

DU VAL CURVES AND THE POINTED BRILL-NOETHER THEOREM

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ABSTRACT. We show that a general curve in an explicit class of what we call Du Val pointed curves satisfies the Brill-Noether Theorem for pointed curves. Furthermore, we prove that a generic pencil of Du Val pointed curves is disjoint from all Brill-Noether divisors on the universal curve. This provides explicit examples of smooth pointed curves of arbitrary genus defined over \mathbb{Q} which are Brill-Noether general. A similar result is proved for 2-pointed curves as well using explicit curves on elliptic ruled surfaces.

The pointed Brill-Noether Theorem concerns the study of linear series on a general pointed algebraic curve $[C, p]$ with prescribed ramification at the marked point p . Recall that for a point $p \in C$ and a linear series $\ell = (L, V) \in G_d^r(C)$, one denotes by

$$\alpha^\ell(p) : 0 \leq \alpha_0^\ell(p) \leq \dots \leq \alpha_r^\ell(p) \leq d - r$$

the *ramification* sequence of ℓ at p . One says that $p \in C$ is a ramification point of ℓ if $\alpha_r^\ell(p) > 0$. For instance, the ramification points of the canonical linear series are precisely the Weierstrass points of C . The total number of ramification points of ℓ , counted with appropriate multiplicities, is given by the *Plücker formula*, see for instance [EH1] Proposition 1.1. Fixing a Schubert index $\alpha : 0 \leq \alpha_0 \leq \dots \leq \alpha_r \leq d - r$, one can ask when a general pointed curve $[C, p]$ of genus g carries a linear series $\ell \in G_d^r(C)$ with ramification sequence $\alpha^\ell(p) \geq \alpha$. The locus $G_d^r(C, p, \alpha)$ of linear series on C satisfying this condition is a generalized determinantal variety of expected dimension

$$\rho(g, r, d, \alpha) := \rho(g, r, d) - w(\alpha),$$

where $\rho(g, r, d) := g - (r + 1)(g - d + r)$ and $w(\alpha) := \alpha_0 + \dots + \alpha_r$ is the weight of α . It is proved in [EH2] Theorem 1.1 that for a general pointed curve $[C, p] \in \mathcal{M}_{g,1}$, each component of $G_d^r(C, p, \alpha)$, if nonempty, has dimension precisely $\rho(g, r, d, \alpha)$. Moreover, [EH2] Proposition 1.2 establishes that $G_d^r(C, p, \alpha) \neq \emptyset$ if and only if

$$\sum_{i=0}^r \max\{\alpha_i + g - d + r, 0\} \leq g.$$

The proofs in [EH2] rely on limit linear series and degeneration to the boundary of the universal curve $\overline{\mathcal{C}}_g := \overline{\mathcal{M}}_{g,1}$. Up to now, no examples whatsoever of *smooth pointed* curves $[C, p] \in \mathcal{C}_g$ verifying the pointed Brill-Noether Theorem have been known. This situation contrasts the classical Brill-Noether Theorem; even though the original proof in [GH] used degeneration to nodal curves, soon afterwards, in his well-known paper [Laz], Lazarsfeld showed that sections of general polarized $K3$ surfaces are Brill-Noether-Petri general.

2010 *Mathematics Subject Classification*. 14H99 (primary), 14J26 (secondary).

Key words and phrases. Brill-Noether general smooth pointed curves, Du Val curves, rational and ruled surfaces.

Since curves in the polarization class of a $K3$ surface have no obvious distinguished marked points, it is far from clear how to extend the results of [Laz] to the case of pointed curves. In [ABFS], an *explicit* specialization of Lazarsfeld's curves emerging from the paper [ABS] is worked out. It is shown that suitably general singular plane curves of degree $3g$ having multiplicity g at eight points in \mathbf{P}^2 and multiplicity $g - 1$ at a further ninth point verify the Brill-Noether-Petri Theorem. Such curves, which belong to the closure in $\overline{\mathcal{M}}_g$ of the locus of curves lying on $K3$ surfaces, are called *Du Val curves* of genus g .

One aim of this paper is to show that the Du Val curves introduced in [ABFS] lead to Brill-Noether general smooth *pointed* curves of any genus defined over \mathbb{Q} . The essential observation is that, unlike curves on general $K3$ surfaces, Du Val curves have a distinguished marked point with respect to which a pointed Brill-Noether Theorem can be established.

We begin by recalling the setting of [ABFS]. Let S' be the blow-up of \mathbf{P}^2 at nine points p_1, \dots, p_9 which are *general* in the sense of [ABFS] (see also Section 1 for the precise definition). Let E_1, \dots, E_9 be the exceptional curves on S' . We denote by $J' \in |-K_{S'}|$ the unique smooth plane cubic passing through p_1, \dots, p_9 and consider the linear system on S'

$$L_g := |3g\ell - gE_1 - \dots - gE_8 - (g-1)E_9|,$$

where $\ell \in \text{Pic}(S')$ is the proper transform of a line in \mathbf{P}^2 . The main result of [ABFS] is that a general curve $C' \in L_g$ verifies the Brill-Noether-Petri Theorem. For each $g \geq 1$, the points p_1, \dots, p_9 determine a 10-th point $p_{10} = p_{10}^{(g)}$ which is the base point of L_g . In fact, $p_{10} \in C' \cdot J'$, for every $C' \in L_g$. The point p_{10} is determined by the relation

$$(1) \quad p_{10} = p_{10}^{(g)} = -gp_1 - \dots - gp_8 - (g-1)p_9 \in J',$$

with respect to the group law of the elliptic curve. Under the genericity assumptions on the points p_1, \dots, p_9 we started with, the points $p_{10}^{(g)}$ are distinct from one another, as well as from p_1, \dots, p_9 , see also Proposition 1. As in [ABFS], we set $S := \text{Bl}_{p_{10}}(S)$ and, by slight abuse of notation, we denote by E_1, \dots, E_{10} the corresponding exceptional curves. If C is the strict transform of C' , then $|\mathcal{O}_S(C)|$ is a base point free linear system of curves of genus g having a section induced by E_{10} (note that $C \cdot E_{10} = 1$).

A *pointed Du Val curve* is a smooth pointed curve $[C, p] \in \overline{\mathcal{C}}_g$, where $C \subset S$ is as above and $\{p\} = C \cdot E_{10}$. Before stating our main results, we recall that for a linear system $\ell \in G_d^r(C)$ and points $p_1, \dots, p_n \in C$, the *pointed Brill-Noether number* is defined as

$$\rho(\ell, p_1, \dots, p_n) := \rho(g, r, d) - w(\alpha^\ell(p_1)) - \dots - w(\alpha^\ell(p_n)).$$

Theorem 1. *A general pointed Du Val curve $[C, p]$ verifies the pointed Brill-Noether Theorem, that is, $\dim G_d^r(C, p, \alpha) = \rho(g, r, d, \alpha)$, when $G_d^r(C, p, \alpha) \neq \emptyset$. In particular, for every linear system ℓ on C , one has $\rho(\ell, p) \geq 0$.*

Since the points p_1, \dots, p_9 can be chosen to have rational coefficients, $p = p_{10}^{(g)} \in \mathbf{P}^2(\mathbb{Q})$ and then $[C, p]$ is also defined over \mathbb{Q} . Hence, paralleling [ABFS] Corollary 1.3, our Theorem 1 provides examples of Brill-Noether general pointed curves of arbitrary genus g defined over \mathbb{Q} .

