|^2 + \frac{\kappa_2}{r}|\nabla\zeta|^r + \delta_{[0,1]}(\zeta) - f(t){\cdot}u\,{\rm d}x - \int_{\Gamma_{\rm Neu}} g(t){\cdot}u\,{\rm d}S,$$

$$\mathcal{R}\Big(\frac{{\rm d}\pi}{{\rm d}t},\frac{{\rm d}\zeta}{{\rm d}t}\Big) \equiv \mathcal{R}_1\Big(\frac{{\rm d}\pi}{{\rm d}t}\Big) + \mathcal{R}_2\Big(\frac{{\rm d}\zeta}{{\rm d}t}\Big) := \int_\Omega \delta_S^*\Big(\frac{\partial\pi}{\partial t}\Big) + a\Big(\frac{\partial\zeta}{\partial t}\Big)^- + b\Big(\frac{\partial\zeta}{\partial t}\Big)^+ {\rm d}x.$$

with the convention $\dot z^+ = \max(\dot z, 0)$ and $\dot z^- = \max(-\dot z, 0) \geq 0$.

The fractional-step algorithm (based on the splitting (u,π) vs ζ):

two convex minimization problems:

first

$$\begin{array}{ll} \text{minimize} & \mathcal{E}(k\tau,u,\pi,\zeta_\tau^{k-1})+\mathcal{R}_1(\pi-\pi_\tau^{k-1}) \\ \text{subject to} & (u,\pi)\in H^1(\Omega;\mathbb{R}^d)\times H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}), \quad u|_{\Gamma_{\rm Dir}}=0, \end{array}$$

and, denoting the unique solution as (u_τ^k,π_τ^k), then

$$\begin{array}{ll} \text{minimize} & \mathcal{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta)+\mathcal{R}_2(\zeta-\zeta_\tau^{k-1}) \\ \text{subject to} & \zeta\in W^{1,r}(\Omega), \quad 0\le\zeta\le 1, \end{array}$$

and denote its (possibly not unique) solution by ζ_τ^k.

Assumptions on the data:

$\mathbb{C}(\cdot), \mathbb{H} \in \mathbb{R}^{d\times d\times d\times d}$ positive definite, symmetric,

$\mathbb{C} : [0,1] \to \mathbb{R}^{d\times d\times d\times d}$ continuous,

$a,\ b,\ \kappa_1, \kappa_2 > 0, \quad S \subset \mathbb{R}^{d\times d}_{\rm dev}$ convex, bounded, closed, $\ \mathrm{int}\, S \ni 0$,

$w_{\rm Dir} \in W^{1,1}(0,T; W^{1/2,2}(\Gamma_{\rm Dir};\mathbb{R}^d))$,

$$f \in W^{1,1}(0,T;L^p(\Omega;\mathbb{R}^d)) \quad \text{with} \quad p \begin{cases} > 1 & \text{for } d=2, \\ = 2d/(d{+}2) & \text{for } d \geq 3 \end{cases}$$

$$g \in W^{1,1}(0,T;L^p(\Gamma_{\rm Neu};\mathbb{R}^d)) \quad \text{with} \quad p \begin{cases} > 1 & \text{for } d=2, \\ = 2{-}2/d & \text{for } d \geq 3. \end{cases}$$

A-priori estimates:

$$\big\|\bar u_\tau\big\|_{L^\infty(I;H^1(\Omega;\mathbb{R}^d))} \leq C,$$

$$\big\|\bar\pi_\tau\big\|_{L^\infty(I;H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))\cap {\rm BV}(I;L^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))} \leq C,$$

$$\big\|\bar\zeta_\tau\big\|_{L^\infty(\Omega)\cap {\rm BV}(I;L^1(\Omega))} \leq C.$$

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$$\big\|\bar{\zeta}_\tau\big\|_{L^\infty(\Omega)\cap {\rm BV}(I;L^1(\Omega))} \leq C.$$

Discrete local solution:

Equilibrium of displacements:

$$\forall t\in I: \quad \partial_u\mathcal{E}\big(t_\tau,\bar u_\tau(t),\bar\pi_\tau(t),\underline{\zeta}_\tau(t)\big)=0 \qquad \text{with } t_\tau:=\min\{k\tau\ge t;\ k\in\mathbb{N}\},$$

two separate semi-stability conditions for $\bar\zeta_\tau$ and $\bar\pi_\tau$:

$$\begin{aligned}&\forall t\in I\ \forall\widetilde\pi\in H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}): \quad \mathcal{E}\big(t_\tau,\bar u_\tau(t),\bar\pi_\tau(t),\underline{\zeta}_\tau(t)\big)\\ &\qquad\qquad\le\mathcal{E}\big(t_\tau,\bar u_\tau(t),\widetilde\pi,\underline{\zeta}_\tau(t)\big)+\mathcal{R}_1\big(\widetilde\pi-\bar\pi_\tau(t)\big),\\ &\forall t\in I\ \forall\widetilde\zeta\in W^{1,r}(\Omega),\ 0\le\widetilde\zeta\le1: \ \mathcal{E}\big(t_\tau,\bar u_\tau(t),\bar\pi_\tau(t),\bar\zeta_\tau(t)\big)\\ &\qquad\qquad\le\mathcal{E}\big(t_\tau,\bar u_\tau(t),\bar\pi_\tau(t),\widetilde\zeta\big)+\mathcal{R}_2\big(\widetilde\zeta-\bar\zeta_\tau(t)\big),\end{aligned}$$

and the energy (im)balance ($\forall 0\le t_1<t_2\le T,\ t_i=k_i\tau,\ k_i\in\mathbb{N}$):

$$\begin{aligned}&\mathcal{E}\big(t_2,\bar u_\tau(t_2),\bar\pi_\tau(t_2),\bar\zeta_\tau(t_2)\big)+\mathrm{Var}_{\mathcal{R}_1}\big(\bar\pi_\tau;[t_1,t_2]\big)+\mathrm{Var}_{\mathcal{R}_2}\big(\bar\zeta_\tau;[t_1,t_2]\big)\\ &\qquad\le\mathcal{E}\big(t_1,\bar u_\tau(t_1),\bar\pi_\tau(t_1),\bar\zeta_\tau(t_1)\big)+\int_{t_1}^{t_2}\mathcal{E}_t'\big(t,\bar u_\tau(t),\bar\pi(t),\bar\zeta(t)\big)\,\mathrm{d}t.\end{aligned}$$

Convergence:

Step 1: a (generalized) Helly's selection principle:

$\exists\,\zeta,\,\zeta_*\in \mathrm{B}(I;W^{1,r}(\Omega))\cap\mathrm{BV}(I;L^1(\Omega))$ and
$\exists\,\pi\in \mathrm{B}(I;H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))\cap\mathrm{BV}(I;L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))$ and a subsequence so that:

$$\bar\zeta_\tau(t)\to\zeta(t)\quad\&\quad \underline{\zeta}_\tau(t)\rightharpoonup\zeta_*(t)\qquad \text{weakly in } W^{1,r}(\Omega)\quad \text{for all } t\in I,$$

$$\bar\pi_\tau(t)\to\pi(t)\qquad \text{weakly in } H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})\ \text{for all } t\in I.$$

Then fix (for a moment) $t\in I$: by Banach's selection principle:

$$\bar u_\tau(t)\to u(t)\qquad \text{weakly in } H^1(\Omega;\mathbb{R}^d).$$

$\bar u_\tau(t)$ minimizes $\mathcal{E}(t_\tau,\cdot,\bar\pi_\tau(t),\underline{\zeta}_\tau(t))$ with $t_\tau:=\min\{k\tau\ge t;\ k\in\mathbb{N}\}$
$\Rightarrow$ $u(t)$ minimizes the strictly convex functional $\mathcal{E}(t,\cdot,\zeta_*(t),\pi(t))$

the compactness in both π and ζ due to the gradient theories involved.

$\Rightarrow$ $u(t)$ uniquely determined by $\zeta_*(t)$ and $\pi(t)$

(i.e. no other t-dependent selection needed).

$u:I\to H^1(\Omega;\mathbb{R}^d)$ is measurable because ζ_* and π are measurable.

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Convergence:

Step 2: strong convergence in u and π:

the discrete momentum equilibrium $\mathrm{div}(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}) + \bar{g}_\tau = 0$
the discrete plastic flow-rule $\bar{\xi}_\tau + \mathbb{H}\bar{\pi}_\tau - \mathrm{dev}\,\bar{\sigma}_\tau = \kappa_1\Delta\bar{\pi}_\tau$ with $\bar{\sigma}_\tau = \mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}$ and $\bar{\xi}_\tau(t) \in \partial\delta_S^*(\frac{\partial\pi_\tau}{\partial t}(t))$ and $\bar{e}_{\mathrm{el},\tau} = e(\bar{u}_\tau - \bar{u}_{\mathrm{Dir},\tau}) - \bar{\pi}_\tau$
at time t with B.C. considered in the weak sense and tested respectively by $\bar{u}_\tau(t) - u(t)$ and $\bar{\pi}_\tau(t) - \pi(t)$.

$$\int_\Omega \mathbb{C}(\underline{\zeta}_\tau(t))\big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) : \big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) + \mathbb{H}\big(\bar{\pi}_\tau(t) - \pi(t)\big) : \big(\bar{\pi}_\tau(t) - \pi(t)\big) + \frac{\kappa_1}{2}\big|\nabla\bar{\pi}_\tau(t) - \nabla\pi(t)\big|^2\,\mathrm{d}x$$
$$\le \int_\Omega -\mathbb{C}(\underline{\zeta}_\tau(t))e_{\mathrm{el}}(t) : \big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) - \big(\mathbb{H}\pi(t) - \bar{\xi}_\tau(t)\big) : \big(\bar{\pi}_\tau(t) - \pi(t)\big)$$
$$+ \frac{\kappa_1}{2}\nabla\pi(t) \vdots \nabla\big(\bar{\pi}_\tau(t) - \pi(t)\big) - \bar{f}_\tau(t)\cdot\big(\bar{u}_\tau(t) - u(t)\big)\,\mathrm{d}x$$
$$- \int_{\Gamma_{\mathrm{Neu}}} \bar{g}_\tau(t)\cdot\big(\bar{u}_\tau(t) - u(t)\big)\mathrm{d}S \to 0.$$

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$$\begin{aligned}
&\int_\Omega \mathbb{C}(\underline{\zeta}_\tau(t))\big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) : \big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) \\
&\qquad\qquad + \mathbb{H}\big(\bar{\pi}_\tau(t) - \pi(t)\big) : \big(\bar{\pi}_\tau(t) - \pi(t)\big) + \frac{\kappa_1}{2}\big|\nabla\bar{\pi}_\tau(t) - \nabla\pi(t)\big|^2\,\mathrm{d}x \\
&\le \int_\Omega -\mathbb{C}(\underline{\zeta}_\tau(t))e_{\mathrm{el}}(t) : \big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) - \big(\mathbb{H}\pi(t) - \bar{\xi}_\tau(t)\big) : \big(\bar{\pi}_\tau(t) - \pi(t)\big) \\
&\qquad\qquad + \frac{\kappa_1}{2}\nabla\pi(t) \,\vdots\, \nabla\big(\bar{\pi}_\tau(t) - \pi(t)\big) - \bar{f}_\tau(t)\cdot\big(\bar{u}_\tau(t) - u(t)\big)\,\mathrm{d}x \\
&\qquad\qquad\qquad\qquad - \int_{\Gamma_{\mathrm{Neu}}} \bar{g}_\tau(t)\cdot\big(\bar{u}_\tau(t) - u(t)\big)\mathrm{d}S \to 0.
\end{aligned}$$

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$$
\begin{aligned}
&\int_\Omega \mathbb{C}(\underline{\zeta}_\tau(t))\big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) : \big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) \\
&\qquad\qquad + \mathbb{H}\big(\bar{\pi}_\tau(t) - \pi(t)\big) : \big(\bar{\pi}_\tau(t) - \pi(t)\big) + \frac{\kappa_1}{2}\big|\nabla\bar{\pi}_\tau(t) - \nabla\pi(t)\big|^2\,\mathrm{d}x \\
&\quad \le \int_\Omega -\mathbb{C}(\underline{\zeta}_\tau(t))e_{\mathrm{el}}(t) : \big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) - \big(\mathbb{H}\pi(t) - \bar{\xi}_\tau(t)\big) : \big(\bar{\pi}_\tau(t) - \pi(t)\big) \\
&\qquad\qquad + \frac{\kappa_1}{2}\nabla\pi(t) \,\vdots\, \nabla\big(\bar{\pi}_\tau(t) - \pi(t)\big) - \bar{f}_\tau(t)\cdot\big(\bar{u}_\tau(t) - u(t)\big)\,\mathrm{d}x \\
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\end{aligned}
$$

$\Rightarrow\quad \bar{e}_{\mathrm{el},\tau}(t) \to e_{\mathrm{el}}(t) \quad \& \quad \bar{\pi}_\tau(t) \to \pi(t)$ strongly in $H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$

$\Rightarrow\quad e(\bar{u}_\tau(t)) = e(u_{\mathrm{Dir},\tau}(t)) + \bar{\pi}_\tau(t) + \bar{e}_{\mathrm{el},\tau}(t) \to e(u(t))$ strongly in $L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$

$\Rightarrow\quad \bar{u}_\tau(t) \to u(t)$ strongly in $H^1(\Omega;\mathbb{R}^d)$.

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Step 2: strong convergence in u and π:

the discrete momentum equilibrium $\mathrm{div}(\mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}) + \bar{g}_\tau = 0$

the discrete plastic flow-rule $\bar{\xi}_\tau + \mathbb{H}\bar{\pi}_\tau - \mathrm{dev}\,\bar{\sigma}_\tau = \kappa_1\Delta\bar{\pi}_\tau$ with $\bar{\sigma}_\tau = \mathbb{C}(\underline{\zeta}_\tau)\bar{e}_{\mathrm{el},\tau}$ and $\bar{\xi}_\tau(t) \in \partial\delta_S^*(\frac{\partial\pi_\tau}{\partial t}(t))$ and $\bar{e}_{\mathrm{el},\tau} = e(\bar{u}_\tau - \bar{u}_{\mathrm{Dir},\tau}) - \bar{\pi}_\tau$ at time t with B.C. considered in the weak sense and tested respectively by $\bar{u}_\tau(t) - u(t)$ and $\bar{\pi}_\tau(t) - \pi(t)$.

$$\int_\Omega \mathbb{C}(\underline{\zeta}_\tau(t))\big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) : \big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) + \mathbb{H}\big(\bar{\pi}_\tau(t) - \pi(t)\big) : \big(\bar{\pi}_\tau(t) - \pi(t)\big) + \frac{\kappa_1}{2}\big|\nabla\bar{\pi}_\tau(t) - \nabla\pi(t)\big|^2\,\mathrm{d}x$$
$$\le \int_\Omega -\mathbb{C}(\underline{\zeta}_\tau(t))e_{\mathrm{el}}(t) : \big(\bar{e}_{\mathrm{el},\tau}(t) - e_{\mathrm{el}}(t)\big) - \big(\mathbb{H}\pi(t) - \bar{\xi}_\tau(t)\big) : \big(\bar{\pi}_\tau(t) - \pi(t)\big) + \frac{\kappa_1}{2}\nabla\pi(t) \vdots \nabla\big(\bar{\pi}_\tau(t) - \pi(t)\big) - \bar{f}_\tau(t)\cdot\big(\bar{u}_\tau(t) - u(t)\big)\,\mathrm{d}x - \int_{\Gamma_{\mathrm{Neu}}} \bar{g}_\tau(t)\cdot\big(\bar{u}_\tau(t) - u(t)\big)\mathrm{d}S \to 0.$$

Important note: $S \subset \mathbb{R}^{d\times d}_{\mathrm{dev}}$ bounded $\Rightarrow$ $(\bar{\xi}_\tau)_{\tau>0} \subset L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ bounded
$\Rightarrow$ relatively compact in $H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})^*$ (here $\nabla\pi$ needed!)
$\Rightarrow \int_\Omega \bar{\xi}_\tau(t) : (\bar{\pi}_\tau(t) - \pi(t))\,\mathrm{d}x \to 0$

Convergence:

