SECANT LOCI ON MODULI OF PRYM VARIETIES

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To Yuri Tschinkel with friendship

ABSTRACT. We present a Prym analogue of Lazarsfeld's result that curves on general polarized K3 surfaces verify the Brill-Noether Theorem, or equivalently, that their canonical embedding has no unexpected secants. We show that the Prym-canonical embedding of a curve on a general Nikulin surface (both of standard and non-standard types) has no unexpected secants and explain how these two geometric facts suffice to determine the class of the difference divisor on the moduli space $\overline{\mathcal{R}}_g$ of stable Prym curves of (odd) genus g.

1. INTRODUCTION

The Brill-Noether Theorem, asserting that for a general curve C of genus g the varieties $W^r_d(C) := \{L \in \mathrm{Pic}^d(C) : h^0(C, L) \geq r + 1\}$ have the expected dimension

$$\rho(g, r, d) = g - (r + 1)(g - d + r)$$

is one of the cornerstones of the theory of algebraic curves, having multiple applications to enumerative geometry, or to the birational geometry of the moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g, see [EH]. Via Riemann-Roch, the Brill-Noether Theorem can be reformulated in terms of secant varieties to the canonical linear system of C. More generally, given a line bundle $L \in \mathrm{Pic}^d(C)$ with $h^0(C, L) = r + 1$ and positive integers $0 \leq f < e$, one denotes by C_e the e-th symmetric product of C and introduces the variety

(1) $$V^{e-f}_e(L) := \big\{Z \in C_e : h^0(C, L(-Z)) \geq r + 1 - e + f\big\}$$

of effective divisors Z of degree e which impose f conditions less than expected on $|L|$. Since $V^{e-f}_e(L)$ can be represented as a degeneracy locus on C_e globalizing the evaluation maps $\mathrm{ev}_Z\colon H^0(C, L) \to L_Z$, one obtains that every irreducible component of $V^{e-f}_e(L)$ has dimension at least equal to $e - f(r + 1 - e + f)$. With this terminology, observing that for $0 \leq r < d$ the variety $V^{d-r}_d(\omega_C)$ is isomorphic to the locus $C^r_d \subseteq C_d$ of effective divisors D such that $\dim |D| \geq r$ and thus $V^{d-r}_d(\omega_C) \to W^r_d(C)$ is generically a $\mathbf{P}^r$-bundle, the Brill-Noether Theorem is equivalent to the statement that for a general curve C of genus g, all varieties $V^{e-f}_e(\omega_C)$ have the expected dimension. In particular, Lazarsfeld's Theorem [La1] can then be formulated as saying that if X is a smooth $K3$ surface with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, where $H^2 = 2g - 2$, then for any smooth curve $C \in |H|$ the secant loci of ω_C have the expected dimension. For further statements on the applications of these loci to the Green-Lazarsfeld Secant Conjecture on syzygies of curves, we refer to [FK], or [F1].

Moving to the Prym setting, let $\mathcal{R}_g$ be the moduli space parameterizing pairs $[C, \eta]$, where C is a smooth curve of genus g and $\eta \in \mathrm{Pic}^0(C)$ is a 2-torsion point. The Prym moduli

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space $\mathcal{R}_g$ admits a Deligne-Mumford compactification $\overline{\mathcal{R}}_g := \overline{\mathcal{M}}_g(\mathcal{B}\mathbb{Z}_2)$ constructed by Cornalba [Cor], whose birational geometry has been described in detail in [FL]. In particular, $\overline{\mathcal{R}}_g$ has negative Kodaira dimension for $g \leq 9$, see [Don], [FV1], [FV2], [FV3], and it is a variety of general type for $g \geq 13$ and $g \neq 16$, see [FL], [CEFS], [FJP] and [Br]. For odd genus $g = 2i+1$, a key role in all these results is played by the (closure in $\overline{\mathcal{R}}_{2i+1}$ of the) *difference Prym divisor*

$$\mathcal{D}_{2i+1} := \Big\{[C,\eta] \in \mathcal{R}_{2i+1} : \eta \in C_i - C_i\Big\}, \tag{2}$$

where $C_i - C_i \subseteq \mathrm{Pic}^0(C)$ denotes the divisorial difference variety (cf. Section 4). The defining condition for points in $\mathcal{D}_{2i+1}$ can be reformulated as $V_i^{i-1}(\omega_C \otimes \eta) \neq \emptyset$, or equivalently, that the Prym-canonical embedding $\varphi_{\omega_C\otimes\eta} \colon C \hookrightarrow \mathbf{P}^{2i-1}$ possesses an i-secant $(i-2)$-plane. Using this perspective, $\mathcal{D}_{2i+1}$ becomes the exact Prym analogue of the *Hurwitz divisor*

$$\mathcal{M}^1_{2i+1,i+1} := \Big\{[C] \in \mathcal{M}_{2i+1} : V^i_{i+1}(\omega_C) \neq \emptyset\Big\} = \Big\{[C] \in \mathcal{M}_{2i+1} : \exists\, C \stackrel{i+1:1}{\longrightarrow} \mathbf{P}^1\Big\} \tag{3}$$

defined by Harris and Mumford [HM] and which proved to be instrumental in showing that $\overline{\mathcal{M}}_g$ is of general type for odd $g > 23$.

The first result we establish determines the dimension of the secant loci for the general Prym-canonical curve of genus g.

Theorem 1.1. *We fix integers $0 \leq f < e < g$. Then for a general Prym curve $[C,\eta] \in \mathcal{R}_g$ the secant locus $V_e^{e-f}(\omega_C \otimes \eta)$ is equidimensional of dimension*

$$\dim V_e^{e-f}(\omega_C \otimes \eta) = e - f(g-1-e+f).$$

In particular, if $e - f(g-1-e+f) < 0$, then $V_e^{e-f}(\omega_C \otimes \eta) = \emptyset$.

The proof of Theorem 1.1 uses degeneration and is presented in Section 2. In what follows, we discuss a more refined version of Theorem 1.1 for Prym curves on a special class of $K3$ surfaces.

1.1. **Nikulin surfaces and Prym varieties.** Lazarsfeld's result [La1] provides explicit examples of smooth curves lying on general $K3$ surfaces which satisfy the Brill-Noether Theorem, in particular, which lie outside the Hurwitz divisor $\overline{\mathcal{M}}^1_{2i+1,i+1}$. This geometric condition alone is sufficient to determine the slope $s\big([\overline{\mathcal{M}}^1_{2i+1,i+1}]\big) = 6 + \frac{12}{g+1}$ of the Hurwitz divisor. This slope is the relevant quantity that ultimately accounts for the bigness of the canonical bundle $K_{\overline{\mathcal{M}}_{2i+1}}$ for $i \geq 12$. In this paper, we show that in Prym setting, one has a similar result for curves lying on Nikulin surfaces and, in the process, we obtain a more precise version of Theorem 1.1.

We recall that, given a point $[C,\eta] \in \mathcal{R}_g$, the 2-torsion line bundle $\eta \in \mathrm{Pic}^0(C)$ defines an étale double cover of C; the specialization to curves C lying on Nikulin surfaces has the advantage of realizing such a cover as the restriction of a double cover of a $K3$ surface whose branch divisor does not intersect C. A *Nikulin surface* is a $K3$ surface X endowed with a double cover

$$f \colon \tilde{X} \longrightarrow X$$

branched over 8 disjoint smooth rational curves $N_1, \ldots, N_8 \subseteq X$. Blowing down the (-1)-curves $f^{-1}(N_j)$, one obtains a minimal $K3$ surface $\sigma \colon \tilde{X} \to Y$, together with an involution $\iota \in \mathrm{Aut}(Y)$ having 8 fixed points corresponding to the images of the curves $\sigma(f^{-1}(N_j))$. The class $\mathcal{O}_X(N_1 + \cdots + N_8)$ is divisible by 2 and we set $\mathfrak{e} \in \mathrm{Pic}(X)$ to be the class characterized by $\mathfrak{e}^{\otimes 2} \cong \mathcal{O}_X(N_1 + \cdots + N_8)$.

Let $\mathfrak{N}$ denote the Nikulin lattice defined as the even lattice of rank 8 generated by the classes of $N_1, \ldots, N_8$ and by $\mathfrak{e}$. A *polarized Nikulin surface* of genus g is a $K3$ surface X together with an embedding $\mathbb{Z} \cdot L \oplus_\perp \mathfrak{N} \hookrightarrow \mathrm{Pic}(X)$, such that L is a big and nef class on X, with $L^2 = 2g-2$. If this embedding is primitive, we say that X is a *standard* Nikulin surface, else we say that X is *non-standard*[1]. There are two irreducible 11-dimensional moduli spaces $\mathcal{F}_g^{\mathfrak{N}}$ respectively $\mathcal{F}_g^{\mathfrak{N},\mathrm{ns}}$ of standard (respectively non-standard) polarized Nikulin surfaces of genus g, see [Dol], [vGS] for details. We consider the following open subset of a $\mathbf{P}^g$-bundle over $\mathcal{F}_g^{\mathfrak{N}}$

$$\mathcal{P}_g^{\mathfrak{N}} := \Big\{(X,C) : C \subseteq X \text{ is a smooth curve such that } \big[X, \mathfrak{N} \oplus_\perp \mathcal{O}_X(C)\big] \in \mathcal{F}_g^{\mathfrak{N}}\Big\},$$

together with forgetful maps

$$\begin{array}{ccc} & \mathcal{P}_g^{\mathfrak{N}} & \\ {}^{p_g}\swarrow & & \searrow^{\chi_g} \\ \mathcal{F}_g^{\mathfrak{N}} & & \mathcal{R}_g \end{array}$$

where $p_g\big([X,C]\big) := [X, \mathcal{O}_X(C)]$ and $\chi_g\big([X,C]\big) := [C, \mathfrak{e}_C^\vee := \mathfrak{e}^\vee \otimes \mathcal{O}_C]$. In particular, χ_g can be used to find a uniform parametrization of $\mathcal{R}_g$ for small g, see [FV1, Theorem 0.2]. Keeping this notation throughout, we can now state our next result:

Theorem 1.2. *Let $[X, \mathfrak{N} \oplus_\perp \mathbb{Z} \cdot L] \in \mathcal{F}_g^{\mathfrak{N}}$ be a general standard Nikulin surface of genus $g \geq 6$. Then for every smooth irreducible curve $C \in |L|$, we have that $V_e^{e-f}(\omega_C \otimes \mathfrak{e}_C) = \emptyset$, whenever $e - f(g-1-e+f) < 0$. In particular, for odd genus $g = 2i+1$, we have that $\mathfrak{e}_C \notin C_i - C_i$.*

Theorem 1.2 thus provides the first examples of explicit smooth Prym curves of odd genus which are generic from the point of view of the Brill-Noether theory of the Prym-canonical system.

If standard Nikulin surfaces are instrumental in parametrizing $\overline{\mathcal{R}}_g$, non-standard Nikulin surfaces can be used to parametrize the boundary divisor Δ_0^{ram} of $\overline{\mathcal{R}}_g$ for low g, as explained in [KLV1] and [KLV2]. As explained in [Bud], the geometry of the divisor Δ_0^{ram} is intimately related to that of the moduli space $\overline{\mathcal{R}}_{g,2}$ of Prym varieties arising as ramified double covers of curves of genus g.