If \mathcal{W}_g denotes the locus of Weierstrass points in \mathcal{C}_g (known to be an irreducible divisor on the universal curve), by direct calculation we show that the image of the family $j : \mathbf{P}^1 \rightarrow \overline{\mathcal{C}}_g$ induced by a Lefschetz pencil of Du Val curves on S satisfies

$$j(\mathbf{P}^1) \cap \overline{\mathcal{W}}_g = \emptyset,$$

that is, for *every* pointed Du Val curve $[C, p]$, the marked point p is not a Weierstrass point of C . As we point out in Corollary 1, this implies that $j(\mathbf{P}^1)$ is disjoint from all pointed Brill-Noether divisors on $\overline{\mathcal{C}}_g$. We refer to Section 1 for detailed background on pointed Brill-Noether divisors on $\overline{\mathcal{C}}_g$.

0.1. Brill-Noether general 2-pointed curves on elliptic ruled surfaces. The Brill-Noether problem can be formulated for n -pointed curves $[C, p_1, \dots, p_n]$ and concerns the variety of linear series $\ell \in G_d^r(C)$ having prescribed ramification $\alpha^\ell(p_i) \geq \alpha^i$ for $i = 1, \dots, n$, given in terms of fixed Schubert indices $\alpha^1, \dots, \alpha^n$. In Section 2, using *decomposable* elliptic ruled surfaces, we exhibit for the first time examples of smooth 2-pointed curves of arbitrary genus verifying the 2-pointed Brill-Noether Theorem. The construction is inspired by a very nice note of Treibich [Tre].

We start with an elliptic curve J and a non-torsion line bundle $\eta \in \text{Pic}^0(J)$. The decomposable ruled surface

$$\phi : Y := \mathbf{P}(\mathcal{O}_J \oplus \eta) \rightarrow J$$

is endowed with two disjoint sections J_0 and J_1 respectively. Pick a point $r \in J$ and denote by $f := \phi^{-1}(r)$ the corresponding ruling of Y . We denote by $s = s^{(g)} \in J$ the point determined by the equation $\mathcal{O}_J(s - r) = \eta^{\otimes g}$. The linear system $|gJ_0 + f|$ consists of curves of genus g and has two base points, namely

$$\{p\} := \phi^{-1}(r) \cdot J_1 \quad \text{and} \quad \{q\} := \phi^{-1}(s) \cdot J_0,$$

respectively. We establish the following result:

Theorem 2. *The 2-pointed curve $[C, p, q] \in \mathcal{M}_{g,2}$, where $C \in |gJ_0 + f|$ is a general element and p and q are as above, verifies the 2-pointed Brill-Noether Theorem. In particular, for every linear series $\ell \in G_d^r(C)$ the inequality $\rho(\ell, p, q) \geq 0$ holds.*

A Brill-Noether general 2-pointed curve supports a Brill-Noether general 1-pointed curve obtained by dropping either marked point. In particular, both 1-pointed curves $[C, p]$ and $[C, q]$ in the statement of Theorem 2 verify the 1-pointed Brill-Noether Theorem as well. For details, we refer to Section 2. The proofs of both Theorems 1 and 2 are intimately related, and rely on a canonical degeneration within the corresponding linear system on the surface to a singular curve with an elliptic tail. This leads to an inductive argument in the genus, which ultimately proves the desired Brill-Noether type theorems.

Arguably, for many applications, the curves constructed in Theorem 2 are the simplest known examples of Brill-Noether general smooth curves of arbitrary genus. They combine two desirable features: (i) The canonical elliptic tail degeneration in $|gJ_0 + f|$ provides a system of Brill-Noether general curves of any genus on the surface Y , which invites inductive proofs and reduction to genus 1 curves and Schubert calculus problems in the spirit of limit linear series, and (ii) The general curve in $|gJ_0 + f|$ being smooth, one need not build-up the degeneration set-up typical for limit linear series applications. A vivid

instance of their use is the recent proof in [FK1] of the Prym-Green Conjecture concerning the naturality of the resolution of a paracanonical curve $\varphi_{K_C \otimes \eta} : C \hookrightarrow \mathbf{P}^{g-2}$, where C is a general curve of odd genus and η is an ℓ -torsion line bundles on C . The conjecture is proven for odd g and arbitrary ℓ using precisely the curves constructed in Theorem 2. For a proof of the Prym-Green Conjecture using special $K3$ surfaces instead — but only in the range $\ell \geq \sqrt{\frac{g+2}{2}}$ — see [FK2].

Both classes of curves constructed in Theorems 1 and 2 lie in the closure of the locus in \mathcal{M}_g of curves contained in a $K3$ surface, see [ABS]. Theorem 4, which is a variation of the construction in Theorem 2 in the sense that Y is replaced by an *indecomposable* ruled surface over an elliptic curve, provides examples of Brill-Noether general pointed curves which are not limits of $K3$ section. We refer to the end of Section 2 for details.

Acknowledgments: Section 1 of this paper uses in an essential way the methods developed in [ABFS]. The first author is grateful to Enrico Arbarello for interesting discussions related to this circle of ideas. The presentation of the paper clearly benefitted from the insightful remarks of two referees, whom we thank.

1. POINTED DU VAL CURVES AND WEIERSTRASS POINTS

We assume familiarity with the theory of limit linear series in the sense of [EH1]. We need a few facts concerning divisor classes on the universal curve $\overline{\mathcal{C}}_g := \overline{\mathcal{M}}_{g,1}$. The rational Picard group $\text{Pic}(\overline{\mathcal{C}}_g)$ is generated by the Hodge class λ , the relative cotangent class ψ , the boundary divisor class $\delta_{\text{irr}} := [\Delta_{\text{irr}}]$ of irreducible pointed stable curves of genus g and by the classes $\delta_i := [\Delta_i]$, where for each $i = 1, \dots, g-1$, the boundary divisor $\Delta_i \subset \overline{\mathcal{C}}_g$ corresponds to a transverse union of two smooth curves of genus i and $g-i$ respectively, meeting in one point, the marked points lying on the genus i component. If $\pi : \overline{\mathcal{C}}_g \rightarrow \overline{\mathcal{M}}_g$ is the morphism forgetting the marked point, the boundary divisors on $\overline{\mathcal{M}}_g$ and those on $\overline{\mathcal{C}}_g$ are related by the following formulas:

$$\pi^*(\delta_{\text{irr}}) = \delta_{\text{irr}}, \quad \pi^*(\delta_i) = \delta_i + \delta_{g-i}, \quad \text{for } 1 \leq i < \frac{g}{2}, \quad \text{and } \pi^*(\delta_{\frac{g}{2}}) = \delta_{\frac{g}{2}}, \quad \text{for } g \text{ even.}$$

