IRRATIONAL COMPONENTS OF THE HILBERT SCHEME OF POINTS

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ABSTRACT. We construct irrational irreducible components of the Hilbert scheme of points of affine space $\mathbb{A}^n$ for $n \geqslant 12$. We start with irrational components of the Hilbert scheme of curves in $\mathbb{CP}^3$ and use methods developed by Jelisiejew to relate these to irreducible components of the Hilbert schemes of points of $\mathbb{A}^n$. The result solves Problem XX of the Hilbert scheme of points problem list [J23].

1. INTRODUCTION

The Hilbert scheme $\mathrm{Hilb}(\mathbb{A}^n)$ of points, parametrizing finite subschemes of $\mathbb{A}^n$, plays an important role in current research in algebraic geometry with connections to many fields including commutative algebra, combinatorics, enumerative geometry, representation theory, singularity theory, topology, and complexity theory. The Hilbert scheme decomposes into connected components,

$$\mathrm{Hilb}(\mathbb{A}^n) = \bigsqcup_{d=0}^{\infty} \mathrm{Hilb}_d(\mathbb{A}^n),$$

where each $\mathrm{Hilb}_d(\mathbb{A}^n)$ parametrizes finite schemes of length d. While $\mathrm{Hilb}_d(\mathbb{A}^n)$ is smooth and irreducible for $n \leqslant 2$, the Hilbert scheme of points is singular and reducible if $n \geqslant 3$ and d is large. For each d, there is a distinguished irreducible component of $\mathrm{Hilb}_d(\mathbb{A}^n)$, called the *smoothable* component, whose general points parametrize reduced subschemes. The smoothable component is generically smooth of dimension dn and is rational, see [LR11, Theorem C]. Very little is known about the geometry of the other components.

We investigate the classical problem of rationality. While much is known about the birational geometry of moduli spaces of varieties of positive dimension, the situation is markedly different for 0-dimensional objects. There are very few results concerning the birational type of non-smoothable components. The lack of intrinsic geometric structure of 0-dimensional objects makes describing them harder. For example, methods for certifying rationality based on Gröbner strata [LR11] are successful in case of Hilbert schemes of varieties of positive dimension, but in the case of points they do not provide information about components other than the smoothable one.

The following problem has been open for a long time, see, for example, [J23, Problem XX] or [AIM10, Problem 1.4].

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Problem 1. *Does there exist an irrational irreducible component of the Hilbert scheme of points* $\mathrm{Hilb}(\mathbb{A}^n)$*?*

Since components of $\mathrm{Hilb}(\mathbb{A}^n)$ may be non-reduced in their natural scheme structure, any birational attribute of a component of $\mathrm{Hilb}(\mathbb{A}^n)$ refers to the underlying reduced algebraic variety. All the components that have been described explicitly by standard constructions, whether reduced or not, are obtained by constructing loci $\mathcal{L} \subseteq \mathrm{Hilb}(\mathbb{A}^n)$ which are isomorphic to open subsets of products of Grassmannians [J19, Remark 6.10]. In particular, all such constructions yield rational components.

Our main result is a positive answer to Problem 1. Specifically, for each $n \geqslant 12$, we show the existence of components in $\mathrm{Hilb}(\mathbb{A}^n)$ that are not rational (and not even rationally connected). In characteristic zero, we prove a stronger result: for each $n \geqslant 12$, the Hilbert scheme $\mathrm{Hilb}(\mathbb{A}^n)$ contains irreducible components with MRC-fibrations of arbitrary large dimension (a measure of irrationality). Our construction is inspired by ideas in the study of Murphy's Law for singularities [V06, E12, J20], where pathologies are transferred across moduli spaces of different types.

In Section 3, we perform a reduction from arbitrary graded Hilbert schemes[1] of a polynomial ring to graded Hilbert schemes parametrizing finite graded algebras. Then, in Section 4, we construct dominant morphisms of components of $\mathrm{Hilb}(\mathbb{A}^{n\geqslant 12})$ to the moduli space of curves, making essential use of Jelisiejew's TNT frames [J20] and the Hilbert scheme of curves in $\mathbb{CP}^3$.