If Δ_0 is the boundary divisor of $\overline{\mathcal{M}}_g$ parametrizing irreducible nodal curves of genus g and their degenerations and $\pi \colon \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ is the forgetful morphism dropping the Prym structure on each curve, then

$$\pi^*(\Delta_0) = \Delta_0' + \Delta_0'' + 2\Delta_0^{\mathrm{ram}},$$

see [FL, §1]. In particular Δ_0^{ram} is the ramification divisor of π and its general point corresponds to a Prym curve $[X = C \cup_{\{x,y\}} \mathbf{P}^1, \eta] \in \overline{\mathcal{R}}_g$, where C is a smooth curve of genus $g-1$ meeting a smooth rational component $\mathbf{P}^1$ at two points x, y, whereas $\eta \in \mathrm{Pic}^0(X)$ satisfies

[1] Occasionally standard (respectively non-standard) Nikulin surfaces are referred to as being of the *first* (respectively of the *second* kind), see [FV1], [FV2]. The terminology used here is in line with [KLV1].

$\eta_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(1)$ and $\eta_C^{\otimes 2} \cong \mathcal{O}_C(-x-y)$. In other words, the line bundle η_C induces a degree 2 cover of C branched over x and y.

Non-standard Nikulin surfaces appear only for odd genus. If X is a non-standard Nikulin surface of genus $g \equiv 3 \bmod 4$, then following [vGS, Proposition 2.7], one can show that up to a permutation of the (-2)-curves $N_1, \ldots, N_8$, there exist classes R, R' such that

$$R \equiv \frac{L - N_1 - N_2}{2}, \qquad R' \equiv \frac{L - N_3 - \cdots - N_8}{2} \in \mathrm{Pic}(X). \tag{4}$$

Moreover, for a general point of $\mathcal{F}_g^{\mathfrak{N},\mathrm{ns}}$, one has that $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot R \oplus \mathfrak{N}$.

We now set $L^2 = 16i - 4$, therefore curves in the linear system $|L|$ have genus $8i-1$. Using (4) an irreducible nodal curve $C \in |R|$ has arithmetic genus $g = 2i$. Moreover

$$C \cdot N_1 = C \cdot N_2 = 1, \quad \text{while} \quad C \cdot N_i = 0 \ \text{ for } i = 3, \ldots, 8.$$

It follows that $f_{|f^{-1}(C)} \colon f^{-1}(C) \to C$ is a double cover ramified over two points $x_1 = C \cdot N_1$ and $x_2 = C \cdot N_2$. Equivalently,

$$\xi(C) := \bigl[C \cup_{\{x_1, x_2\}} \mathbf{P}^1,\ \eta_C := \mathfrak{e}_C^{\vee},\ \eta_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(1)\bigr] \in \Delta_0^{\mathrm{ram}} \subseteq \overline{\mathcal{R}}_{2i+1}. \tag{5}$$

One obtains a rational map $\xi \colon |R| \dashrightarrow \overline{\mathcal{R}}_{2i+1}$ and for a Lefschetz pencil $\Xi \cong \mathbf{P}^1 \subseteq |R|$, a corresponding pencil of Prym curves $\Xi^{\mathrm{ns}} := \xi(\Xi)$ of genus $2i+1$.

We present an analogue of Theorem 1.2 in the setting of non-standard Nikulin surfaces.

Theorem 1.3. *Let $[X, \mathfrak{N} \oplus_{\perp} \mathbb{Z} \cdot L] \in \mathcal{F}_{8i-1}^{\mathfrak{N},\mathrm{ns}}$ be a general non-standard Nikulin surface of genus $8i-1$. Then for every irreducible nodal curve $C \in |R|$, we have that $\xi\bigl([C]\bigr) \notin \overline{\mathcal{D}}_{2i+1}$.*

Thus, also non-standard Nikulin surfaces provide explicit examples of Prym curves that are general from the point of view of Prym-Brill-Noether theory. Theorems 1.2 and 1.3 provide explicit pencils of Prym curves on Nikulin surfaces which are disjoint from the closure $\overline{\mathcal{D}}_{2i+1}$ of the difference Prym divisor in $\overline{\mathcal{R}}_{2i+1}$. These two properties uniquely determine the slope of $[\overline{\mathcal{D}}_{2i+1}] \in \mathrm{Pic}(\overline{\mathcal{R}}_{2i+1})$, as we shall explain.

We denote by $\lambda \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$ the Hodge class and by $\delta_0' := [\Delta_0']$, $\delta_0'' := [\Delta_0'']$ and respectively $\delta_0^{\mathrm{ram}} := [\Delta_0^{\mathrm{ram}}]$ the corresponding boundary divisor classes, see also [FL, 1] for background on the Picard group of $\overline{\mathcal{R}}_g$. Then if $\Xi^{\mathrm{s}} \subseteq \overline{\mathcal{R}}_{2i+1}$ and $\Xi^{\mathrm{ns}} \subseteq \overline{\mathcal{R}}_{2i+1}$ denote pencils on a standard (respectively, non-standard) Nikulin surface of genus $2i+1$ (respectively $8i-1$) as above, then we have the following relations (see Proposition 5.1):

$$\Xi^{\mathrm{s}} \cdot \lambda = 2i+2,\ \ \Xi^{\mathrm{s}} \cdot \delta_0' = 12i+8,\ \ \Xi^{\mathrm{s}} \cdot \delta_0'' = 0 \ \text{ and } \ \Xi^{\mathrm{s}} \cdot \delta_0^{\mathrm{ram}} = 8,$$
$$\Xi^{\mathrm{ns}} \cdot \lambda = 2i+1,\ \ \Xi^{\mathrm{ns}} \cdot \delta_0' = 12i+6,\ \ \Xi^{\mathrm{ns}} \cdot \delta_0'' = 0 \ \text{ and } \ \Xi^{\mathrm{ns}} \cdot \delta_0^{\mathrm{ram}} = 4.$$

Imposing the condition that the intersection numbers of $[\overline{\mathcal{D}}_{2i+1}]$ with both pencils Ξ^{s} and Ξ^{ns} are equal to zero (which amounts to a slight refinement of both Theorems 1.2 and 1.3), we rederive one of the main results of [FL]:

Corollary 1.4. *The class $[\overline{\mathcal{D}}_{2i+1}]$ is given up to a positive constant by the following formula:*

$$[\overline{\mathcal{D}}_{2i+1}] = c\Bigl((3i+1)\lambda - \frac{i}{2}\bigl(\delta_0' + \delta_0''\bigr) - \frac{2i+1}{4}\delta_0^{\mathrm{ram}} - \cdots\Bigr) \in \mathrm{Pic}(\overline{\mathcal{R}}_{2i+1}).$$

The class $[\overline{\mathcal{D}}_{2i+1}]$ has numerous desirable features, above all, its very low slope in the (λ, δ_0')-plane of $\mathrm{Pic}(\overline{\mathcal{R}}_{2i+1})$. As such, the divisor $\overline{\mathcal{D}}_{2i+1}$ is used in the proof that $\overline{\mathcal{R}}_g$ is of general type for odd $g \geq 13$.

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2. Secant divisors of general Prym-canonical curves

For a smooth curve C and an integer $e > 0$, we denote by C_e its e-th symmetric product. We recall the definition of the varieties of secant divisors to a linear system on a curve C. We fix integers $0 \leq f < e$, a line bundle $L \in \mathrm{Pic}^d(C)$ and set $r := h^0(C, L) - 1$. Let us introduce the universal degree e effective divisor

$$\mathcal{Z} := \Big\{(Z, x) \in C_e \times C : x \in \mathrm{supp}(Z)\Big\}$$

and let $\pi_1 \colon C_e \times C \to C_e$ respectively $\pi_2 \colon C_e \times C \to C$ be the two projections. We define the rank e tautological bundle $E_L = L^{[e]} := (\pi_1)_*\big(\pi_2^*(L) \otimes \mathcal{O}_{\mathcal{Z}}\big)$ having fibres $E_{L|Z} \cong L_Z = L \otimes \mathcal{O}_Z$ over a point $Z \in C_e$ and consider the evaluation morphism

$$\mathrm{ev} \colon H^0(C, L) \otimes \mathcal{O}_{C_e} \longrightarrow E_L.$$

Then the variety of secant divisors $V_e^{e-f}(L)$ defined in (1) is defined as the degeneracy locus of those divisors $Z \in C_e$ for which $\mathrm{rk}\big(\mathrm{ev}_Z\big) \leq e - f$. The expected dimension exp.dim $V_e^{e-f}(L)$ is then accordingly equal to $e - f(r + 1 - e + f)$.

The study of the loci $V_e^{e-f}(\omega_C)$ is by definition equivalent to that of the Brill-Noether loci $W_d^r(C)$. For line bundles L of large degree, the Green-Lazarsfeld *Secant Conjecture* predicts a close relation between the non-emptiness of the varieties $V_e^{e-f}(L)$ and the non-vanishing of the Koszul cohomology groups $K_{p,q}(C, L)$ and we refer to [FK] and [F1] for new developments and details.

We take a Prym curve $[C, \eta] \in \mathcal{R}_g$ and let $L := \omega_C \otimes \eta$ be the Prym-canonical linear system. We fix integers $0 \leq f < e < g$. By Riemann-Roch, we then have the following equivalence

$$V_e^{e-f}(\omega_C \otimes \eta) \neq \emptyset \Longleftrightarrow \eta \in C_e - W_e^{f-1}(C) \subseteq \mathrm{Pic}^0(C). \tag{6}$$

We prove Theorem 1.1 by using degeneration. First, we recall the notation for vanishing and ramification sequences of linear series on curves following [EH]. If $\ell = (L, V) \in G_d^r(C)$ is a linear system on a smooth curve C, the *ramification sequence* of ℓ at a point $q \in C$

$$\alpha^\ell(q) : 0 \leq \alpha_0^\ell(q) \leq \cdots \leq \alpha_r^\ell(q) \leq d - r$$

is obtained from the *vanishing sequence*

$$a^\ell(q) : 0 \leq a_0^\ell(q) < \cdots < a_r^\ell(q) \leq d$$

by setting $\alpha_i^\ell(q) := a_i^\ell(q) - i$, for $i = 0, \ldots, r$. The *ramification weight* of q with respect to ℓ is defined as the quantity $\mathrm{wt}^\ell(q) := \sum_{i=0}^r \alpha_i^\ell(q)$.