Step 3: strong convergence in ζ by using the uniform-like monotonicity of

$$\zeta \mapsto \partial\delta_{[0,1]}(\zeta) - \kappa_2\,\mathrm{div}(|\nabla\zeta|^{r-2}\nabla\zeta) : W^{1,r}(\Omega) \rightrightarrows W^{1,r}(\Omega)^*.$$

The discrete damage flow rule:

$$\bar\xi_{\mathrm{dam},\tau} + \mathbb{C}'(\underline{\zeta}_\tau)\bar e_{\mathrm{el},\tau} : \bar e_{\mathrm{el},\tau} = \kappa_2\,\mathrm{div}\big(|\nabla\bar\zeta_\tau|^{r-2}\nabla\bar\zeta_\tau\big) - \bar\eta_\tau$$

$$\text{with some } \bar\xi_{\mathrm{dam},\tau} \in \partial\delta^*_{[-a,b]}\Big(\frac{\partial\zeta}{\partial t}_\tau\Big) \quad\text{and}\quad \bar\eta_\tau \in \partial\delta_{[0,1]}(\bar\zeta_\tau)$$

with the boundary condition $\nabla\bar\zeta_\tau\cdot\vec n = 0$. By Banach selection principle:

$$\bar\xi_{\mathrm{dam},\tau}(t) \to \xi_{\mathrm{dam}}(t) \quad \text{weakly* in} \quad L^\infty(\Omega)$$

for some t-dependent subsequence

here $\bar\xi_{\mathrm{dam},\tau}(t)$ valued in $[-b,a]$ with the (small) healing by $b<\infty$ exploited!

and

$$\mathbb{C}'(\underline{\zeta}_\tau(t))\bar e_{\mathrm{el},\tau}(t){:}\bar e_{\mathrm{el},\tau}(t) \to \mathbb{C}'(\zeta(t))e_{\mathrm{el}}(t){:}e_\tau(t) \quad \text{strongly in } L^1(\Omega)\subset W^{1,r}(\Omega)^*$$

already proved in Step 2 with now exploiting again the gradient concept of ζ

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$$\mathbb{C}'(\underline{\zeta}_\tau(t))\bar{e}_{\mathrm{el},\tau}(t):\bar{e}_{\mathrm{el},\tau}(t) \to \mathbb{C}'(\zeta(t))e_{\mathrm{el}}(t):e_\tau(t) \ \ \text{strongly in } L^1(\Omega) \subset W^{1,r}(\Omega)^*$$

already proved in Step 2 with now exploiting again the gradient concept of ζ.

the limit damage flow rule (at a time t):

$$\xi_{\rm dam}(t)+\mathbb{C}'(\zeta)e_{\rm el}(t):e_{\rm el}(t)=\kappa_2\,{\rm div}\big(|\nabla\zeta(t)|^{r-2}\nabla\zeta(t)\big)-\eta(t)$$
$$\text{with some}\ \ \eta(t)\in\partial\delta_{[0,1]}(\zeta(t)).$$

and, at this t, we can estimate

$$\kappa_2\limsup_{k\to\infty}\big(\|\nabla\bar\zeta_\tau(t)\|_{L^r(\Omega;\mathbb{R}^d)}^{r-1}-\|\nabla\zeta(t)\|_{L^r(\Omega;\mathbb{R}^d)}^{r-1}\big)\big(\|\nabla\bar\zeta_\tau(t)\|_{L^r(\Omega;\mathbb{R}^d)}-\|\nabla\zeta(t)\|_{L^r(\Omega;\mathbb{R}^d)}\big)$$
$$\le\limsup_{k\to\infty}\int_\Omega\kappa_2\big(|\nabla\bar\zeta_\tau(t)|^{r-2}\nabla\bar\zeta_\tau(t)-|\nabla\zeta(t)|^{r-2}\nabla\zeta(t)\big)\cdot\nabla(\bar\zeta_\tau(t)-\zeta(t))$$
$$+(\bar\eta_\tau(t)-\eta(t))(\bar\zeta_\tau(t)-\zeta(t))\,{\rm d}x$$
$$=\lim_{k\to\infty}\int_\Omega\mathbb{C}'(\underline{\zeta}_\tau(t))\bar e_{{\rm el},\tau}(t):\bar e_{{\rm el},\tau}(t)\big(\bar\zeta_\tau(t)-\zeta(t)\big)$$
$$-\kappa_2|\nabla\zeta(t)|^{r-2}\nabla\zeta(t)\cdot\nabla(\bar\zeta_\tau(t)-\zeta(t))-(\xi_{\rm dam}(t)+\eta(t))(\bar\zeta_\tau(t)-\zeta(t))\,{\rm d}x=0.$$

important: $\mathbb{C}'(\underline{\zeta}_\tau(t))\bar e_{{\rm el},\tau}(t):\bar e_{{\rm el},\tau}(t)\big(\bar\zeta_\tau(t)-\zeta(t)\big)\to0$ weakly in $L^1(\Omega)$,
or, in fact, even strongly in $L^1(\Omega)$ – again $r>d$ is exploited.

Thus $\|\nabla\bar\zeta_\tau(t)\|_{L^r(\Omega;\mathbb{R}^d)}\to\|\nabla\zeta(t)\|_{L^r(\Omega;\mathbb{R}^d)}$.
Uniform convexity of the space $L^r(\Omega;\mathbb{R}^d)$ $\Rightarrow$ $\nabla\bar\zeta_\tau(t)\to\nabla\zeta(t)$ strongly.

the t-dependent selection for $\xi_{{\rm dam},\tau}(t)\to\xi_{\rm dam}(t)$ in fact not needed

the limit damage flow rule (at a time t):

$$\xi_{\rm dam}(t) + \mathbb{C}'(\zeta)e_{\rm el}(t) : e_{\rm el}(t) = \kappa_2 \,{\rm div}\big(|\nabla\zeta(t)|^{r-2}\nabla\zeta(t)\big) - \eta(t)$$

with some $\eta(t) \in \partial\delta_{[0,1]}(\zeta(t))$.

and, at this t, we can estimate

$$\kappa_2 \limsup_{k\to\infty} \big(\|\nabla\bar\zeta_\tau(t)\|_{L^r(\Omega;\mathbb{R}^d)}^{r-1} - \|\nabla\zeta(t)\|_{L^r(\Omega;\mathbb{R}^d)}^{r-1}\big)\big(\|\nabla\bar\zeta_\tau(t)\|_{L^r(\Omega;\mathbb{R}^d)} - \|\nabla\zeta(t)\|_{L^r(\Omega;\mathbb{R}^d)}\big)$$

$$\le \limsup_{k\to\infty} \int_\Omega \kappa_2\big(|\nabla\bar\zeta_\tau(t)|^{r-2}\nabla\bar\zeta_\tau(t) - |\nabla\zeta(t)|^{r-2}\nabla\zeta(t)\big)\cdot\nabla(\bar\zeta_\tau(t)-\zeta(t)) + (\bar\eta_\tau(t)-\eta(t))(\bar\zeta_\tau(t)-\zeta(t))\,{\rm d}x$$

$$= \lim_{k\to\infty} \int_\Omega \mathbb{C}'(\underline{\zeta}_\tau(t))\bar e_{{\rm el},\tau}(t) : \bar e_{{\rm el},\tau}(t)\big(\bar\zeta_\tau(t)-\zeta(t)\big) - \kappa_2\,|\nabla\zeta(t)|^{r-2}\nabla\zeta(t)\cdot\nabla(\bar\zeta_\tau(t)-\zeta(t)) - (\xi_{\rm dam}(t)+\eta(t))(\bar\zeta_\tau(t)-\zeta(t))\,{\rm d}x = 0.$$

important: $\mathbb{C}'(\underline{\zeta}_\tau(t))\bar e_{{\rm el},\tau}(t) : \bar e_{{\rm el},\tau}(t)\big(\bar\zeta_\tau(t)-\zeta(t)\big) \to 0$ weakly in $L^1(\Omega)$,
or, in fact, even strongly in $L^1(\Omega)$ – again $r > d$ is exploited.

Thus $\|\nabla\bar\zeta_\tau(t)\|_{L^r(\Omega;\mathbb{R}^d)} \to \|\nabla\zeta(t)\|_{L^r(\Omega;\mathbb{R}^d)}$.
Uniform convexity of the space $L^r(\Omega;\mathbb{R}^d)$ $\Rightarrow$ $\nabla\bar\zeta_\tau(t) \to \nabla\zeta(t)$ strongly.

the t-dependent selection for $\bar\xi_{{\rm dam},\tau}(t) \to \xi_{\rm dam}(t)$ in fact not needed

the limit damage flow rule (at a time t):

$$\xi_{\rm dam}(t) + \mathbb{C}'(\zeta)e_{\rm el}(t) : e_{\rm el}(t) = \kappa_2\,{\rm div}\big(|\nabla\zeta(t)|^{r-2}\nabla\zeta(t)\big) - \eta(t) \qquad \text{with some } \ \eta(t) \in \partial\delta_{[0,1]}(\zeta(t)).$$

and, at this t, we can estimate

$$\begin{aligned}
&\kappa_2 \limsup_{k\to\infty} \Big(\|\nabla\bar\zeta_\tau(t)\|^{r-1}_{L^r(\Omega;\mathbb{R}^d)} - \|\nabla\zeta(t)\|^{r-1}_{L^r(\Omega;\mathbb{R}^d)}\Big)\Big(\|\nabla\bar\zeta_\tau(t)\|_{L^r(\Omega;\mathbb{R}^d)} - \|\nabla\zeta(t)\|_{L^r(\Omega;\mathbb{R}^d)}\Big)\\
&\le \limsup_{k\to\infty} \int_\Omega \kappa_2\big(|\nabla\bar\zeta_\tau(t)|^{r-2}\nabla\bar\zeta_\tau(t) - |\nabla\zeta(t)|^{r-2}\nabla\zeta(t)\big)\cdot\nabla\big(\bar\zeta_\tau(t){-}\zeta(t)\big)\\
&\qquad\qquad + (\bar\eta_\tau(t){-}\eta(t))(\bar\zeta_\tau(t){-}\zeta(t))\,{\rm d}x\\
&= \lim_{k\to\infty}\int_\Omega \mathbb{C}'(\underline{\zeta}_\tau(t))\bar e_{{\rm el},\tau}(t) : \bar e_{{\rm el},\tau}(t)\big(\bar\zeta_\tau(t){-}\zeta(t)\big)\\
&\qquad - \kappa_2\,|\nabla\zeta(t)|^{r-2}\nabla\zeta(t)\cdot\nabla\big(\bar\zeta_\tau(t){-}\zeta(t)\big) - \big(\xi_{\rm dam}(t)+\eta(t)\big)\big(\bar\zeta_\tau(t){-}\zeta(t)\big)\,{\rm d}x = 0.
\end{aligned}$$

important: $\mathbb{C}'(\underline{\zeta}_\tau(t))\bar e_{{\rm el},\tau}(t) : \bar e_{{\rm el},\tau}(t)\big(\bar\zeta_\tau(t){-}\zeta(t)\big) \to 0$ weakly in $L^1(\Omega)$,
or, in fact, even strongly in $L^1(\Omega)$ – again $r > d$ is exploited.

Thus $\|\nabla\bar\zeta_\tau(t)\|_{L^r(\Omega;\mathbb{R}^d)} \to \|\nabla\zeta(t)\|_{L^r(\Omega;\mathbb{R}^d)}$.
Uniform convexity of the space $L^r(\Omega;\mathbb{R}^d)$ $\Rightarrow$ $\nabla\bar\zeta_\tau(t) \to \nabla\zeta(t)$ strongly.
the t-dependent selection for $\bar\xi_{{\rm dam},\tau}(t) \to \xi_{\rm dam}(t)$ in fact not needed.

Convergence:

Step 4: Limit passage in the discrete local solution is then easy:

Equilibrium of displacements:

$$\forall t\in I: \quad \partial_u\mathcal{E}\big(t_\tau,\bar u_\tau(t),\bar\pi_\tau(t),\underline{\zeta}_\tau(t)\big)=0 \qquad \text{with } t_\tau:=\min\{k\tau\ge t;\ k\in\mathbb{N}\},$$

two separate semi-stability conditions for $\bar\zeta_\tau$ and $\bar\pi_\tau$:

$$\begin{aligned}\forall t\in I\ \forall\widetilde\pi\in H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}):\quad &\mathcal{E}\big(t_\tau,\bar u_\tau(t),\bar\pi_\tau(t),\underline{\zeta}_\tau(t)\big)\\ &\le\mathcal{E}\big(t_\tau,\bar u_\tau(t),\widetilde\pi,\underline{\zeta}_\tau(t)\big)+\mathcal{R}_1\big(\widetilde\pi-\bar\pi_\tau(t)\big),\\ \forall t\in I\ \forall\widetilde\zeta\in W^{1,r}(\Omega),\ 0\le\widetilde\zeta\le1:\ &\mathcal{E}\big(t_\tau,\bar u_\tau(t),\bar\pi_\tau(t),\bar\zeta_\tau(t)\big)\\ &\le\mathcal{E}\big(t_\tau,\bar u_\tau(t),\widetilde\zeta,\bar\pi_\tau(t)\big)+\mathcal{R}_2\big(\widetilde\zeta-\bar\zeta_\tau(t)\big),\end{aligned}$$

and the energy (im)balance ($\forall 0\le t_1<t_2\le T,\ t_i=k_i\tau,\ k_i\in\mathbb{N}$):

$$\begin{aligned}&\mathcal{E}\big(t_2,\bar u_\tau(t_2),\bar\pi_\tau(t_2),\bar\zeta_\tau(t_2)\big)+\mathrm{Var}_{\mathcal{R}_1}\big(\bar\pi_\tau;[t_1,t_2]\big)+\mathrm{Var}_{\mathcal{R}_2}\big(\bar\zeta_\tau;[t_1,t_2]\big)\\ &\qquad\le\mathcal{E}\big(t_1,\bar u_\tau(t_1),\bar\pi_\tau(t_1),\bar\zeta_\tau(t_1)\big)+\int_{t_1}^{t_2}\mathcal{E}_t'\big(t,\bar u_\tau(t),\bar\pi(t),\bar\zeta(t)\big)\,\mathrm{d}t.\end{aligned}$$

A physically-justified attempt :

a small ("vanishing" in the limit) viscosity in (u,π) or in ζ:

$$\varepsilon_1\partial\mathcal{R}_1\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)+\partial_u\mathcal{E}(t,u,z)\ni 0 \qquad \text{and} \qquad \varepsilon_2\partial\mathcal{R}_2\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)+\partial\mathcal{R}\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)+\partial_z\mathcal{E}(t,u,z)\ni 0$$

with $z=(\xi,\pi)$ and $\mathcal{R}_1\geq 0$ and $\mathcal{R}_2\geq 0$ convex quadratic.

Here: $\mathcal{R}_1\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)=\int_\Omega\frac{1}{2}\mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big):e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x$ and e.g. $\mathcal{R}_2\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)=\int_\Omega\frac{1}{2}\Big|\frac{\partial\pi}{\partial t}\Big|^2\,\mathrm{d}S$.

Again, semi-implicit time discretisation works efficiently. In the limit $\tau\to 0$:
The energy conservation (if $\mathcal{R}_1>0$ or $\mathcal{R}_2>0$) for $(u_\varepsilon,z_\varepsilon)$ with $\varepsilon:=(\varepsilon_1,\varepsilon_2)$:

$$\mathcal{E}(t_2,u_\varepsilon(t_2),z_\varepsilon(t_2))+\mathrm{Var}_{\mathcal{R}}(z_\varepsilon;[t_1,t_2])+\int_{t_1}^{t_2}2\varepsilon_1\mathcal{R}_1\Big(\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}t}\Big)+2\varepsilon_2\mathcal{R}_2\Big(\frac{\mathrm{d}z_\varepsilon}{\mathrm{d}t}\Big)\,\mathrm{d}t$$
$$=\mathcal{E}(t_1,u_\varepsilon(t_1),z_\varepsilon(t_1))+\int_{t_1}^{t_2}\mathcal{E}_t'(t,u_\varepsilon(t),z_\varepsilon(t))\,\mathrm{d}t.$$

In the vanishing-viscosity limit for $\varepsilon\to 0$ (as subsequences) $\Rightarrow$ "defect measure" μ

$$2\varepsilon_1\mathcal{R}_1\Big(\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}t}\Big)+2\varepsilon_2\mathcal{R}_2\Big(\frac{\mathrm{d}z_\varepsilon}{\mathrm{d}t}\Big)\to\mu\geq 0 \qquad \text{weakly* as a measure on } [0,T].$$

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Here: $\mathcal{R}_1\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)=\int_\Omega\frac{1}{2}\mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big){:}e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x$ and e.g. $\mathcal{R}_2\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)=\int_\Omega\frac{1}{2}\Big|\frac{\partial\pi}{\partial t}\Big|^2\,\mathrm{d}S$.

