LINEAR SYZYGIES OF CURVES WITH PRESCRIBED GONALITY

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Abstract. We prove two statements concerning the linear strand of the minimal free resolution of a k-gonal curve C of genus g. Firstly, we show that a general curve C of genus g of non-maximal gonality $k \leq \frac{g+1}{2}$ satisfies Schreyer's Conjecture, that is, $b_{g-k,1}(C, \omega_C) = g-k$. This statement goes beyond Green's Conjecture and predicts that all highest order linear syzygies in the canonical embedding of C are determined by the syzygies of the $(k-1)$-dimensional scroll containing C. Secondly, we prove an optimal effective version of the Gonality Conjecture for general k-gonal curves, which makes more precise the (asymptotic) Gonality Conjecture proved by Ein-Lazarsfeld and improves results of Rathmann.

0. Introduction

1. The effective gonality conjecture. Let C be a smooth complex algebraic curve and L a very ample line bundle on C inducing an embedding $\varphi_L : C \hookrightarrow \mathbf{P}H^0(C, L)$. In order to describe the equations of this embedding, after setting $r := r(L)$, we consider the finitely generated graded $S := \mathrm{Sym}\ H^0(C, L) \cong \mathbb{C}[x_0, \ldots, x_r]$-module $\Gamma_C(L) := \bigoplus_n H^0(C, L^{\otimes n})$. By the *Hilbert Syzygy Theorem*, one has a minimal free resolution

$$0 \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow \Gamma_C(L) \longrightarrow 0,$$

where

$$F_p = \bigoplus_{q>0} K_{p,q}(C, L) \otimes S(-p-q),$$

with $K_{p,q}(C, L)$ being the Koszul cohomology group of p-th order syzygies of weight q. As usual, the *graded Betti numbers* of (C, L) are defined by $b_{p,q} := \dim K_{p,q}(C, L)$. If L is non-special, then $K_{p,q}(C, L) = 0$ for all $q \geq 3$. Accordingly, the graded Betti diagram of (C, L) consists only of two non-trivial rows: the linear strand $(q = 1)$ and the quadratic strand $(q = 2)$.

The quadratic strand of the resolution is the subject of the Green-Lazarsfeld Secant Conjecture [GL1] and has been studied extensively in [FK], [K2]. The linear row is the subject of the Gonality Conjecture formulated in the same paper [GL1].

Assume C is k-gonal and let L be a line bundle on C of degree $\deg(L) \geq 2g - 1 + k$. By the Green-Lazarsfeld Nonvanishing Theorem [G, Appendix], one has $K_{h^0(L)-k-1,1}(C, L) \neq 0$. In a major breakthrough, generalizing results in [AV] in the case of general k-gonal curves, Ein and Lazarsfeld [EL] proved that for an *arbitrary* smooth curve C of gonality k, if $\deg(L) \gg 0$, then

$$K_{h^0(L)-k,1}(C, L) = 0. \qquad (1)$$

This result has been significantly improved by Rathmann [R], who showed that the vanishing (1) holds for every smooth curve C of genus g, when $\deg(L) \geq 4g - 3$. As already indicated in the original paper [GL1] Conjecture 3.7, one can ask for an effective version of the Gonality Conjecture. We show the following:

Theorem 0.1. *Let C be a general k-gonal curve of genus $g \geq 4$. Then for each line bundle L on C of degree* $\deg(L) \geq 2g - 1 + k$*, one has*

$$K_{h^0(L)-k,1}(C, L) = 0.$$

While the original Gonality Conjecture has been formulated as an asymptotic statement in $\deg(L)$, the bound appearing in Theorem 0.1 is already raised as a possibility in [GL1, page 86]. Clearly Theorem 0.1 implies $K_{p,1}(C,L)=0$, for all $p\geq h^0(C,L)-k$. The bound on $\deg(L)$ appearing in Theorem 0.1 is optimal. Indeed, if $A\in W^1_k(C)$ is a pencil of minimal degree, then

$$K_{g-1,1}(C,\omega_C\otimes A)\neq 0,$$

by the Green–Lazarsfeld Nonvanishing Theorem, that is, on every curve there exist line bundles of degree $2g-2+k$ which do not verify (1).

In the interest of convenience, we say that a smooth curve C of genus g and gonality k satisfies the *Effective Gonality Conjecture* if for each line bundle $L\in \mathrm{Pic}^d(C)$, where $d\geq 2g-1+k$, one has $K_{h^0(L)-k,1}(C,L)=0$. Equivalently, if there exists a line bundle $L\in \mathrm{Pic}^{2g-1+k}(C)$ such that $K_{g,1}(C,L)\neq 0$, then $\mathrm{gon}(C)\leq k-1$. Theorem 0.1 can be reformulated as stating that a general k-gonal curve of genus $g\geq 4$ verifies the Effective Gonality Conjecture.

By Green's $K_{p,1}$-theorem, see [G, Theorem 3.c.1], an *arbitrary* 3-gonal curve of genus $g\geq 4$ satisfies the Effective Gonality Conjecture. The same conclusion holds for each 4-gonal curve of genus $g\geq 7$, see [Te, Proposition 3.8] or [AS]. Note that Theorem 0.1 fails for $g=3$. In this case, the general curve is trigonal and it is easy to see that $K_{3,1}(C,\omega_C^{\otimes 2})\neq 0$, using the fact that the canonical linear system embeds C in the plane.

For curves of maximal gonality of odd genus $g\geq 5$, our results are complete:

Theorem 0.2. *Every smooth curve of odd genus $g\geq 5$ and maximal gonality satisfies the Effective Gonality Conjecture.*

Theorem 0.2, which plays an essential role in the proof of Theorem 0.1 turns out to be intimately related to the divisorial case of the Green-Lazarsfeld Secant Conjecture proved in full generality [FK, Theorem 1.4]. We observe that using [FK], if C is a smooth curve of genus $g=2n+1$ and gonality $n+2$, the following equivalence holds for a line bundle $M\in \mathrm{Pic}^{2g}(C)$:

$$K_{n,1}(C,M)\neq 0 \Longleftrightarrow M-K_C\in C_{n+1}-C_{n-1}. \tag{2}$$

The right hand side denotes the divisorial difference variety $C_{n+1}-C_{n-1}\subseteq \mathrm{Pic}^2(C)$. An argument involving the geometry of secant varieties for line bundles on C then shows that (2) implies the vanishing $K_{g,1}(C,L)=0$, for *every* line bundle $L\in \mathrm{Pic}^{5n+3}(C)$, thus establishing Theorem 0.2. In order to deduce Theorem 0.1, we fix a value for the gonality $k\leq \frac{g+3}{2}$ and perform induction on the genus g; the initial step is Theorem 0.2. By induction, assume that the general smooth curve C of genus g and gonality k satisfies the Effective Gonality Conjecture. The stable curve X of genus $g+1$ obtained by adding an elliptic curve E at a point of ramification of a degree k pencil on C lies in the limit in $\overline{\mathcal{M}}_{g+1}$ of the locus of smooth k-gonal curves of genus $g+1$. An analysis of syzygies of line bundles of bidegree $(2g+k,1)$ on X allows us to deduce the Effective Gonality Conjecture for a smooth deformation of X having gonality k.

2. Schreyer's Conjecture. Consider a general k-gonal curve canonically embedded curve $C\hookrightarrow \mathbf{P}^{g-1}$ of gonality k. Green's Conjecture, known in this case, see [V1], [V2], [Ap2], and asserting that

$$K_{p,1}(C,\omega_C)=0 \quad \text{if and only if} \quad p\geq g-k+1,$$

determines the length of the linear (as well as that of the quadratic) strand of the resolution of C. Schreyer's Conjecture [Sch3, §6] and [SSW] addresses the more refined question of what actually *is* the Betti diagram of C, that is, determine the values $b_{p,1}(C,\omega_C)$ for $k-2\leq p\leq g-k$. Note that in the case when C has the same gonality as a general curve of genus g, that is, $\mathrm{gon}(C)=\lfloor \frac{g+3}{2}\rfloor$, and only in this case, Green's Conjecture determines the entire resolution of C. Indeed, in this case Green's Conjecture is equivalent to the statement that the resolution of

$C \subseteq \mathbf{P}^{g-1}$ is *natural*, or equivalently

$$b_{p,2}(C,\omega_C)\cdot b_{p+1,1}(C,\omega_C)=0$$

for all p. Since the differences $b_{p+1,1}(C,\omega_C)-b_{p,2}(C,\omega_C)$ are known and independent of C, knowing which Betti numbers vanish amounts to knowing the entire Betti diagram.