A *limit linear series* on a curve C of compact type consists of a collection

$$\ell = \Big\{\ell_Y = (L_Y, V_Y) \in G^r_d(Y) : Y \text{ is a component of } C\Big\}$$

satisfying compatibility conditions in terms of the vanishing sequences at the nodes of C of the respective aspects, see [EH]. We denote by $\overline{G}^r_d(C)$ the variety of limit linear series of type g^r_d on C. More generally, if $q_1, \ldots, q_s \in C_{\mathrm{req}}$ are smooth points and

$$\alpha^i = \big(0 \leq \alpha^i_0 \leq \cdots \leq \alpha^i_r \leq d-r\big)$$

are *Schubert indices*, we denote by $\overline{G}^r_d\big(C, (q_1, \alpha^1), \ldots, (q_s, \alpha^s)\big)$ the variety of limit linear series $\ell \in \overline{G}^r_d(C)$ satisfying the conditions $\alpha^{\ell}(q_i) \geq \alpha^i$ for $i = 1, \ldots, s$. Each component of $\overline{G}^r_d\big(C, (q_1, \alpha^1), \ldots, (q_s, \alpha^s)\big)$ has dimension at least equal to $\rho(g, r, d) - \mathrm{wt}(\alpha^1) - \cdots - \mathrm{wt}(\alpha^s)$. Theorem 1.1 of [EH] offers adequate sufficient conditions when the actual dimension of the locus $\overline{G}^r_d\big(C, (q_1, \alpha^1), \ldots, (q_s, \alpha^s)\big)$ equals the expected dimension.

We can now state our first result:

Theorem 2.1. *Let $[C, \eta] \in \mathcal{R}_g$ be a general Prym curve of genus g and we fix integers $0 \leq f < e < g$. If $e - f(g-1-e+f) < 0$, then $V^{e-f}_e(\omega_C \otimes \eta) = \emptyset$.*

Proof. Assume the conclusion to be false, in particular, for every $[C, \eta] \in \mathcal{R}_g$, there exists $Z \in V^{e-f}_e(\omega_C \otimes \eta)$, therefore, by (6), we find $\eta(Z) \in W^{f-1}_e(C)$.

We degenerate to nodal curves and fix general pointed elliptic tails $[E_j, x_j] \in \mathcal{M}_{1,1}$ and non-trivial 2-torsion points $\eta_j \in \mathrm{Pic}^0(E_j)[2]$, for $j = 1, \ldots, g$. We then consider the map

$$j \colon \overline{\mathcal{M}}_{0,g} \longrightarrow \overline{\mathcal{R}}_g, \ \ [R, x_1, \ldots, x_g] \mapsto \big[C := R \cup_{x_1} E_1 \cup \ldots \cup_{x_g} E_g, \ \ \eta_R \cong \mathcal{O}_R, \ \eta_{E_j} \cong \eta_j\big]. \tag{7}$$

By assumption, on each curve C as above, after possibly inserting chains of smooth rational curves at the points $x_1, \ldots, x_g$, or at the nodes of R, there exists a refined limit linear series $\ell \in \overline{G}^{f-1}_e(C)$ and an effective divisor Z of total degree e on C, such that if $Z_j \subseteq Z$ denotes the (possibly empty) subdivisor consisting of all points of Z lying on E_j, then the E_j-aspect of ℓ_{E_j} has $\eta_j\big(Z_j + (e - \deg(Z_j)) \cdot x_j\big) \in \mathrm{Pic}^e(E_j)$ as its underlying line bundle, for $j = 1, \ldots, g$.

Applying [F2, Proposition 2.2], it follows that there exists *one* degeneration to a flag curve C like in (7), such that $R = R_1 \cup_p R_2$, with both R_1 and R_2 being trees of smooth rational curves meeting at the point p and there exists $0 \leq m \leq e$ such that $x_1, \ldots, x_m \in R_1 \setminus \{p\}$ and $x_{m+1}, \ldots, x_g \in R_2 \setminus \{p\}$, and, moreover, the divisor Z lies on $R_1 \cup_{x_1} E_1 \cup \ldots \cup_{x_m} E_m$. In other words, one always finds a degeneration to a flag curve of genus g such that the points in the support of the divisor Z specialize to a subcurve of genus $m \leq e$.

Let $\ell \in \overline{G}^{f-1}_e(C)$ be the corresponding limit linear series and we denote by

$$\alpha^j = \alpha^{\ell_{E_j}}(x_j) = \big(\alpha^j_0 \leq \cdots \leq \alpha^j_{f-1}\big)$$

the ramification sequence of ℓ_{E_j} of ℓ, where $j = 1, \ldots, g$. We denote by $\overline{\alpha}^j := \alpha^{\ell_{R_1}}(x_j)$ the ramification sequence of the R_1-aspect of ℓ at x_j for $j = 1, \ldots, m$ and by $\overline{\alpha}^j := \alpha^{\ell_{R_2}}(x_j)$ the ramification sequence of the R_2-aspect of ℓ at x_j for $j = m+1, \ldots, g$ respectively. Note that

$$\mathrm{wt}(\alpha^j) + \mathrm{wt}(\overline{\alpha}^j) = f(e-f+1), \quad \text{for } \ j = 1, \ldots, g. \tag{8}$$

We now estimate the dimension of the space of such limit linear series $\ell \in \overline{G}^{f-1}_e(C)$. For $j = m+1,\ldots,g$, the line bundle underlying ℓ_{E_j} is equal to $\eta_j(e\cdot x_j)$ and ℓ_j is determined by the choice of an f-dimensional subspace of sections $V_j \subseteq H^0\big(E_j, \eta_j(e\cdot x_j)\big)$, satisfying the condition $\alpha^{\ell_{E_j}}(x_j) \geq \alpha^j$. This being a Schubert condition on the Grassmannian $\mathrm{Gr}(f,e) \cong \mathrm{Gr}\big(f, H^0(E_j, \eta_j(e\cdot x_j))\big)$, it follows that the aspect ℓ_{E_j} moves in a parameter space of dimension,

$$\dim \mathrm{Gr}(f,e) - \mathrm{w}(\alpha^j) = f(e-f) - \mathrm{wt}(\alpha^j). \tag{9}$$

For $j = 1,\ldots,m$, the aspect ℓ_{E_j} belongs to the moduli space $\overline{G}^{f-1}_e\big(E_j,(x_j,\alpha^j)\big)$. In particular, ℓ_{E_j} moves in a moduli space of dimension at most

$$\rho(1,f-1,e) - \mathrm{wt}(\alpha^j) = 1 + f(e-f) - \mathrm{wt}(\alpha^j), \tag{10}$$

see also [EH, Theorem 1.1]. Furthermore, using once more [EH, Theorem 1.1], the R_1-aspects of ℓ (that is, the collection of the aspect of ℓ corresponding to the components of the subcurve R_1 of C) move in a space of dimension

$$\dim \overline{G}^{f-1}_e\Big(R_1,(x_1,\overline{\alpha}^1),\ldots,(x_m,\overline{\alpha}^m),(p,\alpha^{\ell_{R_1}}(p))\Big) = f(e+1-f) - \sum_{j=1}^{m}\mathrm{wt}(\overline{\alpha}^j) - \mathrm{wt}\big(\alpha^{\ell_{R_1}}(p)\big). \tag{11}$$

Similarly, the R_2-aspect of ℓ move in a family of dimension

$$\dim \overline{G}^{f-1}_e\Big(R_2,(x_{m+1},\overline{\alpha}^{m+1}),\ldots,(x_g,\overline{\alpha}^g),(p,\alpha^{\ell_{R_2}}(p))\Big) = f(e+1-f) - \sum_{j=m+1}^{g}\mathrm{wt}(\overline{\alpha}^j) - \mathrm{wt}\big(\alpha^{\ell_{R_2}}(p)\big). \tag{12}$$

Adding the dimensions in (9), (10), (11) and (12), and using also (8), as well as the compatibility relation $\mathrm{wt}\big(\alpha^{\ell_{R_1}}(p)\big) + \mathrm{wt}\big(\alpha^{\ell_{R_2}}(p)\big) = f(e-f+1)$, the limit linear series $\ell \in \overline{G}^{f-1}_e(C)$ moves in a family of dimension at most equal to

$$\begin{aligned} m + mf(e-f) - \sum_{j=1}^{m}\mathrm{wt}(\alpha^j) + (g-m)f(e-f) - \sum_{j=m+1}^{g}\mathrm{wt}(\alpha^j)\\ + f(e+1-f) + f(e+1-f) - \sum_{j=1}^{g}\mathrm{wt}(\overline{\alpha}^j) - f(e+1-f)\\ = m - f(g-1-e+f) \leq e - f(g-1-e+f). \end{aligned}$$

In particular, if $e - f(g-1-e+f) < 0$, no such limit linear series ℓ can exist, which finishes the proof. $\square$

We are now prepared to prove Theorem 1.1.

Proof of Theorem 1.1. We assume by contradiction that for every Prym curve $[C,\eta] \in \mathcal{R}_g$, there exists a component of $V^{e-f}_e(\omega_C \otimes \eta)$ having dimension at least $e - f(g-1-e+f)+1$.

We degenerate C to a curve C_0 of compact type $Y \cup_p E$, where both $[Y,p] \in \mathcal{M}_{g-1,1}$ and $[E,p] \in \mathcal{M}_{1,1}$ are general pointed curves of genera $g-1$ and 1 respectively. We also fix a Prym structure η on C_0, by choosing *non-trivial* 2-torsion points $\eta_Y \in \mathrm{Pic}^0(Y)[2]$ and $\eta_E \in \mathrm{Pic}^0(E)[2]$. By semicontinuity, there exists a family of dimension at least $e - f(g-1-e+f)+1$ of effective divisors Z of degree e on C_0, and such that for any line bundle $L_{C_0} \in \mathrm{Pic}^e(C_0)$ which is obtained by a twist at p of the line bundle $\eta(Z)$, one has

$$h^0\big(C_0, L_{C_0}\big) \geq f.$$

By possibly having to insert chains of rational curves at p and working with the resulting curve instead of C_0, we may also assume that the general such Z does not contain p in its support, that is, $\mathcal{O}_{C_0}(Z)$ is a line bundle.

Claim: One can find a degeneration $C_0 = Y \cup_p E$ as above and such a divisor Z on C_0 moving in a family of dimension larger than $e - f(g-1-e+f)$, such that $\mathrm{supp}(Z) \subseteq Y$.

To that end, we invoke once more Proposition 2.2 from [F2]. Precisely, we obtain that there exists a flag curve of genus g

$$C = R \cup_{x_1} E_1 \cup \ldots \cup_{x_g} E_g$$

consisting of a tree R of smooth rational curves meeting a fixed smooth elliptic curve E_j at the point x_j, for $j = 1, \ldots, g$, such that all points of Z specialize on a connected subcurve Y_1 of C of arithmetic genus at most e with $\big|Y_1 \cap \overline{C \setminus Y_1}\big| = 1$, see also the proof of [F2, Theorem 0.1]. Since we assumed that $e < g$, denoting by x_1 and x_g the extremal points on R among the set $\{x_1, \ldots, x_g\}$, it follows that no point in $\mathrm{supp}(Z)$ specializes on at least one of the curves E_1 or E_g, for instance on E_1. This proves our contention, by smoothing all the nodes of C with the exception of x_1.