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$$=\mathcal{E}(t_1,u_\varepsilon(t_1),z_\varepsilon(t_1))+\int_{t_1}^{t_2}\mathcal{E}_t'(t,u_\varepsilon(t),z_\varepsilon(t))\,\mathrm{d}t.$$

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with $z=(\xi,\pi)$ and $\mathcal{R}_1\ge 0$ and $\mathcal{R}_2\ge 0$ convex quadratic.

Here: $\mathcal{R}_1\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)=\int_\Omega\frac12\mathbb{D}e\Big(\frac{\partial u}{\partial t}\Big):e\Big(\frac{\partial u}{\partial t}\Big)\,\mathrm{d}x$ and e.g. $\mathcal{R}_2\Big(\frac{\mathrm{d}z}{\mathrm{d}t}\Big)=\int_\Omega\frac12\Big|\frac{\partial\pi}{\partial t}\Big|^2\,\mathrm{d}S.$

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$$2\varepsilon_1\mathcal{R}_1\Big(\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}t}\Big)+2\varepsilon_2\mathcal{R}_2\Big(\frac{\mathrm{d}z_\varepsilon}{\mathrm{d}t}\Big)\to\mu\ge 0 \qquad \text{weakly* as a measure on } [0,T].$$

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with $z=(\xi,\pi)$ and $\mathcal{R}_1\geq 0$ and $\mathcal{R}_2\geq 0$ convex quadratic.

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$$\mathcal{E}(t_2,u_\varepsilon(t_2),z_\varepsilon(t_2))+\mathrm{Var}_{\mathcal{R}}(z_\varepsilon;[t_1,t_2])+\int_{t_1}^{t_2}2\varepsilon_1\mathcal{R}_1\Big(\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}t}\Big)+2\varepsilon_2\mathcal{R}_2\Big(\frac{\mathrm{d}z_\varepsilon}{\mathrm{d}t}\Big)\,\mathrm{d}t$$
$$=\mathcal{E}(t_1,u_\varepsilon(t_1),z_\varepsilon(t_1))+\int_{t_1}^{t_2}\mathcal{E}_t'(t,u_\varepsilon(t),z_\varepsilon(t))\,\mathrm{d}t.$$

In the vanishing-viscosity limit for $\varepsilon\to 0$ (as subsequences) $\Rightarrow$ "defect measure" μ

$$2\varepsilon_1\mathcal{R}_1\Big(\frac{\mathrm{d}u_\varepsilon}{\mathrm{d}t}\Big)+2\varepsilon_2\mathcal{R}_2\Big(\frac{\mathrm{d}z_\varepsilon}{\mathrm{d}t}\Big)\to\mu\geq 0 \qquad \text{weakly* as a measure on } [0,T].$$

Illustration of a vanishing (or rather very small) viscosity solution:

two nontrivial 2D symmetry-broken computational experiments with a surface damage (=delamination or debonding of an adhesive):

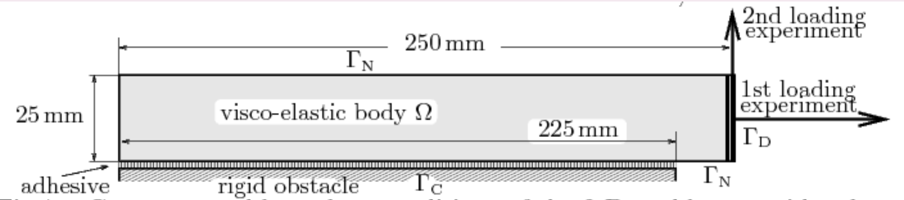

Fig. 7. Geometry and boundary conditions of the 2-D problem considered.

The defect measure distribution (the horizontal-loading experiment):

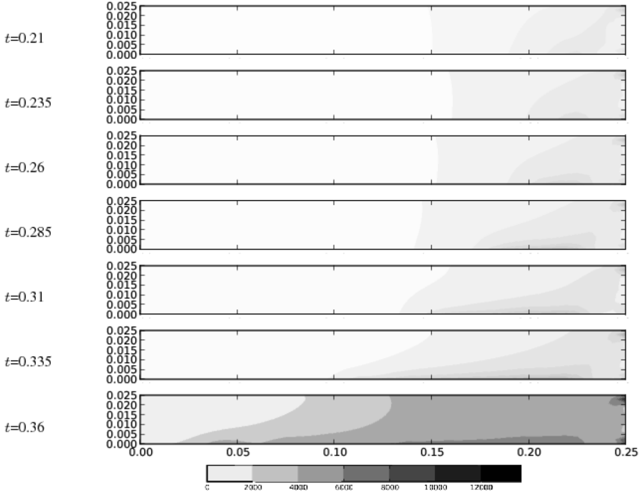

Fig. 9. The spatial distribution of the energy dissipated by (even very small) viscosity over the time interval $[0,t]$, i.e. $\int_0^t \chi\mathbb{C}e(\dot{u}_{\chi,\tau}){:}e(\dot{u}_{\chi,\tau})\,\mathrm{d}t$ depicted in a gray scale at 6 selected time instances as also used on Fig. 8.

(BEM implementation, calculations, visualisation: C.G.Panagiotopoulos, U. of Seville)

The defect measure distribution (the vertical-loading experiment):

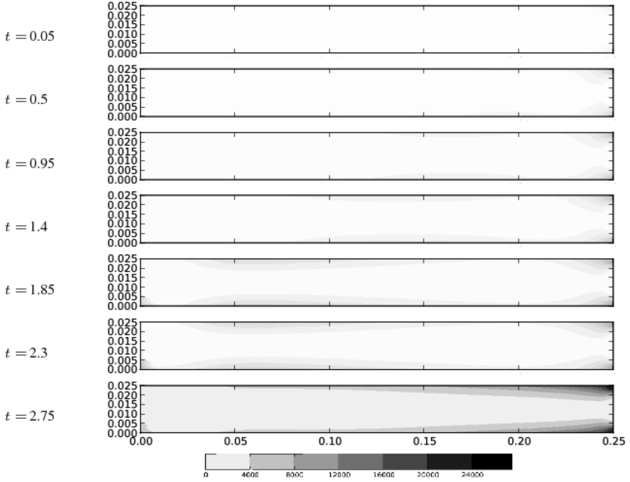

Fig. 12. The spatial distribution of the energy dissipated by viscosity over $[0,t]$, i.e. $\int_0^t \chi\mathbb{C}e(\dot{u}_{\chi,\tau}){:}e(\dot{u}_{\chi,\tau})\,\mathrm{d}t$ depicted at 6 selected time instances as on Fig. 11. Surprising tendency to a symmetry even under nonsymmetry loading can be observed.

(BEM implementation, calculations, visualisation: C.G.Panagiotopoulos, U. of Seville)

Comparison on the 1st delamination experiment on the force response:

Left: a vanishing-viscosity solution
– in fact, a very small viscosity, energy (approximately) conserved.
Right: a maximally-dissipative local solution (by fractional-step algorithm).

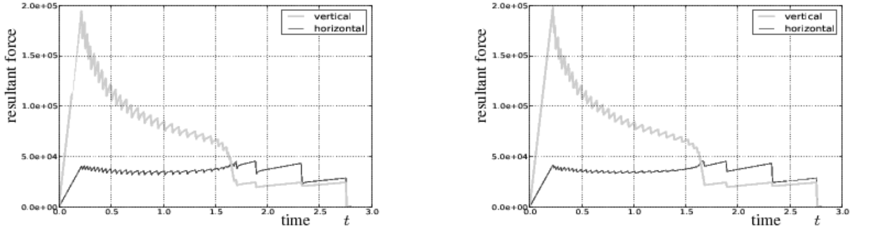

Fig. 13. Vertical and horizontal components of the reaction force on the Dirichlet loading (left) and its comparison with the simplified inviscid algorithm from Remark 4.2 (right), again showing a surprising match as on Figures 6 and 10.

(BEM implementation, calculations, visualisation: C.G.Panagiotopoulos, U. of Seville)

– a surprisingly good match of the mechanical response also in 2D simulations.
– a certain justification of the maximally-dissipative local sln concept.

Convergence: most important modifications in Steps 1-4:

Step 2: Strong convergence in u and π:

the "viscous" momentum equilibrium $\operatorname{div}(\varepsilon_1\mathbb{D}e(\frac{\partial u_\varepsilon}{\partial t}) + \mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}) + g = 0$
the "viscous" plastic flow-rule $\varepsilon_2\frac{\partial \pi_\varepsilon}{\partial t} + \xi_\varepsilon + \mathbb{H}\pi_\varepsilon - \operatorname{dev}\sigma_\varepsilon = \kappa_1\Delta\pi_\varepsilon$ with $\sigma_\varepsilon = \mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}$ and $\xi_\varepsilon \in \partial\delta_S^*(\frac{\partial \pi_\varepsilon}{\partial t})$ and $e_{\mathrm{el},\varepsilon} = e(u_\varepsilon - u_{\mathrm{Dir}}) - \pi_\varepsilon$ with B.C. considered in the weak sense and tested respectively by $u_\varepsilon - u$ and $\pi_\varepsilon - \pi$. Integrated over $[0,T]$ and using $\|e(\frac{\partial u_\varepsilon}{\partial t})\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})} = \mathscr{O}(1/\sqrt{\varepsilon_1})$ and $\|\frac{\partial \pi_\varepsilon}{\partial t}\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})} = \mathscr{O}(1/\sqrt{\varepsilon_2})$, it yields:

$$\int_Q \mathbb{C}(\zeta_\varepsilon)(e_{\mathrm{el},\varepsilon} - e_{\mathrm{el}}) : (e_{\mathrm{el},\varepsilon} - e_{\mathrm{el}}) + \mathbb{H}(\pi_\varepsilon - \pi) : (\pi_\varepsilon - \pi) + \frac{\kappa_1}{2}\left|\nabla\pi_\varepsilon - \nabla\pi\right|^2 \mathrm{d}x\mathrm{d}t$$
$$\leq \int_\Omega -\Big(\varepsilon_1\mathbb{D}e\Big(\frac{\partial u_\epsilon}{\partial t}\Big) + \mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el}}\Big) : (e_{\mathrm{el},\varepsilon} - e_{\mathrm{el}}) - \Big(\varepsilon_2\frac{\partial \pi_\varepsilon}{\partial t} + \mathbb{H}\pi - \xi_\varepsilon\Big) : (\pi_\varepsilon - \pi)$$
$$+ \frac{\kappa_1}{2}\nabla\pi : \nabla(\pi_\varepsilon - \pi) - f_\varepsilon\cdot(u_\varepsilon - u)\,\mathrm{d}x - \int_{\Gamma_{\mathrm{Neu}}} g(t)\cdot(u_\varepsilon - u)\mathrm{d}S\mathrm{d}t \to 0.$$

Convergence: most important modifications in Steps 1-4:

Step 2: Strong convergence in u and π:

the "viscous" momentum equilibrium $\mathrm{div}(\varepsilon_1\mathbb{D}e(\frac{\partial u_\varepsilon}{\partial t}) + \mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}) + g = 0$
the "viscous" plastic flow-rule $\varepsilon_2\frac{\partial \pi_\varepsilon}{\partial t} + \xi_\varepsilon + \mathbb{H}\pi_\varepsilon - \mathrm{dev}\,\sigma_\varepsilon = \kappa_1\Delta\pi_\varepsilon$ with $\sigma_\varepsilon = \mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}$ and $\xi_\varepsilon \in \partial\delta_S^*(\frac{\partial \pi_\varepsilon}{\partial t})$ and $e_{\mathrm{el},\varepsilon} = e(u_\varepsilon - u_{\mathrm{Dir}}) - \pi_\varepsilon$ with B.C. considered in the weak sense and tested respectively by $u_\varepsilon - u$ and $\pi_\varepsilon - \pi$. Integrated over $[0, T]$ and using $\|e(\frac{\partial u_\varepsilon}{\partial t})\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})} = \mathscr{O}(1/\sqrt{\varepsilon_1})$ and $\|\frac{\partial \pi_\varepsilon}{\partial t}\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})} = \mathscr{O}(1/\sqrt{\varepsilon_2})$, it yields:

$$\int_Q \mathbb{C}(\zeta_\varepsilon)\big(e_{\mathrm{el},\varepsilon} - e_{\mathrm{el}}\big) : \big(e_{\mathrm{el},\varepsilon} - e_{\mathrm{el}}\big) + \mathbb{H}\big(\pi_\varepsilon - \pi\big) : \big(\pi_\varepsilon - \pi\big) + \frac{\kappa_1}{2}\big|\nabla\pi_\varepsilon - \nabla\pi\big|^2\,\mathrm{d}x\mathrm{d}t$$
$$\leq \int_\Omega -\Big(\varepsilon_1\mathbb{D}e\big(\frac{\partial u_\epsilon}{\partial t}\big) + \mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el}}\Big) : \big(e_{\mathrm{el},\varepsilon} - e_{\mathrm{el}}\big) - \Big(\varepsilon_2\frac{\partial \pi_\varepsilon}{\partial t} + \mathbb{H}\pi - \xi_\varepsilon\Big) : \big(\pi_\varepsilon - \pi\big)$$
$$+ \frac{\kappa_1}{2}\nabla\pi : \nabla\big(\pi_\varepsilon - \pi\big) - f_\varepsilon\cdot(u_\varepsilon - u)\,\mathrm{d}x - \int_{\Gamma_{\mathrm{Neu}}} g(t)\cdot(u_\varepsilon - u)\mathrm{d}S\mathrm{d}t \to 0.$$

Convergence: most important modifications in Steps 1-4:

Step 2: Strong convergence in u and π:

the "viscous" momentum equilibrium $\mathrm{div}(\varepsilon_1 \mathbb{D} e(\frac{\partial u_\varepsilon}{\partial t}) + \mathbb{C}(\zeta_\varepsilon) e_{\mathrm{el},\varepsilon}) + g = 0$
the "viscous" plastic flow-rule $\varepsilon_2 \frac{\partial \pi_\varepsilon}{\partial t} + \xi_\varepsilon + \mathbb{H}\pi_\varepsilon - \mathrm{dev}\, \sigma_\varepsilon = \kappa_1 \Delta \pi_\varepsilon$ with $\sigma_\varepsilon = \mathbb{C}(\zeta_\varepsilon) e_{\mathrm{el},\varepsilon}$ and $\xi_\varepsilon \in \partial\delta_S^*(\frac{\partial \pi_\varepsilon}{\partial t})$ and $e_{\mathrm{el},\varepsilon} = e(u_\varepsilon - u_{\mathrm{Dir}}) - \pi_\varepsilon$ with B.C. considered in the weak sense and tested respectively by $u_\varepsilon - u$ and $\pi_\varepsilon - \pi$. Integrated over $[0, T]$ and using $\|e(\frac{\partial u_\varepsilon}{\partial t})\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})} = \mathscr{O}(1/\sqrt{\varepsilon_1})$ and $\|\frac{\partial \pi_\varepsilon}{\partial t}\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})} = \mathscr{O}(1/\sqrt{\varepsilon_2})$, it yields:

$$\int_Q \mathbb{C}(\zeta_\varepsilon)(e_{\mathrm{el},\varepsilon} - e_{\mathrm{el}}) : (e_{\mathrm{el},\varepsilon} - e_{\mathrm{el}}) + \mathbb{H}(\pi_\varepsilon - \pi) : (\pi_\varepsilon - \pi) + \frac{\kappa_1}{2}\big|\nabla\pi_\varepsilon - \nabla\pi\big|^2 \,\mathrm{d}x\mathrm{d}t$$
$$\leq \int_\Omega -\Big(\varepsilon_1 \mathbb{D} e\Big(\frac{\partial u_\epsilon}{\partial t}\Big) + \mathbb{C}(\zeta_\varepsilon) e_{\mathrm{el}}\Big) : (e_{\mathrm{el},\varepsilon} - e_{\mathrm{el}}) - \Big(\varepsilon_2 \frac{\partial \pi_\varepsilon}{\partial t} + \mathbb{H}\pi - \xi_\varepsilon\Big) : (\pi_\varepsilon - \pi)$$
$$+ \frac{\kappa_1}{2}\nabla\pi : \nabla(\pi_\varepsilon - \pi) - f_\varepsilon\cdot(u_\varepsilon - u)\,\mathrm{d}x - \int_{\Gamma_{\mathrm{Neu}}} g(t)\cdot(u_\varepsilon - u)\mathrm{d}S\mathrm{d}t \to 0.$$

$\Rightarrow$ $\forall_{\mathrm{a.a.}} t$: $e_{\mathrm{el},\varepsilon}(t) \to e_{\mathrm{el}}(t)$ & $\pi_\varepsilon(t) \to \pi(t)$ strongly in $H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$

$\Rightarrow$ $\forall_{\mathrm{a.a.}} t$: $e(u_\varepsilon(t)) = e(u_{\mathrm{Dir}}(t)) + \pi_\varepsilon(t) + e_{\mathrm{el},\varepsilon}(t) \to e(u(t))$ strongly in $L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$

$\Rightarrow$ $u_\varepsilon(t) \to u(t)$ strongly in $H^1(\Omega;\mathbb{R}^d)$.