Assume now $\mathrm{gon}(C)\leq \frac{g+1}{2}$, that is, C has non-maximal gonality. In this case, Green's Conjecture predicts the following resolution, where we observe that $b_{p,1}(C,\omega_C)\cdot b_{p,2}(C,\omega_C)\neq 0$ for $k-2\leq p\leq g-k$.

1	2	...	$k-3$	$k-2$	...	$g-k$	$g-k+1$	...	$g-2$
$b_{1,1}$	$b_{2,1}$	...	$b_{k-3,1}$	$b_{k-2,1}$	...	$b_{g-k,1}$	0	...	0
0	0	...	0	$b_{k-2,2}$	...	$b_{g-k,2}$	$b_{g-k+1,2}$	...	$b_{g-2,2}$

TABLE 1. The Betti table of a general canonical k-gonal curve of genus g.

It is known [AC] that such a curve C carries a unique pencil $A\in W^1_k(C)$ of minimal degree, inducing a $(k-1)$-dimensional scroll $X\subseteq \mathbf{P}^{g-1}$ swept out by the fibres of $|A|$. The Betti numbers of $(X,\mathcal{O}_X(1))$ are determined by the Eagon-Northcott complex, see [Sch1]. Since $C\subseteq X\subseteq \mathbf{P}^{g-1}$, one has the following inequality (see also Section 4)

$$b_{p,1}(C,\omega_C)\geq b_{p,1}\big(X,\mathcal{O}_X(1)\big)=p\cdot\binom{g-k+1}{p+1}. \tag{3}$$

It was originally expected that the inequality (3) is always an equality for $p\geq \lceil\frac{g-1}{2}\rceil$. This, however, is now known to fail. Indeed, Bopp [B] showed that the for a general 5-gonal curve of sufficiently high genus, if $m:=\lceil\frac{g-1}{2}\rceil$, then $b_{m,1}(C,\omega_C)>b_{m,1}(X,\mathcal{O}_X(1))$. Schreyer's Conjecture [SSW] concerns the value of the highest non-zero Betti number in the linear strand and predicts that in this case, under suitable generality assumptions, inequality (3) is an equality.

Conjecture 0.3 (Schreyer's Conjecture)**.** *Let C be a curve of genus g and non-maximal gonality $3\leq k\leq\frac{g+1}{2}$. Assume $W^1_k(C)=\{A\}$ is a reduced single point and A is the unique line bundle of degree at most $g-1$ achieving the Clifford index. Then*

$$b_{g-k,1}(C,\omega_C)=g-k.$$

The converse statement is straightforward. Indeed, if $W^1_k(C)$ does *not* consist of a reduced single point, then $b_{g-k,1}(C,\omega_C)>g-k$, see [SSW, Proposition 4.10]. The condition $b_{g-k,1}(C,\omega_C)=g-k$ automatically implies the vanishing statements $b_{p,1}(C,\omega_C)=0$, for $p>g-k$. Conjecture 0.3 is known to hold for a *general* k-gonal curve provided $(k-1)^2<g$, see [Sch2]. An important piece of evidence for the conjecture is the case of general k-gonal curves of odd genus $2k-1$. Such curves form a divisor $\mathfrak{Hur}$ in the moduli space $\mathcal{M}_{2k-1}$, much studied by Harris and Mumford in [HM]. Combining results in [HR] and those in [V2], it follows that Conjecture 0.3 holds in this case. Outside this divisorial range, little has been known. The main result of this paper is the following:

Theorem 0.4. *Schreyer's Conjecture holds for a general k-gonal curve C of genus $g\geq 2k-1$:*

$$b_{g-k,1}(C,\omega_C)=g-k.$$

In fact we give a Brill-Noether theoretic sufficient condition for Schreyer's conjecture to hold for a given curve. In what follows, $G^{1,\mathrm{bpf}}_d(C)\subseteq G^1_d(C)$ denotes the subvariety of base point free pencils of degree d on a curve C. We establish the following result implying Theorem 0.4.

Theorem 0.5. *Assume C is a k-gonal curve C of genus $g \geq 2k-1$ satisfying bpf-linear growth:*

$$\dim G^1_{k+m}(C) \leq m, \ \ \textit{for } 0 \leq m \leq g-2k+1$$

$$\textit{and} \ \ \dim G^{1,\mathrm{bpf}}_{k+m}(C) < m, \ \ \textit{for } 0 < m \leq g-2k+1.$$

Assume there is a unique pencil in $A \in G^1_k(C)$ having simple ramification and with $h^0(C, A^{\otimes 2}) = 3$. Then Schreyer's Conjecture holds for C.

Part of Theorem 0.5 is that there is a canonical identification

$$K_{g-k,1}(C,\omega_C) \cong \bigwedge^{g-k+1} H^0(C, K_C \otimes A^{\vee}) \otimes \mathrm{Sym}^{g-k-1} H^0(C,A) \otimes \bigwedge^2 H^0(C,A),$$

where A is the unique degree k pencil on C. *All* the $(g-k)$-th syzygies linear syzygies of the canonical curve $C \subseteq \mathbf{P}^{g-1}$ are of *Eagon-Northcott type* and can be written down explicitly. Precisely, if $(\tau_0, \ldots, \tau_{g-k})$ is a basis of $H^0(C, \omega_C \otimes A^{\vee})$ and $\sigma \in H^0(C,A)$, then the syzygy corresponding to the power $\sigma^{g-k-1} \in \mathrm{Sym}^{g-k-1} H^0(C,A)$ has the form

$$\sum_{j=0}^{g-k} (-1)^j (\sigma\tau_1) \wedge \ldots \widehat{(\sigma\tau_j)} \wedge \ldots \wedge (\sigma\tau_{g-k}) \wedge \big\{(\sigma\tau_0) \otimes (\sigma'\tau_j) - (\sigma'\tau_0) \otimes (\sigma\tau_j)\big\} \in \bigwedge^{g-k} H^0(\omega_C) \otimes H^0(\omega_C),$$

where $\sigma' \in H^0(C,A)$ is another section such that (σ, σ') form a basis of $H^0(C,A)$.

It is tempting to interpolate between and link the two main results of this paper, namely Theorems 0.1 and 0.5, and conjecture that a statement analogous to Schreyer's Conjecture holds not only for the canonical bundle, but for every sufficiently positive line bundle on C. We fix a general k-gonal curve C of genus $g \geq 2k-1$ and a line bundle L on C with $\deg(L) \geq 2g+k$.

Conjecture 0.6. *If $r = r(L)$, one has* $\dim K_{r-k,1}(C,L) = r-k$.

We expect that all syzygies in $K_{r-k,1}(C,L)$ are again of Eagon-Northcott type, being induced by the k-dimensional scroll induced by the unique pencil $A \in W^1_k(C)$ and which contains the embedded curve $\varphi_L : C \hookrightarrow \mathbf{P}^r$.