Having established the claim, we return to the curve $C_0 = Y \cup_p E$ and to the divisor $Z \in Y_e$. We write down the Mayer-Vietoris sequence on C_0

$$0 \longrightarrow H^0(C_0, \eta(Z)) \longrightarrow H^0(Y, \eta_Y(Z)) \oplus H^0(E, \eta_E) \longrightarrow \mathbb{C}_p \longrightarrow \cdots$$

to obtain that $h^0(Y, \eta_Y(Z-p)) \geq f$, for a family Z of effective divisors in Y_e of dimension at least $e - f(g-1-e+f)+1$. The divisor $Y_{e-1} \cong p + Y_{e-1} \hookrightarrow Y_e$ being ample, see e.g. [ACGH, p. 310], it follows that there exists a non-empty family of effective divisors $Z_1 \in Y_{e-1}$ moving in a family of dimension at least $e - f(g-1-e+f)$ such that $h^0(Y, \eta_Y(Z_1)) \geq f$, or equivalently by Riemann-Roch

$$h^0\big(Y, \omega_Y \otimes \eta_Y(-Z_1)\big) \geq g - 2 - e + f.$$

We obtain that $\dim V_{e-1}^{e-1-f}(\omega_Y \otimes \eta_Y) \geq e - 1 - f(g-1-e+f)$, that is, the secant locus $V_{e-1}^{e-1-f}(\omega_Y \otimes \eta_Y)$ has excessive dimension. By continuing the descending induction on g, we let Y degenerate to a union of a curve of genus $g-2$ and an elliptic tail such that all the secant divisors specialize on the component of genus $g-2$ and at some point we obtain that a general Prym curve $[Y_1, \eta_1] \in \mathcal{R}_{g_1}$, for some $g_1 \leq g$, satisfies $V_{e_1-1}^{e_1-f}(\omega_{Y_1} \otimes \eta_1) \neq \emptyset$, even though the expected dimension $e_1 - f(g_1 - e_1 + f)$ is negative. This on the other hand contradicts Theorem 2.1, which finishes the proof. $\square$

3. PRYM-CANONICAL SECANTS VIA NIKULIN SURFACES

In this Section we prove Theorem 1.2. In order to obtain divisors on $\overline{\mathcal{R}}_g$, we focus on the case when the expected dimension of $V_e^{e-f}(\omega_C \otimes \eta)$ equals -1, that is, when

$$e - f(g-1-e+f) = -1. \tag{13}$$

In each such case when (13) is satisfied, we define the locus

$$\mathcal{D}_g^f := \Big\{[C, \eta] \in \mathcal{R}_g : V_e^{e-f}(\omega_C \otimes \eta) \neq \emptyset\Big\}. \tag{14}$$

As explained in the Introduction, we regard the loci $\mathcal{D}_g^f$ as the Prym analogues of the Brill-Noether subvarieties $\mathcal{M}_{g,d}^r := \big\{[C] \in \mathcal{M}_g : W_d^r(C) \neq \emptyset\big\}$ of the moduli space of curves.

When $f = 1$, setting $e = i$, we obtain from (13) that $g = 2i + 1$ and $\mathcal{D}_{2i+1}^1 = \mathcal{D}_{2i+1}$ is the Prym difference divisor considered in [FL]. From our results, it will follow that $\mathcal{D}_g^f$ is always an effective divisor on $\mathcal{R}_g$ whenever (13) is satisfied. When, on the other hand, $e = g$, then via (13) we obtain $g = f^2$ and the condition defining the locus $\mathcal{D}_g^f$ becomes

$$h^0(C, A \otimes \eta) \neq 0,$$

for a linear system $A \in W_{f^2-1}^{f-1}(C)$. Note that the Brill-Noether number of such a linear system A equals zero. Since this condition amounts to saying that $A \otimes \eta$ belongs to te theta divisor of $\mathrm{Pic}^0(C)$, the divisor $\mathcal{D}_{f^2}^f$ can be regarded as a global version of a translate of a multi-theta theta divisor on the moduli space $\mathcal{R}_g$.

Proof of Theorem 1.2. Let X be a standard Nikulin surface of such that $\mathrm{Pic}(X) \cong \mathfrak{N} \oplus_\perp \mathbb{Z} \cdot L$ with $L^2 = 2g - 2$. Recall from the Introduction that $N_1, \ldots, N_8$ are the disjoint (-2)-curves on X and that $L \cdot N_j = 0$, for $j = 1, \ldots, 8$. We fix a smooth curve $C \in |L|$ and set $\eta := \mathfrak{e}_C^\vee$. Using [ACGH, p. 356], since $\omega_C \otimes \eta$ is a non-special linear system, we obtain that $V_e^{e-f}(\omega_C \otimes \eta) \neq \emptyset$, whenever $e - f(g - 1 - e + f) \geq 0$. Therefore it suffices to show that $V_e^{e-f}(\omega_C \otimes \eta) = \emptyset$, when $\mathrm{exp.dim}\ V_e^{e-f}(\omega_C \otimes \eta) < 0$.

Suppose, by contradiction, that $Z = x_1 + \cdots + x_e$ is an element of $V_e^{e-f}(\omega_C \otimes \eta)$. Note that we do not require the points x_j to be distinct. By definition, Z may be regarded as a length e curvilinear subscheme of X. Then $h^0(C, \omega_C \otimes \eta(-Z)) \geq h^0(C, \omega_C \otimes \eta) - e + f = g - 1 - e + f$.

Observe first that $H^i(X, \mathfrak{e}) = 0$ for $i = 0, 1, 2$. By twisting the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{I}_{Z/X} \longrightarrow \mathcal{O}_C(-Z) \longrightarrow 0$$

by $L \otimes \mathfrak{e}^\vee$ and then taking cohomology, we obtain the exact sequence

$$0 \longrightarrow H^0(X, \mathfrak{e}^\vee) \longrightarrow H^0\big(X, \mathcal{I}_{Z/X} \otimes L \otimes \mathfrak{e}^\vee\big) \longrightarrow H^0\big(C, \omega_C \otimes \mathfrak{e}_C(-Z)\big) \longrightarrow H^1(X, \mathfrak{e}^\vee) \longrightarrow \cdots,$$

which implies that

$$H^0\big(X, \mathcal{I}_{Z/X} \otimes L \otimes \mathfrak{e}^\vee\big) \cong H^0\big(C, \omega_C \otimes \eta(-Z)\big). \tag{15}$$

We consider the class $H := L \otimes \mathfrak{e}^\vee \in \mathrm{Pic}(X)$. Note that $H^2 = 2g - 6$, therefore curves $D \in |H|$ have arithmetic genus $g - 2$. If $g \geq 6$, using [GS, Lemma 3.1] we know that the linear system $|H|$ is very ample. In particular, using (15), we can choose a smooth curve $D \in |H|$ passing through the points $x_1, \ldots, x_e$.

We tensor the exact sequence $0 \to \mathcal{O}_X(-D) \to \mathcal{I}_{Z/X} \to \mathcal{O}_D(-Z) \to 0$ by the line bundle H and taking cohomology, we obtain the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0\big(X, \mathcal{I}_{Z/X} \otimes L \otimes \mathfrak{e}^\vee\big) \longrightarrow H^0\big(D, \omega_D(-Z)\big) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \cdots,$$

from which we compute via the isomorphism (15) that $h^0(D, \omega_D(-Z)) \geq g - 2 - e + f$. Therefore, we can write

$$h^0\big(D, \mathcal{O}_D(Z)\big) = h^0\big(D, \omega_D(-Z)\big) + e + 1 - g + 2 \geq g - 2 - e + f + e - g + 3 = f + 1,$$

that is $\mathcal{O}_D(Z) \in W_e^f(D)$. Computing the Brill-Noether number for $\mathcal{O}_D(Z)$, we observe

$$\rho(g-2, f, e) = g - 2 - (f+1)(g - 2 - e + f) = e - f(g - 1 - e + f) = \mathrm{exp.dim}\ V_e^{e-f}(\omega_C \otimes \eta) < 0,$$

that is, the curve D is Brill-Noether special, precisely $W^f_e(D) \neq \emptyset$.

We show this is not possible. Indeed, using the arguments in Lazarsfeld's proof [La1, Lemma 1.3] of Petri's theorem for curves on surfaces $K3$ (see also [La2]), to rule out this possibility, it suffices to show that there is no decomposition $L \otimes \mathfrak{e}^\vee = A_1 \otimes A_2 \in \mathrm{Pic}(X)$, where $h^0(X, A_i) \geq 2$ for $i = 1, 2$.

Suppose there is such a decomposition and write $A_1 \equiv a_1 L + b_1 N_1 + \cdots + b_8 N_8$, respectively $A_2 = a_2 L + c_1 N_1 + \cdots + c_8 N_8$, where $a_1, a_2 \in \mathbb{Z}$ while $b_i, c_j \in (1/2)\mathbb{Z}$ for $i, j = 1, \ldots, 8$. Then a_1 and a_2 must be non-negative and since $a_1 + a_2 = 1$, without loss of generality we may assume $a_1 = 0$, that is, $A_1 \in \mathfrak{N}$. But then $|A_1|$ can consists only of sums of the rational curves $N_1, \ldots, N_8$, that is, $h^0(X, A_1) \leq 1$, which yields the desired contradiction. Therefore $V^{e-f}_e(\omega_C \otimes \eta) = \emptyset$, when $e - f(g - 1 - e + f) < 0$.

□

Remark 3.1. The conclusion of Theorem 1.2 can be extended to any irreducible 1-nodal curve $C \in |L|$, while keeping the same assumptions. Indeed, suppose C is such a 1-nodal curve with $\mathrm{Sing}(C) = \{o\}$, and denote by $\nu \colon C' \to C$ the normalization map with $x, y \in C'$ such that $\nu(x) = \nu(y) = o$. Assuming $[C, \mathfrak{e}_C]$ is a limit of smooth Prym curves $[C_t, \eta_t] \in \mathcal{R}_g$, with $V^{e-f}_e(\omega_{C_t} \otimes \eta_t) \neq \emptyset$, then if Z is the limit on C of the e-secant divisors on C_t, in the case $o \notin \mathrm{supp}(Z)$ the proof remains unchanged. Assume for simplicity that *one* of the points in $\mathrm{supp}(Z)$, say x_1, specializes to o, while $x_2, \ldots, x_e \in C_{\mathrm{reg}}$. (The case when several points in $\mathrm{supp}(Z)$ specialize to o is similar). We denote by $\epsilon \colon X' \to X$ the blow-up of X at o, by E the exceptional divisor and by $C' \subseteq X'$ the proper transform of C. Set $Y = \epsilon^*(C) = C' + E$. Then $C' \cap E = \{x, y\}$. We denote by $Z' = x_1 + x_2 + \cdots + x_e$ the limiting e-secant divisor, where $x_1 \in E \setminus \{x, y\}$ and $x_2, \ldots, x_e \in C' \setminus \{x, y\}$. We then have

$$H^0\big(X', \mathcal{I}_{Z'/X'} \otimes \epsilon^*(L \otimes \mathfrak{e}^\vee)\big) \cong H^0\big(Y, \epsilon^*(L \otimes \mathfrak{e}^\vee)(-Z')\big),$$

and pushing down to X, we obtain again that $h^0\big(X, \mathcal{I}_{Z/X} \otimes (L \otimes \mathfrak{e}^\vee)\big) \geq g - 1 - e + f$. The rest of the proof is identical to that of Theorem 1.2.