Convergence: most important modifications in Steps 1-4:

Step 2: Strong convergence in u and π:

the "viscous" momentum equilibrium $\mathrm{div}(\varepsilon_1\mathbb{D}e(\frac{\partial u_\varepsilon}{\partial t})+\mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon})+g=0$
the "viscous" plastic flow-rule $\varepsilon_2\frac{\partial\pi_\varepsilon}{\partial t}+\xi_\varepsilon+\mathbb{H}\pi_\varepsilon-\mathrm{dev}\,\sigma_\varepsilon=\kappa_1\Delta\pi_\varepsilon$ with $\sigma_\varepsilon=\mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}$ and $\xi_\varepsilon\in\partial\delta_S^*(\frac{\partial\pi_\varepsilon}{\partial t})$ and $e_{\mathrm{el},\varepsilon}=e(u_\varepsilon-u_{\mathrm{Dir}})-\pi_\varepsilon$ with B.C. considered in the weak sense and tested respectively by $u_\varepsilon-u$ and $\pi_\varepsilon-\pi$. Integrated over $[0,T]$ and using $\|e(\frac{\partial u_\varepsilon}{\partial t})\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})}=\mathscr{O}(1/\sqrt{\varepsilon_1})$ and $\|\frac{\partial\pi_\varepsilon}{\partial t}\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})}=\mathscr{O}(1/\sqrt{\varepsilon_2})$, it yields:

$$\int_Q\mathbb{C}(\zeta_\varepsilon)(e_{\mathrm{el},\varepsilon}-e_{\mathrm{el}}):(e_{\mathrm{el},\varepsilon}-e_{\mathrm{el}})+\mathbb{H}(\pi_\varepsilon-\pi):(\pi_\varepsilon-\pi)+\frac{\kappa_1}{2}\big|\nabla\pi_\varepsilon-\nabla\pi\big|^2\,\mathrm{d}x\mathrm{d}t$$
$$\le\int_\Omega-\Big(\varepsilon_1\mathbb{D}e\Big(\frac{\partial u_\epsilon}{\partial t}\Big)+\mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el}}\Big):(e_{\mathrm{el},\varepsilon}-e_{\mathrm{el}})-\Big(\varepsilon_2\frac{\partial\pi_\varepsilon}{\partial t}+\mathbb{H}\pi-\xi_\varepsilon\Big):(\pi_\varepsilon-\pi)$$
$$+\frac{\kappa_1}{2}\nabla\pi:\nabla(\pi_\varepsilon-\pi)-f_\varepsilon\cdot(u_\varepsilon-u)\,\mathrm{d}x-\int_{\Gamma_{\mathrm{Neu}}}g(t)\cdot(u_\varepsilon-u)\mathrm{d}S\mathrm{d}t\to0.$$

Important note: $S\subset\mathbb{R}^{d\times d}_{\mathrm{dev}}$ bounded $\Rightarrow$ $(\xi_\varepsilon)_{\varepsilon>0}\subset L^\infty(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ bounded
& $\pi_\varepsilon\to\pi$ in $L^1(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ by Aubin-Lions' lemma (here $\nabla\pi$ needed!)
$\Rightarrow\int_Q\xi_\varepsilon:(\pi_\varepsilon-\pi)\,\mathrm{d}x\mathrm{d}t\to0.$

Convergence: most important modifications in Steps 1-4:

Step 2: Strong convergence in u and π:

the "viscous" momentum equilibrium $\mathrm{div}(\varepsilon_1\mathbb{D}e(\frac{\partial u_\varepsilon}{\partial t}) + \mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}) + g = 0$
the "viscous" plastic flow-rule $\varepsilon_2\frac{\partial \pi_\varepsilon}{\partial t} + \xi_\varepsilon + \mathbb{H}\pi_\varepsilon - \mathrm{dev}\,\sigma_\varepsilon = \kappa_1\Delta\pi_\varepsilon$ with $\sigma_\varepsilon = \mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}$ and $\xi_\varepsilon \in \partial\delta^*_S(\frac{\partial \pi_\varepsilon}{\partial t})$ and $e_{\mathrm{el},\varepsilon} = e(u_\varepsilon{-}u_{\mathrm{Dir}}) - \pi_\varepsilon$ with B.C. considered in the weak sense and tested respectively by $u_\varepsilon{-}u$ and $\pi_\varepsilon{-}\pi$. Integrated over $[0,T]$ and using $\|e(\frac{\partial u_\varepsilon}{\partial t})\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})} = \mathscr{O}(1/\sqrt{\varepsilon_1})$ and $\|\frac{\partial \pi_\varepsilon}{\partial t}\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})} = \mathscr{O}(1/\sqrt{\varepsilon_2})$, it yields:

$$\int_Q \mathbb{C}(\zeta_\varepsilon)\big(e_{\mathrm{el},\varepsilon}{-}e_{\mathrm{el}}\big):\big(e_{\mathrm{el},\varepsilon}{-}e_{\mathrm{el}}\big) + \mathbb{H}\big(\pi_\varepsilon{-}\pi\big):\big(\pi_\varepsilon{-}\pi\big) + \frac{\kappa_1}{2}\big|\nabla\pi_\varepsilon{-}\nabla\pi\big|^2\,\mathrm{d}x\mathrm{d}t$$
$$\leq \int_\Omega -\Big(\varepsilon_1\mathbb{D}e\big(\frac{\partial u_\epsilon}{\partial t}\big){+}\mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el}}\Big):\big(e_{\mathrm{el},\varepsilon}{-}e_{\mathrm{el}}\big) - \Big(\varepsilon_2\frac{\partial \pi_\varepsilon}{\partial t}{+}\mathbb{H}\pi{-}\xi_\varepsilon\Big):\big(\pi_\varepsilon{-}\pi\big)$$
$$+\frac{\kappa_1}{2}\nabla\pi:\nabla\big(\pi_\varepsilon{-}\pi\big) - f_\varepsilon\cdot(u_\varepsilon{-}u)\,\mathrm{d}x - \int_{\Gamma_{\mathrm{Neu}}} g(t)\cdot(u_\varepsilon{-}u)\mathrm{d}S\mathrm{d}t \to 0.$$

Strong convergence in ζ in $W^{1,r}(\Omega)$ even for all t the same as before.

Step 4: Limit passage in the momentum equilibrium

$\mathrm{div}(\varepsilon_1\mathbb{D}e(\frac{\partial u_\varepsilon}{\partial t})+\mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon})+g=0$ towards $\mathrm{div}(\mathbb{C}(\zeta)e_{\mathrm{el}})+g=0$
easy again due to $\|e(\frac{\partial u_\varepsilon}{\partial t})\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})}=\mathscr{O}(1/\sqrt{\varepsilon}_1)$.

Limit passage in the plastic flow rule:
$\varepsilon_2\frac{\partial\pi_\varepsilon}{\partial t}+\xi_\varepsilon+\mathbb{H}\pi_\varepsilon-\mathrm{dev}\,\sigma_\varepsilon=\kappa_1\Delta\pi_\varepsilon$ with $\sigma_\varepsilon=\mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}$ and
$\xi_\varepsilon\in\partial\delta_S^*(\frac{\partial\pi_\varepsilon}{\partial t})$ and $e_{\mathrm{el},\varepsilon}=e(u_\varepsilon-u_{\mathrm{Dir}})-\pi_\varepsilon$ in the weak form:

$$\int_Q\underbrace{\varepsilon_2\Big|\frac{\partial\pi_\varepsilon}{\partial t}\Big|^2}_{\geq0}+\delta_S^*\Big(\frac{\partial\pi_\varepsilon}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$
$$\leq\int_Q(\mathbb{H}\pi_\varepsilon-\mathrm{dev}\,\sigma_\varepsilon):(\widetilde\pi-\pi_\varepsilon)+\kappa_1\nabla\pi_\varepsilon\,\vdots\,\nabla(\widetilde\pi-\pi_\varepsilon)+\underbrace{\varepsilon_2|\widetilde\pi|^2}_{\to0}+\delta_S^*(\widetilde\pi)\,\mathrm{d}x\mathrm{d}t$$

for any $\widetilde\pi$. After $\varepsilon\to0$, use 1-homogeneity of $\mathcal{R}$ + convexity of $\mathcal{E}(t,\cdot)$ to the get semi-stability.

Limit passage in the damage flow rule the same (no viscosity in ζ), and limit passage in the energy balance: strong convergence in $\mathcal{E}(t,\cdot)$ + weak* convergence to the defect measure μ on $\bar{Q}$.

Step 4: Limit passage in the momentum equilibrium

$\mathrm{div}(\varepsilon_1\mathbb{D}e(\frac{\partial u_\varepsilon}{\partial t}) + \mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}) + g = 0$ towards $\mathrm{div}(\mathbb{C}(\zeta)e_{\mathrm{el}}) + g = 0$ easy again due to $\|e(\frac{\partial u_\varepsilon}{\partial t})\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})} = \mathscr{O}(1/\sqrt{\varepsilon_1})$.

Limit passage in the plastic flow rule:

$\varepsilon_2\frac{\partial\pi_\varepsilon}{\partial t} + \xi_\varepsilon + \mathbb{H}\pi_\varepsilon - \mathrm{dev}\,\sigma_\varepsilon = \kappa_1\Delta\pi_\varepsilon$ with $\sigma_\varepsilon = \mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}$ and $\xi_\varepsilon \in \partial\delta_S^*(\frac{\partial\pi_\varepsilon}{\partial t})$ and $e_{\mathrm{el},\varepsilon} = e(u_\varepsilon - u_{\mathrm{Dir}}) - \pi_\varepsilon$ in the weak form:

$$\int_Q \underbrace{\varepsilon_2\Big|\frac{\partial\pi_\varepsilon}{\partial t}\Big|^2}_{\geq 0} + \delta_S^*\Big(\frac{\partial\pi_\varepsilon}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$
$$\leq \int_Q (\mathbb{H}\pi_\varepsilon - \mathrm{dev}\,\sigma_\varepsilon):(\widetilde\pi - \pi_\varepsilon) + \kappa_1\nabla\pi_\varepsilon \,\vdots\, \nabla(\widetilde\pi - \pi_\varepsilon) + \underbrace{\varepsilon_2|\widetilde\pi|^2}_{\to 0} + \delta_S^*(\widetilde\pi)\,\mathrm{d}x\mathrm{d}t$$

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easy again due to $\|e(\frac{\partial u_\varepsilon}{\partial t})\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})} = \mathscr{O}(1/\sqrt{\varepsilon_1})$.

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$\varepsilon_2\frac{\partial \pi_\varepsilon}{\partial t} + \xi_\varepsilon + \mathbb{H}\pi_\varepsilon - \mathrm{dev}\,\sigma_\varepsilon = \kappa_1\Delta\pi_\varepsilon$ with $\sigma_\varepsilon = \mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}$ and $\xi_\varepsilon \in \partial\delta^*_S(\frac{\partial \pi_\varepsilon}{\partial t})$ and $e_{\mathrm{el},\varepsilon} = e(u_\varepsilon - u_{\mathrm{Dir}}) - \pi_\varepsilon$ in the weak form:

$$\int_Q \underbrace{\varepsilon_2\Big|\frac{\partial \pi_\varepsilon}{\partial t}\Big|^2}_{\geq 0} + \delta^*_S\Big(\frac{\partial \pi_\varepsilon}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$

$$\leq \int_Q (\mathbb{H}\pi_\varepsilon - \mathrm{dev}\,\sigma_\varepsilon):(\widetilde{\pi} - \pi_\varepsilon) + \kappa_1\nabla\pi_\varepsilon \vdots \nabla(\widetilde{\pi} - \pi_\varepsilon) + \underbrace{\varepsilon_2|\widetilde{\pi}|^2}_{\to 0} + \delta^*_S(\widetilde{\pi})\,\mathrm{d}x\mathrm{d}t$$

for any $\widetilde{\pi}$. After $\varepsilon \to 0$, use 1-homogeneity of $\mathcal{R}$ + convexity of $\mathcal{E}(t,\cdot)$ to the get semi-stability.

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$\mathrm{div}(\varepsilon_1\mathbb{D}e(\frac{\partial u_\varepsilon}{\partial t})+\mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon})+g=0$ towards $\mathrm{div}(\mathbb{C}(\zeta)e_{\mathrm{el}})+g=0$
easy again due to $\|e(\frac{\partial u_\varepsilon}{\partial t})\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})}=\mathscr{O}(1/\sqrt{\varepsilon_1})$.

Limit passage in the plastic flow rule:
$\varepsilon_2\frac{\partial \pi_\varepsilon}{\partial t}+\xi_\varepsilon+\mathbb{H}\pi_\varepsilon-\mathrm{dev}\,\sigma_\varepsilon=\kappa_1\Delta\pi_\varepsilon$ with $\sigma_\varepsilon=\mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}$ and
$\xi_\varepsilon\in\partial\delta_S^*(\frac{\partial \pi_\varepsilon}{\partial t})$ and $e_{\mathrm{el},\varepsilon}=e(u_\varepsilon-u_{\mathrm{Dir}})-\pi_\varepsilon$ in the weak form:

$$\int_Q\underbrace{\varepsilon_2\Big|\frac{\partial\pi_\varepsilon}{\partial t}\Big|^2}_{\geq 0}+\delta_S^*\Big(\frac{\partial\pi_\varepsilon}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$

$$\leq\int_Q(\mathbb{H}\pi_\varepsilon-\mathrm{dev}\,\sigma_\varepsilon):(\widetilde{\pi}-\pi_\varepsilon)+\kappa_1\nabla\pi_\varepsilon\vdots\nabla(\widetilde{\pi}-\pi_\varepsilon)+\underbrace{\varepsilon_2|\widetilde{\pi}|^2}_{\to 0}+\delta_S^*(\widetilde{\pi})\,\mathrm{d}x\mathrm{d}t$$

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easy again due to $\|e(\frac{\partial u_\varepsilon}{\partial t})\|_{L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})}=\mathscr{O}(1/\sqrt{\varepsilon}_1)$.

Limit passage in the plastic flow rule:
$\varepsilon_2\frac{\partial\pi_\varepsilon}{\partial t}+\xi_\varepsilon+\mathbb{H}\pi_\varepsilon-\mathrm{dev}\,\sigma_\varepsilon=\kappa_1\Delta\pi_\varepsilon$ with $\sigma_\varepsilon=\mathbb{C}(\zeta_\varepsilon)e_{\mathrm{el},\varepsilon}$ and
$\xi_\varepsilon\in\partial\delta_S^*(\frac{\partial\pi_\varepsilon}{\partial t})$ and $e_{\mathrm{el},\varepsilon}=e(u_\varepsilon-u_{\mathrm{Dir}})-\pi_\varepsilon$ in the weak form:

$$\int_Q\underbrace{\varepsilon_2\Big|\frac{\partial\pi_\varepsilon}{\partial t}\Big|^2}_{\ge 0}+\delta_S^*\Big(\frac{\partial\pi_\varepsilon}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t$$

$$\le\int_Q(\mathbb{H}\pi_\varepsilon-\mathrm{dev}\,\sigma_\varepsilon):(\widetilde{\pi}-\pi_\varepsilon)+\kappa_1\nabla\pi_\varepsilon\,\vdots\,\nabla(\widetilde{\pi}-\pi_\varepsilon)+\underbrace{\varepsilon_2|\widetilde{\pi}|^2}_{\to 0}+\delta_S^*(\widetilde{\pi})\,\mathrm{d}x\mathrm{d}t$$

for any $\widetilde{\pi}$. After $\varepsilon\to 0$, use 1-homogeneity of $\mathcal{R}$ + convexity of $\mathcal{E}(t,\cdot)$ to the get semi-stability.