The bound $n \geqslant 12$ arises for us as follows. We start with an irrational component of the Hilbert scheme of $\mathbb{CP}^m$ for some m. All such Hilbert schemes are rational for $m \leqslant 2$, so $\mathbb{CP}^3$ (with irrational components obtained from Hilbert schemes of curves) is the smallest dimensional projective space that we can use. The Hilbert scheme of curves in $\mathbb{CP}^3$ is locally a graded Hilbert scheme in 4 homogeneous variables x_0, x_1, x_2, x_3. In order to use Jelisiejew's TNT frames, we must first add two more variables x_4 and x_5 and then double the total number of variables. So, 12 arises as $2 \cdot (4+2)$. Whether irrational components occur in $\mathrm{Hilb}(\mathbb{A}^{n<12})$ is an interesting question.

Another question concerns the number of points d required for $\mathrm{Hilb}_d(\mathbb{A}^{12})$ to have an irrational component. The genus and the degree of the curves parameterized by the irrational component of the Hilbert scheme of $\mathbb{CP}^3$ determines a corresponding point number d for $\mathbb{A}^{12}$. We have made no effort to optimize (or even to bound) the very large d required for our construction.

[1] By *graded Hilbert scheme*, we mean the Hilbert scheme of homogeneous ideals of a polynomial ring with respect to the standard degree [HS04]. Detailed definitions can be found in Section 2.

2. Preliminaries

Let $\Bbbk$ denote an algebraically closed field. Let $S = \Bbbk[x_0, \dots, x_m]$ be the polynomial ring in $m+1$ variables. If M is a finitely generated graded S-module, the Hilbert function

$$\mathrm{HF}(M)\colon \mathbb{N} \to \mathbb{N}$$

is defined by $\mathrm{HF}(M; d) = \dim_{\Bbbk}[M]_d$, where $[M]_d$ denotes the graded component of degree d. The Hilbert function agrees with the Hilbert polynomial $\mathrm{HP}(M; z) \in \mathbb{Q}[z]$ for sufficiently large $d \in \mathbb{N}$.

Let $m \in \mathbb{N}$ and $\mathfrak{p}(z) \in \mathbb{Q}[z]$. We denote by $\mathrm{Hilb}_{\mathfrak{p}(z)}(\mathbb{P}^m)$ the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^m$ with Hilbert polynomial $\mathfrak{p}(z)$. Equivalently, $\mathrm{Hilb}_{\mathfrak{p}(z)}(\mathbb{P}^m)$ parametrizes saturated homogeneous ideals

$$I \subseteq S = \Bbbk[x_0, \dots, x_m]$$

such that the algebra S/I has Hilbert polynomial $\mathfrak{p}(z)$. For simplicity, we use the same symbol I to denote both the ideal and the associated $\Bbbk$-point of the Hilbert scheme,

$$I \in \mathrm{Hilb}_{\mathfrak{p}(z)}(\mathbb{P}^m).$$

Let S be a polynomial ring over $\Bbbk$, and let $\mathfrak{h}\colon \mathbb{N} \to \mathbb{N}$ be a function. We denote by $\mathcal{H}^{\mathfrak{h}}(S)$ the graded Hilbert scheme parametrizing homogeneous ideals $I \subseteq S$ such that the algebra S/I has Hilbert function equal to $\mathfrak{h}$ [HS04].

A smooth and proper variety X is *rationally connected* if, through every pair of points of X, there exists a rational curve contained in X. Rational varieties are rationally connected.

Let X be a smooth proper variety over an algebraically closed field $\Bbbk$ of characteristic zero. The maximal rationally connected fibration, or *MRC-fibration* [K96, Definition IV.5.3], of X is a dominant rational map

$$\varphi\colon X \dashrightarrow \mathrm{MRC}(X)$$

with rationally connected fibers which is maximal (in the appropriate sense) with respect to the property of having rationally connected fibers. The variety $\mathrm{MRC}(X)$ is called the MRC-quotient of X. The MRC-fibration is uniquely determined up to birational equivalence and is functorial for dominant maps. The variety $\mathrm{MRC}(X)$ is a point if and only if X is rationally connected, and $\mathrm{MRC}(X) = X$ if and only if X is not uniruled. We define the MRC dimension of X to be $\dim(\mathrm{MRC}(X))$, clearly a birational invariant of X. If X is not smooth or proper, we extend the notions above by taking a desingularization of a compactification of X. See [K96, Section IV.5] for details on MRC-fibrations.