If $\alpha : 0 \leq \alpha_0 \leq \dots \leq \alpha_r \leq d-r$ is a Schubert index of type (r, d) , we introduce the *complementary* Schubert index $\alpha^c : 0 \leq d-r-\alpha_r \leq \dots \leq d-r-\alpha_0 \leq d-r$. When $\alpha_i = 0$ for $i = 0, \dots, r$, we say that α is the trivial Schubert index. We recall the definition of *pointed Brill-Noether divisors* on $\overline{\mathcal{C}}_g$. Fix integers $r, d \geq 1$ and a Schubert index $\alpha : 0 \leq \alpha_0 \leq \dots \leq \alpha_r \leq d-r$ such that the expected dimension of the locus of linear series of type \mathbf{g}_d^r on a curve of genus g with prescribed ramification α at a given point equals -1 . In other words,

$$\rho(g, r, d, \alpha) := g - (r+1)(g-d+r) - w(\alpha) = -1.$$

Let $\mathcal{C}_{g,d}^r(\alpha) := \{[C, p] \in \mathcal{C}_g : G_d^r(C, p, \alpha) \neq \emptyset\}$ be the corresponding pointed Brill-Noether locus. For instance,

$$\mathcal{W}_g := \mathcal{C}_{g,2g-2}^{g-1}(0, \dots, 0, 1) = \left\{ [C, p] \in \mathcal{C}_g : H^0(C, \omega_C(-gp)) \neq 0 \right\}$$

is the divisor of Weierstrass points. Since \mathcal{W}_g can be parametrized by the Hurwitz space of g -fold covers of \mathbf{P}^1 having a point of total ramification, it follows from [Arb] Theorem 2.5

that \mathcal{W}_g is an irreducible divisor. If $\rho(g, r, d) = -1$, then $\mathcal{C}_{g,d}^r(0, \dots, 0)$ is the pull-back to \mathcal{C}_g of the Brill-Noether divisor $\mathcal{M}_{g,d}^r$ consisting of curves carrying a \mathfrak{g}_d^r .

Cukierman [Cuk] computed the class of the closure $\overline{\mathcal{W}}_g$ of the Weierstrass divisor in $\overline{\mathcal{C}}_g$:

$$(2) \quad [\overline{\mathcal{W}}_g] = -\lambda + \binom{g+1}{2} \psi - \sum_{i=1}^{g-1} \binom{g-i+1}{2} \delta_i \in \text{Pic}(\overline{\mathcal{C}}_g).$$

We also recall [EH2] that the class of the pull-back to $\overline{\mathcal{C}}_g$ of the Brill-Noether divisors $\overline{\mathcal{M}}_{g,d}^r$ is given by the formula

$$(3) \quad [\overline{\mathcal{C}}_{g,d}^r(0, \dots, 0)] = c_{g,d,r} \cdot \mathcal{BN}_g,$$

where $c_{g,d,r} \in \mathbb{Q}_{>0}$ and

$$\mathcal{BN}_g := (g+3)\lambda - \frac{g+1}{6} \delta_{\text{irr}} - \sum_{i=1}^{g-1} i(g-i) \delta_i \in \text{Pic}(\overline{\mathcal{C}}_g).$$

Remarkably, the pointed Brill-Noether divisors only span a 2-dimensional cone in $\text{Pic}(\overline{\mathcal{C}}_g)$. It is shown in [EH3] Theorem 1.2 that $\overline{\mathcal{C}}_{g,d}^r(\alpha)$ is a proper subvariety of $\overline{\mathcal{C}}_g$, having a unique divisorial component. The class of this component, which we shall denote by $[\overline{\mathcal{C}}_{g,d}^r(\alpha)] \in CH^1(\overline{\mathcal{C}}_g)$, can be written as a linear combination

$$[\overline{\mathcal{C}}_{g,d}^r(\alpha)] = \mu \cdot [\overline{\mathcal{W}}_g] + \nu \cdot \mathcal{BN}_g,$$

for non-negative rational constants μ and ν , which are determined in [FT].

Definition 1. We say that a pointed curve $[C, p] \in \mathcal{C}_g$ is *Brill-Noether general*, if for every choice of integers r, d and a corresponding Schubert index α of type (r, d) , we have

$$\dim G_d^r(C, p, \alpha) = \rho(g, r, d, \alpha) \quad \text{or} \quad G_d^r(C, p, \alpha) = \emptyset.$$

In particular, for every linear series $\ell \in G_d^r(C)$, the inequality $\rho(\ell, p) \geq 0$ holds.

If $[C, p]$ is a Brill-Noether general pointed curve, by letting α be the trivial Schubert index, we obtain that C is a Brill-Noether general (unpointed) curve.

Lemma 1. *A pointed curve $[C, p] \in \mathcal{C}_g$ carries no linear series ℓ with $\rho(\ell, p) < 0$ if and only if it does not belong to any locus $\mathcal{C}_{g,d}^r(\alpha)$, where $\rho(g, r, d, \alpha) = -1$.*

Proof. One implication being obvious, assume first there exists a linear series $\ell \in G_d^r(C)$ with $w(\alpha^\ell(p)) > \rho(g, r, d) \geq 0$. Then we can find a Schubert index

$$\alpha' : 0 \leq \alpha'_0 \leq \dots \leq \alpha'_r \leq d - r$$

with $w(\alpha') = \rho(g, r, d) + 1 \leq w(\alpha^\ell(p))$, such that $\alpha' \leq \alpha^\ell(p)$ (lexicographically). Hence $\rho(g, r, d, \alpha') = -1$ and $[C, p] \in \mathcal{C}_{g,d}^r(\alpha')$. Finally, assume we are in the case when there exists a linear series $\ell \in G_d^r(C)$ with $\rho(g, r, d) < -1$. Then we can find $d' > d$ and a Schubert index $\alpha' : 0 \leq \alpha'_0 \leq \dots \leq \alpha'_r \leq d' - r$ with $\alpha'_i \leq d' - d$ and $w(\alpha') = \rho(g, r, d') + 1$. Hence $[C, p] \in \mathcal{C}_{g,d'}^r(\alpha')$, which finishes the proof. \square

We now turn to Du Val surfaces. In what follows, we denote by \equiv linear equivalence of divisors on varieties. Following [ABFS] Proposition 2.3, we recall that a set of nine distinct points p_1, \dots, p_9 in \mathbf{P}^2 is said to be *general* if on the blown-up plane $S' := \text{Bl}_{\{p_1, \dots, p_9\}}(\mathbf{P}^2)$, every effective divisor

$$D' \equiv d\ell - \nu_1 E_1 - \dots - \nu_9 E_9$$

with $\nu_i \geq 0$ and satisfying $D \cdot J' = 0$ is necessarily a multiple of J' . In particular, if p_1, \dots, p_9 are general points, then the sum $p_1 + \dots + p_9 \in J'$ is not torsion.