The proof of Theorem 0.5 begins in Section 3 with the already mentioned observation that via [HR] and [V2], a smooth curve C of genus $2k-1$ and gonality k satisfies $b_{k-1,1}(C,\omega_C) = k-1$, provided $W^1_k(C)$ is integral of dimension zero. Consider the Hurwitz space $\mathcal{H}_{2k-1,k}$ of smooth curves of genus g which are k-fold covers of $\mathbf{P}^1$. We define the *Eagon-Northcott* divisor $\mathcal{EN}$ on $\mathcal{H}_{2k-1,k}$ parametrizing moduli points $[f : C \to \mathbf{P}^1] \in \mathcal{H}_{2k-1,k}$ with $b_{k-1,1}(C,\omega_C) > k-1$. In other words, points of $\mathcal{EN}$ correspond to canonical curves $C \subseteq \mathbf{P}^{g-1}$ having a $(g-k)$-th order linear syzygy which is *not* of Eagon-Northcott type. We also consider the Brill-Noether type divisor $\mathfrak{BN}$ on $\mathcal{H}_{2k-1,k}$ consisting of points $[f : C \to \mathbf{P}^1]$ such that C has an extra pencil of degree k. By the above discussion these two divisors coincide set-theoretically, that is,

$$\mathcal{EN} = \mathfrak{BN}.$$

Now suppose we are no longer in the divisorial case and choose $k \leq \frac{g+1}{2}$. We follow a strategy reminiscent of [Ap2]. Starting with a general k-gonal curve C of genus g, we form the irreducible nodal curve $[D] \in \overline{\mathcal{M}}_{2g-2k+1}$ obtained by identifying $g-2k+1$ general pairs of points on C. Clearly $p_a(D) = 2g-2k+1$ and $\mathrm{gon}(D) \leq g-k+1$, that is, $[D]$ belongs to the closure $\overline{\mathfrak{Hur}} = \overline{\mathcal{M}}^1_{2g-2k+1,g-k+1}$ of the Hurwitz divisor, already considered in [HM], [HR] and [FK]. Let

$$\pi : \overline{\mathcal{H}}_{2g-2k+1,g-k+1} \to \overline{\mathcal{M}}_{2g-2k+1}$$

denote the forgetful map from the space of admissible covers of degree $g-k+1$ compactifying the Hurwitz space $\mathcal{H}_{2g-2k+1,g-k+1}$. Assuming the curve C we started with is sufficiently general, one checks directly that *set-theoretically* $W^1_{g-k+1}(D)$ consists of one point (that is, $\pi^{-1}([D])$ consists of one admissible cover $[f]$). This point corresponds to the torsion free sheaf on D given by pushing forward the unique degree k pencil on C. In the last section of the paper we show that $[f] \notin \overline{\mathfrak{BN}}$, therefore $[f] \notin \overline{\mathcal{EN}}$. To conclude $b_{g-k,1}(C,\omega_C) = g-k$, we extend in Section 4 the determinantal structure of the Eagon-Northcott divisor $\mathcal{EN}$ over a partial compactification of $\mathcal{H}_{2g-2k+1,g-k+1}$ containing the moduli point of $[f]$. In Section 5, we then use $K3$ surfaces to show that this extended Eagon-Northcott divisor does not contain the unique boundary component of $\overline{\mathcal{H}}_{2g-2k+1,g-k+1}$ containing $[f]$. Since always $K_{g-k,1}(C,\omega_C) \hookrightarrow K_{g-k,1}(D,\omega_D)$, this completes the proof of Theorem 0.5.[1]

The organisation of the paper is as follows: We first review some background on syzygies of curves in Section 1. In Section 2, we prove Theorem 0.1. We prove Theorem 0.5 in Sections 3, 4 and 5.

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1. Background on Syzygies

We recall a few definitions and collect some basic results on syzygies that will be used throughout the paper. Let X be a (possibly singular) projective variety and let $L, M \in \mathrm{Pic}(X)$ be line bundles. Consider the graded $S := \mathrm{Sym}\, H^0(X,L)$-module

$$\Gamma_X(M,L) := \bigoplus_{n\in\mathbb{Z}_{\geq 0}} H^0(X, L^{\otimes n}\otimes M).$$

One defines the Koszul cohomology groups $K_{p,q}(X,M;L)$ of p-th syzygies of weight q by resolving the module $\Gamma_X(M,L)$ and computes them via the Koszul complex, see [G]. When $M = \mathcal{O}_X$, we write $K_{p,q}(X,L) := K_{p,q}(X,\mathcal{O}_X;L)$. The following fact is surely well-known:

Lemma 1.1 (Semicontinuity)**.** *Let $\pi:\ \mathcal{X}\to S$ be a flat, projective morphism of schemes over an integral base. Let $\mathcal{L}\in \mathrm{Pic}(\mathcal{X})$ be a line bundle such that $h^0(X_s,\mathcal{L}_s) = c$, for each $s\in S$. Let $\mathcal{M}\in\mathrm{Pic}(\mathcal{X})$ be a second line bundle, and assume*

$$h^0(X_s,\mathcal{L}_s^{\otimes(q-1)}\otimes\mathcal{M}_s) = r_1,\ h^0(X_s,\mathcal{L}_s^{\otimes q}\otimes\mathcal{M}_s) = r_2,\ h^0(X_s,\mathcal{L}_s^{\otimes(q+1)}\otimes\mathcal{M}_s) = r_3$$

are also independent of $s\in S$. Then the function

$$\psi : s\mapsto \dim K_{p,q}(X_s,\mathcal{M}_s;\mathcal{L}_s)$$

is upper semicontinuous on S.

We collect some results on syzygies of curves which, taken together, reduce Theorem 0.1 to the extremal case of line bundles of degree $d = 2g-1+\mathrm{gon}(C)$. We quote from [AN], Theorem 4.27:

[1]It might be tempting to carry out this argument at the level of $\overline{\mathcal{M}}_{2g-2k+1}$ rather than pass to the Hurwitz space. However, the scheme structure of $W^1_{g-k+1}(D)$ is difficult to analyse, in particular $[D]$ is a singular point of $\mathrm{Im}(\pi) = \overline{\mathfrak{Hur}}$. Thus a degenerate version of results in [HR], does not actually lead to a proof of Conjecture 0.3.

Lemma 1.2. *Let C be a smooth curve of genus g and L a line bundle of degree $d \geq g$ with $h^1(C,L)=0$. Assume $K_{p,1}(C,L)=0$. Then $K_{p+1,1}(C,L(x))=0$, for any point $x \in C$ such that $L(x)$ is base point free.*

It is standard, see e.g. [AN], Corollary 2.13, that if $L \ncong \mathcal{O}_C$ is a globally generated line bundle on a smooth curve C, if $K_{p,1}(C,L)=0$, then $K_{p+1,1}(C,L)=0$. Accordingly, there are several natural invariants which one can read directly off the Betti table of an embedded curve $C \stackrel{|L|}{\hookrightarrow} \mathbf{P}^{r(L)}$ and which measure the length of the linear and the quadratic strand respectively:

$$\ell_1(C,L) := \max\bigl\{p \in \mathbb{N}_{>0} : b_{p,1}(C,L) \neq 0\bigr\} \quad \text{and}$$

$$\ell_2(C,L) := \min\bigl\{p \in \mathbb{N}_{>0} : b_{p,2}(C,L) \neq 0\bigr\}.$$

Recalling that $K_{p,q}(C,L)=0$ for $p \geq r(L)$, the invariants $\ell_1(C,L)$ are encoded in the more classical properties (N_p) and (M_q) defined in [GL1]. Precisely, $\ell_2(C,L)$ is the smallest integer such that (C,L) fails property $\bigl(N_{\ell_2(C,L)}\bigr)$, whereas $\ell_1(C,L)$ is the smallest integer such that L fails property $\bigl(M_{r(L)-\ell_1(C,L)}\bigr)$.

2. The Effective Gonality Conjecture for generic curves

We start by proving Theorem 0.2. It turns out that our proof of the generic Green-Lazarsfeld Secant Conjecture [FK] takes us a long distance towards finding a complete solution.

Proof of Theorem 0.2. Let C be a curve of genus $2n+1$ and gonality $n+2$. Then using e.g. [HR, Remark 6.3], we observe that $\mathrm{Cliff}(C)=n$, that is, C has maximal Clifford index as well. We need to prove that for any line bundle $L \in \mathrm{Pic}(C)$ of degree at least $5n+3$, we have $K_{i,1}(C,L)=0$ for $i \geq h^0(C,L)-n-2$. We may assume $n \geq 2$ and as explained in the previous section, it is enough to prove that for any line bundle $L \in \mathrm{Pic}^{5n+3}(C)$, we have $K_{2n+1,1}(C,L)=0$.