It is natural to ask whether the conclusion of Theorem 1.2 can be extended to reducible curves on X containing one of the (-2)-curves N_i. We have the following:

Proposition 3.2. *Let X be a standard Nikulin surface of odd genus $g = 2i + 1 \geq 9$ such that $\mathrm{Pic}(X) \cong \mathfrak{N} \oplus_\perp \mathbb{Z} \cdot L$. Let $C_0 \in |L|$ be a reducible curve consisting of two components, namely, one the curves N_i and a smooth irreducible component $C'_0 \in |L - N_i|$, and set $\eta_0 := \mathfrak{e}^\vee_{C_0}$.*

If $e < \frac{g}{2}$, then $[C_0, \eta_0]$ does not lie in the closure in $\overline{\mathcal{R}}_g$ of the locus of Prym curves $[C, \eta]$ such that $V^{e-1}_e(\omega_C \otimes \eta) \neq \emptyset$. In particular, for odd genus $g = 2i + 1$, it lies outside the closure $\overline{\mathcal{D}}_{2i+1}$ in $\overline{\mathcal{R}}_{2i+1}$ of the difference Prym divisor defined in (2).

Proof. Without loss of generality, we may assume $C_0 = C'_0 + N_1$, where $C'_0 \in |L - N_1|$ is smooth and irreducible and denote by x, y the two intersection points between C'_0 and N_1. The restrictions of η to the components of C_0 are given by $\eta_{C'_0} \cong \mathfrak{e}^\vee_{C'_0}$, respectively $\eta_{N_1} \cong \mathcal{O}_{\mathbf{P}^1}(1)$.

By contradiction, assume C_0 lies in the closure of the locus of Prym curves $[C, \eta] \in \overline{\mathcal{R}}_g$ such that $V^{e-1}_e(\omega_C \otimes \eta) \neq \emptyset$, which implies the existence of a divisor $Z \in V^{e-1}_e(\omega_{C_0} \otimes \eta_0)$. We denote by Z' and Z_1 the restriction of Z to C'_0 and N_1 respectively, and set $e' := \deg Z'$ and $e_1 := \deg Z_1$. Without loss of generality we may assume that Z is disjoint from x and y, so that $e = e' + e_1$. We distinguish two cases according to the parity of e_1.

Assume first e_1 is even and write $e_1 = 2a$. By considering the twist by $\eta_0^{\otimes 2a}$ of the Prym-canonical line bundle, by semicontinuity we also have $Z \in V_e^{e-f}(\omega_{C_0} \otimes {\eta_0}^{\otimes(2a+1)})$. The restrictions of the line bundle $\omega_{C_0} \otimes \eta_0^{\otimes(2a+1)}(-Z)$ to the components N_1 respectively C_0' are isomorphic to $\mathcal{O}_{N_1}(1)$ respectively $\omega_{C_0'} \otimes \mathfrak{e}_{C_0'}^{\vee}(-Z_2)$, where $Z_2 = Z' + (a-1)\cdot x + (a-1)\cdot y$. Note that $\deg(Z_2) = e-2$.

The condition $h^0\big(C_0, \omega_{C_0} \otimes \eta_0^{\otimes(2a+1)}(-Z)\big) \geq g-e$ translates then into

$$h^0\big(C_0', \omega_{C_0'} \otimes \mathfrak{e}_{C_0'}^{\vee}(-Z_2)\big) \geq g-e \Longleftrightarrow h^0\big(X, \mathcal{I}_{Z_2/X}(C_0' \otimes \mathfrak{e}^{\vee})\big) \geq g-e. \tag{16}$$

We take a general element D of the linear system $\big|\mathcal{I}_{Z_2/X}(C_0' \otimes \mathfrak{e}^{\vee})\big|$. Then D is a smooth curve of genus $g-4 \geq 5$ and proceeding precisely along the lines of the proof of Theorem 1.2, we conclude from (16) that $h^0\big(D, \mathcal{O}_D(Z_2)\big) \geq 2$, that is, $W^1_{e-2}(D) \neq \emptyset$. Observe now that the Brill-Noether number for such linear systems equals

$$\text{exp.dim } W^1_{e-2}(D) = \rho(g-4,1,e-2) = -g+2e-2 = \text{exp.dim } V_e^{e-1}(\omega_{C_0} \otimes \eta) - 2 \leq -3,$$

that is, D is a Brill-Noether special curve. This is ruled out like in the proof of Theorem 1.2.

Assume now e_1 is odd and write $e_1 = 2a+1$. Then, also $Z \in V_e^{e-f}(\omega_{C_0} \otimes \eta_0^{\otimes(2a+1)})$. The restrictions of $\omega_{C_0} \otimes \eta_0^{\otimes(2a+1)}(-Z)$ to N_1, respectively to C_0', are isomorphic to $\mathcal{O}_{N_1}$, respectively $\omega_{C_0'} \otimes \mathfrak{e}_{C_0'}^{\vee}\big(-(Z' + (a-1)x + (a-1)y)\big)$. Setting $Z_3 := Z' + a\cdot x + a \cdot y$, we obtain that

$$H^0\big(C', \omega_{C_0'} \otimes \mathfrak{e}_{C_0'}^{\vee}(-Z_3)\big) \geq g-e-1 \Longleftrightarrow h^0\big(X, \mathcal{I}_{Z_3/X}(C_0' - \mathfrak{e})\big) \geq g-e-1. \tag{17}$$

Choosing a general curve $D \in \big|\mathcal{I}_{Z_3/X}(C_0' \otimes \mathfrak{e}^{\vee})\big|$, we obtain via the equivalence (17) that $\mathcal{O}_D(Z_3) \in W^1_{e-1}(D)$. We now compute the Brill-Noether number

$$\rho(g-4,1,e-1) = -g+2e = \text{exp.dim } V_e^{e-1}(\omega_{C_0} \otimes \eta_0) < 0, \tag{18}$$

and we conclude like at the end of the proof of Theorem that this situation does not appear. □

Remark 3.3. The above proposition cannot be generalized to $f > 1$. Indeed, following the proof, equation (18) for arbitrary f becomes

$$\rho(g-4,f,e-1) = f-1+e-f(g-1+e+f) = f-1+\text{exp.dim } V_e^{e-f}(\omega_{C_0} \otimes \eta_0) \leq f-2,$$

and one cannot conclude this is negative if $f \geq 2$.

4. The Prym difference divisor via Raynaud theta divisors

In this Section we present an alternative approach to both Theorem 1.2 (in the case $f=1$) and Theorem 1.3 (in full generality) using the representation of difference varieties on Jacobians as Raynaud theta divisors provided in [FMP] in the context of the resolution of the Minimal Resolution Conjecture for points on canonical curves.

We begin by setting some notation. For a smooth curve C and for integers $a, b > 0$, we denote by $C_a - C_b \subseteq \text{Pic}^{a-b}(C)$ its *difference variety* defined as the image of the difference map

$$\phi_{a,b}\colon C_a \times C_b \longrightarrow \text{Pic}^{a-b}(C), \ \ (D_a, D_b) \mapsto \mathcal{O}_C(D_a - D_b).$$

Of particular interest is the case when $a = b = i$ and $g = 2i+1$, in which case $C_i - C_i \subseteq \text{Pic}^0(C)$ is a divisor. Using [FMP] this divisor can be identified with the *Raynaud theta divisor* [R] associated to the kernel (syzygy) vector bundle on C. Precisely, we let M_{ω_C} be the *kernel bundle* defined by the exact sequence

$$0 \longrightarrow M_{\omega_C} \longrightarrow H^0(C, \omega_C) \otimes \mathcal{O}_C \longrightarrow \omega_C \longrightarrow 0. \tag{19}$$

Note that the exact sequence (19) makes sense for every nodal curve C for which ω_C is globally generated. We set $Q_{\omega_C} := M_{\omega_C}^{\vee}$ and observe that the slope of Q_{ω_C} equals $\mu(Q_{\omega_C}) = 2$. Therefore $\mu(\bigwedge^i Q_{\omega_C}) = 2i = g-1$. The main results of [FMP] establishes the following equality of cycles on $\mathrm{Pic}^0(C)$:

$$C_i - C_i = \Theta_{\bigwedge^i Q_{\omega_C}} := \Big\{\xi \in \mathrm{Pic}^0(C) : h^0\big(C, \bigwedge^i Q_{\omega_C} \otimes \xi\big) \geq 1\Big\}. \tag{20}$$

It is advantageous to observe that, whereas the left-hand side of (20) is hard to understand for singular stable curves, the definition of the right-hand side can be extended to cover those stable curves C for which ω_C is globally generated.

We are now prepared to provide a second proof of Theorem 1.2 in the case $f=1$. Note that the assumption $g \geq 6$ is not used in this proof and thus the statement is slightly stronger, as it also covers the cases $g = 3, 5$. Recall that $\overline{\mathcal{D}}_{2i+1}$ denotes the closure in $\overline{\mathcal{R}}_{2i+1}$ of the difference Prym divisor defined via (2).

Theorem 4.1. *Let $[X, \mathfrak{N} \oplus_{\perp} \mathbb{Z}\cdot L] \in \mathcal{F}_{2i+1}^{\mathfrak{N}}$ be a general standard Nikulin surface of genus $g = 2i+1$. Then for every integral nodal curve $C \in |L|$, we have that $[C, \mathfrak{e}_C^{\vee}]$ lies outside $\overline{\mathcal{D}}_{2i+1}$.*

Proof. Using [vGS] we may assume that in this case $\mathrm{Pic}(X) \cong \mathfrak{N} \oplus_{\perp} \mathbb{Z}\cdot L$. The line bundle L is nef and curves $C \in |L|$ have genus g and $C \cdot N_j = 0$, for $j = 1, \ldots, 8$.

