The kodaira dimension of $\overline{M}_{22}$ and $\overline{M}_{23}$

(joint w. D. Jensen and S. Payne)

[Thm: (FJP): Both $\overline{M}_{22}$ and $\overline{M}_{23}$ are of gen'l type

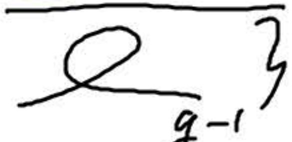
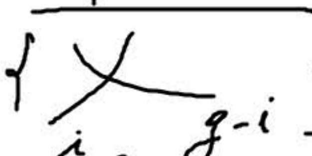
- Severi '15: M_g unirational $g \le 10$. / Severi conjectured M_g unirational $\forall g$
- Sernesi, Chang-Ran, Verra, Schreyer: M_g unirational $g \le 14$; rat'l connected $g = 15$
- '91: Chang-Ran: ~~M_{16} uniruled~~ : $D \subseteq \overline{M}_{16}$ uniruled divs : $K_{\overline{M}_{16}} \cdot R < 0$, ~~$D \cdot R \ge 0$~~, R sweeping curve

2019: D. Tseng: not true

[Thm: (Harris-Mumford-Eisenbud): $\overline{M}_g$ is of gen'l type $g \ge 24$

Open: $g = 16, \dots, 21$

Brief recap on H-M-E: $\overline{M}_g \setminus M_g = \underbrace{\Delta_0 \cup \dots \cup \Delta_{[g/2]}}_{\text{bdry div.}}$

$\Delta_0 = \{$  $\}$ $\qquad \Delta_i = \{$  $\}$

$$\boxed{K_{\overline{M}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \dots - 2\delta_{[g/2]}}$$

Geometric divisors: $D \subseteq \overline{M}_g$ effective div.

$[D] = a\lambda - b_0\delta_0 - \dots - b_{[g/2]}\delta_{[g/2]}$ s.t. slope $s(D) = \frac{a}{\min b_i} < s(K_{\overline{M}_g}) = \frac{13}{2}$

$\Rightarrow \exists \alpha > 0, \beta > 0 : K_{\overline{M}_g} = \alpha \cdot \underbrace{\lambda}_{\text{big}, >0} + \beta \cdot D + (\ge 0 \text{ bdry}) \Rightarrow \overline{M}_g$ of gen'l type

$\exists D$ w/ $\boxed{s(D) < \frac{13}{2} \to \overline{\mathcal{M}}_g \text{ of gen'l type}}$

Brill-Noether divisors : C - gen'l curve , $W^r_d(C) = \{ L \in \mathrm{Pic}^d(C) : h^0(L) \geq r+1 \}$

$$\boxed{\dim W^r_d(C) = \rho(g,r,d) = g - (r+1)(g-d+r)}$$

If $\rho(g,r,d) = -1 \Rightarrow \mathcal{M}^r_{g,d} = \{ [C] \in \mathcal{M}_g : W^r_d(C) \neq \phi \}$ irreducible divisor

$$\boxed{[\overline{\mathcal{M}}^r_{g,d}] = c \cdot \left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i \geq 1} i(g-i)\delta_i \right)} \Rightarrow s(\overline{\mathcal{M}}^r_{g,d}) = 6 + \frac{12}{g+1} < \frac{13}{2}$$

$\Leftrightarrow g \geq 24$

Fact (F, Popa) : If $D \subseteq \overline{\mathcal{M}}_g$ eff. divisor, $s(D) < 6 + \frac{12}{g+1}$ then $D \supseteq \boxed{\mathcal{K}_g = \{ [C] \in \mathcal{M}_g : C \subseteq S \hookrightarrow K3 \}}$ $\hookrightarrow$ 19+g. dim'l.

• Mukai, Lazarsfeld, Voisin ... : curves on K3 have interesting line bdl that fail SYZYGY $\leftarrow$ (proj. normal)

Maximal Rank conjecture : Fix r, d $\rho(g,r,d) \geq 0$.

$$\mathcal{G}^r_d = \{ [C, L] : [C] \in \mathcal{M}_g, L \in W^r_d(C) \} \text{ irr.}$$

$$\sigma \downarrow$$

$$\mathcal{M}_g$$

• If (C,L) general $\boxed{\phi^k_{C,L} : S^k H^0(L) \longrightarrow H^0(L^k) \text{ are all of max. rank}}$

• Hirschowitz, Ballico-Ellia, E. LARSON $\nearrow$ thm. Jensen-Payne

($k=2$ relevant for applications to $\mathcal{M}_g$)

3

Strong MRC: (Aprodu-F)