Theorem 1.4 of [FK] establishes the following equivalence for any line bundle $M \in \mathrm{Pic}^{4n+2}(C)$:

$$K_{n-1,2}(C,M) \neq 0 \Longleftrightarrow M - K_C \in C_{n+1} - C_{n-1}.$$

For any line bundle $M \in \mathrm{Pic}^{4n+2}(C)$ one has cf. [FK, formula (8)]

$$\dim K_{n,1}(C,M) = \dim K_{n-1,2}(C,M).$$

Thus, for any $M \in \mathrm{Pic}^{4n+2}(C)$, the equivalence

$$K_{n,1}(C,M) \neq 0 \Longleftrightarrow M - K_C \in C_{n+1} - C_{n-1}$$

holds. Using Lemma 1.2 again, it thus suffices to show that for any line bundle L of degree $5n+3$, there exists an effective divisor $D \in C_{n+1}$ such that

$$L - D - K_C \notin C_{n+1} - C_{n-1}.$$

Suppose this were not the case, that is,

$$L - K_C - C_{n+1} \subseteq C_{n+1} - C_{n-1}.$$

Then for every $D \in C_{n+1}$ there exists a divisor $E \in C_{n+1}$ such that $H^1(C, L(-D-E)) \neq 0$, that is, $D+E$ is an element of the (determinantal) secant variety $V^{2n+1}_{2n+2}(L)$ of effective divisors failing to impose independent conditions on $|L|$. In particular,

$$\dim V^{2n+1}_{2n+2}(L) \geq n+1,$$

which is one higher than the expected dimension n. We observe that the morphism

$$\psi: \ V^{2n+1}_{2n+2}(L) \to C_{n-1},$$
$$A \mapsto K_C - L + A$$

is well-defined, since $h^0(C, K_C - L + A) = 1$, for $\mathrm{gon}(C) > n-1$. Let I be any component of $V^{2n+1}_{2n+2}(L)$ of dimension $n+1$ and set $r := n - 1 - \dim\ \psi(I)$. Then $\psi_{|_I}$ must have fibres of dimension at least $2+r$. As all divisors in the inverse image $\psi^{-1}(B)$ are clearly linearly equivalent, we have $h^0(C, A) \geq 3 + r$ for all $A \in V^{2n+1}_{2n+2}(L)$ such that $\psi(A) = B \in \psi(I)$. By Riemann–Roch, this implies $h^1(C, A) \geq 1 + r$, or $h^0(C, K_C - A) = h^0(2K_C - L - B) \geq 1 + r$. The latter inequality holds for *any* effective divisor $B \in \psi(I)$, so we must have

$$\dim |2K_C - L| \geq r + \dim \psi(I) = n - 1.$$

This implies $h^1(C, 2K_C - L) \geq 3$, or equivalently $L - K_C \in W^2_{n+3}(C)$. But then $\mathrm{Cliff}(C) \leq n-1$ (if $n = 2$, then compute the Clifford index of $2K_C - L$ rather than $L - K_C$). Since we have $\mathrm{Cliff}(C) = n$, this is a contradiction. □

The proof of Theorem 0.2 gives a characterisation of those line bundles $L \in \mathrm{Pic}^{2g-2+\mathrm{gon}(C)}(C)$, such that $K_{h^0(L)-\mathrm{gon}(C),1}(C, L) \neq 0$, in the case where C has odd genus and maximal gonality.

Proposition 2.1. *Let C be a smooth curve of odd genus $2n+1$ and gonality $n+2$. Let $L \in \mathrm{Pic}^{5n+2}(C)$ be such that $K_{2n,1}(C, L) \neq 0$. Then $L - K_C \in W^1_{n+2}(C)$.*

Proof. Following the proof of Theorem 0.2, we obtain $\dim V^{2n}_{2n+1}(L) \geq n$. By studying the morphism

$$\psi : V^{2n}_{2n+1}(L) \to C_{n-1}, \quad A \mapsto K_C - L + A.$$

and arguing as in Theorem 0.2, we are again led to the statement $h^0(C, 2K_C - L) \geq n$. The Riemann–Roch theorem gives $h^0(C, L - K_C) \geq 2$, as required. □

We shall prove Theorem 0.1 by induction on the genus, fixing the gonality. To perform the induction step, let C be a smooth genus g curve of gonality k and denote by $f : C \to \mathbf{P}^1$ the induced degree k cover. We assume that C verifies the Effective Gonality Conjecture. Let $p \in C$ be a branch point of f, and consider the stable curve $X = C \cup_p E$ obtained by glueing a smooth, genus 1 curve at p. A standard argument with admissible covers or limit linear series shows that X is a limit of smooth k-gonal curves of genus $g+1$, see [HM, §3.G].

Proposition 2.2. *Let $X = C \cup_p E$ be the genus $g+1$ stable curve as above and L a line bundle on X with $\deg(L_C) = 2g + k$ and $\deg(L_E) = 1$. Then, for a general point $q \in E \setminus \{p\}$, we have*

$$K_{g,1}(X, L(-q)) = 0.$$

Further, for such a point, $h^1(X, L(-q)) = h^1(X, L^{\otimes 2}(-2q)) = 0$.

Proof. We have the Mayer-Vietoris sequence on X

$$0 \longrightarrow L_C(-p) \longrightarrow L(-q) \longrightarrow L_E(-q) \longrightarrow 0.$$

For a general point $q \in E \setminus \{p\}$, we have $h^0(E, L_E^{\otimes j}(-jq)) = h^1(E, L_E^{\otimes j}(-jq)) = 0$ for $j = 1, 2$, which implies $h^1(X, L(-q)) = h^1(X, L^{\otimes 2}(-2q)) = 0$. Further, we have a natural isomorphism $H^0(C, L(-p)) \cong H^0(X, L(-q))$, and we know, by the assumptions on C, that

$$K_{g,1}(C, L(-p)) = 0.$$

We will use this to deduce $K_{g,1}(X, L(-q)) = 0$.

We have a natural commutative diagram

$$\begin{array}{ccccc}
\bigwedge^{g+1} H^0(C, L(-p)) & \xrightarrow{d} & \bigwedge^{g} H^0(C, L(-p)) \otimes H^0(C, L(-p)) & \xrightarrow{d} & \bigwedge^{g-1} H^0(C, L(-p)) \otimes H^0(C, L^{\otimes 2}(-2p)) \ , \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
\bigwedge^{g+1} H^0(X, L(-q)) & \xrightarrow{d} & \bigwedge^{g} H^0(X, L(-q)) \otimes H^0(X, L(-q)) & \xrightarrow{d} & \bigwedge^{g-1} H^0(X, L(-q)) \otimes H^0(X, L^{\otimes 2}(-2q))
\end{array}$$

where α, β are isomorphisms, and γ is induced from the natural composition

$$H^0(C, L^{\otimes 2}(-2p)) \hookrightarrow H^0(C, L^{\otimes 2}(-p)) \cong H^0(X, L^{\otimes 2}(-2q)).$$

As $K_{g,1}(C, L(-p)) = 0$, the top row is exact and since β is surjective and γ is injective, the bottom row must also be exact, as required. $\square$

From Proposition 2.2 we readily deduce Theorem 0.1, that is, the effective version of the Gonality Conjecture.

Proof. Fix $k \geq 4$. Assume that for the general k-gonal curve C of genus g one has $K_{g,1}(C, L) = 0$, for any line bundle $L \in \mathrm{Pic}^{2g-1+k}(C)$. We claim there exists a smooth curve C' of genus $g+1$ and gonality k, such that $K_{g+1,1}(C', L') = 0$, for each line bundle $L' \in \mathrm{Pic}^{2g+1+k}(C')$. By performing induction on g and noting that the initial step is Theorem 0.2, this suffices to prove the theorem. By Lemma 1.2, it further suffices to prove that there exists a smooth curve C' of genus $g+1$ and gonality k such that, for each line bundle $L' \in \mathrm{Pic}^{2g+1+k}(C')$, there exists a point $q \in C'$ such that $K_{g,1}(C', L'(-q)) = 0$.