Limit passage in the damage flow rule the same (no viscosity in ζ), and limit passage in the energy balance: strong convergence in $\mathcal{E}(t,.)$ + weak* convergence to the defect measure μ on $\bar{Q}$.

Approximate maximum-dissipation principle (AMDP): Recall:

$$\int_0^T \bar\xi_\tau(t)\mathrm{d}\bar z_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R}(\bar z_\tau;[0,T]) \qquad \text{with} \qquad \bar\xi_\tau(t) \in -\partial_z \bar{\mathcal E}_\tau(t,\bar u_\tau(t),\bar z_\tau(t))$$

where we can explicitly evaluate the left-hand side as

$$\int_0^T \bar\xi_\tau(t)\mathrm{d}\bar z_\tau(t) = \sum_{k=1}^{T/\tau}\langle \xi_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}\rangle \quad \text{with} \quad \xi_\tau^{k-1} \in -\partial_z \mathcal E((k-1)\tau, u_\tau^{k-1}, z_\tau^{k-1}).$$

Here (denoting $z=(\pi,\zeta)$):

$$\int_0^T \bar\xi_\tau(t)\mathrm{d}\bar z_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R}(\bar\zeta_\tau,\bar\pi_\tau;[0,T]) \quad \text{for some}$$

$$\bar\xi_\tau(t) \in -\partial_\zeta \bar{\mathcal E}_\tau(t,\bar u_\tau(t),\bar\pi_\tau(t),\bar\zeta_\tau(t)) \times \big\{ -\big[\bar{\mathcal E}_\tau\big]'_\pi(t,\bar u_\tau(t),\bar\pi_\tau(t),\underline{\zeta}_\tau(t))\big\},$$

or written for plasticity and damage separately:

$$\int_0^T \bar\xi_{\mathrm{plast},\tau}(t)\mathrm{d}\bar\pi_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R_1}(\bar\pi_\tau;[0,T])$$

$$\text{for} \quad \bar\xi_{\mathrm{plast},\tau}(t) = -\big[\bar{\mathcal E}_\tau\big]'_\pi(t,\bar u_\tau(t),\bar\pi_\tau(t),\underline{\zeta}_\tau(t)),$$

$$\int_0^T \bar\xi_{\mathrm{dam},\tau}(t)\mathrm{d}\bar\zeta_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R_2}(\bar\zeta_\tau;[0,T])$$

$$\text{for some} \quad \bar\xi_{\mathrm{dam},\tau}(t) \in -\partial_\zeta \bar{\mathcal E}_\tau(t,\bar u_\tau(t),\bar\pi_\tau(t),\bar\zeta_\tau(t)).$$

Approximate maximum-dissipation principle (AMDP): Recall:

$$\int_0^T \bar\xi_\tau(t)\mathrm{d}\bar z_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R}(\bar z_\tau;[0,T]) \qquad \text{with} \qquad \bar\xi_\tau(t)\in -\partial_z\bar{\mathcal E}_\tau(t,\bar u_\tau(t),\bar z_\tau(t))$$

where we can explicitly evaluate the left-hand side as

$$\int_0^T \bar\xi_\tau(t)\mathrm{d}\bar z_\tau(t) = \sum_{k=1}^{T/\tau}\langle \xi_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}\rangle \quad \text{with} \quad \xi_\tau^{k-1}\in -\partial_z\mathcal E((k-1)\tau, u_\tau^{k-1}, z_\tau^{k-1}).$$

Here (denoting $z=(\pi,\zeta)$):

$$\int_0^T \bar\xi_\tau(t)\mathrm{d}\bar z_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R}(\bar\zeta_\tau,\bar\pi_\tau;[0,T]) \quad \text{for some}$$

$$\bar\xi_\tau(t)\in -\partial_\zeta\bar{\mathcal E}_\tau(t,\bar u_\tau(t),\bar\pi_\tau(t),\bar\zeta_\tau(t))\times\big\{-\big[\bar{\mathcal E}_\tau\big]'_\pi(t,\bar u_\tau(t),\bar\pi_\tau(t),\underline\zeta_\tau(t))\big\},$$

or written for plasticity and damage separately:

$$\int_0^T \bar\xi_{\mathrm{plast},\tau}(t)\mathrm{d}\bar\pi_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R_1}(\bar\pi_\tau;[0,T])$$

$$\text{for} \quad \bar\xi_{\mathrm{plast},\tau}(t) = -\big[\bar{\mathcal E}_\tau\big]'_\pi(t,\bar u_\tau(t),\bar\pi_\tau(t),\underline\zeta_\tau(t)),$$

$$\int_0^T \bar\xi_{\mathrm{dam},\tau}(t)\mathrm{d}\bar\zeta_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R_2}(\bar\zeta_\tau;[0,T])$$

$$\text{for some} \quad \bar\xi_{\mathrm{dam},\tau}(t)\in -\partial_\zeta\bar{\mathcal E}_\tau(t,\bar u_\tau(t),\bar\pi_\tau(t),\bar\zeta_\tau(t)).$$

Approximate maximum-dissipation principle (AMDP): Recall:

$$\int_0^T \bar\xi_\tau(t)\mathrm{d}\bar z_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R}(\bar z_\tau;[0,T]) \qquad \text{with} \qquad \bar\xi_\tau(t) \in -\partial_z\bar{\mathcal E}_\tau(t,\bar u_\tau(t),\bar z_\tau(t))$$

where we can explicitly evaluate the left-hand side as

$$\int_0^T \bar\xi_\tau(t)\mathrm{d}\bar z_\tau(t) = \sum_{k=1}^{T/\tau}\langle \xi_\tau^{k-1}, z_\tau^k - z_\tau^{k-1}\rangle \quad \text{with} \quad \xi_\tau^{k-1} \in -\partial_z\mathcal E((k-1)\tau, u_\tau^{k-1}, z_\tau^{k-1}).$$

Here (denoting $z=(\pi,\zeta)$):

$$\int_0^T \bar\xi_\tau(t)\mathrm{d}\bar z_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R}(\bar\zeta_\tau,\bar\pi_\tau;[0,T]) \quad \text{for some}$$

$$\bar\xi_\tau(t) \in -\partial_\zeta\bar{\mathcal E}_\tau(t,\bar u_\tau(t),\bar\pi_\tau(t),\bar\zeta_\tau(t)) \times \big\{-\big[\bar{\mathcal E}_\tau\big]'_\pi(t,\bar u_\tau(t),\bar\pi_\tau(t),\underline\zeta_\tau(t))\big\},$$

or written for plasticity and damage separately:

$$\int_0^T \bar\xi_{\mathrm{plast},\tau}(t)\mathrm{d}\bar\pi_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R_1}(\bar\pi_\tau;[0,T])$$

$$\text{for} \quad \bar\xi_{\mathrm{plast},\tau}(t) = -\big[\bar{\mathcal E}_\tau\big]'_\pi(t,\bar u_\tau(t),\bar\pi_\tau(t),\underline\zeta_\tau(t)),$$

$$\int_0^T \bar\xi_{\mathrm{dam},\tau}(t)\mathrm{d}\bar\zeta_\tau(t) \overset{?}{\sim} \mathrm{Var}_{\mathcal R_2}(\bar\zeta_\tau;[0,T])$$

$$\text{for some} \quad \bar\xi_{\mathrm{dam},\tau}(t) \in -\partial_\zeta\bar{\mathcal E}_\tau(t,\bar u_\tau(t),\bar\pi_\tau(t),\bar\zeta_\tau(t)),$$

The residua can be evaluated more specifically as:

$$\int_\Omega R_{\pi,\tau}\mathrm{d}x = \int_\Omega \bigg(\sum_{k=1}^{T/\tau} \sigma_{\mathrm{y}} \big|\pi_\tau^k - \pi_\tau^{k-1}\big| - \mathbb{C}(\zeta_\tau^{k-2})\big(\pi_\tau^{k-1} - e(u_\tau^{k-1} + u_{\mathrm{Dir},\tau}^{k-1})\big) \\ - \mathbb{H}\pi_\tau^{k-1} : (\pi_\tau^k - \pi_\tau^{k-1}) - \kappa_2 \nabla\pi_\tau^{k-1} \vdots \nabla(\pi_\tau^k - \pi_\tau^{k-1}) \bigg)\mathrm{d}x \ \geq \ 0,$$

and

$$\int_\Omega R_{\zeta,\tau}\mathrm{d}x = \int_\Omega \bigg(\sum_{k=1}^{T/\tau} a(\zeta_\tau^k - \zeta_\tau^{k-1})^- + b(\zeta_\tau^k - \zeta_\tau^{k-1})^+ - \xi_{\mathrm{const},\tau}^{k-1}(\zeta_\tau^k - \zeta_\tau^{k-1}) \\ - \frac{1}{2}\mathbb{C}'(\zeta_\tau^{k-1})\big(e(u_\tau^{k-1} + u_{\mathrm{Dir},\tau}^{k-1}) - \pi_\tau^{k-1}\big) : \big(e(u_\tau^{k-1} + u_{\mathrm{Dir},\tau}^{k-1}) - \pi_\tau^{k-1}\big) \\ - \kappa_1 |\nabla\zeta_\tau^{k-1}|^{r-2}\nabla\zeta_\tau^{k-1} \cdot \nabla(\zeta_\tau^k - \zeta_\tau^{k-1}) \bigg)\,\mathrm{d}x \ \geq \ 0,$$

with some multiplier $\xi_{\mathrm{const},\tau}^k \in N_{[0,1]}(\zeta_\tau^k)$.

It allows for a spatial localization over Ω.

The residua can be evaluated more specifically as:

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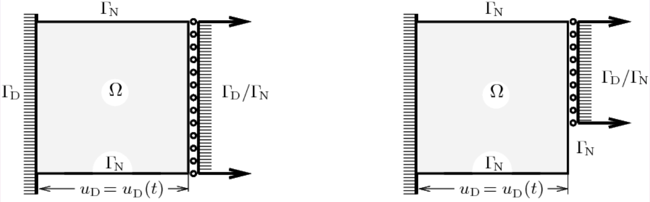
and

$$\int_\Omega R_{\zeta,\tau}\mathrm{d}x = \int_\Omega \bigg(\sum_{k=1}^{T/\tau} a(\zeta_\tau^k - \zeta_\tau^{k-1})^- + b(\zeta_\tau^k - \zeta_\tau^{k-1})^+ - \xi_{\mathrm{const},\tau}^{k-1}(\zeta_\tau^k - \zeta_\tau^{k-1}) \\ - \frac{1}{2}\mathbb{C}'(\zeta_\tau^{k-1})\big(e(u_\tau^{k-1} + u_{\mathrm{Dir},\tau}^{k-1}) - \pi_\tau^{k-1}\big) : \big(e(u_\tau^{k-1} + u_{\mathrm{Dir},\tau}^{k-1}) - \pi_\tau^{k-1}\big) \\ - \kappa_1 |\nabla\zeta_\tau^{k-1}|^{r-2}\nabla\zeta_\tau^{k-1} \cdot \nabla(\zeta_\tau^k - \zeta_\tau^{k-1}) \bigg) \mathrm{d}x \ \geq \ 0,$$

with some multiplier $\xi_{\mathrm{const},\tau}^k \in N_{[0,1]}(\zeta_\tau^k)$.

It allows for a spatial localization over Ω.

Numerical simulations with bulk damage + plasticity
(max-diss. local solutions by fractional step algorithm):

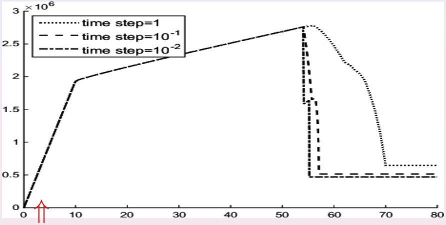


Two variants of geometry of a 2-dimensional square-shaped specimen to be plastified and damaged under a tension-loading experiment.
The right-hand side of Ω is free in tangential direction.

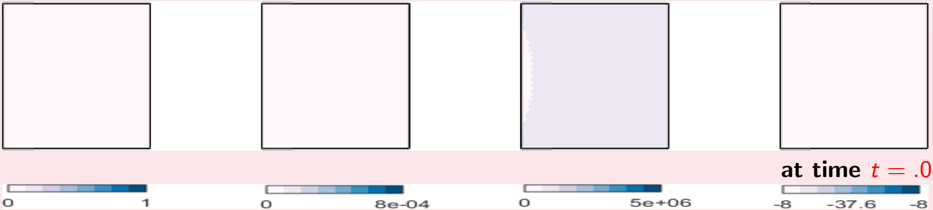
Material: isotropic, homogeneous, $\mathbb{C} = \mathbb{C}(\zeta)$ affine in ζ, $\mathbb{C}(1) = 1000\mathbb{C}(0)$,
$\mathbb{C}(1) \sim$ Young modulus 27 GPa, Poisson ration 0.2, $\mathbb{H} = \mathbb{C}(1)/4$,
$S = \{\sigma \in \mathbb{R}^{d\times d}_{\rm dev},\ |\sigma| \le \sigma_{\rm y}\}$ with $\sigma_{\rm y} = 2\,$MPa,
the damage energy $a = 1\,$kPa, $\kappa_1 = 10^{-9}\,$J/m.

Some implementation shortcuts: $\kappa_2 = 0$ and $r = 2$ (instead of $\kappa_2 > 0$ and $r > 2$)
$\Rightarrow$ after triangulation of Ω: P1-elements have been used for u and ζ
P0-elements suffices for π.

Numerical simulations: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

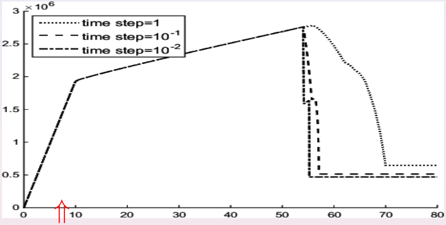


at time $t = .04$

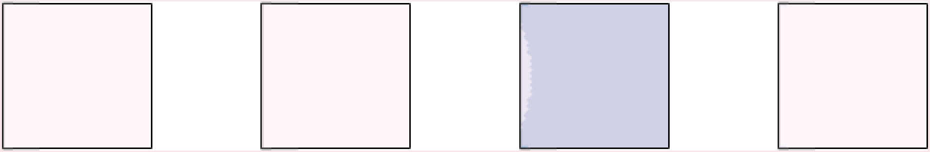
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 200×.