$r=2$: $\phi_L = \phi_{C,L} : S^2H^0(L) \longrightarrow H^0(L^2)$

$$\Sigma_d^r(C) := \{ L \in W_d^r(C) : \phi_L \text{ not of max. rk} \}$$

$$\binom{r+2}{2} \qquad \qquad 2d+1-g$$

$$\begin{tikzcd} S^2H^0(L) \arrow[rr, "\phi_L"] \arrow[dr] & & H^0(L^2) \arrow[dl] \\ & W_d^r(C) & \end{tikzcd}$$

ρ

If C generic then $\Sigma_d^r(C)$ is of exp. dimension $(\rho < r-2)$

If $\rho(g,r,d) - (2d+1-g) + \frac{r(r+3)}{2} < 0 \Rightarrow \Sigma_d^r(C) = \phi \iff \phi_L$ inj $\forall L \in W_d^r(C)$

F, Ortega: Strong MRC for $g=11$ (Mukai theory)

Divisors on $\overline{M}_{22}$ and $\overline{M}_{23}$: $r=6$, $d=g+3$

$g=22, 23$

$$\dim W_{g+3}^6(C) = g-21 = \begin{cases} 1, & g=22 \\ 2, & g=23. \end{cases}$$

$$\begin{tikzcd} S^2H^0(L) \arrow[rr, "\phi_L"] \arrow[dr] & & H^0(L^2) \arrow[dl] \\ & W_{g+3}^6(C) & \end{tikzcd}$$

$28 \qquad g+7$

$g-21$

Exp-codim for non-inj $g-20$

$$\left.\begin{cases} \mathfrak{D}_{22} = \{ [C] \in M_{22} : \exists L = g_{25}^6 \ \phi_L \text{ not injective} \} \\ \mathfrak{D}_{23} = \{ [C] \in M_{23} : \exists L = g_{26}^6 \ \phi_L \ \text{-- '' --} \} \end{cases}\right\} \text{expected divisors}$$

$K_g \leq \mathfrak{D}_g$ ok.

4

Thm 1: (FJP) The slope of the classes of closure these divisors are:

$$s([\widetilde{\mathcal{D}}_{22}]^{virt}) = \frac{17121}{2636} = 6.495\ldots < \frac{13}{2}$$

$$s([\widetilde{\mathcal{D}}_{23}]^{virt}) = \frac{470749}{72725} = 6.473\ldots < \frac{13}{2}$$

$$\begin{array}{ccl} G^r_d & \hookrightarrow & \widetilde{G}^r_d = \text{(generalized) limit } g^r_d\text{'s} \\ \downarrow & & \text{proper} \downarrow \sigma : \text{ρ-dim'l fibres} \\ M_g & \hookrightarrow & \widetilde{M}_g \end{array}$$

$\widetilde{M}_g = M_g \cup \Delta_0 \cup \Delta_1 \setminus$ (codim 2 stuff)

$[\widetilde{\mathcal{D}}_g]^{virt} = a\lambda - b_0\delta_0 - b_1\delta_1$

$\exists$ locally free sheaves $\mathcal{E}, \mathcal{F} / \widetilde{G}^r_d$

$rk(\mathcal{E}) = 7 \quad rk(\mathcal{F}) = g+7$

$\phi : S^2\mathcal{E} \to \mathcal{F}$ s.t.

$\forall [C, L] \in G^r_d \quad \phi_{(C,L)} : S^2H^0(L) \to H^0(L^2)$

ordinary multiplication

$$\boxed{[\widetilde{\mathcal{D}}_g]^{virt} = \sigma_*\left(c_{g-20}(\mathcal{F} - S^2\mathcal{E})\right) \in CH^1(\widetilde{M}_g)}$$

Test curves: $g = 23$: $[C] \in M_{22}$.

$C_1 = \left\{ \cancel{\Big/_y} \ \text{ell} \right\} \subseteq \Delta_1 \subseteq \overline{M}_{23}$: $\underset{\text{3-fold}}{\sigma^*(C_1)} \supseteq \boxed{Y = \left\{ (y, L) : y \in C,\ L = g^6_{26} \text{ w/ cusp at } y \right\}}$

$\{y \in C\}$ $\qquad \underbrace{\qquad\qquad}_{\text{3-fold}} \subseteq C \times W^6_{26}(C)$

$[\widetilde{\mathcal{D}}_{23}]^{virt} \cdot C_1 = c_3(\mathcal{F} - S^2\mathcal{E})|_Y$

$$= c_3\left(\left\{ \underbrace{\overset{28}{S^2H^0(L)}}_{\mathcal{E}} \longrightarrow \underbrace{\overset{30}{H^0(L^2-2y) \oplus \langle M^2\rangle}}_{\mathcal{F}} \right\} \right)$$

$H^0(L)/H^0(L-2y) = \langle M \rangle$ ↓ cokernel bundle.

