Addendum to "Asymptotic Inference for a Linear Stochastic Differential Equation with Time Delay"

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Running title: Addendum

Abstract

We strengthen the convergence result proving the local asymptotic mixed normality property in one of the eleven cases considered in Gushchin and Küchler (1999).

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In Gushchin and Küchler [GK] (1999) we considered the stochastic differential equation

$$dX(t) = aX(t)\,dt + bX(t-1)\,dt + dW(t), \qquad t \geq 0, \tag{1}$$

with an initial condition $X(t) = X_0(t)$, $t \in [-1,0]$, where $(X_0(t),\ t \in [-1,0])$ is a given continuous stochastic process independent of a standard Wiener process $(W(t),\ t \geq 0)$. The parameter $\vartheta = (a,b)^*$ is unknown and supposed to be estimated based on $(X(t),\ t \leq T)$. (The sign * denotes matrix or vector transposition.) The solutions $(X(t),\ t \in [-1,T])$ of the differential equation (1) generate a family $(P_T^\vartheta,\ \vartheta \in \mathbb{R}^2)$ of distributions on $\mathbb{C}([-1,T])$. In [GK] we studied local asymptotic properties of this family and corresponding properties of the maximum likelihood estimators of ϑ for $T \to \infty$.

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In particular, for a fixed $\vartheta_0 = (a,b)^* \in \mathbb{R}^2$ the limit behaviour of $\frac{dP_T^\vartheta}{dP_T^{\vartheta_0}}(X)$, $\vartheta \in \mathbb{R}^2$, was studied. It turned out that this behaviour depends strongly on the position of ϑ_0, and one has to consider eleven different cases. The family (P_T^ϑ) is locally asymptotically normal in one of these cases, it is locally asymptotically mixed normal (LAMN) in other three cases, and it is only locally asymptotically quadratic in five of the remaining cases. As for the last cases called P1 and P2, we showed that the family (P_T^ϑ) is still locally asymptotically quadratic but has an additional property that we called periodic local asymptotic mixed normality (PLAMN).
Namely, assume that we are in case P2, i.e. $a < 1$ and $b < \xi/\sin\xi$, where ξ is the unique number in $(0,\pi)$ with $\xi = a\tan\xi$ ($\xi = \pi/2$ if $a = 0$), or $a \geq 1$ and $b < -e^{a-1}$. We refer to [GK] for the definition of the random vector V_T, the random matrix I_T, and the numbers $v_0 > 0$, $\xi_0 > 0$, A_0 and B_0. It was shown that the following properties hold ([GK], Proposition 2.6):

(1)
$$\log \frac{dP_T^{\vartheta_0 + e^{-v_0 T}\mu}}{dP_T^{\vartheta_0}}(X) = \mu^* V_T - \frac{1}{2}\mu^* I_T \mu, \qquad \mu \in \mathbb{R}^2; \tag{2}$$

(2) for $T_n = u + n\Delta$, where $u \in [0,\Delta)$ is fixed, $\Delta = \pi/\xi_0$, $n \geq 0$,

$$(V_{T_n}, I_{T_n}) \xrightarrow{d} (V_\infty(u), I_\infty(u)), \qquad n \to \infty,$$

where $(V_\infty(u), I_\infty(u)) \stackrel{d}{=} (I_\infty^{1/2}(u)Z, I_\infty(u))$ and Z is an independent of $I_\infty(u)$ and $\mathcal{N}(0,\mathcal{I}_2)$-distributed random vector. The matrix $I_\infty(u)$ is given by

$$I_\infty(u) = \begin{pmatrix} \displaystyle\int_0^\infty e^{-2v_0 t} U_0^2(u-t)\,dt & \displaystyle\int_0^\infty e^{-2v_0 t} U_0(u-t)U_2(u-t)\,dt \\ \displaystyle\int_0^\infty e^{-2v_0 t} U_0(u-t)U_2(u-t)\,dt & \displaystyle\int_0^\infty e^{-2v_0 t} U_2^2(u-t)\,dt \end{pmatrix}, \tag{3}$$

where

$$U_i(t) = X_0(0)\phi_i(t) + b\int_{-1}^0 \phi_i(t-s-1)e^{-v_0(s+1)}X_0(s)\,ds + \int_0^\infty \phi_i(t-s)e^{-v_0 s}\,dW(s), \tag{4}$$

and

$$\phi_i(t) = A_i\cos(\xi_0 t) + B_i\sin(\xi_0 t), \qquad i = 0,2, \tag{5}$$

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = e^{-v_0}\begin{pmatrix} \cos\xi_0 & -\sin\xi_0 \\ \sin\xi_0 & \cos\xi_0 \end{pmatrix}\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \tag{6}$$

We have called this asymptotic behaviour by PLAMN to indicate that the cluster points of (V_T, I_T) have the same structure as in the LAMN case but (V_T, I_T) converges in distribution only if T runs to infinity through a grid in such a way that the fractional part of T/Δ tends to a limit.