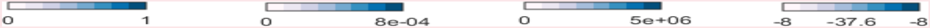
Numerical simulations: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ	**plastic strain** $\lvert\pi\rvert$	**von-Mises stress** $\lvert\mathrm{dev}\,\sigma\rvert$	**residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



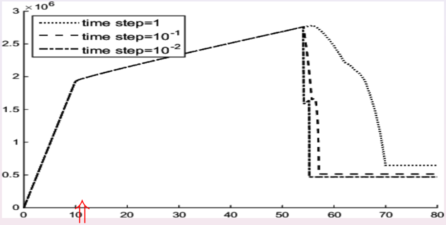
at time $t = .08$



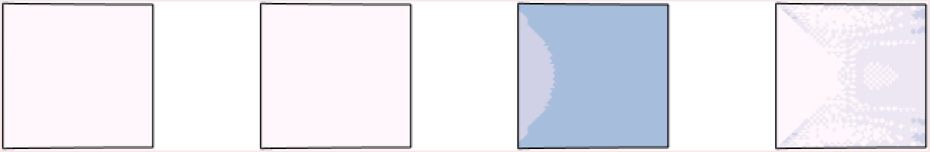
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

Numerical simulations: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

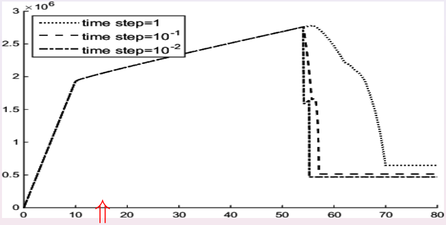


at time $t = .12$

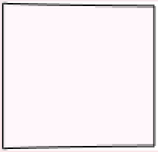
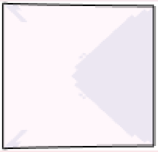
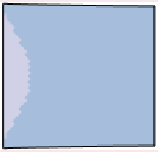
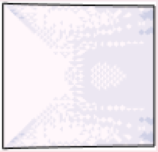
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

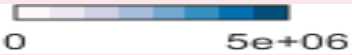
Numerical simulations: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ	plastic strain $\lvert\pi\rvert$	von-Mises stress $\lvert\mathrm{dev}\,\sigma\rvert$	residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

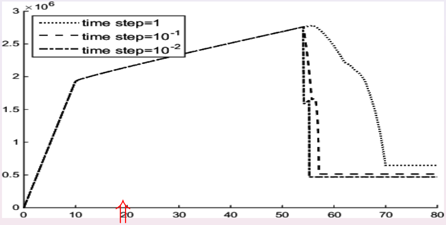
at time $t = .16$



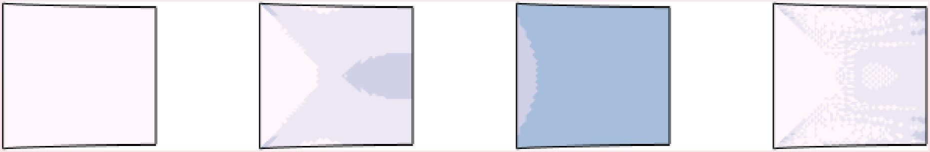
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

Numerical simulations: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

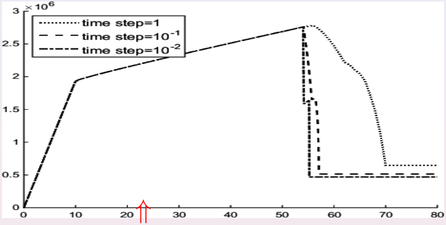


at time $t = .20$

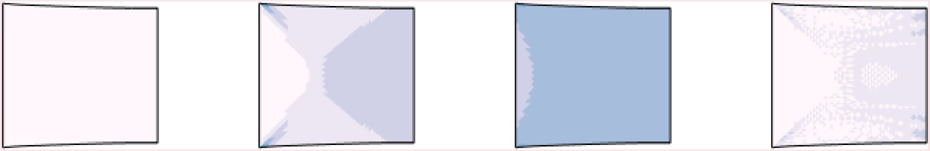
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

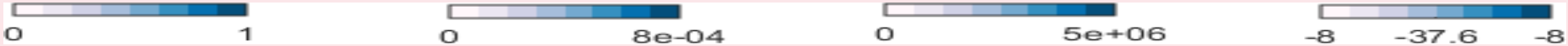
Numerical simulations: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



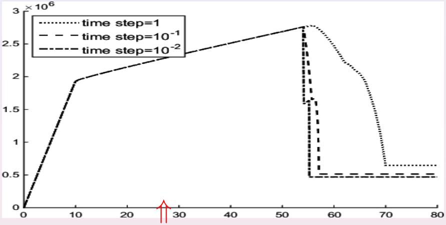
at time $t = .24$

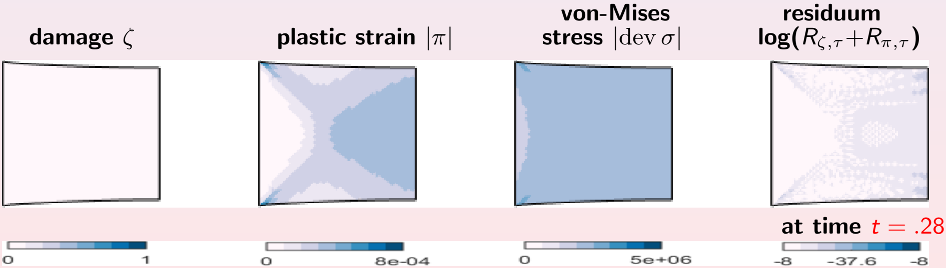


Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

Numerical simulations: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

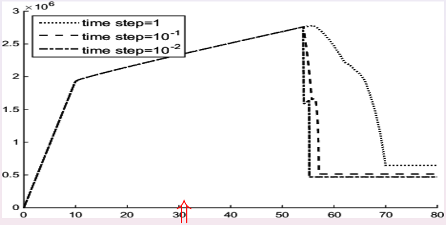




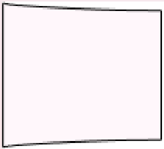
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 200×.

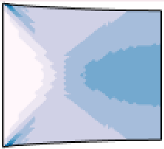
Numerical simulations: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



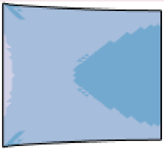
damage ζ



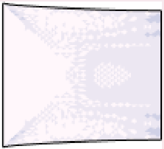
plastic strain $|\pi|$



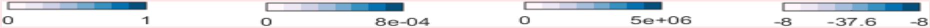
von-Mises stress $|\mathrm{dev}\,\sigma|$



residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



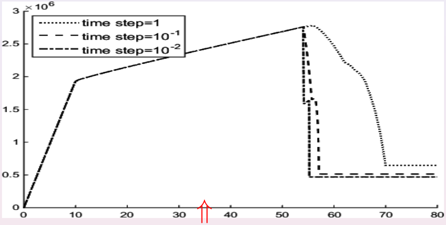
at time $t=.32$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

Numerical simulations: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

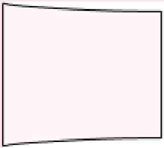


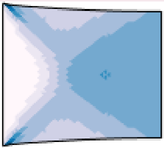
damage ζ

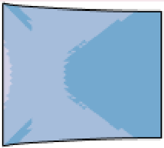
plastic strain $|\pi|$

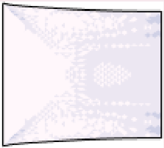
von-Mises stress $|\mathrm{dev}\,\sigma|$

residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

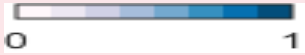


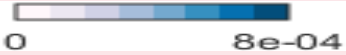






at time $t = .36$

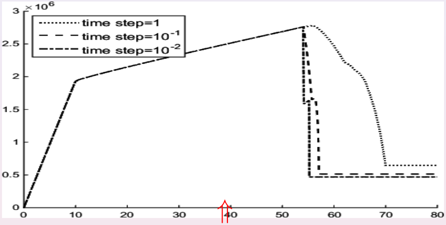




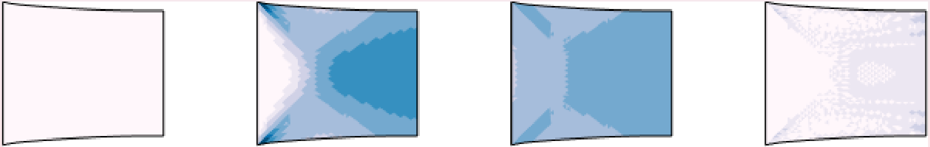
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

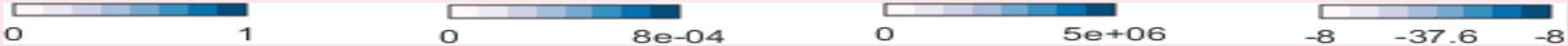
Numerical simulations: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



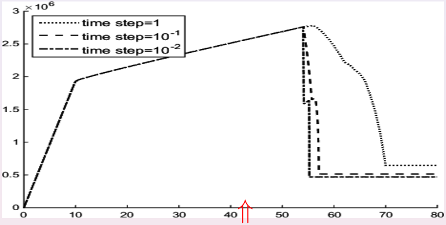
at time $t = .40$



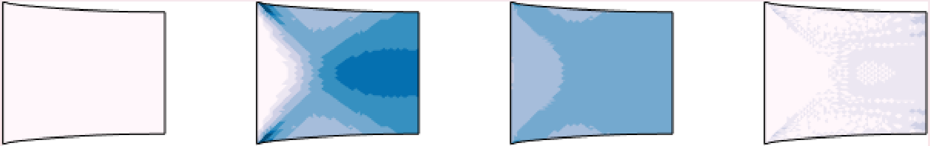
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

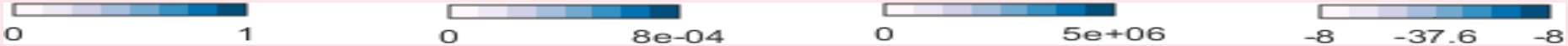
Numerical simulations: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



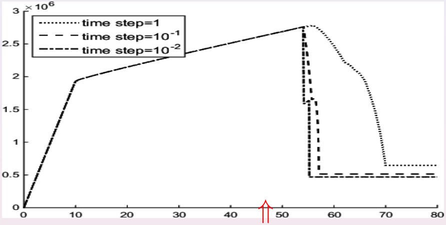
at time $t = .44$



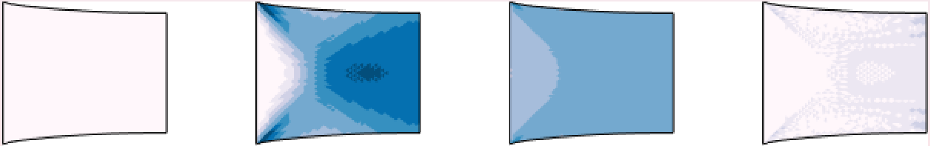
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

Numerical simulations: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ | **plastic strain $|\pi|$** | **von-Mises stress $|\mathrm{dev}\,\sigma|$** | **residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$**

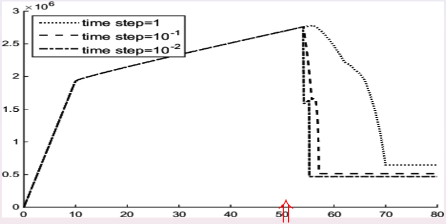


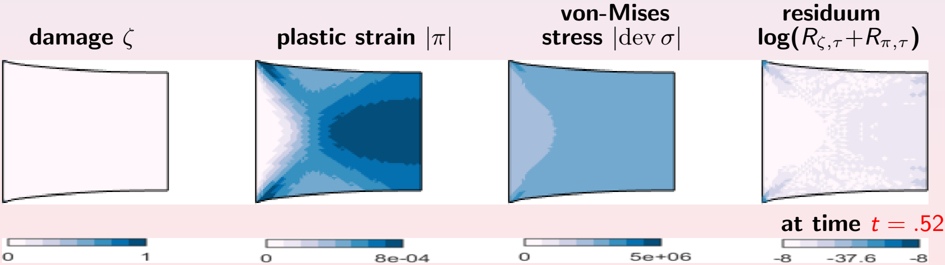
at time $t = .48$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

Numerical simulations: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

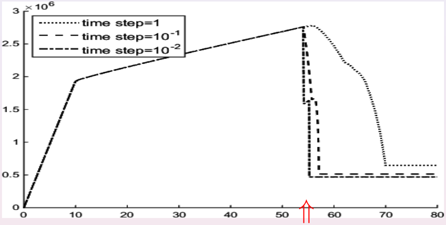


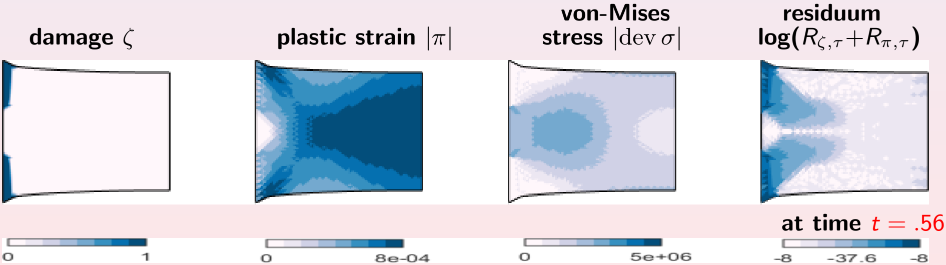


Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 200×.

Numerical simulations: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

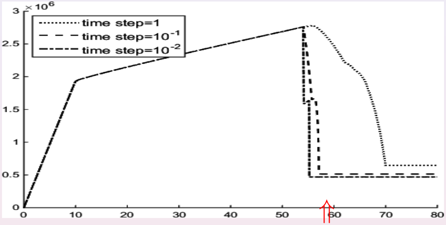


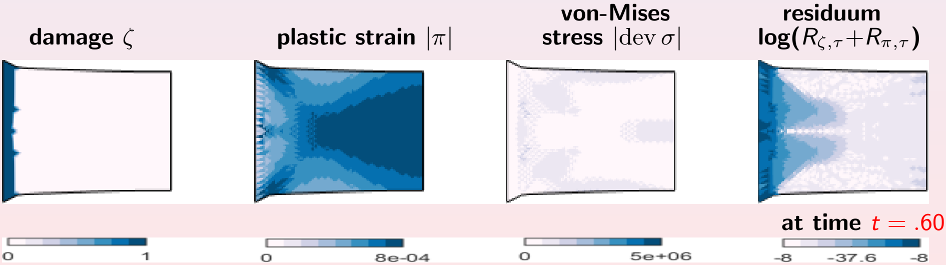


Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 200×.

Numerical simulations: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

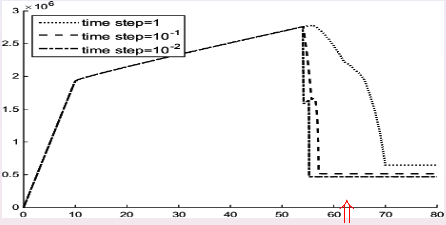




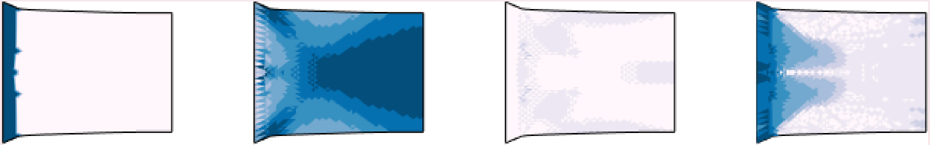
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 200×.

Numerical simulations: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

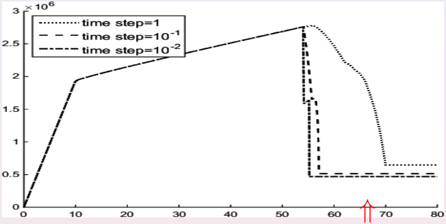


at time $t = .64$

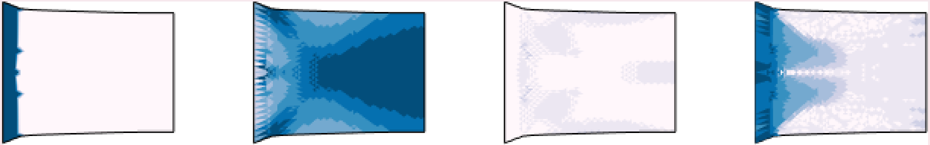
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

Numerical simulations: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

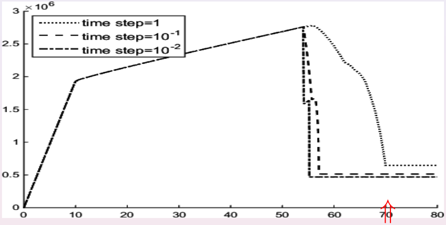


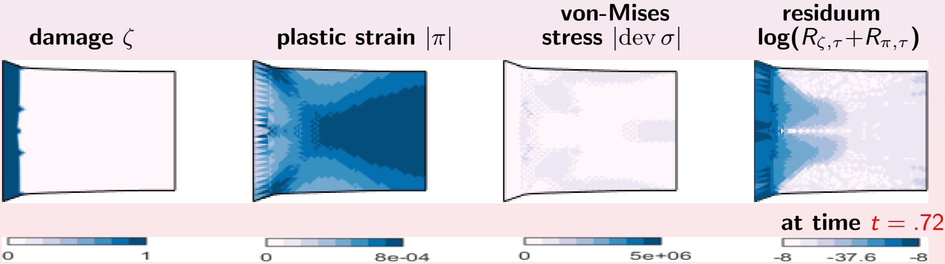
at time $t = .68$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified $200\times$.