$\searrow Y : 3\text{-fold} \swarrow$: Harris–Tu

G^6_{25} ↓ 2-dim. $\mathcal{M}_{23}$

$C_0 = \{$  $\}$ $q \sim y$ $y \in C$ moves $\subseteq \Delta_0 \subseteq \overline{\mathcal{M}}_{23} \longrightarrow$

'04: syzygy divs on $\overline{\mathcal{M}}_{22}$: $L = g^{10}_{30}$, ρ^{10} $\quad K_{2,2}(C,L) = 0$

$G^{10}_{30} \xrightarrow{\text{finite}} \mathcal{M}_{22}$ $\quad s(\ldots) = 6.5003\ldots$

Strong MRC holds in these cases on $\mathcal{M}_{22}$ and $\mathcal{M}_{23}$:

Thm 2: (FJP): For generic C: $\phi_L : S^2H^0(L) \longrightarrow H^0(L^2)$ inj. $\forall L \in W^6_{g+3}(C)$

$S^2\mathcal{E} \xrightarrow{\phi} \mathcal{F}$ $\quad \mathcal{U} = $ deg. locus of ϕ : exp. codim $= \begin{cases} 2 & g=22 \\ 3 & g=23 \end{cases}$

$\searrow \swarrow$ $\quad \sigma(\mathcal{U}) \subsetneqq \widetilde{\mathcal{M}}_g$: to rule out $\exists\ \mathcal{U}' \subseteq \mathcal{U}$ component

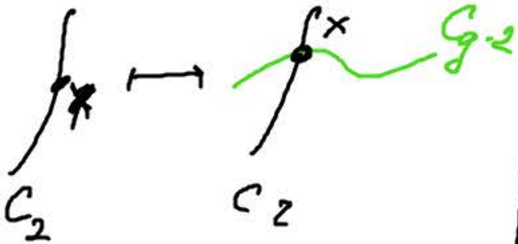
$\widetilde{G}^6_{g+3} \xrightarrow{\sigma} \widetilde{\mathcal{M}}_g$ $\quad$ mapping $\mathcal{U}' \xrightarrow{\sigma} Z$ w/ pos. dim. fibres.

still to do.. (2018)

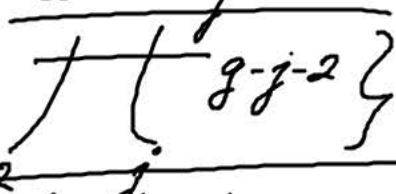
Strategy to rule out existence of $Z \subseteq \overline{\mathcal{M}}_g$:

$\mathcal{U} \xrightarrow{\ \sigma\ } Z \subseteq \overline{\mathcal{M}}_g$: $[C] \in Z$, $\exists\, L \in W^6_{g+3}(C)$, L ramified at p s.t. ϕ_L is not injective.
$\forall p \in C$

$j_2 : \overline{\mathcal{M}}_{2,1} \longrightarrow \overline{\mathcal{M}}_g$



$j_3 : \overline{\mathcal{M}}_{3,1} \longrightarrow \overline{\mathcal{M}}_g$
defined similarly

$\Delta_{2,j} = \Big\{$  $\Big\} \subseteq \overline{\mathcal{M}}_g$

- $j_2^*(Z) = 0$
- $\Delta_{2,j} \not\subseteq Z \quad \forall j$
- $j_3^*(Z) \subseteq$ Weierstrass div. on $\overline{\mathcal{M}}_{3,1}$ $\cup$ Hyperelliptic divisor

$\Longrightarrow [Z] = 0 \in CH^1(\overline{\mathcal{M}}_g) \Longrightarrow \boxed{Z = 0.}$

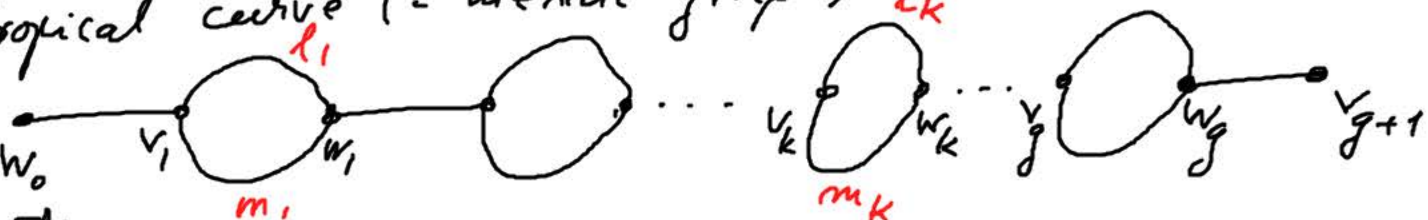
(strong MRC along divisors)

The proof of strong MRC using tropical geometry:

Recall: $g = 22, 23$, C-generic : $\phi_L : S^2H^0(L) \longrightarrow H^0(L^2)$ inj $\forall L \in W^6_{g+3}(C)$. (To do!)