3. Reduction to finite graded algebras

We show here a general result of independent interest: every graded Hilbert scheme of a polynomial ring is isomorphic to a graded Hilbert scheme that parametrizes finite graded algebras.

We begin by recalling some definitions and facts about Hilbert functions and syzygies. Let $S = \Bbbk[x_0, \dots, x_m]$ be a polynomial ring as before. A monomial ideal $L \subseteq S$ is a *lexsegment ideal* if every graded component of L is spanned by an initial segment of monomials of S, where monomials are ordered lexicographically. By Macaulay's Theorem [BH98, Theorem 4.2.10], there is a bijection between Hilbert functions of quotient algebras of S and lexsegment ideals of S. If $\mathfrak{h} = \mathrm{HF}(S/I)$ for some homogeneous ideal $I \subseteq S$, we denote by $\mathrm{Lex}(\mathfrak{h})$ the unique lexsegment ideal of S with $\mathfrak{h} = \mathrm{HF}(S/\mathrm{Lex}(\mathfrak{h}))$.

Let $I \subseteq S$ be a homogeneous ideal. The integers $\beta_{i,j}(I) = \dim_{\Bbbk}[\mathrm{Tor}_i(I, \Bbbk)]_j$ are the graded Betti numbers of I. The Castelnuovo-Mumford regularity is

$$\mathrm{reg}(I) = \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

The degrees of the minimal generators of I are bounded by $\mathrm{reg}(I)$. We have

$$\mathrm{HF}(S/I; d) = \mathrm{HP}(S/I; d)$$

for all $d \geqslant \mathrm{reg}(I)$. By [P96, Theorem 31], the lexsegment ideal $\mathrm{Lex}(\mathfrak{h})$ attains the largest Betti numbers $\beta_{i,j}$ among all homogeneous ideals with Hilbert function $\mathfrak{h}$, for every i, j.

For an integer $B \in \mathbb{N}$, we denote by $I_{\geqslant B}$, respectively, by $I_{\leqslant B}$, the ideal generated by all the homogeneous polynomials of I of degree at least B, respectively, at most B.

Our main result concerning the reduction to finite graded algebra is the following.

Theorem 2. *Let* $S = \Bbbk[x_0, \dots, x_m]$, $\mathbf{m} = (x_0, \dots, x_m)$, *and* $\mathfrak{h}\colon \mathbb{N} \to \mathbb{N}$ *be a function. For* $D \in \mathbb{N}$, *let* $\overline{\mathfrak{h}} : \mathbb{N} \to \mathbb{N}$ *be the function defined by*

$$\overline{\mathfrak{h}}(d) = \begin{cases} \mathfrak{h}(d) & \text{if } d < D, \\ 0 & \text{if } d \geqslant D. \end{cases}$$

Then, for all $D \gg 0$, *there is an isomorphism between the graded Hilbert schemes* $\mathcal{H}^{\mathfrak{h}}(S)$ *and* $\mathcal{H}^{\overline{\mathfrak{h}}}(S)$.

Proof. By [P96, Theorem 31], there exists an upper bound $C \in \mathbb{N}$ for the Castelnuovo-Mumford regularity of all ideals $I \in \mathcal{H}^{\mathfrak{h}}(S)$.

By applying [DM93, Proposition 3.8] or [CS05, Corollary 2.7], there exists an upper bound $D_1 \in \mathbb{N}$, depending only on C and m, for the regularity of all homogeneous ideals of S generated by polynomials of degree at most C. Choose an integer $D \geqslant D_1 + m + 1$.

Consider the morphisms $\theta\colon \mathcal{H}^{\mathfrak{h}}(S) \to \mathcal{H}^{\bar{\mathfrak{h}}}(S)$ and $\rho\colon \mathcal{H}^{\bar{\mathfrak{h}}}(S) \to \mathcal{H}^{\mathfrak{h}}(S)$ defined by the assignments

$$\theta(I) = I + \mathbf{m}^D \quad \text{and} \quad \rho(J) = J_{\leqslant C}.$$

We claim that these morphisms are inverse to each other.