Numerical simulations: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

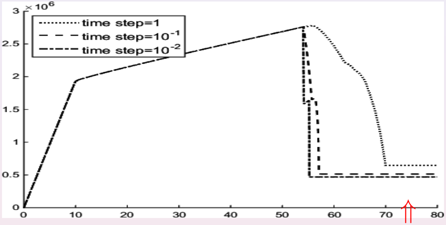




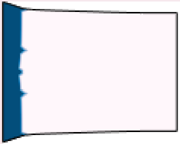
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 200×.

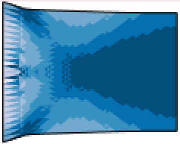
Numerical simulations: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



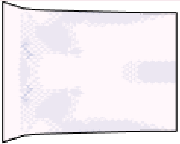
damage ζ



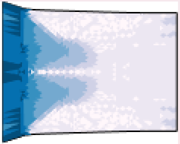
plastic strain $|\pi|$



von-Mises stress $|\mathrm{dev}\,\sigma|$



residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

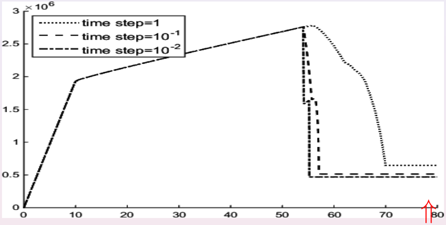


at time $t = .76$

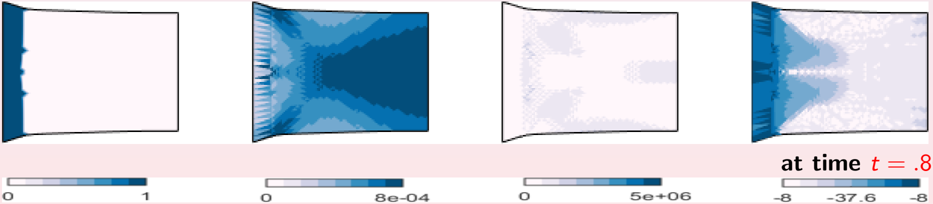
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 200 ×.

Numerical simulations: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum log**$(R_{\zeta,\tau}+R_{\pi,\tau})$

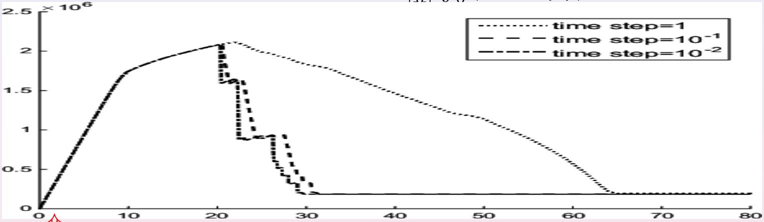


at time $t = .80$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 200×.

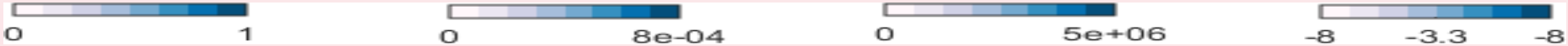
Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



⇑

damage ζ **plastic strain $|\pi|$** **von-Mises stress $|\mathrm{dev}\,\sigma|$** **residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$**

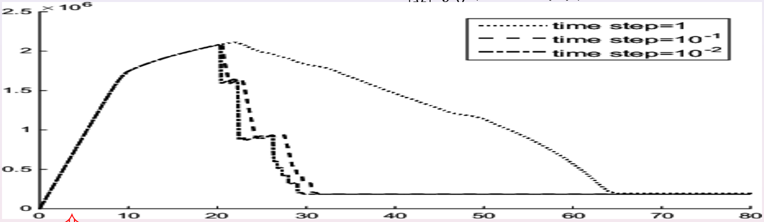
at time $t = .02$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

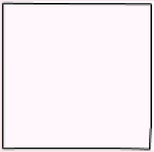
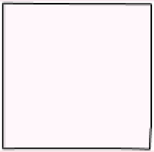
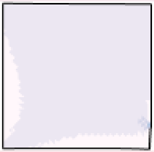
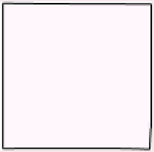
Deformation of the specimen depicted by displacement u magnified $200\times$.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

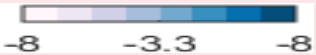


⇑

damage ζ	plastic strain $\lvert\pi\rvert$	von-Mises stress $\lvert\mathrm{dev}\,\sigma\rvert$	residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

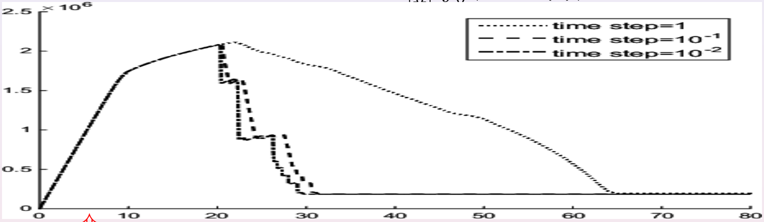
at time $t = .04$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

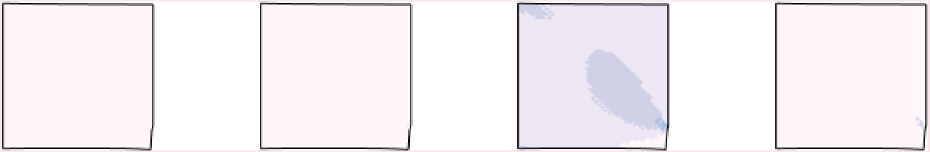
Deformation of the specimen depicted by displacement u magnified 200×.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

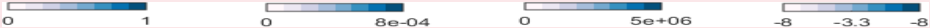


⇑

damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



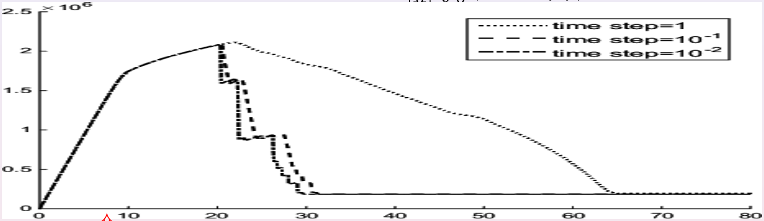
at time $t = .06$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

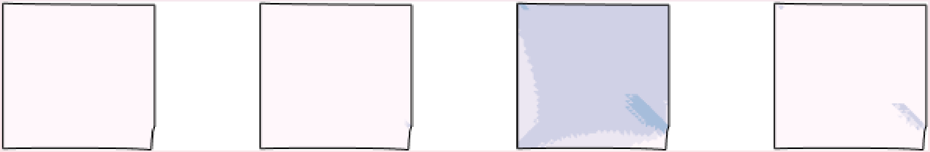
Deformation of the specimen depicted by displacement u magnified 200 ×.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

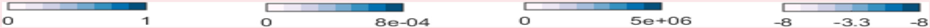


⇑

damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



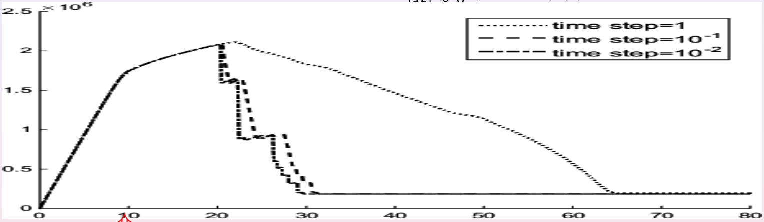
at time $t = .08$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

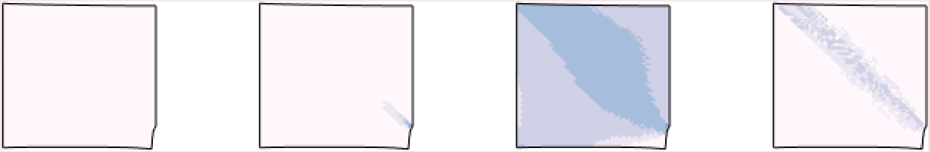
Deformation of the specimen depicted by displacement u magnified 200 ×.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



⇑

damage ζ | **plastic strain $|\pi|$** | **von-Mises stress $|\mathrm{dev}\,\sigma|$** | **residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$**

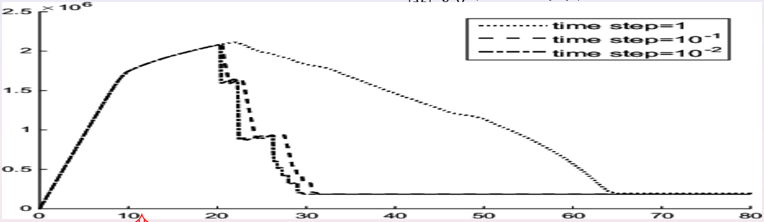


at time $t = .10$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

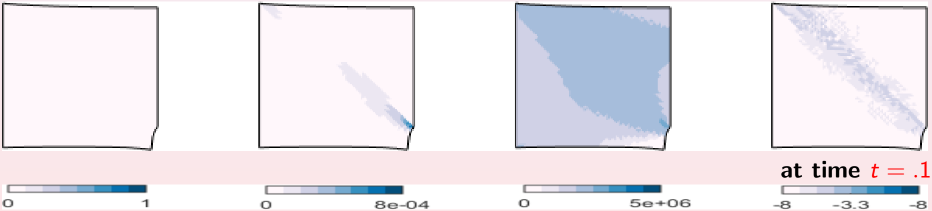
Deformation of the specimen depicted by displacement u magnified $200\times$.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



⇑

damage ζ | plastic strain $|\pi|$ | von-Mises stress $|\mathrm{dev}\,\sigma|$ | residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

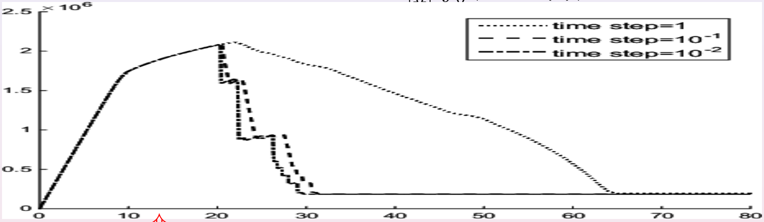


at time $t = .12$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

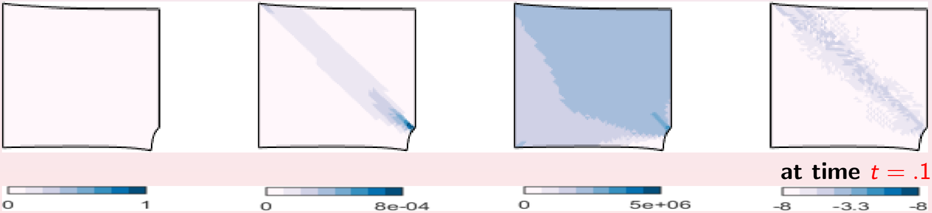
Deformation of the specimen depicted by displacement u magnified 200×.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



⇑

damage ζ **plastic strain $|\pi|$** **von-Mises stress $|\mathrm{dev}\,\sigma|$** **residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$**

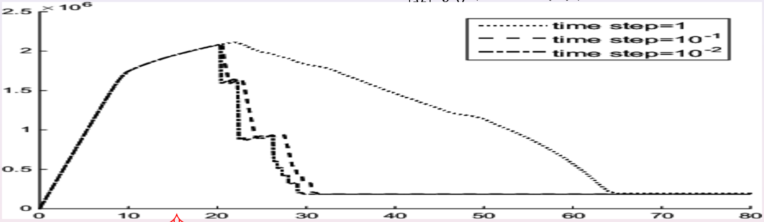


at time $t = .14$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

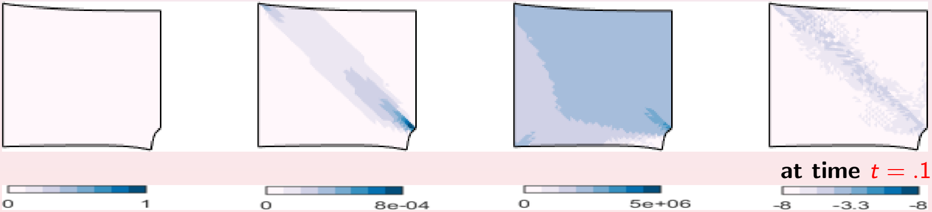
Deformation of the specimen depicted by displacement u magnified 200 ×.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



⇑

damage ζ **plastic strain $|\pi|$** **von-Mises stress $|\mathrm{dev}\,\sigma|$** **residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$**

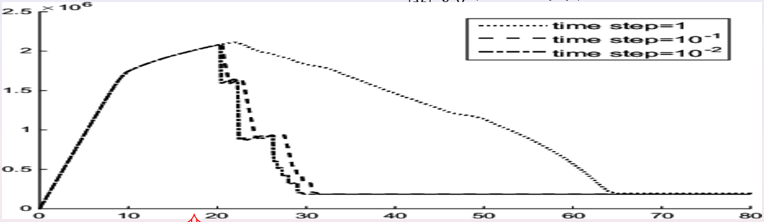


at time $t = .16$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

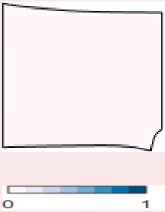
Deformation of the specimen depicted by displacement u magnified $200\times$.

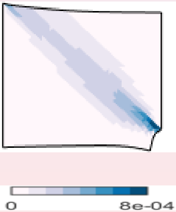
Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

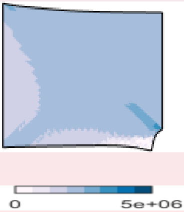


⇑

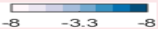
damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$







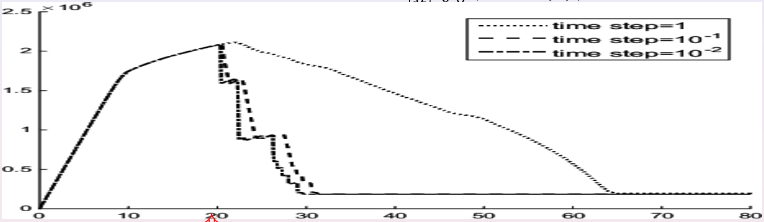
at time $t = .18$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

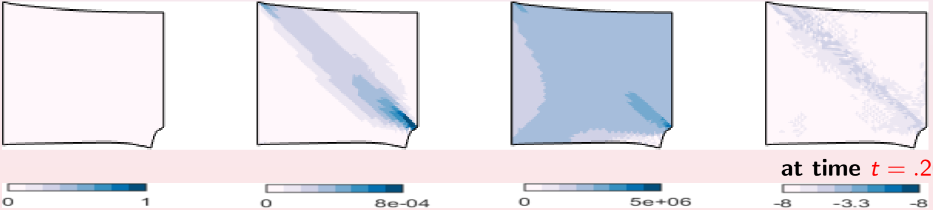
Deformation of the specimen depicted by displacement u magnified $200\times$.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



⇑

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$ **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

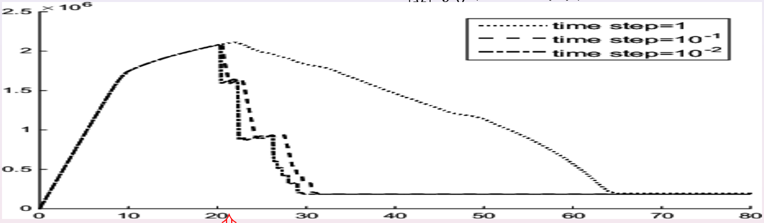


at time $t = .20$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

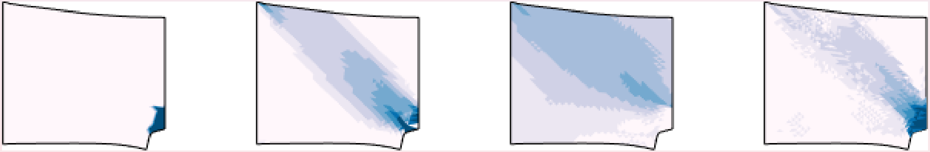
Deformation of the specimen depicted by displacement u magnified 200 ×.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

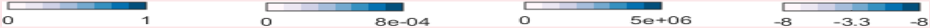


$\Uparrow$

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$ **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



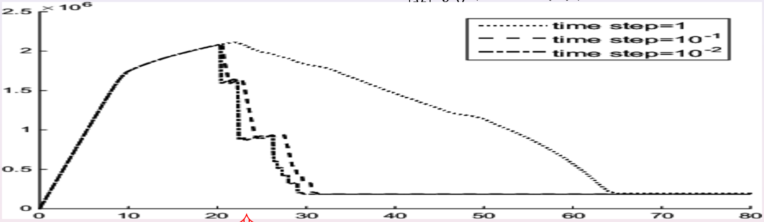
at time $t = .22$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

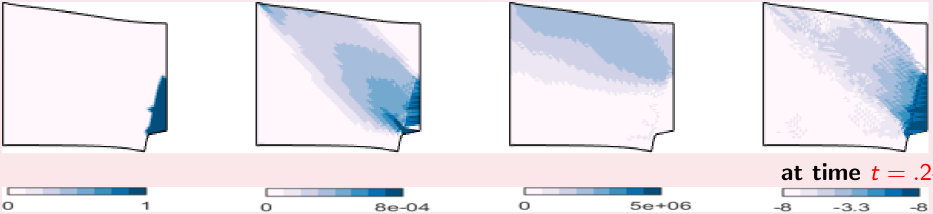
Deformation of the specimen depicted by displacement u magnified 200 ×.