We choose a nodal integral curve $C \in |L|$ and recall that $Q_{\omega_C} = M_{\omega_C}^{\vee}$. By requiring the 2-torsion line bundle of a stable Prym curve to lie in the left hand side of (20), one obtains a codimension one subvariety of $\overline{\mathcal{R}}_{2i+1}$ that contains the closure $\overline{\mathcal{D}}_{2i+1}$ of $\mathcal{D}_{2i+1}$, as well as possibly other boundary divisors of $\overline{\mathcal{R}}_{2i+1}$. Throughout this proof we set $\eta := \mathfrak{e}_C \in \mathrm{Pic}^0(C)[2]$. In order to conclude that $[C, \eta] \notin \overline{\mathcal{D}}_{2i+1}$, it suffices thus to show that

$$H^0\big(C, \bigwedge^i Q_{\omega_C} \otimes \mathfrak{e}_C\big) = 0. \tag{21}$$

We now specialize X to a *hyperelliptic* Nikulin surface X', that is, a $K3$ surface with

$$\mathrm{Pic}(X') \cong \mathfrak{N} \oplus_{\perp} \mathbb{Z}\cdot L \oplus \mathbb{Z}\cdot E,$$

where $E^2 = 0$, $E \cdot L = 2$ and $E \cdot N_j = 0$, for $j = 1, \ldots, 8$. By the Torelli theorem for $K3$ surfaces, see e.g. [Dol], there exists a $K3$ surface X' having this Picard lattice. Moreover, $[X', \mathfrak{N} \oplus_{\perp} \mathbb{Z}\cdot L] \in \mathcal{F}_g^{\mathfrak{N}}$, that is, X' is a standard Nikulin surface of genus g. We may assume that both C and E are nef classes on X'. Since $|E|$ is an elliptic pencil and $C \cdot E = 2$, it follows that any $C \in |L|$ is hyperelliptic. Furthermore, $\omega_C = E_C^{\otimes(g-1)}$ and from (19) we obtain that $M_{\omega_C} \cong \big(E_C^{\vee}\big)^{\oplus(g-1)}$ **(cf.** [FMP, Proposition 3.5]**)**, therefore $Q_{\omega_C} \cong E_C^{\oplus(g-1)}$. It follows that

$$\bigwedge^i Q_{\omega_C} \cong \Big(E_C^{\otimes i}\Big)^{\oplus \binom{g-1}{i}},$$

therefore condition (21), amounts to

$$H^0\big(C, E_C^{\otimes i} \otimes \mathfrak{e}_C\big) = 0. \tag{22}$$

By tensoring the exact sequence $0 \to \mathcal{O}_{X'}(-C) \to \mathcal{O}_{X'} \to \mathcal{O}_C \to 0$ by $\mathcal{O}_{X'}(iE - \mathfrak{e})$ and taking cohomology, then (22) follows once we show that

$$H^0\big(X', \mathcal{O}_{X'}(iE - \mathfrak{e})\big) = 0 \text{ and } H^1\big(X', \mathcal{O}_{X'}(iE - \mathfrak{e} - C)\big) = 0.$$

The first statement follows from Lemma 5.8 in [FK] (note that the assumption $g \geq 11$ in *loc.cit.* is not used anywhere in the proof). For the second statement, since $(iE - \mathfrak{e} - C)^2 = -4$, while clearly $H^0\big(X', \mathcal{O}_{X'}(iE - \mathfrak{e} - C)\big) = 0$ (intersect with the nef class E), it follows by Riemann-Roch coupled with Serre duality that

$$h^1\big(X', \mathcal{O}_{X'}(iE - C - \mathfrak{e})\big) = h^2\big(X', \mathcal{O}_{X'}(iE - C - \mathfrak{e})\big) = h^0\big(X', \mathcal{O}_{X'}(C + \mathfrak{e} - iE)\big).$$

Thus we are left with proving that $H^0\big(X', \mathcal{O}_{X'}(C + \mathfrak{e} - iE)\big) = 0$. Assuming by contradiction that $C + \mathfrak{e} - iE$ is effective, by intersecting with N_i, since $N_i \cdot (C + \mathfrak{e} - iE) = -1$, we obtain that also $C + \mathfrak{e} - iE - N_1 - \cdots - N_8 = C - \mathfrak{e} - iE$ is effective. Below we shall show that this is not possible, thus completing the proof. □

Lemma 4.2. *Let X' be a general standard hyperelliptic Nikulin surface of genus $2i+1$ as above. Then*

$$H^0\big(X', \mathcal{O}_{X'}(C - iE - \mathfrak{e})\big) = 0.$$

Proof. We set $B \equiv L - iE - \mathfrak{e} \in \text{Pic}(X')$. Since $B^2 = (C - iE)^2 + \mathfrak{e}^2 = -4$, if B is effective, there must exist a (-2)-curve $D \subseteq X'$ such that $B - D$ is effective and $D \cdot B < 0$ (indeed, any divisor in $|B|$ has at least one irreducible component whose intersection with B is negative and such a component must be a (-2)-curve). We write

$$D \equiv aL + bE + c_1N_1 + \cdots + c_8N_8 \in \text{Pic}(X'), \tag{23}$$

where $a, b \in \mathbb{Z}$ and $c_j \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

Since $B \cdot N_j = 1$, it follows that $D \neq N_j$, therefore $D \cdot N_j \geq 0$, that is, $c_j \leq 0$, for $j = 1, \ldots, 8$. Intersecting (23) with the nef class E, we obtain $a \geq 0$. On the other hand, also $(B - D) \cdot E \geq 0$, which yields $a \leq 1$, that is, $a \in \{0, 1\}$, since from (23) we have $a \in \mathbb{Z}$.

Assume first $a = 1$. From the inequality $(B - D) \cdot C \geq 0$, we obtain $i + b \leq 0$. On the other hand, then $D^2 = (C + bE)^2 - 2(c_1^2 + \cdots + c_8^2) \leq 4(i + b)$, which forces $b = -i$, since $b \in \mathbb{Z}$. Since $D^2 = -2$, from (23) we obtain that $c_1^2 + \cdots + c_8^2 = 1$. Observe that since $c_1N_1 + \cdots + c_8N_8 \in \mathfrak{N}$, the integers $2c_1, \ldots, 2c_8$ are all of the same parity. Therefore we conclude that there exists $j \in \{1, \ldots, 8\}$ such that $c_j = -1$, while $c_\ell = 0$, for $\ell \neq 8$. We obtain that the class $B - D = N_j - e$ is effective, which is obviously impossible.

Assume now $a = 0$. Since $D \cdot C \geq 0$, we obtain $b \geq 0$. Since $D^2 = -2$, we again obtain $c_1^2 + \cdots + c_8^2 = 1$, and the same reasoning as above implies that there exists $j \in \{1, \ldots, 8\}$ such that $c_j = -1$, while $c_\ell = 0$, for $\ell \neq j$. We then have $0 > B \cdot D = 2b + 1 \geq 0$, therefore, we have reached a contradiction. □

4.1. **Prym curves on non-standard Nikulin surfaces.** We start with a general non-standard Nikulin surface of genus $h = 8i - 1$, where $i \geq 2$. Then using [vGS, Proposition 2.7] there exist disjoint (-2)-curves $N_1, N_2 \in \mathfrak{N} \subseteq \text{Pic}(X)$, such that if $\mathfrak{e} \cong \frac{N_1 + \cdots + N_8}{2} \in \text{Pic}(X)$, where $\mathfrak{N} = \langle N_1, \ldots, N_8, \mathfrak{e} \rangle$, then

$$R \equiv \frac{L - N_1 - N_2}{2} \in \text{Pic}(X).$$

We denote by $N_3, \ldots, N_8$ the remaining effective (-2)-curves in $\text{Pic}(X)$.

Then $R^2 = 4i - 2$, that is, curves $C \in |R|$ have genus $2i$. Note that $R \cdot N_1 = R \cdot N_2 = 1$, while $L \cdot N_i = 0$, for $i = 3, \ldots, 8$. We fix an irreducible such curve C and assume it to be smooth at the points $C\cdot N_1$ and $C\cdot N_2$. Retaining the notation from the Introduction, if $C\cdot N_i=\{x_i\}$, for $i=1,2$ and $f\colon \widetilde{X}\to X$ is the double cover induced by e, then $\mathfrak{e}_C^2\cong \mathcal{O}_C(x_1+x_2)$, that is, the double cover $f_{|f^{-1}(C)}\colon f^{-1}(C)\to C$ is branched only over the points x_1 and x_2. We set

$$Y:=C\cup_{\{x_1,x_2\}}\mathbf{P}^1,$$

where $\mathbf{P}^1$ meets C transversally at x_1 and x_2 and we let η be the line bundle on Y having restrictions $\eta_{\mathbf{P}^1}\cong \mathcal{O}_{\mathbf{P}^1}(1)$ and $\eta_C\cong \mathfrak{e}_C^{\vee}$. By the discussion above (5), recall that $\xi(C):=[Y,\eta]\in \Delta_0^{\mathrm{ram}}$ may be regarded as a stable Prym curve of genus $2i+1$.

We are now in a position to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Retaining the notation above, in order to conclude that $[Y,\eta]\notin \overline{\mathcal{D}}_{2i+1}$, it suffices to show that

$$H^0\big(Y,\bigwedge^i Q_{\omega_Y}\otimes \eta^{\vee}\big)=0,\quad \text{or equivalently,}\quad H^0\big(Y,\bigwedge^i M_{\omega_Y}\otimes \omega_Y\otimes \eta\big)=0. \tag{24}$$

The equivalence between the two vanishing statements in(24) follows via Riemann-Roch from the fact that Q_{ω_Y} is a vector bundle of slope $\mu(Q_{\omega_Y})=2$.

We have $M_{\omega_Y|C}\cong M_{\omega_C(x_1+x_2)}$ and $M_{\omega_Y|\mathbf{P}^1}\cong \mathcal{O}_{\mathbf{P}^1}^{\oplus 2i}$. One has the following Mayer-Vietoris type exact sequence on Y

$$0\longrightarrow \bigwedge^i M_Y\longrightarrow \bigwedge^i M_{\omega_C(x_1+x_2)}\bigoplus \mathcal{O}_{\mathbf{P}^1}^{\oplus\binom{2i}{i}}\longrightarrow \bigwedge^i M_{\omega_Y|x_1+x_2}\longrightarrow 0.$$

We twist this sequence by $\omega_Y\otimes\eta$. We use the isomorphisms $\big(\omega_Y\otimes\eta\big)_{|\mathbf{P}^1}\cong \mathcal{O}_{\mathbf{P}^1}(1)$ and $\big(\omega_Y\otimes\eta\big)_{|C}\cong \omega_C\otimes\mathfrak{e}_C$, as well as the isomorphism

$$H^0\big(\mathbf{P}^1,\mathcal{O}_{\mathbf{P}^1}(1)\big)^{\oplus\binom{2i}{i}}\stackrel{\cong}{\longrightarrow}\bigwedge^i M_{\omega_Y}\otimes(\omega_Y\otimes\eta)_{|x_1+x_2}.$$

This map is given by evaluation at x_1 and x_2 and the isomorphism holds due to the fact that since $\mathrm{rk}\big(\bigwedge^i M_{\omega_Y}\otimes(\omega_Y\otimes\eta)\big)=\binom{2i}{i}$, both sides are vector spaces of dimension $2\binom{2i}{i}$, coupled with the obvious fact $H^0(\mathbf{P}^1,\mathcal{O}_{\mathbf{P}^1}(1))\cong \big(\mathcal{O}_{\mathbf{P}^1}(1)\big)_{|x_1+x_2}$.