Let $X = C \cup_p E$ be the genus $g+1$ stable curve introduced in Proposition 2.2. Consider a flat family $\pi : \mathcal{C} \to \Delta$ of stable curves over a smooth, pointed, one dimensional base $(\Delta, 0)$, such that the central fibre is X and $\pi^{-1}(s)$ is a smooth curve of gonality k for all $0 \neq s \in \Delta$. As X is a curve of compact type, after shrinking Δ and performing a finite base change if necessary, we have a relative Picard scheme

$$v : \mathcal{P}ic^{2g+1+k}(\mathcal{C}/\Delta) \to \Delta,$$

with central fibre consisting of all line bundles of multidegree $(2g+k, 1)$ on $X = C \cup_p E$; this scheme is flat and proper over Δ, see [D, §4] and [EH], proof of Theorem 3.3.

Let $\mathcal{C}_0 := \mathcal{C} \setminus \{p\}$ be the open set of all points which are smooth in the fibres over Δ. By Proposition 2.2 together with semicontinuity for the dimension of Koszul groups, there is an open subset $U \subseteq \mathcal{P}ic^{2g+1+k}(\mathcal{C}/\Delta) \times_\Delta \mathcal{C}_0$ such that for each pair $(L', q') \in U$, one has $K_{g,1}(C', L'(-q)) = 0$, where $C' = \pi^{-1}(v(L'))$, and such that

$$0 \notin v\Big(\mathcal{P}ic^{2g+1+k}(\mathcal{C}/\Delta) \setminus \mathrm{pr}_1(U)\Big),$$

where $\mathrm{pr}_1 : \mathcal{P}ic^{2g+1+k}(\mathcal{C}/\Delta) \times_\Delta \mathcal{C}_0 \to \mathcal{P}ic^{2g+1+k}(\mathcal{C}/\Delta)$ is the projection. As flat morphisms are open, $\mathrm{pr}_1(U)$ is open, and since v is proper, the image

$$V := v\big(\mathcal{P}ic^{2g+1+k}(\mathcal{C}/\Delta) \setminus \mathrm{pr}_1(U)\big)$$

is closed. Thus if $0 \neq t \in \Delta \setminus V$ and $C_t := \pi^{-1}(t)$, then, for each $L \in \mathrm{Pic}^{2g+1+k}(C_t)$ there exists $q \in C_t$ with $K_{g,1}(C_t, L(-q)) = 0$, as required.

$\square$

3. Schreyer's Conjecture for general curves of non-maximal gonality

In this section, we begin discussing Schreyer's Conjecture for general k-gonal curves of genus $g \geq 2k-1$. We start by explaining the relevance of [HR] for Conjecture 0.3.

For $g = 2k-1$, we consider two divisors on $\mathcal{M}_g$, which already played a role in [Ap2] or [FK]:

$$\mathfrak{Syz} := \Big\{[C] \in \mathcal{M}_g : K_{k-1,1}(C, \omega_C) \neq 0\Big\}$$
$$\mathfrak{Hur} := \Big\{[C] \in \mathcal{M}_g : W^1_k(C) \neq \emptyset\Big\}.$$

Recall that $\mathfrak{Syz}$ has a structure of degeneracy locus, whereas $\mathfrak{Hur}$ is the push-forward of the smooth Hurwitz space $\mathcal{H}_{2k-1,k}$ of degree k covers of $\mathbf{P}^1$. We view both $\mathfrak{Syz}$ and $\mathfrak{Hur}$ as divisors on the *moduli stack* of smooth curves of genus $2k-1$, rather than on the associated coarse moduli space. It is proved in [HR], that one has the following relation at stack level:

$$[\mathfrak{Syz}] = (k-1)[\mathfrak{Hur}] \in CH^1(\mathcal{M}_{2k-1}).$$

Theorem 3.1. *(*[HR]*) Let C be a curve of genus $2k-1$ and gonality k such that the point $W^1_k(C)$ consists of a reduced single point. Then $b_{k-1,1}(C,\omega_C) = k-1$.*

Proof. For a smooth curve C, we denote by $\phi : X \to (S,0)$ its versal deformation space, hence the associated moduli map $m(\phi) : S \to \mathcal{M}_g$ is an étale neighbourhood of the point $[C] \in \mathcal{M}_g$. For $s \in S$, set $C_s := \phi^{-1}(s)$, thus $C_0 = C$. From [HR], there exist two vector bundles V and W of the same rank over S together with a morphism $\chi : V \to W$, such that, for any $s \in S$, we may identify $K_{k-1,1}(C_s,\omega_{C_s}) = \mathrm{Ker}(\chi_s)$. Then the divisor $\mathfrak{Syz}(\phi) \subseteq S$ is defined by $\det(\chi)$. Suppose $b_{k-1,1}(C,\omega_C) \geq k$. Thus $\det(\chi)$ vanishes to order at least k, cf. [HR, Lemma 6.1]. By the equality of cycles $\mathfrak{Syz}(\phi) = (k-1)\mathfrak{Hur}(\phi)$ on S, the function defining $\mathfrak{Hur}(\phi)$ must vanish to order at least two. Thus $\mathfrak{Hur}(\phi)$ is not smooth at the point $0 \in S$. On the other hand it is well-known, see [C], that $\mathfrak{Hur}(\phi)$ is smooth at a point $0 \in S$ corresponding to a curve C if and only if $W^1_k(C)$ consists of a single pencil A and, moreover, $h^0(C, A^{\otimes 2}) = 3$. □

Remark 3.2. One can generalise Theorem 3.1 as follows. For an integral nodal curve D, we define $W^1_k(D) \subseteq \overline{\mathrm{Pic}}^k(D)$ to be the closed subset of the compactified Jacobian of rank one, torsion free sheaves A of degree k on C with $h^0(D,A) \geq 2$. Suppose D is integral, nodal of genus $2k-1$ and assume $W^1_k(D) = \{A\}$, where A is *locally-free* and base-point free. Then the proof of Theorem 3.1 shows $b_{k-1,1}(D,\omega_D) = k-1$.

We now turn our attention to curves of genus g and non-maximal gonality $k \leq \frac{g+1}{2}$. Let $G^{1,\mathrm{bpf}}_d(C) \subseteq G^1_d(C)$ be the subvariety of base point free pencils of degree d on C and further let $W^1_d(C)$ denote the Brill–Noether variety of line bundles of degree d with at least two sections. Note that there is a morphism $G^1_d(C) \to W^1_d(C)$, with fibre over a point $[L] \in W^1_d(C)$ equal to the Grassmannian of pencils $V \subseteq H^0(C,L)$. The following observation is a slight modification of the linear growth condition of [Ap2, Theorem 2]:

Lemma 3.3. *A general curve C of genus g and gonality $k \leq \frac{g+1}{2}$ satisfies* bpf-linear growth*:*

$$\dim G^1_{k+m}(C) \leq m, \quad \textit{for } 0 \leq m \leq g-2k+1$$
$$\textit{and, further,} \quad \dim G^{1,\mathrm{bpf}}_{k+m}(C) < m, \quad \textit{for } 0 < m \leq g-2k+1.$$

Proof. From [Ap2], we have dim $W^1_{k+m}(C) = m$, for $0 \leq m \leq g-2k+1$. We observe that if $Z \subseteq W^r_d(C)$ is an irreducible component, then $Z \cap W^{r+1}_d(C)$ has codimension at least two in Z, provided $g-r+d \geq 0$. This follows from the fact that no component of C^r_d is entirely contained in C^{r+1}_d, where C^r_d is the variety parametrizing divisors D of degree d on C with $\dim |D| \geq r$, see [ACGH, §IV.1].