The purpose of this note is to show that in case P2 one can prove the usual LAMN property. To this end, we look for matrices $\Phi_T, T \geq 0$, such that $\Phi_T^* \begin{pmatrix} U_0(T-t) \\ U_2(T-t) \end{pmatrix}$ does not depend on T. In view of (4) and (5), this will be the case if $\Phi_T^* \begin{pmatrix} A_0 & A_2 \\ B_0 & B_2 \end{pmatrix} \begin{pmatrix} \cos(T-t) \\ \sin(T-t) \end{pmatrix}$ is independent of T. Using (6), after elementary calculations one gets that the last property takes place if and only if

$$\Phi_T = \Phi_0 \begin{pmatrix} \cos(\xi_0 T) - \sin(\xi_0 T)\cot\xi_0 & -e^{-v_0}\sin(\xi_0 T)\csc\xi_0 \\ e^{v_0}\sin(\xi_0 T)\csc\xi_0 & \cos(\xi_0 T) + \sin(\xi_0 T)\cot\xi_0 \end{pmatrix}. \tag{7}$$

Define Φ_T according to (7) with $\Phi_0 := \mathcal{I}_2$, where $\mathcal{I}_2$ is the identity matrix. Then

$$\begin{pmatrix} \widetilde{U}_0(t) \\ \widetilde{U}_2(t) \end{pmatrix} := \Phi_T^* \begin{pmatrix} U_0(T-t) \\ U_2(T-t) \end{pmatrix} = X_0(0) \begin{pmatrix} A_0\cos(\xi_0 t) - B_0\sin(\xi_0 t) \\ e^{-v_0}(A_0\cos(\xi_0(t+1)) - B_0\sin(\xi_0(t+1))) \end{pmatrix}$$

$$+b\int_{-1}^{0} \begin{pmatrix} A_0\cos(\xi_0(t+s+1)) - B_0\sin(\xi_0(t+s+1)) \\ e^{-v_0}(A_0\cos(\xi_0(t+s+2)) - B_0\sin(\xi_0(t+s+2))) \end{pmatrix} e^{-v_0(s+1)} X_0(s)\,ds$$

$$+\int_0^{\infty} \begin{pmatrix} A_0\cos(\xi_0(t+s)) - B_0\sin(\xi_0(t+s)) \\ e^{-v_0}(A_0\cos(\xi_0(t+s+1)) - B_0\sin(\xi_0(t+s+1))) \end{pmatrix} e^{-v_0 s}\,dW(s).$$

Extend the definition of $I_\infty(u)$ to $u \in \mathbb{R}_+$ by the same formula (3). Then

$$\Phi_T^* I_\infty(T)\Phi_T = \int_0^\infty e^{-2v_0 t}\Phi_T^* \begin{pmatrix} U_0(T-t) \\ U_2(T-t) \end{pmatrix} \begin{pmatrix} U_0(T-t) \\ U_2(T-t) \end{pmatrix}^* \Phi_T\,dt,$$

and, therefore, the matrix $\widetilde{I}_\infty := \Phi_T^* I_\infty(T)\Phi_T$ does not depend on T and is given by

$$\widetilde{I}_\infty = \begin{pmatrix} \displaystyle\int_0^\infty e^{-2v_0 t}\widetilde{U}_0^2(t)\,dt & \displaystyle\int_0^\infty e^{-2v_0 t}\widetilde{U}_0(t)\widetilde{U}_2(t)\,dt \\ \displaystyle\int_0^\infty e^{-2v_0 t}\widetilde{U}_0(t)\widetilde{U}_2(t)\,dt & \displaystyle\int_0^\infty e^{-2v_0 t}\widetilde{U}_2^2(t)\,dt \end{pmatrix}.$$

Put

$$\widetilde{V}_T := \Phi_T^* V_T, \qquad \widetilde{I}_T := \Phi_T^* I_T \Phi_T.$$

Proposition 1 *In Case* P2 *the family* $(P^\vartheta,\ \vartheta \in \mathbb{R}^2)$ *is locally asymptotically mixed normal at every* ϑ_0:

$$\log \frac{dP_T^{\vartheta_0 + e^{-v_0 T}\Phi_T \mu}}{dP_T^{\vartheta_0}}(X) = \mu^* \widetilde{V}_T - \frac{1}{2}\mu^* \widetilde{I}_T \mu \tag{8}$$

and

$$(\widetilde{V}_T, \widetilde{I}_T) \xrightarrow{d} (\widetilde{V}_\infty, \widetilde{I}_\infty), \tag{9}$$

where $(\widetilde{V}_\infty, \widetilde{I}_\infty) \stackrel{d}{=} (\widetilde{I}_\infty^{1/2} Z, \widetilde{I}_\infty)$ *and* Z *is an independent of* $\widetilde{I}_\infty$ *and* $\mathcal{N}(0, \mathcal{I}_2)$*-distributed vector.*

Proof: (8) is obvious from (2).
It was shown in the proof of Proposition 2.6 that

$$I_T - I_\infty(T) \xrightarrow{P} 0.$$

Thus

$$\widetilde{I}_T \xrightarrow{P} \widetilde{I}_\infty$$

because the matrices Φ_T are uniformly bounded. Now the joint convergence (9) follows from the stable limit theorem for martingales as in the proofs of Propositions 2.2–2.4 in [GK].

Remark: On the contrary to the case just considered, in Case P1 one cannot avoid periodic behaviour of the likelihood and prove the LAMN property by choosing an appropriate matrix norming. For a general statement of this type for delay processes see Putschke (2000).

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