Remark 1. Examples of sets of nine general points in $\mathbf{P}^2(\mathbb{Q})$ are easy to produce, if one starts with a concrete elliptic curve defined over \mathbb{Q} . For instance, it is shown in [ABFS] that the following points lying on the elliptic curve $E : y^2 = x^3 + 17$ are general: $p_1 = (-2, 3)$, $p_2 = (-1, -4)$, $p_3 = (2, 5)$, $p_4 = (4, 9)$, $p_5 = (52, 375)$, $p_6 = (5234, 37866)$, $p_7 = (8, -23)$, $p_8 = (43, 282)$, and $p_9 = \left(\frac{1}{4}, -\frac{33}{8}\right)$.

Recall the definition (1) of the points $p_{10}^{(g)} \in J'$, where $g \geq 1$.

Proposition 1. *Assume that the points p_1, \dots, p_9 are general. Then for $k = 2, \dots, g$, the difference $p_{10}^{(k)} - p_{10}^{(k-1)} \in \text{Pic}^0(J')$ is not torsion.*

Proof. Using (1), we obtain that $p_{10}^{(k-1)} - p_{10}^{(k)} = p_1 + \dots + p_9$ (with respect to the group law of J'), for each $k \geq 2$. As pointed out, this is not a torsion point on J' . \square

We now introduce the pointed Du Val pencil in $\overline{\mathcal{C}}_g$, which is a lift under the forgetful map $\pi : \overline{\mathcal{C}}_g \rightarrow \overline{\mathcal{M}}_g$ of the pencil of unpointed curves introduced in Section 4 of [ABFS]. Recall that $S := \text{Bl}_{p_{10}^{(g)}}(S')$ and we denote by L_g the proper transform of the linear system on S' denoted by the same symbol in the Introduction. The linear system of Du Val curves of genus $g - 1$ on S , that is,

$$\Lambda_{g-1} := |3(g-1)\ell - (g-1)E_1 - \dots - (g-1)E_8 - (g-2)E_9|$$

appears as a hyperplane in the g -dimensional linear system L_g . It consists precisely of the curves of the form $D + J \in L_g$, where $J \subset S$ denotes the proper transform of J' and $D \in \Lambda_{g-1}$. Since $J \equiv 3\ell - E_1 - \dots - E_{10}$, note that $D \cdot J = 1$.

We now choose a Lefschetz pencil in L_g , which has $2g - 2 = C^2$ base points. Let $X := \text{Bl}_{2g-2}(S)$ be the blow-up of S at those points. Since $C \cdot E_{10} = 1$ for $C \in L_g$, the corresponding fibration $f : X \rightarrow \mathbf{P}^1$ has a section induced by the proper transform of E_{10} on X . This induces a pencil in the universal curve

$$j : \mathbf{P}^1 \rightarrow \overline{\mathcal{C}}_g.$$

In what follows it will be convenient to use the notation $C_1 \cup_p C_2$, for a stable curve consisting of two irreducible components C_1 and C_2 respectively, meeting transversally at a point p .

Proposition 2. *The intersection numbers of the pointed Du Val pencil with the generators of $\text{Pic}(\overline{\mathcal{C}}_g)$ are as follows:*

$$j^*(\lambda) = g, \quad j^*(\psi) = 1, \quad j^*(\delta_{\text{irr}}) = 6(g+1), \quad j^*(\delta_1) = 1, \quad j^*(\delta_i) = 0 \quad \text{for } i = 2, \dots, g-1.$$

Proof. One has $j^*(\psi) = -E_{10}^2 = 1$. The restrictions of the classes $\lambda, \delta_{\text{irr}}, \delta_2, \dots, \delta_{g-2}$ follow from [ABFS] Theorem 4.1 and are copied here for the sake of completeness. There exists precisely one element of the pencil f of the type $D + J$, for some $D \in \Lambda_{g-1}$. Since $E_{10} \cdot J = 1$ while $E_{10} \cdot D = 0$, the marked point lies on the elliptic component of this singular element. The corresponding pointed stable curve is $[D \cup_{p_{10}^{(g-1)}} J', p_{10}^{(g)}] \in \overline{\mathcal{C}}_g$. Hence $j^*(\delta_1) = 1$, and since $\pi^*(\delta_1) = \delta_1 + \delta_{g-1}$, it follows that $j^*(\delta_{g-1}) = 0$. \square

By direct computation, using (2) and (3), it follows that the pencil $j(\mathbf{P}^1) \subset \overline{\mathcal{C}}_g$ has intersection number zero with the Brill-Noether class \mathcal{BN}_g as well as with the Weierstrass divisor $\overline{\mathcal{W}}_g$, that is,

$$j^*(\mathcal{BN}_g) = (g+3)g - \frac{g+1}{6}(6g+6) - (g-1) = 0, \quad \text{and}$$

$$j^*(\overline{\mathcal{W}}_g) = -g + \binom{g+1}{2} - \binom{g}{2} = 0.$$

Since the class of any pointed Brill-Noether divisor lies in the cone spanned by these classes [EH3] Theorem 1.2, it follows that the intersection number of $j(\mathbf{P}^1)$ with the closure of any pointed Brill-Noether divisor is zero as well.

We are now in a position to complete the proof of our main result.

Proof of Theorem 1. We shall establish by induction on g that the general member of the Du Val pencil satisfies the pointed Brill-Noether Theorem. For $g = 1$, we have that $[C, p] \in \mathcal{C}_1$ and it is well-known that each smooth pointed elliptic curve is Brill-Noether general, see e.g. [EH2] Theorem 1.1. Assuming the statement for Du Val curves of genus $g-1$, suppose by contradiction that there exist $r, d \geq 1$ and a Schubert index α such that $\dim G_d^r(C, p, \alpha) > \rho(g, r, d, \alpha)$, for each $C \in L_g$, where $\{p\} = C \cap E_{10}$.

Let $j : \mathbf{P}^1 \rightarrow \overline{\mathcal{C}}_g$ be a Lefschetz pencil of Du Val curves on S . As explained in Proposition 2, the pencil contains a unique elliptic tail degeneration $[D \cup_{p_{10}^{(g-1)}} J', p_{10}^{(g)}]$, where D is an element of Λ_{g-1} . Then the variety

$$\overline{G}_d^r(D \cup J', p_{10}^{(g)}, \alpha)$$

of limit linear series $\ell = (\ell_D, \ell_{J'}) \in G_d^r(D) \times G_d^r(J')$ on $D \cup_{p_{10}^{(g-1)}} J'$ satisfying the ramification condition $\alpha^\ell(p_{10}^{(g)}) \geq \alpha$ is of dimension at least $\rho(g, r, d, \alpha) + 1$. Note that $[D, p_{10}^{(g-1)}]$ can be assumed to be a general Du Val curve of genus $g-1$, for every curve from Λ_{g-1} appears as an elliptic tail degeneration in a genus g Du Val pencil.