For every ideal $I \in \mathcal{H}^{\mathfrak{h}}(S)$, $S/\theta(I)$ certainly has Hilbert function equal to $\bar{\mathfrak{h}}$, so we have $\theta(I) \in \mathcal{H}^{\bar{\mathfrak{h}}}(S)$. Moreover, since I is generated in degrees at most equal to C,

$$I = I_{\leqslant C} = (I + \mathbf{m}^D)_{\leqslant C},$$

since $D > C$. Therefore, $\rho(\theta(I)) = I$.

Consider, in particular, the lexsegment ideal $L = \mathrm{Lex}(\mathfrak{h}) \in \mathcal{H}^{\mathfrak{h}}(S)$ with Hilbert function $\mathfrak{h}$. Then,

$$\bar{L} = \theta(L) = L + \mathbf{m}^D \in \mathcal{H}^{\bar{\mathfrak{h}}}(S)$$

is again a lexsegment ideal (with Hilbert function now $\bar{\mathfrak{h}}$). Since L is generated in degrees up to C, $\bar{L}$ is generated in degrees up to C and in degree D. By [P96, Theorem 31], every ideal $J \in \mathcal{H}^{\bar{\mathfrak{h}}}(S)$ is generated, at most, in degrees up to C and in degree D.

Consider now any ideal $J \in \mathcal{H}^{\bar{\mathfrak{h}}}(S)$. By the conclusion of the previous paragraph, the graded components of the ideals J and $\rho(J) = J_{\leqslant C}$ agree up to degree $D-1$. In other words,

$$\mathrm{HF}(S/\rho(J); d) = \bar{\mathfrak{h}}(d) = \mathfrak{h}(d)$$

for all $d \leqslant D-1$ (in particular, for all $d \leqslant D_1 + m$). Since $\rho(J)$ is generated in degrees up to C, it follows, by choice of D_1, that the regularity of $\rho(J)$ is at most equal to D_1. Therefore,

$$\mathrm{HP}(S/\rho(J); d) = \mathrm{HF}(S/\rho(J); d)$$

for all $d \geqslant D_1$. Let $\mathfrak{p}(z)$ be the Hilbert polynomial of the Hilbert function $\mathfrak{h}$. By choice of C, we have $\mathfrak{p}(d) = \mathfrak{h}(d)$ for all $d \geqslant C$. In conclusion, we have

$$\mathrm{HP}(S/\rho(J); d) = \mathrm{HF}(S/\rho(J); d) = \bar{\mathfrak{h}}(d) = \mathfrak{h}(d) = \mathfrak{p}(d)$$

for all $d = D_1, D_1+1, \ldots, D_1+m$. Since $\mathrm{HP}(S/\rho(J); d)$ and $\mathfrak{p}(d)$ are univariate polynomials of degree at most m, they must be the same polynomial. We deduce

$$\mathrm{HF}(S/\rho(J); d) = \mathfrak{p}(d) = \mathfrak{h}(d)$$

for all $d \geqslant D_1$, in other words, $S/\rho(J)$ has Hilbert function equal to $\mathfrak{h}$, and $\rho(J) \in \mathcal{H}^{\mathfrak{h}}(S)$.

Since J is generated in degrees up to C and in degree D and $J_{\geqslant D} = \mathbf{m}^D$, we obtain

$$J = J_{\leqslant C} + \mathbf{m}^D.$$

Therefore, $\theta(\rho(J)) = J$ as desired. □

Theorem 2 improves [E12, Proposition 3.1], which yields an isomorphism of complete local rings $\hat{\mathcal{O}}_{\mathcal{H}^{\mathfrak{h}}(S), I} \cong \hat{\mathcal{O}}_{\mathcal{H}^{\bar{\mathfrak{h}}}(S), I+\mathbf{m}^D}$ for saturated homogeneous ideals $I \subseteq S$.