Alternative geometry: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



⇑

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$ **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

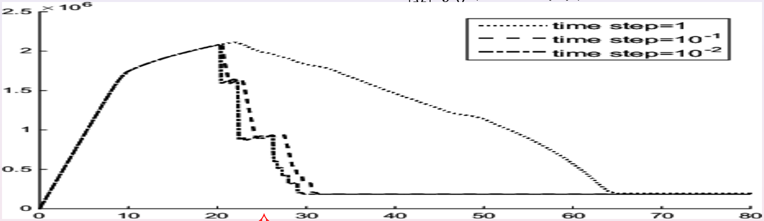


at time $t = .24$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

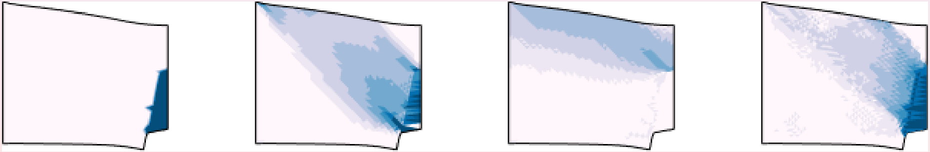
Deformation of the specimen depicted by displacement u magnified 200 ×.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



⇑

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$ **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

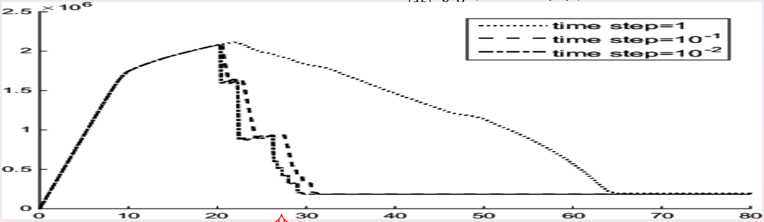


at time $t = .26$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

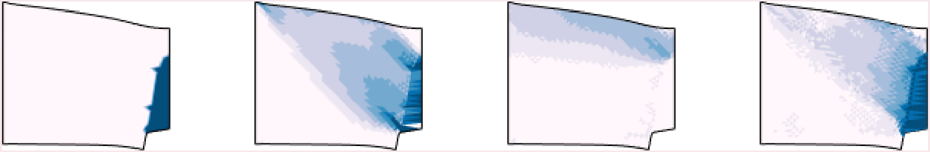
Deformation of the specimen depicted by displacement u magnified 200 ×.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

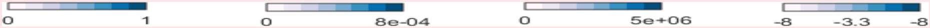


⇑

damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



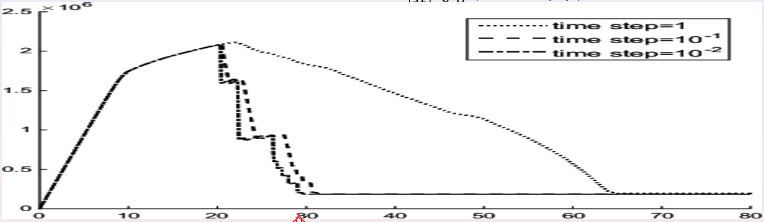
at time $t = .28$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

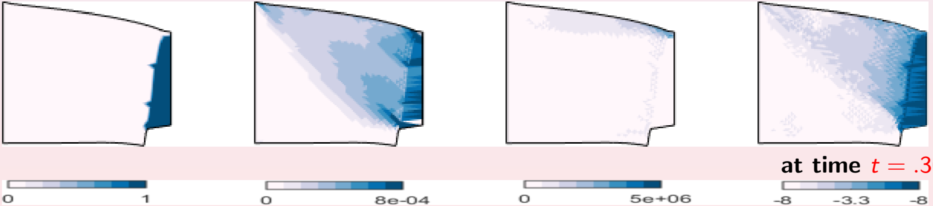
Deformation of the specimen depicted by displacement u magnified 200 ×.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



⇑

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$ **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

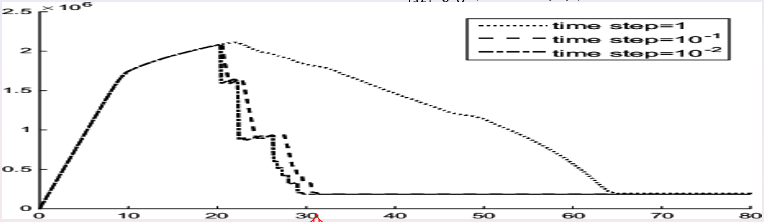


at time $t = .30$

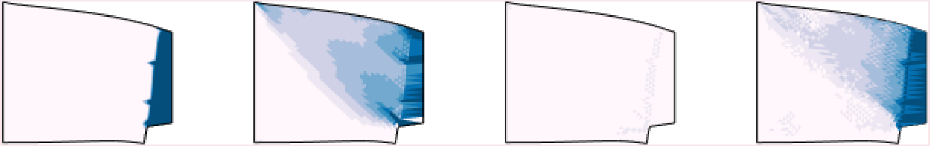
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 200 ×.

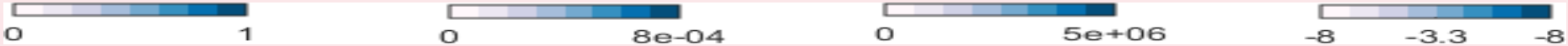
Alternative geometry: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$ **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



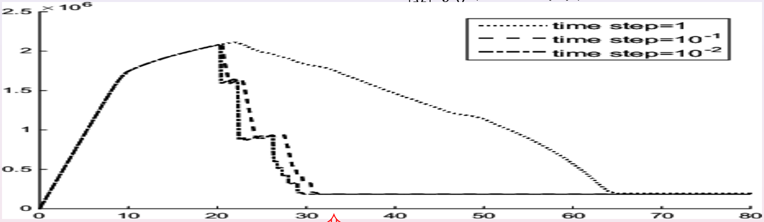
at time $t = .32$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

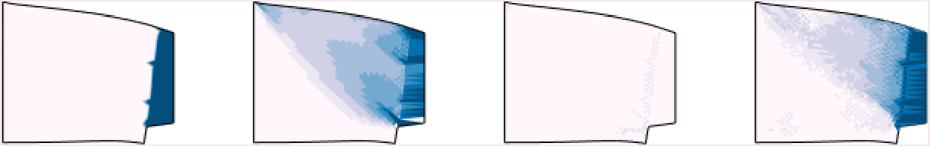
Deformation of the specimen depicted by displacement u magnified $200\times$.

Alternative geometry: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

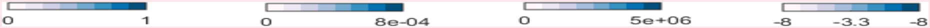


⇑

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$ **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



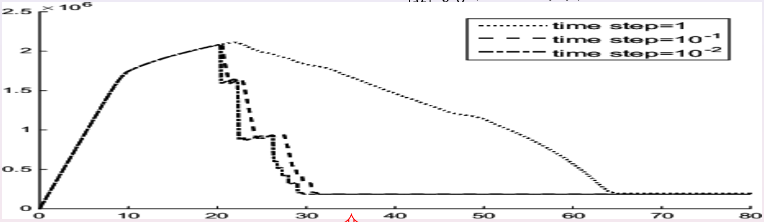
at time $t = .34$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

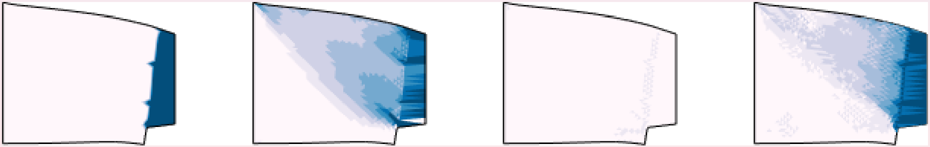
Deformation of the specimen depicted by displacement u magnified 200 ×.

Alternative geometry: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.

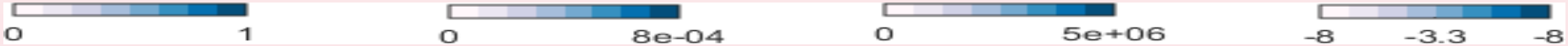


⇑

damage ζ | **plastic strain** $|\pi|$ | **von-Mises stress** $|\mathrm{dev}\,\sigma|$ | **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$



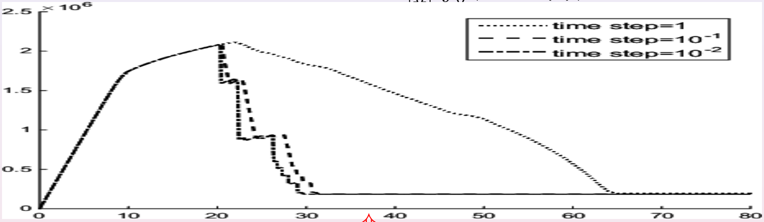
at time $t = .36$



Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

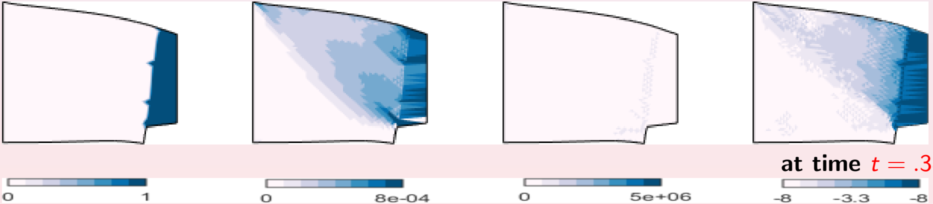
Deformation of the specimen depicted by displacement u magnified 200 ×.

Alternative geometry: Overall von-Mises stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



⇑

damage ζ **plastic strain** $|\pi|$ **von-Mises stress** $|\mathrm{dev}\,\sigma|$ **residuum** $\log(R_{\zeta,\tau}+R_{\pi,\tau})$

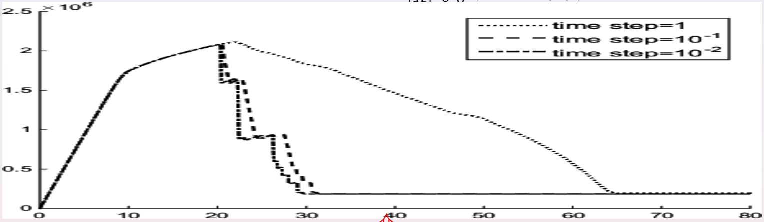


at time $t=.38$

Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

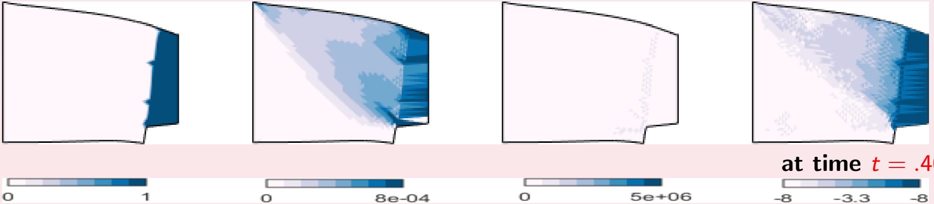
Deformation of the specimen depicted by displacement u magnified 200 ×.

Alternative geometry: Overall von-Mizes stress $\frac{1}{|\Omega|}\int_\Omega |\mathrm{dev}\,\sigma(t)|\,\mathrm{d}x$ depending on t.



⇑

damage ζ | **plastic strain $|\pi|$** | **von-Mises stress $|\mathrm{dev}\,\sigma|$** | **residuum $\log(R_{\zeta,\tau}+R_{\pi,\tau})$**

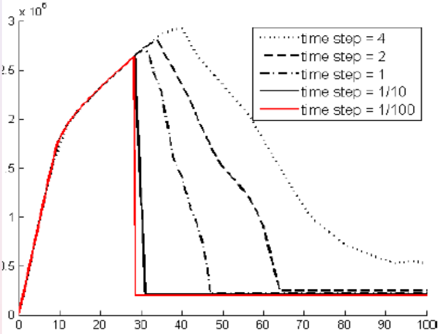


at time $t=.40$

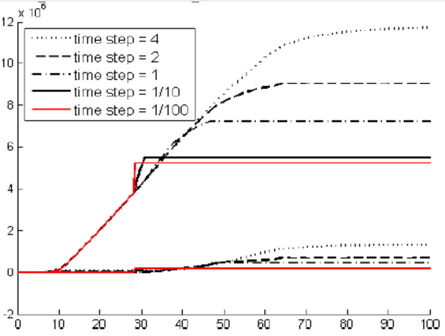
Calculations and visualization: courtesy of Jan Valdman (Czech Acad. Sci.).

Deformation of the specimen depicted by displacement u magnified 200 ×.

Convergence of evolution of the overall von Mises stress



The dissipated energy and the residuum in the approx. max.-diss. principle



Note: the residual stress resulted from
the nonuniform plastification of the specimen.

During plasticizing phase: residuum is small,
Hill's maximum dissipation principle always well satisfied.

During damaging phase: residuum is possibly larger,
it may not mean that the evolution is not stress driven.

Some open problems:

Purely unidirectional damage known only for energetic solution.
For stress-driven type solutions open.

Complete damage known only for energetic solution without plasticity
(G. Bouchitté, A.Mielke, T.R., 2009)
with plasticity and/or for stress-driven type solutions open.

A limit with a big elasticity moduli $\mathbb{C} \to \infty$ towards plastic-rigid model open.

Etc.

Some references:

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- A.Mielke, T.Roubíček, U.Stefanelli: Γ-limits and relaxations for rate--independent evolutionary problems. *Calc. Var. P.D.E.* **31** (2008), 387-416.
- T.Roubíček: Rate independent processes in viscous solids at small strains. *Math. Methods Appl. Sci.* **32** (2009), 825-862.
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- T.Roubíček, C.G.Panagiotopoulos, V.Mantič: Quasistatic adhesive contact of visco-elastic bodies and its numerical treatment for very small viscosity. *Zeitschrift angew. Math. Mech.* **93** (2013), 823-840.
- T.Roubíček, J.Valdman: Stress-driven solution to rate-independent damage with plasticity at small strains and its computer implementation. (Preprint arXiv no.1506.01421) *Math. Mech. of Solids*, printed on line.

More on: `www.karlin.mff.cuni.cz/~roubicek/trpublic.htm`
or:
`https://www.researchgate.net/profile/Tomas_`*Roubicek*2

Thanks a lot for your attention.

More on: www.karlin.mff.cuni.cz/~roubicek/trpublic.htm
or:
https://www.researchgate.net/profile/Tomas_*Roubicek*2

Vielen Dank für Ihre Aufmerksamkeit.