Let ℓ be a general point of an irreducible component Z of $\overline{G}_d^r(D \cup J', p_{10}^{(g)}, \alpha)$ of maximal dimension, and set $\beta := \alpha^{\ell_D}(p_{10}^{(g-1)})$. By the additivity of the Brill-Noether number with respect to marked points, we write

$$\rho(g, r, d, \alpha) = \rho(\ell, p_{10}^{(g)}) \geq \rho(\ell_D, p_{10}^{(g-1)}) + \rho(\ell_{J'}, p_{10}^{(g)}, p_{10}^{(g-1)}).$$

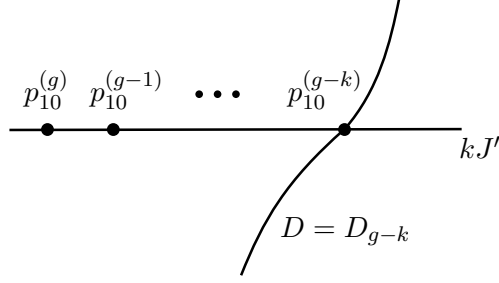


FIGURE 1. The k -th step of the elliptic tail specialization in a Du Val pencil.

By the construction in [EH1] Theorem 3.3 of the variety of limit linear series, Z is birational to an irreducible component of the product

$$G_d^r(D, p_{10}^{(g-1)}, \beta) \times G_d^r(J', (p_{10}^{(g-1)}, \beta^c), (p_{10}^{(g)}, \alpha)).$$

By assumption, each component of $G_d^r(D, p_{10}^{(g-1)}, \beta)$ has dimension $\rho(g-1, r, d, \beta)$.

Moving to J' , first observe that $\rho(\ell_{J'}, p_{10}^{(g)}, p_{10}^{(g-1)}) \geq 0$. Indeed, assuming otherwise, we denote by (a_0, \dots, a_r) and (b_0, \dots, b_r) the vanishing sequences of $\ell_{J'}$ at the points $p_{10}^{(g-1)}$ and $p_{10}^{(g)}$ respectively, and obtain that there exist indices $0 \leq i < j \leq r$ such that

$$a_i + b_{r-i} = a_j + b_{r-j} = d.$$

In particular, the underlying line bundle of the linear series $\ell_{J'}$ corresponds to the divisors $a_i \cdot p_{10}^{(g-1)} + b_{r-i} \cdot p_{10}^{(g)} \equiv a_j \cdot p_{10}^{(g-1)} + b_{r-j} \cdot p_{10}^{(g)}$, from which it follows that $p_{10}^{(g-1)} - p_{10}^{(g)}$ is a torsion class in $\text{Pic}^0(J')$, which contradicts Proposition 1.

Furthermore, it implicitly follows from [EH1], and it is spelled-out explicitly in [Oss] Lemma 2.1, that every 2-pointed elliptic curve $[E, x, y] \in \mathcal{M}_{1,2}$, where the difference $\mathcal{O}_E(x-y)$ is not a torsion class, is Brill-Noether general. This follows from the observation that for a line bundle $L \in \text{Pic}^d(E)$ which is not given by a divisor on E supported only at x and y , the flags in $H^0(E, L) \cong \mathbb{C}^d$ given by the vanishing of sections at x and y respectively, are transversal. In particular, Schubert cycles in $G(r+1, H^0(E, L))$ defined in terms of these flags intersect in the expected dimension. Applying this fact to the case at hand, we find

$$\dim G_d^r(J', (p_{10}^{(g-1)}, \beta^c), (p_{10}^{(g)}, \alpha)) = \rho(1, r, d, \beta^c, \alpha) := \rho(1, r, d) - w(\beta^c) - w(\alpha).$$

Putting all together, we obtain that

$$\begin{aligned} \rho(g, r, d, \alpha) < \dim Z &= \dim G_d^r(D, p_{10}^{(g-1)}, \beta) + \dim G_d^r(J', (p_{10}^{(g-1)}, \beta^c), (p_{10}^{(g)}, \alpha)) \\ &= \rho(g-1, r, d, \beta) + \rho(1, r, d, \beta^c, \alpha) \leq \rho(g, r, d, \alpha), \end{aligned}$$

which is a contradiction. Therefore, the singular pointed curve $(D \cup J', p_{10}^{(g)})$ is Brill-Noether general. \square

Corollary 1. *The image of a Du Val pencil $j : \mathbf{P}^1 \rightarrow \overline{\mathcal{C}}_g$ is disjoint from all pointed Brill-Noether divisors $\overline{\mathcal{C}}_{g,d}^r(\alpha)$.*

Proof. As noted in Proposition 2, we have $j(\mathbf{P}^1) \cdot \overline{\mathcal{C}}_{g,d}^r(\alpha) = 0$. Either $j(\mathbf{P}^1) \cap \overline{\mathcal{C}}_{g,d}^r(\alpha) = \emptyset$, or else, $j(\mathbf{P}^1) \subset \overline{\mathcal{C}}_{g,d}^r(\alpha)$. The proof of Theorem 1 rules out the second possibility. \square

In general it is not known whether $\mathcal{C}_{g,d}^r(\alpha)$ is pure of codimension 1. However, when this happens, for instance in the case of the Weierstrass divisor \mathcal{W}_g , Corollary 1 shows that *every* pointed Du Val curve is Brill-Noether general with respect to linear series of that type.

1.1. **Towards the effective cone of $\overline{\mathcal{C}}_g$.** The Slope Conjecture [HM] on effective divisors on $\overline{\mathcal{M}}_g$ used to predict that the Brill-Noether divisors $\overline{\mathcal{M}}_{g,d}^r$ of curves with a linear series \mathfrak{g}_d^r where $\rho(g, r, d) = -1$ are extremal. Via Lazarsfeld's result [Laz], an equivalent formulation of the Slope Conjecture is that the rational curve $R \subset \overline{\mathcal{M}}_g$ induced by a Lefschetz pencil of genus g curves on a general polarized $K3$ surface (X, H) , with $H^2 = 2g - 2$ is *nef*, that is, it intersects every effective divisor on $\overline{\mathcal{M}}_g$ non-negatively. Note that the intersection numbers of R with the generators of $\text{Pic}(\overline{\mathcal{M}}_g)$ are given as follows, see for instance [FP]:

$$R \cdot \lambda = g + 1, \quad R \cdot \delta_{\text{irr}} = 6g + 18 \quad \text{and} \quad R \cdot \delta_i = 0, \quad \text{for } i = 1, \dots, \left\lfloor \frac{g}{2} \right\rfloor.$$

Although the Slope Conjecture is false for high g , see [FP] and [Far], it is known to hold for $g \leq 9$ and $g = 11$. The statement played an important role in Mukai's work on alternative birational models of $\overline{\mathcal{M}}_g$ for $g = 7, 8, 9$ and has guided the search for geometric divisors D on $\overline{\mathcal{M}}_g$ having small slope, that is, satisfying $R \cdot D < 0$, which necessarily contain the locus in $\overline{\mathcal{M}}_g$ of curves that lie on $K3$ surfaces.