Start w/ tropical curve (= metric graph):

Γ :

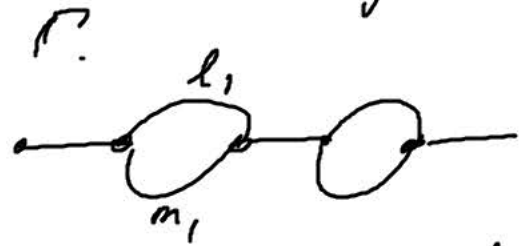


$\hookrightarrow$ hyperelliptic

$\mathbb{K}$ = alg. closed valued field
$\cup\!\!|$
R valuation ring

$$\begin{array}{ccc} X & \subseteq & \mathcal{X} \\ \downarrow & & \downarrow \\ \underbrace{\mathbb{K}}_{\text{smooth curve}} & & R \end{array}$$

: model whose degeneration is Γ:



$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ R \end{array}$$

$xy = t^{l_1}$ at

$xy = t^{m_1}$ at

$\underbrace{X^{an}}_{\text{analytification}} \longrightarrow \Gamma$: pts of Γ can be viewed as valuations of $k(X)$

Tropicalization in 2 guises:

$\mathrm{Trop}: \mathrm{Div}(X) \longrightarrow \mathrm{Div}(\Gamma)$

$\mathrm{trop}: k(X)^* \longrightarrow PL(\Gamma)$ $\quad f \mapsto \mathrm{trop}(f)(v) = \mathrm{val}_v(f)$ piecewise linear w/ $\mathbb{Z}$-slopes

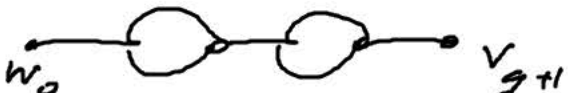
$\mathrm{Trop}(\mathrm{div}(f)) = \mathrm{div}(\mathrm{trop}(f))$:

$$\left.\begin{array}{l} D_X \in \mathrm{Div}(X), \ D = \mathrm{Trop}(D_X). \\ V^{r+1} \subseteq H^0(X, D_X) \\ g^r_d \end{array}\right\} \Longrightarrow \begin{array}{l} \mathrm{Trop}(V^{r+1}) \subseteq \mathcal{L}(D) = \{\varphi \in PL(\Gamma) : \mathrm{div}\,\varphi + D \geq 0\} \\ \mathrm{Trop}(W^r_d(X)) \subseteq W^r_d(\Gamma) \end{array}$$

<u>Thm</u>: (FJP): $g = 22, 23$: $X/\mathbb{K}$ smooth curve w/ minimal skeleton of X^{an} is Γ. Then φ_L inj $\forall L \in W^6_{g+3}(X)$ (+ condition of lengths of Γ)

Ingredients: 1) Description of $W^r_d(\Gamma)$

$d \ge g$. $\forall D \in \mathrm{Div}^d(\Gamma)$, $r(D) \ge r$, $\exists!\, D'$



$D' = (d-g)w_0 + x_1 + \dots + x_i$

$x_i \in \mathrm{Loop}_i$

$$\boxed{W^r_d(\Gamma) = \bigcup_{\substack{\text{standard tableau} \\ \lambda : (r+1) \times (g-d+r) \to \{1,\dots,g\}}} T^\lambda \to \text{torus}}$$

free coordinates are those x_i, $i \notin \mathrm{Im}\,\lambda$.

$\dim W^r_d(\Gamma) = \rho$. { generic case is vertex avoiding }

$\leftarrow$ Cools, Payne, Draisma ...

2) Tropical independence: $\phi_{\mathcal{L}} : S^2 H^0(X,\mathcal{L}) \to H^0(X,\mathcal{L}^2)$

$f_0 \dots f_6$ basis of $H^0(\mathcal{L})$ $\quad \sum a_{ij} f_i f_j = 0 \in H^0(X,\mathcal{L}^2)$

$\forall v \in \Gamma$: $\min_{ij}\{ v(a_{ij}) + \mathrm{trop}(f_i)(v) + \mathrm{trop}(f_j)(v) \}$ attained twice

$\mathrm{trop}(f_i) \in PL(\Gamma)$ explicit

$\Updownarrow$ $\exists\, b_{ij} \in \mathbb{R}$ s.t. $\mathrm{trop}(f_i) + \mathrm{trop}(f_j) + b_{ij}$ each achieves minimum alone for some $v \in \Gamma$ $\to$ construct tropical independence.

In vertex avoiding case: $\forall i$: $D \sim D_i$ ($\exists!$) $\quad D_i \ge i\, w_0 + (r-i)\, v_{g+1}$ algorithm is simpler ...