4. Dominant maps between components of Hilbert schemes

We prove here the main result of the paper by constructing dominant rational maps between irreducible components of various Hilbert schemes. An important tool for us is the theory of TNT frames of [J20], which produces dominant rational maps in the form of local retractions. A *retraction* is a morphism of schemes $\pi\colon X \to Y$ with a section,

$$\iota\colon Y \to X \quad \text{such that} \quad \pi \circ \iota = \mathrm{id}_Y .$$

A *local retraction* of pointed schemes $(X,x) \to (Y,y)$ is a retraction $(U,x) \to (V,y)$ for open subsets $x \in U \subseteq X$ and $y \in V \subseteq Y$.

Theorem 3. *There exist irreducible components of* $\mathrm{Hilb}(\mathbb{A}^{12})$ *that are not rationally connected. If* $\mathrm{char}(\mathbb{k}) = 0$*, there exist irreducible components of* $\mathrm{Hilb}(\mathbb{A}^{12})$ *of arbitrarily large MRC dimension.*

Proof. Let $\mathscr{M}_g$ be the moduli space of curves of genus g. We will prove the claim by constructing a dominant rational map

$$\mathcal{C} \dashrightarrow \mathscr{M}_g ,$$

where $\mathcal{C} \subseteq \mathrm{Hilb}(\mathbb{A}^{12})$ is an irreducible component. We will realize the map as a composition of a series of dominant morphisms $(H_{i+1}, I_{i+1}) \to (H_i, I_i)$, where each H_i is an irreducible open subscheme of a certain Hilbert scheme and $I_i \in H_i$ is an ideal parametrized by that Hilbert scheme.

Fix an arbitrary $g \geqslant 22$ and consider smooth curves in $\mathbb{P}^3$ of genus g and sufficiently large degree $d \geqslant \frac{3g+12}{4}$ embedded by a complete linear system. There exists a unique irreducible component

$$Y \subseteq \mathrm{Hilb}_{dz+1-g}(\mathbb{P}^3)$$

dominating the moduli space $\mathscr{M}_g$, see [E86]. For a general curve $[C \subseteq \mathbb{P}^3]$ parametrized by Y, the Hilbert function of the coordinate ring is determined by the *Maximal Rank Theorem* [BE85, L17] asserting that all multiplication maps

$$\mathrm{Sym}^a H^0(C, \mathcal{O}_C(1)) \to H^0(C, \mathcal{O}_C(a))$$

are all of maximal rank for all integers $a \geqslant 1$. By comparing dimensions, it follows that the curve C lies on no quadric surface. There exists an open subset $H_1 \subseteq Y$ where the Hilbert function of S/I is constant for all $I \in H_1$. Let $\mathfrak{h}$ denote this Hilbert function, and let $I_1 \in H_1$ be an ideal that belongs to no other component of $\mathrm{Hilb}_{dz+1-g}(\mathbb{P}^3)$.

Let $S = \mathbb{k}[x_0, x_1, x_2, x_3]$ and consider the saturated ideal $I_2 = I_1$ as a point in the graded Hilbert scheme $\mathcal{H}^{\mathfrak{h}}(S)$. The natural map $\mathcal{H}^{\mathfrak{h}}(S) \to \mathrm{Hilb}_{dz+1-g}(\mathbb{P}^3)$ restricts to an isomorphism $(H_2, I_2) \cong (H_1, I_1)$ for some $H_2 \subseteq \mathcal{H}^{\mathfrak{h}}(S)$ open and irreducible.

Let $P = S[x_4, x_5] = \Bbbk[x_0, \dots, x_5]$, let $I_3 = I_2 \cdot P$ be the extended ideal, and let $\mathfrak{h}' = \mathrm{HF}(P/I_3)$. Then, there is a local retraction

$$(\mathcal{H}^{\mathfrak{h}'}(P), I_3) \to (\mathcal{H}^{\mathfrak{h}'}(S), I_2)$$

proceeding as in [J20, Proof of Theorem 1.3]. Thus, there is a dominant morphism

$$(H_3, I_3) \to (H_2, I_2),$$

for some open irreducible $H_3 \subseteq \mathcal{H}^{\mathfrak{h}'}(P)$. Since I_2 is saturated, we have $\mathrm{depth}(S/I_2) \geqslant 1$ and, therefore, $\mathrm{depth}(P/I_3) \geqslant 3$. Moreover, I_3 contains no quadrics by construction.