It is an interesting question to find an adequate definition of the notion of slope for effective divisors on the universal curve and an analogue of the Slope Conjecture on $\overline{\mathcal{C}}_g$.

Problem 1. *For what values of g is the Du Val pencil $j : \mathbf{P}^1 \rightarrow \overline{\mathcal{C}}_g$ nef, that is, $j^*(D) \geq 0$, for every effective divisor D on $\overline{\mathcal{C}}_g$? For which g does this inequality hold for all effective divisors D on $\overline{\mathcal{C}}_g$ such that $\pi(D) = \overline{\mathcal{M}}_g$?*

In light of Corollary 1, a closely related question is whether the Weierstrass divisor $\overline{\mathcal{W}}_g$ is extremal in the effective cone $\text{Eff}(\overline{\mathcal{C}}_g)$. The hypothesis that $\overline{\mathcal{W}}_g$ is extremal has recently received further credence due to [Pol]. Note that for the pull-backs to $\overline{\mathcal{C}}_g$ of the effective divisors on $\overline{\mathcal{M}}_{6i+10}$ constructed in [Far], Problem 1 has a negative answer. For instance, when $g = 10$, the divisor in question is

$$\mathcal{Z}_{10} := \{[C, p] \in \mathcal{C}_{10} : C \text{ lies on a } K3 \text{ surface}\},$$

and $[\overline{\mathcal{Z}}_{10}] = 7\lambda - 5\delta_{\text{irr}} - \delta_1 - \delta_9 - 12\delta_2 - 12\delta_8 - \dots \in \text{Pic}(\overline{\mathcal{C}}_{10})$, see [FP] Theorem 1.6. By applying Proposition 2, we compute $j^*([\overline{\mathcal{Z}}_{10}]) = -1 < 0$. We are unaware of any example of an effective divisor D on $\overline{\mathcal{C}}_g$ that is *not* a pull-back of an effective divisor from $\overline{\mathcal{M}}_g$ and which satisfies $j^*(D) < 0$.

2. BRILL-NOETHER GENERAL TWO-POINTED CURVES VIA ELLIPTIC SURFACES

In this section we construct explicit smooth 2-pointed curves of arbitrary genus verifying the Brill-Noether Theorem. Given a smooth curve C , distinct points $p, q \in C$ and two Schubert indices

$$\alpha : 0 \leq \alpha_0 \leq \dots \leq \alpha_r \leq d - r \quad \text{and} \quad \beta : 0 \leq \beta_0 \leq \dots \leq \beta_r \leq d - r,$$

we consider the variety $G_d^r(C, (p, \alpha), (q, \beta))$ of linear series $\ell \in G_d^r(C)$ verifying ramification conditions at two points:

$$\alpha^\ell(p) \geq \alpha \quad \text{and} \quad \alpha^\ell(q) \geq \beta.$$

We say that $[C, p, q]$ satisfies the 2-pointed Brill-Noether Theorem if for any α and β ,

$$\dim G_d^r(C, (p, \alpha), (q, \beta)) = \rho(g, r, d, \alpha, \beta) := \rho(g, r, d) - w(\alpha) - w(\beta),$$

unless $G_d^r(C, (p, \alpha), (q, \beta)) = \emptyset$. Eisenbud and Harris [EH2] Theorem 1.1 established the 2-pointed Brill-Noether Theorem for general 2-pointed curves by use of degeneration. As in the case of 1-pointed curves, up to now no explicit example of a smooth Brill-Noether general 2-pointed curve has been known. We construct such curves using decomposable elliptic ruled surfaces.

We start with an elliptic curve J and consider a non-torsion line bundle $\eta \in \text{Pic}^0(J)$. Let

$$\phi : Y := \mathbf{P}(\mathcal{O}_J \oplus \eta) \rightarrow J$$

be the ruled surface corresponding to a *decomposable* rank 2 vector bundle. We denote by J_0 and J_1 the disjoint sections of Y such that

$$N_{J_0/Y} = \eta \quad \text{and} \quad N_{J_1/Y} = \eta^\vee.$$

In particular, $J_0^2 = J_1^2 = 0$. Observe that $J_1 \equiv J_0 - \phi^*(\eta)$. We fix a point $r \in J$ and let $f = f_r := \phi^{-1}(r)$ be the corresponding ruling. For each $g \geq 1$, we denote by $s = s^{(g)}$ the point on the base elliptic curve J determined by

$$\mathcal{O}_J(s^{(g)} - r) = \eta^{\otimes g}.$$

Since η is not a torsion class, we have $s^{(g)} \neq r$, for all $g \geq 1$. Furthermore, the difference $s^{(g)} - s^{(g-1)} \in \text{Pic}^0(J)$ is not a torsion class. As explained in the Introduction, we set

$$\{p\} = J_1 \cdot f_r \quad \text{and} \quad \{q^{(g)}\} := J_0 \cdot f_{s^{(g)}}.$$

Lemma 2. *We have that $h^0(Y, \mathcal{O}_Y(gJ_0 + f_r)) = g + 1$. The general point of the linear system $|gJ_0 + f_r|$ is a smooth curve of genus g passing through the points p and $q^{(g)}$.*

Proof. By direct calculation, using Riemann-Roch, we find that

$$h^0(Y, \mathcal{O}_Y(gJ_0 + f_r)) = h^0(\mathcal{O}_J(r) \otimes \text{Sym}^g(\mathcal{O}_J \oplus \eta)) = \deg(\mathcal{O}_J(r) \otimes \text{Sym}^g(\mathcal{O}_J \oplus \eta)) = g + 1.$$

Furthermore, since $K_Y \equiv -2J_0 + \phi^*(\eta) \equiv -2J_1 + \phi^*(\eta^\vee)$, from the adjunction formula we obtain that a smooth curve $C \in |gJ_0 + f_r|$ has genus g .