We claim $\dim G^1_{d+m}(C) \leq m$, for $0 \leq m \leq g-2k+1$. Take an irreducible component $J \subseteq G^1_{d+m}(C)$ and consider the restriction to J of the surjection $c : G^1_{k+m}(C) \to W^1_{k+m}(C)$. Assume $c(J) \subseteq W^{1+j}_{d+m}(C)$ and choose $j \geq 0$ maximal with this property. Then by the above, $\dim \psi(J) \leq m-2j$. Since the general fibre of $c_{|J}$ is isomorphic to the Grassmannian $G(2,2+j)$, it follows $\dim J \leq 2j + \dim c(J) \leq m$. By an identical argument and using [AC, Theorem 2.6], we similarly obtain that $\dim G^{1,\mathrm{bpf}}_{k+m}(C) < m$, in the range $0 < m \leq g-2k+1$. □

The next Proposition is similar to Theorem 2 in [Ap2] and we skip the details.

Proposition 3.4. *Let C be a smooth curve of genus g and gonality $k \leq \frac{g+1}{2}$. Assume C satisfies bpf-linear growth and $W^1_k(C)$ consists of a single point A. If (x_i, y_i) are general pairs of points* *on C, where $1 \leq i \leq g-2k+1$, let D be the nodal curve obtained by glueing x_i to y_i for all i. Then $W^1_{g-k+1}(D) = \{\nu_*(A)\}$, where $\nu : C \to D$ is the normalisation morphism. Furthermore,* $\mathrm{gon}(D) = g-k+1$.

Consider the moduli space $\overline{\mathcal{H}}_{g,k}$ of degree k admissible covers of genus g. Precisely,

$$\overline{\mathcal{H}}_{g,k} = \overline{\mathcal{M}}_{0,2g+2k-2}(\mathcal{B}\mathfrak{S}_k)/\mathfrak{S}_{2g+2k-2}$$

is the space of twisted stable maps from genus zero curves into the classifying stack $\mathcal{B}\mathfrak{S}_k$ of the symmetric group $\mathfrak{S}_k$ and which are simply branched over $2g+2k-2$ points which we do not order. We refer to [ACV] for the construction of this space. It is known that $\overline{\mathcal{H}}_{g,k}$ is the normalisation of the space of admissible covers constructed by Harris and Mumford in [HM]. There is a morphism $\pi : \overline{\mathcal{H}}_{g,k} \to \overline{\mathcal{M}}_g$ given by stabilisation of the source curve of each admissible cover and then $\mathrm{Im}(\pi) = \overline{\mathfrak{Hur}}$. The following result is the translation of Proposition 3.4 to the moduli space of admissible covers.

Proposition 3.5. *Let C be a smooth curve of genus g and gonality $k \leq \frac{g+1}{2}$. Assume C satisfies bpf-linear growth and that $W^1_k(C)$ consists of a single point A, which we assume to have only simple ramification. For $1 \leq i \leq g-2k+1$, we choose general pairs of points (x_i, y_i) on C and let $[D] \in \overline{\mathcal{M}}_{2g-2k+1}$ be the nodal curve obtained by glueing x_i to y_i. If*

$$\pi : \overline{\mathcal{H}}_{2g-2k+1,g-k+1} \to \overline{\mathcal{M}}_{2g-2k+1}$$

is the forgetful map, then $\pi^{-1}([D])$ consists of a unique point $[f' : B' \to T]$.

Proof. We show that the construction described in [HM, Theorem 5] is *unique* in our case. Let $[f' : B' \to T] \in \overline{\mathcal{H}}_{2g-2k+1,g-k+1}$ be an admissible cover, where $p_a(T) = 0$ and B' is a nodal curve whose stable model is isomorphic to D. There exists a unique component C_0 of B' having positive genus. The restriction $f_0 := f'_{|C_0}$ gives a morphism $f_0 : C_0 \to \mathbf{P}^1_0$ onto a smooth rational component $\mathbf{P}^1_0$ of T. By admissibility, $C_0 \cong C$ and $\deg(f_0) \geq k$.

Assume that $f_0(x_i) = f_0(y_i)$ if and only if $1 \leq i \leq j$. For $i = j+1, \ldots, g-2k+1$, we denote by R_{x_i} and R_{y_i} the irreducible components of B' meeting C at x_i and y_i respectively. As the stabilisation of B' is D and $f'(R_{x_i}) \cap f'(R_{y_i}) = \emptyset$, for each such i there must be a component $\widetilde{R}_i$ of the subcurve $\overline{B' - C_0}$ of B', such that $f'(\widetilde{R}_i) = \mathbf{P}^1_0$, or else T contains a loop. As $\deg(f') = g-k+1$, this implies that $d := \deg(f_0) \leq k+j$.

Since the pairs $(x_1, y_1), \ldots, (x_j, y_j)$ are general and f_0 gives rise to an element of $G^{1,\mathrm{bpf}}_d(C)$, it follows $\dim G^{1,\mathrm{bpf}}_d(C) \geq j$. If $d > k$, this contradicts the bpf-linear condition on C, which implies that $\deg(f_0) = k$ and f_0 is the map induced by the pencil of minimal degree $A \in W^1_k(C)$. Each $\widetilde{R}_i$ maps isomorphically onto $\mathbf{P}^1_0$. Clearly $\deg(f'_{R_{x_i}}) \geq 2$ and $\deg(f'_{R_{y_i}}) \geq 2$, in particular $f'_{R_{x_i}}$ and $f'_{R_{y_i}}$ will both contain at least two ramification points of f', for each $i = 1, \ldots, g-2k+1$ (Note that being general points, x_i, y_i are not among the ramification points of f_0). Counting the total number of ramification points of the cover f', it follows that $\deg(f'_{R_{x_i}}) = \deg(f'_{R_{y_i}}) = 2$. The morphism f' is now uniquely determined, for $f'^{-1}(\mathbf{P}^1_0) = C \cup \widetilde{R}_1 \cup \ldots \cup \widetilde{R}_{g-2k+1}$ and all the components of $f'^{-1}(f(R_{x_i}))$ and $f'^{-1}(f(R_{y_i}))$ other than R_{x_i} and R_{y_i} respectively are mapped isomorphically onto their images. $\square$

Definition 3.6. *Let $\overline{\mathfrak{BN}}' \subseteq \overline{\mathcal{H}}_{2g-2k+1,g-k+1} \times_{\overline{\mathcal{M}}_{2g-2k+1}} \overline{\mathcal{H}}_{2g-2k+1,g-k+1}$ be the closure of the locus of pairs of covers $\big([g_1 : C \to \boldsymbol{P}^1], [g_2 : C \to \boldsymbol{P}^1]\big)$, where C is a smooth curve of genus $2g-2k+1$ and $g_1 \ncong g_2$. We introduce the Brill-Noether divisor of curves possessing an extra pencil $\overline{\mathfrak{BN}} := \mathrm{pr}_1(\overline{\mathfrak{BN}}') \subseteq \overline{\mathcal{H}}_{2g-2k+1,g-k+1}$, where pr_1 denotes the first projection. We set $\mathfrak{BN} := \overline{\mathfrak{BN}} \cap \mathcal{H}_{2g-2k+1,g-k+1}$.*

Applying [AC, Proposition 2.4], we know that $\dim \overline{\mathfrak{BN}}' = \dim \mathcal{H}_{2g-2k+1,g-k+1} - 1$. Since $\overline{\mathfrak{BN}}'$ is birational to the Severi variety of nodal curves of type $(g-k+1, g-k+1)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ having geometric genus $2g-2k+1$, using [Ty], we conclude that $\overline{\mathfrak{BN}}$ is an irreducible divisor. We also recall Coppens' result [C] saying that if a curve $[C] \in \mathcal{M}_{2g-2k+1}$ has a pencil $A \in W^1_{g-k+1}(C)$ with $h^0(C, A^{\otimes 2}) \geq 4$, then $[C, A] \in \overline{\mathfrak{BN}}$. The locus of such pairs $[C, A] \in \mathcal{H}_{2g-2k+1,g-k+1}$ is of pure codimension one in $\overline{\mathfrak{BN}}$.