Therefore, the statement (24) is equivalent to the following vanishing on C

$$H^0\Big(C,\bigwedge^i M_{\omega_C(x_1+x_2)}\otimes\omega_C\otimes\mathfrak{e}_C\Big)=0. \tag{25}$$

We then have the following exact sequence on C

$$0\longrightarrow M_{\omega_C}\longrightarrow M_{\omega_C(x_1+x_2)}\longrightarrow \mathcal{O}_C(-x_1-x_2)\longrightarrow 0,$$

which after taking exterior products leads to the following exact sequence

$$0\longrightarrow \bigwedge^i M_{\omega_C}\longrightarrow \bigwedge^i M_{\omega_C(x_1+x_2)}\longrightarrow \bigwedge^{i-1} M_{\omega_C}(-x_1-x_2)\longrightarrow 0.$$

Twisting this sequence by $\omega_C\otimes\mathfrak{e}_C$ and taking cohomology, we conclude that the vanishing (25) holds if both statements below hold

$$H^0\big(C,\bigwedge^i M_{\omega_C}\otimes\omega_C\otimes\mathfrak{e}_C\big)=0\quad\text{and}\quad H^0\big(C,\bigwedge^{i-1} M_{\omega_C}\otimes\omega_C\otimes\mathfrak{e}_C^{\vee}\big)=0. \tag{26}$$

We specialize X to a non-standard Nikulin surface X' having the Picard lattice

$$\mathrm{Pic}(X') \cong \mathfrak{N} \oplus \mathbb{Z} \cdot R \oplus \mathbb{Z} \cdot E,$$

where $E^2 = 0$, $E \cdot L = 4$ and $E \cdot N_j = 0$, for $j = 1, \ldots, 8$. By [Dol] it follows that a Nikulin surface X' with this Picard lattice exists, furthermore both R and E are nef classes. Observe that $E \cdot R = 2$, that is, a curve $C \in |R|$ is hyperelliptic. Accordingly, $M_{\omega_C} \cong \big(E_C^{\vee}\big)^{\oplus(2i-1)}$ and since $\omega_C \cong E_C^{\otimes(2i-1)}$, we conclude that (26) is equivalent to the following vanishing statements

$$H^0\big(C, E_C^{\otimes(i-1)} \otimes \mathfrak{e}_C\big) = 0 \quad \text{and} \quad H^0\big(C, E_C^{\otimes i} \otimes \mathfrak{e}_C^{\vee}\big) = 0. \tag{27}$$

Twisting the exact sequence $0 \to \mathcal{O}_{X'}(-C) \to \mathcal{O}_{X'} \to \mathcal{O}_C \to 0$ by $\mathcal{O}_{X'}(iE - \mathfrak{e})$ respectively by $\mathcal{O}_{X'}((i-1)E + \mathfrak{e})$, we conclude that (27) are implied by the following vanishing statements:

$$H^0\big(X', \mathcal{O}_{X'}((i-1)E + \mathfrak{e})\big) = 0 \quad \text{and} \quad H^0\big(X', \mathcal{O}_{X'}(C - (i-1)E - \mathfrak{e})\big) = 0, \tag{28}$$

respectively

$$H^0\big(X', \mathcal{O}_{X'}(iE - \mathfrak{e})\big) = 0 \quad \text{and} \quad H^0\big(X', \mathcal{O}_{X'}(C - iE + \mathfrak{e})\big) = 0. \tag{29}$$

Statements (28) and (29) are established in the next lemma, bringing the proof to an end. □

Lemma 4.3. *Let X' be a non-standard hyperelliptic Nikulin surface with* $\mathrm{Pic}(X') \cong \mathbb{Z} \cdot R \oplus \mathfrak{N} \oplus \mathbb{Z} \cdot E$, *where $R^2 = 4i-2$, $R \cdot E = 2$ and $E \perp \mathfrak{N}$ as above. Then the following hold:*

$$H^0\big(X', \mathcal{O}_{X'}((i-1)E + \mathfrak{e})\big) = 0 \quad \textit{and} \quad H^0\big(X', \mathcal{O}_{X'}(iE - \mathfrak{e})\big) = 0,$$
$$H^0\big(X', \mathcal{O}_{X'}(R - (i-1)E - \mathfrak{e})\big) = 0 \quad \textit{and} \quad H^0\big(X', \mathcal{O}_{X'}(R - iE + \mathfrak{e})\big) = 0.$$

Proof. Observe that each of the divisor classes appearing in the statement above has self-intersection (-4) and the proofs are quite similar. We will provide details only for the last statement, that is, $H^0\big(X', \mathcal{O}_{X'}(R - iE + \mathfrak{e})\big) = 0$.

We proceed by contradiction and assume $H^0\big(X', \mathcal{O}_{X'}(R - iE + \mathfrak{e})\big) \neq 0$. By intersecting with each of the curves $N_3, \ldots, N_8$ and recalling that $R \equiv \frac{L - N_1 - N_2}{2}$, we obtain that

$$H^0\Big(X', \mathcal{O}_{X'}\Big(\frac{L}{2} - iE - \frac{N_3}{2} - \cdots - \frac{N_8}{2}\Big)\Big) \neq 0.$$

We set

$$B := \frac{L}{2} - iE - \frac{N_3}{2} - \cdots - \frac{N_8}{2} \in \mathrm{Pic}(X').$$

Since $B^2 = -4$, just like in the proof of Lemma 4.2, there exists an effective (-2)-curve

$$D \equiv aL + bE + c_1 N_1 + \cdots + c_8 N_8 \in \mathrm{Pic}(X'), \tag{30}$$

such that $B - D$ is effective and $B \cdot D < 0$.

Since $B \cdot N_j \geq 0$, it follows that $D \cdot N_j \geq 0$, therefore $c_j \leq 0$, for $j = 1, \ldots, 8$. Furthermore, $a \geq 0$, since $D \cdot E \geq 0$. The class E being nef and $B - D$ being effective, one has $(B - D) \cdot E \geq 0$, which yields $a \leq \frac{1}{2}$, that is, $a \in \{0, \frac{1}{2}\}$, since from (30) it follows that $a \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

Assume first $a = \frac{1}{2}$. From the inequality $(B - D) \cdot L \geq 0$, we obtain $i + b \leq 0$. Furthermore, we have $-2 = D^2 = 4(b+i) - 1 - 2(c_1^2 + \cdots + c_8^2)$. Since $b \in \mathbb{Z}$, it follows $b + i = 0$ and $c_1^2 + \cdots + c_8^2 = \frac{1}{2}$. The only solution of this equality compatible with the description (30), is when $c_1 = c_2 = -\frac{1}{2}$ and $c_3 = \cdots = c_8 = 0$. We obtain that $\frac{1}{2}\big(N_1 + N_2 - N_3 - \cdots - N_8\big) \in \mathrm{Pic}(X')$ is effective, which is impossible.

We are left with the case $a = 0$. Since $D \cdot L \geq 0$, we obtain $b \geq 0$. From $D^2 = -2$, we find $c_1^2 + \cdots + c_8^2 = 1$. Note that the integers $2c_1$ and $2c_2$, respectively $2c_3, \ldots, 2c_8$ must have the same parity, which leaves as only cases possible $c_1 = -1$ and $c_2 = 0$ (respectively $c_1 = 0$ and $c_2 = -1$) and $c_3 = \ldots = c_8 = 0$. In both cases we then compute that $D \cdot B = 2b \geq 0$, which yields the desired contradiction. □

5. THE CLASS OF $\overline{\mathcal{D}}_{2i+1}$

We now explain how the geometric information contained in Theorem 1.2 and 1.3 suffices to determine the class $[\overline{\mathcal{D}}_{2i+1}]$ at least in the $\langle \lambda, \delta_0', \delta_0^{\mathrm{ram}} \rangle$-subspace of $\mathrm{Pic}(\overline{\mathcal{R}}_{2i+1})$. As explained in [FL], it is precisely these three coefficients of $[\overline{\mathcal{D}}_{2i+1}]$ that are relevant for Kodaira dimension calculations of $\overline{\mathcal{R}}_g$.

Recall that if $\pi \colon \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ is the finite morphism forgetting the Prym structure of every Prym curve, then we have a relation

$$\pi^*(\delta_0) = \delta_0' + \delta_0'' + 2\delta_0^{\mathrm{ram}} \in \mathrm{Pic}(\overline{\mathcal{R}}_g).$$

We briefly describe the modular meaning of these divisor classes. If we fix a point $[C_{xy}] \in \Delta_0$ induced by a smooth 2-pointed curve $[C, x, y]$ of genus $g-1$ and the normalization map $\nu \colon C \to C_{xy}$, where $\nu(x) = \nu(y)$, a general point of the irreducible divisor Δ_0' (respectively of Δ_0'') corresponds to a stable Prym curve $[C_{xy}, \eta]$, where $\eta \in \mathrm{Pic}^0(C_{xy})[2]$ and $\nu^*(\eta) \in \mathrm{Pic}^0(C)$ is non-trivial (respectively trivial). A general point of Δ_0^{ram} is the Prym curve $[Y, \eta]$, where $Y := C \cup_{\{x,y\}} \mathbf{P}^1$ is a quasi-stable curve and $\eta \in \mathrm{Pic}^0(Y)$ satisfies $\eta_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(1)$ and $\eta_C^{\otimes 2} \cong \mathcal{O}_C(-x-y)$. Then $\delta_0^{'} := [\Delta_0']$, $\delta_0^{''} := [\Delta_0'']$, respectively $\delta_0^{\mathrm{ram}} := [\Delta_0^{\mathrm{ram}}]$. For details we refer to [FL, 1].

We now let $[X, \mathfrak{N} \oplus_\perp \mathbb{Z} \cdot L] \in \mathcal{F}_g^{\mathfrak{N}}$ be a general standard Nikulin surface of genus g. A Lefschetz pencil in the linear system $|L|$ on X induces a family of stable Prym curves

$$\Xi^{\mathrm{s}} := \Bigl\{ [C_t, \mathfrak{e}_{C_t}^\vee] : t \in \mathbf{P}^1 \Bigr\} \subseteq \overline{\mathcal{R}}_g.$$

Observe that in this pencil there exists precisely one curve having N_j as one of its components, for $j = 1, \ldots, 8$. In this case, the corresponding curve in Ξ^s looks like $C_j' + N_j$, where C_j' is a smooth curve of genus $g-1$ and $C_j' \cdot N_j = 2$. Furthermore, $\mathfrak{e}_{N_j}^\vee \cong \mathcal{O}_{N_j}(1)$, while $\deg\bigl(\mathfrak{e}_{C_j'}^\vee\bigr) = -1$. These eight points correspond to the intersection of Ξ^s with the divisor Δ_0^{ram}. For a general choice of Ξ^{s}, the intersection with Δ_0^{ram} is transversal, therefore $\Xi^s \cdot \delta_0^{\mathrm{ram}} = 8$, see also [FV1, Proposition 1.4] for details. Furthermore, $\Xi^s \cdot \lambda = g+1$, $\Xi^s \cdot \delta_0'' = 0$ and accordingly

$$\Xi^s \cdot \delta_0' = \bigl(\pi_*(\Xi^{\mathrm{s}})\bigr) \cdot \delta_0 - 2 \cdot 8 = 6g + 18 - 2 \cdot 8 = 6g + 2.$$

Let us now assume that X is a non-standard Nikulin surface of genus $4g-5$ such that $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot R \oplus \mathfrak{N}$, where as above

$$R \equiv \frac{L - N_1 - N_2}{2} \in \mathrm{Pic}(X).$$

We choose a Lefschetz pencil $\{C_t\}_{t \in \mathbf{P}^1}$ in the linear system $|R|$, thus the genus of C_t equals $g-1$. Let $\Xi \subseteq \overline{\mathcal{M}}_{g-1}$ be the induced curve in moduli. and we denote by $x_{j,t}$ the point of intersection of C_t with N_j for $j = 1, 2$. We denote by $\Xi^{\mathrm{ns}} \subseteq \Delta_0^{\mathrm{ram}} \subseteq \overline{\mathcal{R}}_g$ the family of stable

Prym curves

$$\Xi^{\rm ns} := \Big\{ \big[C_t \cup_{\{x_{1,t}, x_{2,t}\}} \mathbf{P}^1, \eta_{C_t} = \mathfrak{e}_{C_t}^{\vee},\ \eta_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(1)\big] : t \in \mathbf{P}^1 \Big\}.$$

The intersection numbers of $\Xi^{\rm ns}$ with the generators of $\mathrm{Pic}(\overline{\mathcal{R}}_g)$ are as follows:

Proposition 5.1. *One has* $\Xi^{\rm ns} \cdot \lambda = g,\ \Xi^{\rm ns} \cdot \delta_0^{\rm ram} = 4,\ \Xi^{\rm ns} \cdot \delta_0'' = 0$ *and* $\Xi^{\rm ns} \cdot \delta_0' = 6g$.