From [FGP] Proposition 11, since η is non-torsion, the base points of $|gJ_0 + f_r|$ lie on $J_0 + J_1 = |-K_Y|$. Since $\mathcal{O}_{J_1}(gJ_0 + f_r) = \mathcal{O}_{J_1}(p)$, the point p must lie in the base locus of

$|gJ_0 + f_r|$. Finally, since $\mathcal{O}_{J_0}(gJ_0 + f_r) = \eta^{\otimes g} \otimes \mathcal{O}_{J_0}(f_r) = \mathcal{O}_{J_0}(q^{(g)})$, it follows that $q^{(g)}$ belongs to each curve $C \in |gJ_0 + f_r|$. Hence, the base locus of $|gJ_0 + f_r|$ consists of the points p and $q^{(g)}$. \square

Therefore, on each curve from the linear system $|gJ_0 + f_r|$ we can single out two marked points, p and $q = q^{(g)}$. These are precisely the points for which the Brill-Noether Theorem will be proved.

Theorem 3. *The 2-pointed curve $[C, p, q] \in \mathcal{M}_{g,2}$, where $C \in |gJ_0 + f_r|$ is general and p and $q := q^{(g)}$ are as above, verifies the 2-pointed Brill-Noether Theorem, that is,*

$$\dim G_d^r\left(C, (p, \alpha), (q, \beta)\right) = \rho(g, r, d, \alpha, \beta) \quad \text{or} \quad G_d^r\left(C, (p, \alpha), (q, \beta)\right) = \emptyset,$$

for all Schubert indices α and β .

Proof. Assume by contradiction that for a 2-pointed curve $[C, p, q^{(g)}]$, where $C \in |gJ_0 + f_r|$ is a general element, the Brill-Noether Theorem fails for certain Schubert indices α and β , that is, there exists a component of $G_d^r\left(C, (p, \alpha), (q, \beta)\right)$ whose dimension exceeds $\rho(g, r, d, \alpha, \beta)$. Then, similarly to the proof of Theorem 1, we consider a specialization of C to the sublinear system $\{J_0\} + |(g-1)J_0 + f_r| \cong \mathbf{P}^{g-1}$, which appears as a hyperplane in $|gJ_0 + f_r| \cong \mathbf{P}^g$. The 2-pointed curve corresponding to the general element of this subsystem is a curve of the form

$$[D \cup J_0, p \in D, q^{(g)} \in J_0] \in \overline{\mathcal{M}}_{g,2},$$

where $D \in |(g-1)J_0 + f_r|$ is a smooth curve of genus $g-1$ passing through p and the point $q^{(g-1)} \in J_0 \cdot f_{s^{(g-1)}}$. Note that $D \cap J_0 = \{q^{(g-1)}\}$. Observe moreover that under the isomorphism $\phi = \phi|_{J_0} : J_0 \xrightarrow{\cong} J$, we have

$$q^{(g)} - q^{(g-1)} = \phi^*(s^{(g)}) - \phi^*(s^{(g-1)}) = \phi^*(\eta) \in \text{Pic}^0(J_0),$$

that is, the difference $q^{(g)} - q^{(g-1)}$ is not torsion on J_0 .

The proof now follows by induction. By semicontinuity, the variety of limit linear series ℓ of type \mathfrak{g}_d^r on $D \cup J_0$ verifying the ramification conditions $\alpha^\ell(p) \geq \alpha$ and $\alpha^\ell(q^{(g)}) \geq \beta$ must have a component Z of dimension strictly greater than $\rho(g, r, d, \alpha, \beta)$. Denote by $\ell = (\ell_D, \ell_{J_0})$ a general point of Z . We may assume that ℓ is a refined limit linear series. Set $\gamma := \alpha^{\ell_D}(q^{(g-1)})$. Then Z is birationally isomorphic to the product

$$G_d^r\left(D, (p, \alpha), (q^{(g-1)}, \gamma)\right) \times G_d^r\left(J_0, (q^{(g-1)}, \gamma^c), (q^{(g)}, \beta)\right).$$

By induction on the genus, we may assume that $[D, p, q^{(g-1)}] \in \mathcal{M}_{g-1,2}$ satisfies the 2-pointed Brill-Noether Theorem, in particular

$$\dim G_d^r\left(D, (p, \alpha), (q^{(g-1)}, \gamma)\right) = \rho(g-1, r, d, \alpha, \gamma).$$

Since $q^{(g)} - q^{(g-1)} \in \text{Pic}^0(J_0)$ is not torsion, as we have observed $[J_0, q^{(g-1)}, q^{(g)}] \in \mathcal{M}_{1,2}$ is a Brill-Noether general 2-pointed curve, hence

$$\dim G_d^r\left(J_0, (q^{(g-1)}, \gamma^c), (q^{(g)}, \beta)\right) = \rho(1, r, d, \gamma^c, \beta).$$

Using the additivity of the Brill-Noether number, we have

$$\dim Z = \rho(g-1, r, d, \alpha, \gamma) + \rho(1, r, d, \gamma^c, \beta) = \rho(g, r, d, \alpha, \beta),$$

a contradiction. \square

Remark 2. Since a Brill-Noether general n -pointed curve supports a Brill-Noether general m -pointed curve for all $m < n$ obtained by dropping $n - m$ of the marked points, it follows that the curve $C \in |gJ_0 + f_r|$ satisfies the (unpointed) Brill-Noether Theorem as well.

2.1. Brill-Noether general pointed curves which are not limits of $K3$ sections.

The Du Val curves considered in [ABFS] and in Section 1 of this paper are known to lie in the closure in \mathcal{M}_g of the locus of curves of genus g lying on a $K3$ surface. Algebraic surfaces $\bar{S} \subset \mathbf{P}^g$ having canonical hyperplane sections have been classified by Epema [Epe]. All such surfaces are potentially limits in \mathbf{P}^g of smooth polarized $K3$ surfaces of degree $2g - 2$. A criterion for when such surfaces smooth to $K3$ surfaces is given in [ABS] Corollary 26. Du Val surfaces, as well as the decomposable elliptic ruled surfaces considered in Theorem 2, are minimal models of corresponding instances of such objects, see [Epe], as well as [ABS] Proposition 29.

It is natural to ask whether there are explicit examples of Brill-Noether general pointed curves, other than those which are limits of curves on $K3$ surfaces. The affirmative answer to this question, provided in Theorem 4 below, uses *indecomposable* elliptic ruled surfaces and is inspired by [Tre].

We fix again an elliptic curve J and denote by \mathcal{E} the unique indecomposable vector bundle on J defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_J \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_J \longrightarrow 0.$$

Let $\varphi : X' := \mathbf{P}(\mathcal{E}) \rightarrow J$ be the induced ruled surface. We fix a point $r \in J$ and set $f := \varphi^{-1}(r)$, therefore $f^2 = 0$. Let $J_0 \subset X'$ be the unique section of φ having $N_{J_0/Y'} = \mathcal{O}_{J_0}$ and set $\{q\} := J_0 \cdot f_r$. In a way similar to the proof of Lemma 2, one can show that the general element of the linear system $|gJ_0 + f|$ is a curve of genus g passing through the point q .