Our goal is to show that, in the notation of Proposition 3.5, the unique point of $\pi^{-1}([D])$ does not lie in $\overline{\mathfrak{BN}}$ provided the normalisation C is sufficiently general. To ease the notation, set $a := g-k+1$ and assume $a \geq 3$. To carry out the argument, it is convenient to work with stable maps rather than admissible covers. Let

$$\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a} := \widetilde{\mathcal{M}}^{\mathrm{ns}}_{2a-1}(\mathbf{P}^1, a)$$

denote the moduli space of *finite* stable maps $f : C \to \mathbf{P}^1$ of degree a such that C has genus $2a-1$, has only *non-separating* nodes and with $h^0(C, f^*\mathcal{O}_{\mathbf{P}^1}(1)) = 2$. Then $\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$ is an open subset of the projective moduli space $\overline{\mathcal{M}}_{2a-1}(\mathbf{P}^1, a)$ of stable maps $f : C \to \mathbf{P}^1$ with $f_*[C] = a[\mathbf{P}^1]$. Let

$$\widetilde{\pi} : \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a} \to \overline{\mathcal{M}}_{2a-1}$$

denote the natural projection. The Hurwitz space $\mathcal{H}_{2a-1,a}$ can be realized as the quotient of an open set of $\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$ by $PGL(2)$. We associate a stable map $[f : B \to \mathbf{P}^1] \in \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$ to the unique point $[f' : B' \to T] \in \pi^{-1}([D])$ by letting B be the curve obtained from B' by contracting all components of B' whose image is different from $f'(C)$, and then letting $[f : B \to \mathbf{P}^1]$ be the map which, on each component of B, agrees with f' on the corresponding component of B' (this is only determined up to the $PGL(2)$ action). Then

$$B = C \cup R_1 \cup \ldots \cup R_{a-k},$$

where $R_i \cong \mathbf{P}^1$ meets C at two general points (x_i, y_i) and $\deg(f_{R_i}) = 1$ for $i = 1, \ldots, a-k$.

Let $\mathfrak{B}^{\mathrm{ns}}_a \subseteq \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a} \times_{\overline{\mathcal{M}}_{2a-1}} \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$ be the closure of the locus of pairs

$$\Big([g_1 : X \to \mathbf{P}^1], [g_2 : X \to \mathbf{P}^1]\Big),$$

where X is a smooth curve of genus $2a-1$ and there is no automorphism $\sigma \in PGL(2)$ such that $[g_1] \cong \sigma \cdot [g_2]$. In order to prove that the unique point of $\pi^{-1}([D])$ does not lie in $\overline{\mathfrak{BN}} \subseteq \overline{\mathcal{H}}_{2a-1,a}$ it is sufficient to prove that

$$([f], [f]) \notin \mathfrak{B}^{\mathrm{ns}}_a.$$

For $0 \leq n \leq 2a-5$, let $\widetilde{\mathcal{M}}^{\mathrm{ns}}_{2a-1-n}(\mathbf{P}^1, a-n; 2n)$ denote the moduli space of finite stable maps $f : C \to \mathbf{P}^1$ of degree $a-n$ with $2n$ markings and such that C has genus $2a-1-n$, non-separating nodes and $h^0(C, f^*\mathcal{O}_{\mathbf{P}^1}(1)) = 2$. Then $\widetilde{\mathcal{M}}^{\mathrm{ns}}_{2a-1-n}(\mathbf{P}^1, a-n; 2n)$ is smooth of dimension $\dim \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a} - 2n$. Define

$$q_n : \widetilde{\mathcal{M}}^{\mathrm{ns}}_{2a-1-n}(\mathbf{P}^1, a-n; 2n) \to \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$$

by sending a map $f : C \to \mathbf{P}^1$ with markings $x_1, \ldots, x_n, y_1, \ldots, y_n$ to the stable map

$$\widetilde{f} : X \to \mathbf{P}^1,$$

where $X := C \cup R_1 \cup \ldots \cup R_n$, with $R_i \cong \mathbf{P}^1$, $R_i \cap C = \{x_i, y_i\}$, and with $\tilde{f}_C = f$ and $\deg(\tilde{f}_{R_i}) = 1$ for $i = 1, \ldots, n$. We let $\widetilde{X}$ be the stabilization of X. Set

$$Z_n := q_n^{-1}(\mathrm{pr}_1(\mathfrak{B}^{\mathrm{ns}}_a)). \tag{4}$$

Each component of Z_n has dimension at least $\dim \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a} - 2n - 1$.

The following theorem, giving a classification of points in Z_n, will be proved in the last section of the paper and is crucial in completing the proof of Schreyer's conjecture.

Theorem 3.7. *Fix $a \geq 3$ and $0 \leq n \leq 2a-5$, and let C be an integral nodal curve of genus $2a-1-n$ with a unique pencil $f : C \to \boldsymbol{P}^1$ of degree $a-n$. Choose pairs (x_i, y_i) of smooth distinct points of C for $i = 1, \ldots, n$. Assume $f(x_i) \neq f(y_i)$ and that for any subset $S \subseteq \{x_i, y_i\}_{i=1}^n$ of cardinality at most n, one has $h^0\big(C, f^*(\mathcal{O}_{\boldsymbol{P}^1}(2))(\sum_{s\in S} s)\big) = 3$. Then if $\pi^{-1}\big([\widetilde{X}]\big)$ consists of a unique point, then $[f, x_1, y_1, \ldots, x_n, y_n]$ does not lie in Z_n.*

The hypothesis that C has a unique pencil of degree $a-n$ amounts to $W^1_{a-n}(C) = \{A\}$, with A being a base-point free line bundle with $h^0(A) = 2$, which implies C has gonality $a-n$. If C is an integral, nodal curve of genus $2a-1-n$ having a pencil f of degree $a-n$ satisfying $h^0(f^*\mathcal{O}_{\mathbf{P}^1}(2)) = 3$ and if $\{x_i, y_i\}_{i=1}^n$ is a general set of points on C, then

$$h^0\Big(C, f^*(\mathcal{O}_{\mathbf{P}^1}(2))(\sum_{s\in S} s)\Big) = 3$$

for a subset $S \subseteq \{x_i, y_i\}_{i=1}^n$ of cardinality at most $n+1$. Indeed, $h^0\big(C, \omega_C \otimes f^*\mathcal{O}_{\mathbf{P}^1}(-2)\big) = n+1$, and since (x_i, y_i) are general, we find $h^0\big(C, \omega_C \otimes f^*\mathcal{O}_{\mathbf{P}^1}(-2)(-\sum_{s\in S} s)\big) = n + 1 - |S|$. Thus $h^0\big(C, f^*\mathcal{O}_{\mathbf{P}^1}(2)(\sum_{s\in S} s)\big) = 3$.

In particular, the following is an immediate corollary of Theorem 3.7.

Theorem 3.8. *Let C be a curve of genus g and gonality $k \leq \frac{g+1}{2}$ satisfying bpf-linear growth. Assume that there is a unique $A \in G^1_k(C)$ and that we have $h^0(C, A^{\otimes 2}) = 3$. Choose general pairs of points (x_i, y_i) on C for $i = 1, \ldots, g-2k+1$ and let D be the nodal curve obtained by identifying x_i and y_i for all i. Then $\pi^{-1}([D]) \cap \overline{\mathfrak{BN}} = \emptyset$.*

The proof of Theorem 3.7 (and thus that of Theorem 3.8) is surprisingly involved and takes up the last section of the paper. To avoid disrupting the logical flaw of the paper, we assume Theorem 3.7 and proceed towards the proof of Theorems 0.4 and 0.5.

4. Extending the Eagon-Northcott Divisor

We begin by recalling the definition of the Eagon-Northcott from the Introduction.