Proof. Clearly $\Xi^{\rm ns} \cdot \lambda = \big(\Xi \cdot \lambda\big)_{\overline{\mathcal{M}}_{g-1}} = g$. Furthermore, $\big(\Xi \cdot \delta_0\big)_{\overline{\mathcal{M}}_{g-1}} = 6(g+2)$. The pencil Ξ contains *six* reducible curves of type $N_j + C_j'$ for $j = 3, \ldots, 8$, where C_j' is a smooth curve of genus $g-2$ intersecting N_j at two points and having no further intersection with the remaining curve N_ℓ, with $\ell \neq j$. These six points will clearly contribute to the intersection $\Xi^{\rm ns} \cdot \delta_0^{\rm ram}$. Since $\Xi^{\rm ns} \cdot \delta_0'' = 0$, it follows that $\Xi^{\rm ns} \cdot \delta_0' = 6(g+2) - 2 \cdot 6 = 6g$. Finally, from the adjunction formula, we first observe that the contribution to the intersection number $\pi_*(\Xi^{\rm ns}) \cdot \delta_0$ coming from $\delta_0^{\rm ram}$ equals $N_1^2 + N_2^2 + \#\big\{\text{nodes in } C_3' + N_3, \ldots, C_8' + N_8\big\} = -4 + 2 \cdot 6$. Since $\pi^*(\delta_0) = 2\delta_0^{\rm ram} + \cdots$, we then compute

$$\Xi^{\rm ns} \cdot \delta_0^{\rm ram} = 6 + \frac{1}{2}\big(N_1^2 + N_2^2\big) = 6 - \frac{4}{2} = 4,$$

which finishes the proof. □

We now explain how Theorem 1.2 and 1.3 determines the part of the class of the divisor $\overline{\mathcal{D}}_{2i+1}$ that is relevant to birational geometry applications for $\overline{\mathcal{R}}_{2i+1}$. This provides an alternative proof of Theorem 0.2 from [FL].

Proposition 5.2. *One has* $\Xi^{\rm s} \cap \overline{\mathcal{D}}_{2i+1} = \emptyset$. *Accordingly, also* $\Xi^{\rm s} \cdot \overline{\mathcal{D}}_{2i+1} = 0$.

Proof. We have already explained in the course of proving Theorem 1.2 that no irreducible Prym curve in the pencil $\Xi^{\rm s}$ lies in $\overline{\mathcal{D}}_{2i+1}$. It remains to show that also the reducible Prym curves $\big[C_j' + N_j,\ \mathfrak{e}_{C_j'}^{\vee},\ \mathcal{O}_{N_j}(1)\big] \in \Delta_0^{\rm ram}$ lie outside $\overline{\mathcal{D}}_{2i+1}$. But this is precisely the conclusion of Proposition 3.2. □

Proposition 5.3. *One has* $\Xi^{\rm ns} \cap \overline{\mathcal{D}}_{2i+1} = \emptyset$. *Accordingly, also* $\Xi^{\rm ns} \cdot \overline{\mathcal{D}}_{2i+1} = 0$.

Proof. In the course of the proof of Theorem 1.3 we have established that no Prym curve $\xi(C)$ in $\Xi^{\rm ns}$, where C is irreducible, lies in $\overline{\mathcal{D}}_{2i+1}$. It remains to extend this conclusion to the six reducible curves $C = C_j' + N_j$, where $C_j' \in |R - N_j|$ and $j = 3, \ldots, 8$. This amounts either to an inspection that each step of the proof of Theorem 1.3 also works when $C = C_j + N_j$. Alternatively, one can carry out the steps in Proposition 3.2. □

Remark 5.4. We point out that Propositions 5.2 and 5.3 determine the λ, δ_0' and $\delta_0^{\rm ram}$-coefficients of $[\overline{\mathcal{D}}_{2i+1}]$ up to multiplication by a non-zero constant. Even though in this paper we stop short of computing the class of the diviors $\overline{\mathcal{D}}_g^f$, we point out that for $f \geq 2$ it is no longer the case $\Xi^{\rm ns} \cdot \overline{\mathcal{D}}_g^f = 0$. Indeed, setting $e = g$, then equation 13 implies $g = f^2$ and in this case $\mathcal{D}_g^f$ is the locus of pairs $[C, \eta] \in \mathcal{R}_g$ such that $h^0(C, A \otimes \eta) \neq 0$, for one of the linear systems $A \in W_{f^2-1}^{f-1}(C)$. The class of $[\overline{\mathcal{D}}_g^f]$ is computed in [FL, Theorem 0.4] and by direct calculation we observe that $\Xi^{\rm ns} \cdot \overline{\mathcal{D}}_g^f \neq 0$ (while, $\Xi^{\rm s} \cdot \overline{\mathcal{D}}_g^f = 0$). Note that when $g = 9$ and $f = 3$, the divisor $\overline{\mathcal{D}}_9^3$ is instrumental in showing that $\overline{\mathcal{R}}_9$ in uniruled, see [FV3].

5.1. **Open questions.** We end the paper with several questions. Using in an essential way the work of Arbarello-Bruno-Sernesi [ABS] describing certain rational surfaces of Du Val type (that is, blow-ups of $\mathbf{P}^2$ at 9 points) as limits of polarized $K3$ surfaces, explicit Brill-Noether general curves of every genus g defined over $\mathbb{Q}$ have been constructed in [ABFS].

Question 5.5. Can one construct Prym-canonical curves $[C,\eta]\in\mathcal{R}_g$ defined over $\mathbb{Q}$ that are Brill-Noether generic in Prym sense, that is, satisfy

$$\dim V_e^{e-f}(\omega_C\otimes\eta)=e-f(g-1-e+f)$$

for every $0\le f<e<g$? Is there an analogue of *Du Val* surfaces in Nikulin setting, that is, a class of explicit rational surfaces of degree $2g-2$ in $\mathbf{P}^g$ that are limits of general polarized standard Nikulin surfaces $X\stackrel{|L|}{\hookrightarrow}\mathbf{P}^g$? The same question can be asked for non-standard Nikulin surfaces.

Question 5.6. Because of the low slope of the λ and δ_0'-coefficients of the class $[\overline{\mathcal{D}}_{2i+1}]$, this divisor has been instrumental in showing that $\overline{\mathcal{R}}_g$ is of general type for odd $g\ge 13$, see [Br], [FJP], [FL]. Is the class $[\overline{\mathcal{D}}_{2i+1}]$ an extremal point in the effective cone of divisors $\mathrm{Eff}(\overline{\mathcal{R}}_{2i+1})$? If so, is there a (modular) birational model of $\overline{\mathcal{R}}_{2i+1}$ in which the divisor $\overline{\mathcal{D}}_{2i+1}$ gets contracted?

The last question is of a more speculative nature. Let us denote by $\chi\colon\overline{\mathcal{R}}_g\to\overline{\mathcal{M}}_{2g-1}$ the map assigning to a Prym curve $[C,\eta]$ the source curve $[\widetilde{C}]$ of the double cover $f\colon\widetilde{C}\to C$ induced by η. The pull-back map $\chi^*\colon\mathrm{Pic}(\overline{\mathcal{M}}_{2g-1})\to\mathrm{Pic}(\overline{\mathcal{R}}_g)$ has been described in [FL, Proposition 4.1]. Setting $g=2i+1$ and recalling that $\overline{\mathcal{M}}^1_{2g-1,g}$ is the corresponding Hurwitz divisor on $\overline{\mathcal{M}}_{2g-1}$ of curves having gonality at most g, putting together results from [HM] and [FL, Theorem 0.2], we observe that has the following equality of divisor classes on $\overline{\mathcal{R}}_{2i+1}$:
(31)

$$\chi^*\Bigl([\overline{\mathcal{M}}^1_{2g-1,g}]\Bigr)-\frac{\binom{4i}{2i-1}}{\binom{2i}{i-1}}\cdot\frac{i-1}{4i-1}\cdot\pi^*\Bigl([\overline{\mathcal{M}}^1_{2i+1,i+1}]\Bigr)-\frac{\binom{4i}{2i-1}}{\binom{2i}{i}}\cdot\frac{3}{4i-1}\cdot[\overline{\mathcal{D}}_{2i+1}]\in\mathbb{Q}\Bigl\langle\delta_j,\delta_{j,g-j}:j\ge 1\Bigr\rangle.$$

In other words, in the $\mathbb{Q}\langle\lambda,\delta_0',\delta_0'',\delta_0^{\mathrm{ram}}\rangle$-subspace of $\mathrm{Pic}(\overline{\mathcal{R}}_{2i+1})$ an effective linear combination of $\pi^*(\overline{\mathcal{M}}_{2i+1,i+1})$ and $\overline{\mathcal{D}}_{2i+1}$ is linearly equivalent to the pullback of the Hurwitz divisor $\overline{\mathcal{M}}^1_{2g-1,g}$ under the map χ. This invites the following question:

Question 5.7. Fix $i\ge 1$. Does one have a set-theoretic equality on $\mathcal{R}_{2i+1}$

$$\chi^{-1}\bigl(\mathcal{M}^1_{2g-1,g}\bigr)=\pi^{-1}(\mathcal{M}_{2i+1,i+1})\cup\mathcal{D}_{2i+1}?$$

Equivalently, given a smooth curve C of genus $g=2i+1$ and of maximal gonality $i+2$, for an étale double cover $f\colon\widetilde{C}\to C$ such that $f_*\mathcal{O}_{\widetilde{C}}\cong\mathcal{O}_C\oplus\eta$, is the following equivalence true:

$$\mathrm{gon}(\widetilde{C})\le 2i+1\Longleftrightarrow\eta\in C_i-C_i? \tag{32}$$

Note that one implication is obviously true. If $\eta\cong\mathcal{O}_C(D-D')\in C_i-C_i$, with D and D' being effective divisors of degree i, then $h^0\bigl(\widetilde{C},f^*(\mathcal{O}_C(D))\bigr)=h^0(C,\mathcal{O}_C(D))+h^0(C,\mathcal{O}_C(D'))\ge 2$, therefore $\mathrm{gon}(\widetilde{C})\le 2i$. In particular, a positive answer to (32) would imply that $\mathrm{gon}(\widetilde{C})$ always has to be even! Question (32) has been answered in the affirmative for $g=3,5$ in [FL, Theorems 5.1 and 5.2]. Other questions concerning the Brill-Noether theory of the curve $\widetilde{C}$ covering C, have been investigated in [DLC].

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