Each curve $C \in |gJ_0 + f|$ has a distinguished marked point, namely $q \in C \cdot J_0$. In [Tre], Treibich considers curves in the linear system $|gJ_0 + f|$ and sketches an argument using Fay's trisecant formula for showing that a general curve $C \in |gJ_0 + f|$ is Brill-Noether general. Reasoning in a way very similar to the proof of Theorem 3, we prove the stronger fact that the general curve $[C, q]$ satisfies the pointed Brill-Noether Theorem.

Since the linear system $|gJ_0 + f|$ has a base point, we denote by $\epsilon : X := \text{Bl}_q(X') \rightarrow X'$ the blow-up of X' at q and by E the exceptional divisor. We keep denoting by J_0 and f , the strict transforms of the curves denoted by the same symbols on X' . Finally, let $C \subset X$ be the strict transform of a curve in the linear system $|gJ_0 + f|$. Then $C \cdot E = 1$ and $C^2 = 2g - 1$. Since $|C|$ is base point free, a Lefschetz pencil in this linear system induces a family of pointed curves

$$\iota : \mathbf{P}^1 \rightarrow \bar{\mathcal{C}}_g.$$

Proposition 3. *The numerical features of the pencil $\iota : \mathbf{P}^1 \rightarrow \overline{\mathcal{C}}_g$ are as follows:*

$$\begin{aligned} \iota^*(\lambda) = g - 1, \quad \iota^*(\psi) = 1, \quad \iota^*(\delta_{\text{irr}}) = 6(g - 1), \quad \iota^*(\delta_1) = 1, \quad \iota^*(\delta_{g-1}) = 1, \\ \text{and } \iota^*(\delta_i) = 0 \text{ for } i = 2, \dots, g - 2. \end{aligned}$$

Proof. We blow-up X at the $2g - 1$ base points of a Lefschetz pencil in $|C|$ and denote by $h : \tilde{X} \rightarrow \mathbf{P}^1$ the induced fibration. Clearly $h^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ and $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^1(J, \mathcal{O}_J)$ is 1-dimensional, therefore $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. Accordingly,

$$\iota^*(\lambda) = \chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) + g - 1 = g - 1.$$

By the Noether formula, the total number of singular fibres in the pencil ι is given by

$$\iota^*(\delta) = c_2(\tilde{X}) + 4g - 4 = 12\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) - K_{\tilde{X}}^2 + 4g - 4 = 6g - 4.$$

In the pencil ι there exists a unique curve from the linear system $|(g - 1)J_0 + f| + J_0$, which is viewed as a hyperplane inside $|gJ_0 + f|$. This singular curve is of the type

$$(4) \quad t = [D \cup E \cup J_0, \tilde{q} := f \cdot E \in E] \in \overline{\mathcal{C}}_g,$$

where $D \in |(g - 1)J_0 + f|$ is a smooth curve of genus $g - 1$ with $D \cap J_0 = \emptyset$ (on X). Note that the rational curve E intersects both D and J_0 at one point. Forgetting the marked point \tilde{q} , the stable model of this curve is $[D \cup_q J_0] \in \overline{\mathcal{M}}_g$. The point t lies on both boundary divisors Δ_1 and Δ_{g-1} , which implies $\iota^*(\delta_1) = \iota^*(\delta_{g-1}) = 1$, therefore $\iota^*(\delta_{\text{irr}}) = 6(g - 1)$. \square

Corollary 2. *The numerical features of the pencil $\bar{\iota} := \pi \circ \iota : \mathbf{P}^1 \rightarrow \overline{\mathcal{M}}_g$ obtained by forgetting the marked point, are given by:*

$$\bar{\iota}^*(\lambda) = g - 1, \quad \bar{\iota}^*(\delta_{\text{irr}}) = 6(g - 1), \quad \bar{\iota}^*(\delta_1) = 2, \quad \bar{\iota}^*(\delta_i) = 0, \text{ for } i = 2, \dots, \left\lfloor \frac{g}{2} \right\rfloor.$$

Proof. The only thing which has to be observed is that $\bar{\iota}^*(\delta_1) = \iota^*(\delta_1) + \iota^*(\delta_{g-1}) = 2$. \square

Using Proposition 3 it is now immediate to check that the pencil ι , just like the Du Val pencil, satisfies the relations

$$\iota^*(\mathcal{BN}_g) = 0 \quad \text{and} \quad \iota^*(\overline{\mathcal{W}}_g) = 0.$$

Theorem 4. *The general pointed curve $[C, q]$, where $C \in |gJ_0 + f|$ and $\{q\} = J_0 \cdot f$, verifies the pointed Brill-Noether Theorem.*

Proof. The proof proceeds by induction on g in a way mirroring the proofs of Theorems 1 and 3. Assume by contradiction that the pointed Brill-Noether Theorem fails for every smooth curve $[C, q]$, where $C \in |gJ_0 + f|$. By choosing a Lefschetz pencil ι in $|gJ_0 + f|$ as above, the same conclusion holds for the degenerate pointed curve $t = [D \cup E \cup J_0, \tilde{q}]$. That is, the variety of limit linear series ℓ of type \mathbf{g}_d^r on $D \cup E \cup J_0$ such that $a^\ell(\tilde{q}) \geq \alpha$ has a component Z of dimension strictly greater than $\rho(g, r, d, \alpha)$, for some r, d , and α . For $\ell = (\ell_D, \ell_E, \ell_{J_0})$ a general point of Z , let $\gamma_D := \alpha^{\ell_D}(D \cdot E)$ and $\gamma_{J_0} := \alpha^{\ell_{J_0}}(J_0 \cdot E)$. Then Z is birationally isomorphic to

$$G_d^r \left(D, (D \cdot E, \gamma_D) \right) \times G_d^r \left(E, (E \cdot D, \gamma_D^c), (E \cdot J_0, \gamma_{J_0}^c), (\tilde{q}, \alpha) \right) \times G_d^r \left(J_0, (J_0 \cdot E, \gamma_{J_0}) \right).$$

Both the 3-pointed rational curve $[E, E \cdot D, E \cdot J_0, \tilde{q}] \in \mathcal{M}_{0,3}$, as well as the 1-pointed elliptic curve $[J_0, J_0 \cdot E] \in \mathcal{M}_{1,1}$ verify the pointed Brill-Noether Theorem. By induction the same can be assumed for $[D, D \cdot E] \in \mathcal{C}_{g-1}$. It follows that

$$\dim Z = \rho(g-1, r, d, \gamma_D) + \rho(0, r, d, \gamma_D^c, \gamma_{J_0}^c, \alpha) + \rho(1, r, d, \gamma_{J_0}) = \rho(g, r, d, \alpha),$$

a contradiction. \square

Remark 3. Using [Epe] page 34, we observe that the indecomposable elliptic surface $\mathbf{P}(\mathcal{E})$ is not the desingularization of a surface in \mathbf{P}^g with canonical curve sections. This can also be seen from the self-intersection formula $C^2 = 2g - 1$ (rather than $2g - 2$) on the blown-up surface $X = \text{Bl}_q(X')$.

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