Let $D \in \mathbb{N}$ be an integer obtained by applying Theorem 2 to the graded Hilbert scheme $\mathcal{H}^{\mathfrak{h}'}(P)$, and let

$$I_4 = I_3 + (x_0, \dots, x_5)^D \subseteq P.$$

By Theorem 2, there exists an isomorphism $(H_4, I_4) \cong (H_3, I_3)$ for some open irreducible $H_4 \subseteq \mathcal{H}^{\overline{\mathfrak{h}}'}(P)$, where $\overline{\mathfrak{h}}' = \mathrm{HF}(P/I_4)$.

Let $T = P[y_0, \dots, y_5] = \Bbbk[x_0, \dots, x_5, y_0, \dots, y_5]$, and let $I_5 \subseteq T$ be a frame-like ideal for the ideal $I_3 \subseteq P$, following [J20, Definition 4.2]. Then, T/I_5 is a finite $\Bbbk$-algebra, so

$$I_5 \in \mathrm{Hilb}(\mathbb{A}^{12}).$$

Since $\mathrm{depth}(P/I_3) \geqslant 3$ and I_3 contains no quadrics, we can apply [J20, Proposition 4.10] and obtain a local retraction

$$(\mathrm{Hilb}(\mathbb{A}^{12}), I_5) \to (\mathcal{H}^{\overline{\mathfrak{h}}'}(P), I_4).$$

Therefore, there exists a dominant morphism $(H_5, I_5) \to (H_4, I_4)$ for some open irreducible $H_5 \subseteq \mathrm{Hilb}(\mathbb{A}^{12})$.

Finally, consider the irreducible component $\mathcal{C} = \overline{H}_5 \subseteq \mathrm{Hilb}(\mathbb{A}^{12})$. The composition of the maps

$$(H_5, I_5) \to (H_4, I_4) \cong (H_3, I_3) \to (H_2, I_2) \cong (H_1, I_1)$$

together with the dominant rational map $Y \dashrightarrow \mathscr{M}_g$ yields a dominant rational map

$$\mathcal{C} \dashrightarrow \mathscr{M}_g.$$

We obtain a dominant rational map $\mathcal{C}_{red} \dashrightarrow \mathscr{M}_g$ on the reduced structure.

By [EH87, HM82, FJP20], the moduli space $\mathscr{M}_g$ is of general type for $g \geqslant 22$. Therefore, $\mathcal{C}_{red}$ is not rationally connected. If $\mathrm{char}(\Bbbk) = 0$, by the functoriality of MRC-fibrations [K96, Theorem IV.5.5], we obtain a dominant rational map

$$\mathrm{MRC}(\mathcal{C}_{red}) \dashrightarrow \mathrm{MRC}(\mathscr{M}_g).$$

Since $\mathscr{M}_g$ is of general type, we have $\mathrm{MRC}(\mathscr{M}_g) = \mathscr{M}_g$. Thus, the MRC-dimension of $\mathcal{C}_{red}$ is at least $\dim \mathscr{M}_g = 3g - 3$. □

Corollary 4. *The conclusion of Theorem 3 holds for* $\mathrm{Hilb}(\mathbb{A}^n)$ *for all* $n \geqslant 12$.

Proof. Let $R = T[z_{13}, \dots, z_n]$ and $I_6 = I_5 \cdot R + (z_{13}, \dots, z_n) \subseteq R$, where $I_5 \subseteq T$ is as in the proof of Theorem 3. Let $\mathbb{G}_m = \Bbbk^*$ act on R with weight 1 on the variables z_i and weight 0 on the remaining variables, and consider the corresponding $\mathbb{Z}$-grading on R. We denote the positive[2] Bialynicki-Birula decomposition [JS19] for $\mathrm{Hilb}(\mathbb{A}^n)$ by $\mathrm{Hilb}(\mathbb{A}^n)^+$.

Since I_6 is generated in degrees 0 and 1 and R/I_6 is concentrated in degree 0, the tangent space $T_{\mathrm{Hilb}(\mathbb{A}^n), I_6} = \mathrm{Hom}(I_6, R/I_6)$ has no graded components of positive degree. It follows by [JS19, Proposition 1.6] that the natural map $\mathrm{Hilb}(\mathbb{A}^n)^+ \to \mathrm{Hilb}(\mathbb{A}^n)$ is an open immersion near I_6, and that we have a local retraction

$$(\mathrm{Hilb}(\mathbb{A}^n), I_6) \to (\mathrm{Hilb}(\mathbb{A}^n)^{\mathbb{G}_m}, I_6)$$

to the fixed locus. The connected component of $\mathrm{Hilb}(\mathbb{A}^n)^{\mathbb{G}_m}$ containing I_6 parametrizes homogeneous ideals of R with the same Hilbert function as I_6 and hence is isomorphic to the connected component of $\mathrm{Hilb}(\mathbb{A}^{12})$ containing I_5. We therefore have a local retraction

$$(\mathrm{Hilb}(\mathbb{A}^n), I_6) \to (\mathrm{Hilb}(\mathbb{A}^{12}), I_5),$$

and hence a dominant morphism $(H_6, I_6) \to (H_5, I_5)$ for some open irreducible $H_6 \subseteq \mathrm{Hilb}(\mathbb{A}^n)$. The conclusion follows as in the last paragraph of the proof of Theorem 3, by taking the composition with the dominant rational map $\mathcal{C} \dashrightarrow \mathscr{M}_g$. □

5. Questions

It is unclear to us whether the components $\mathcal{C}$ produced in Theorem 3 are reduced. A natural approach to answering this question, by [J19, Theorem 4.6], would involve showing the vanishing of the obstruction space $T^2(T/I_5)_{\geqslant 0}$ to prove the smoothness of the point $I_5 \in \mathcal{C}$. However, controlling this obstruction space is in general difficult. See, for example, the delicate arguments of [SS23] in the case of some considerably simpler ideals.

Question 5. Are any of the irreducible components of $\mathrm{Hilb}(\mathbb{A}^{12})$ constructed in the proof of Theorem 3 reduced?

It is natural to ask about the smallest ambient dimension in which irrational components exist. Though dimension 12 in Theorem 3 is required for our method, we would expect that $\mathrm{Hilb}(\mathbb{A}^n)$ contains irrational components also for smaller n. It would be especially interesting to see if irrational components exist already in $\mathrm{Hilb}(\mathbb{A}^3)$: dimension 3 is not only the smallest dimension where the question is open, it is an exceptional boundary case for Hilbert schemes in many respects [BBS13, BF08, GGGL23, JKS23, MNOP06, RS25, R23]. Thus, we ask the following upgraded version of Problem 1.

[2]This is the Bialynicki-Birula decomposition associated to limits $t \to 0$ (as opposed to the one in [J20], associated to limits $t \to \infty$).

Question 6. Does $\mathrm{Hilb}(\mathbb{A}^3)$ contain irrational components?

Following [BDELU17], the degree of irrationality $\mathrm{irr}(X)$ of an n-dimensional projective variety X is the minimal degree $\delta > 0$ of a dominant map $X \dashrightarrow \mathbb{P}^n$. We can ask the following question:

Question 7. Does $\mathrm{Hilb}(\mathbb{A}^n)$ contain components of unbounded degree of irrationality?

The answer to the parallel question for the moduli space $\mathscr{M}_g$ is not known, but somehow expected to be out of reach at present, see also [BDELU17, Problem 4.4]. It is conceivable that the case of $\mathrm{Hilb}(\mathbb{A}^n)$ is more approachable.

On a more speculative note, one can ask about the array of different birational types found in Hilbert schemes of points. It is clear that not all birational types can occur: for example, every component contains lots of rational curves, and therefore cannot be of general type. This is analogous to Murphy's law for singularities [V06]: we cannot expect to find all singularities on a moduli space, but rather, all singularity types, that is, equivalence classes induced by smooth morphisms. Inspired by the statement of Theorem 3, we ask the following birational analogue of Murphy's law:

Question 8. Do all birational types of MRC quotients occur in the Hilbert scheme of points?

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