Definition 4.1. *The Eagon-Northcott divisor $\mathcal{EN} \subseteq \mathcal{H}_{2g-2k+1,g-k+1}$ is defined as the locus of covers $[f : C \to \boldsymbol{P}^1]$ such that $\dim K_{g-k,1}(C, \omega_C) > g-k$.*

We extend $\mathcal{EN}$ as a determinantal locus over a partial compactification of $\mathcal{H}_{2g-2k+1,g-k+1}$. From Theorem 3.1 and [SSW, Proposition 4.10], observe that we have the following equality of subsets of $\mathcal{H}_{2g-2k+1,g-k+1}$:

$$\mathfrak{BN} = \mathcal{EN}.$$

We construct an extension of $\mathcal{EN}$ on the moduli space $\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a} := \widetilde{\mathcal{M}}^{\mathrm{ns}}_{2a-1}(\mathbf{P}^1, a)$ of stable maps from the previous section. Precisely, we construct the extended Eagon-Northcott divisor

$$\widetilde{\mathcal{EN}} \subseteq \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$$

by studying the minimal free resolutions of the scrolls attached to a cover $[f : C \to \mathbf{P}^1] \in \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$. Set $A := f^*(\mathcal{O}_{\mathbf{P}^1}(1)) \in W^1_a(C)$. Since f is finite and flat, $f_*\mathcal{O}_C$ is locally free and we write $f_*\mathcal{O}_C \cong \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{E}_f^\vee$, where $\mathcal{E}_f$ is the so-called *Tschirnhausen bundle* of f, admitting a splitting

$$\mathcal{E}_f = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_{a-1}),$$

where $e_1 \leq \ldots \leq e_{a-1}$ are the *scrollar invariants* of f and satisfy $e_1 + \cdots + e_{a-1} = 3a-2$. Dualising the morphism $\mathcal{O}_{\mathbf{P}^1} \to f_*\mathcal{O}_C$ leads to an exact sequence

$$0 \longrightarrow \mathcal{E}_f \longrightarrow f_*\omega_f \longrightarrow \mathcal{O}_{\mathbf{P}^1} \longrightarrow 0.$$

We tensor the morphism $f^*(\mathcal{E}_f) \to \omega_f$ by $f^*\omega_{\mathbf{P}^1}$ and produce a morphism $f^*(\mathcal{E}_f(-2)) \twoheadrightarrow \omega_C$, inducing a closed immersion, see [Sch1], or [CE]

$$j : C \to \mathbf{P}\big(\mathcal{E}_f(-2)\big).$$

Note that $\mathcal{E}_f(-2)$ is a globally generated vector bundle on $\mathbf{P}^1$ with $\deg(\mathcal{E}_f(-2)) = a$. Denoting by $\varphi : X := \mathbf{P}\big(\mathcal{E}_f(-2)\big) \to \mathbf{P}^1$ the associated $(a-1)$-dimensional scroll, we have a morphism

$$\iota : X \to \mathbf{P}\big(H^0(\mathbf{P}^1, \mathcal{E}_f(-2))\big) \cong \mathbf{P}^{2a-2},$$

such that $\iota \circ j : C \to \mathbf{P}^{2a-2}$ is the canonical morphism of C, cf. [Sch1]. Observe that since C has no disconnected nodes, ω_C is globally generated. Also observe that if $h^0(C, A^{\otimes 2}) = 3$, then $e_1 \geq 3$ and ι is a closed immersion.

The Picard group of the scroll X is generated by the class of a ruling $R := \varphi^*(\mathcal{O}_{\mathbf{P}^1}(1))$ together with $H := \mathcal{O}_X(1)$. Note that $H^0(X, H) \cong H^0(C, \omega_C)$, whereas $H^0(X, R) \cong H^0(C, A)$ and $H^0(X, \mathcal{O}_X(H-R)) \cong H^0(C, \omega_C \otimes A^\vee)$. As already mentioned in the Introduction, the Eagon-Northcott complex, explicitly describes the minimal free resolution of

$$\Gamma_X(H) := \bigoplus_{q \in \mathbb{Z}} H^0(X, H^{\otimes q}),$$

as a Sym $H^0(X, H)$-module, see [Sch1]. This gives that

$$K_{p,0}(X, H) = 0 \text{ for } p > 0, \text{ whereas } \ K_{p,q}(X, H) = 0, \text{ for } q \geq 2 \text{ and any } p,$$

as well as the canonical identifications

$$K_{p,1}(X, H) \cong \bigwedge^{p+1} H^0(X, H-R) \otimes \mathrm{Sym}^{p-1} H^0(X, R) \otimes \bigwedge^{2} H^0(X, R)$$

$$\cong \bigwedge^{p+1} H^0(C, \omega_C \otimes A^\vee) \otimes \mathrm{Sym}^{p-1} H^0(C, A) \otimes \bigwedge^{2} H^0(C, A).$$

In particular, $b_{p,1}(X, H) = p\binom{a}{p+1}$.

We record the following lemma, while skipping the proof:

Lemma 4.2. *We have the vanishing $H^i(X, H^{\otimes q}) = 0$, for $i \geq 1$ and $q \geq 0$. Furthermore, $H^i(X, \mathcal{O}_X(-H)) = 0$, for $i \geq 2$.*

Define the kernel bundles M_H and M_{ω_C} on X and C respectively by the exact sequences

$$0 \longrightarrow M_H \longrightarrow H^0(X, H) \otimes \mathcal{O}_X \longrightarrow H \longrightarrow 0$$
$$0 \longrightarrow M_{\omega_C} \longrightarrow H^0(C, \omega_C) \otimes \mathcal{O}_C \longrightarrow \omega_C \longrightarrow 0.$$

As $C \subseteq X$ is linearly normal, $j^* M_H \cong M_{\omega_C}$. Note that $H^0\big(X, \bigwedge^p M_H\big) = H^0\big(C, \bigwedge^p M_{\omega_C}\big) = 0$, for $p \geq 1$. Further, we record the following short exact sequences:

(5)

$$0 \longrightarrow \bigwedge^{p+1} M_H \otimes \mathcal{O}_X\big((q-1)H\big) \longrightarrow \bigwedge^{p+1} H^0(X, H) \otimes \mathcal{O}_X\big((q-1)\big) \longrightarrow \bigwedge^{p} M_H \otimes \mathcal{O}_X\big(qH\big) \longrightarrow 0,$$

(6)

$$0 \longrightarrow \bigwedge^{p+1} M_{\omega_C} \otimes \omega_C^{\otimes(q-1)} \longrightarrow \bigwedge^{p+1} H^0(C, \omega_C) \otimes \omega_C^{\otimes(q-1)} \longrightarrow \bigwedge^{p} M_{\omega_C} \otimes \omega_C^{\otimes q} \longrightarrow 0.$$

We shall make use of the following vanishing statement.

Lemma 4.3. *We have $H^i\big(X, \bigwedge^p M_H \otimes H^{\otimes q}\big) = 0$ for $i \geq 2$ and arbitrary $p, q \geq 0$.*

Proof. By the sequence (5) and Lemma 4.2, it suffices to show $H^{i-1}\big(\bigwedge^{p-1} M_H \otimes H^{\otimes(q+1)})\big) = 0$. Continuing in this fashion, it suffices to show $H^1\big(X, \bigwedge^{p-i+1} M_H \otimes H^{\otimes(q+i-1)})\big) = 0$. Since $H^1\big(X, H^{\otimes(q+i-1)}\big) = 0$, this amounts to $K_{p,q+i}(X,H) = 0$, which holds as $q+i \geq 2$. □

Lemma 4.4. *There is an injective restriction map of linear syzygies*

$$\alpha_f : K_{a-1,1}(X,H) \to K_{a-1,1}(C,\omega_C).$$

The map α_f is surjective if and only if the restriction map

$$\beta_f : \ H^0\Big(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}\Big) \to H^0\Big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\Big)$$

is injective.

Proof. The map α_f fits into a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^a H^0(X,H) \arrow[r] \arrow[d, "\cong"] & H^0\big(X, \bigwedge^{a-1} M_H \otimes H\big) \arrow[r] \arrow[d, "\mathrm{res}_C"] & K_{a-1,1}(X,H) \arrow[r] \arrow[d, "\alpha_f"] & 0\ . \\
0 \arrow[r] & \bigwedge^a H^0(C,\omega_C) \arrow[r] & H^0\big(C, \bigwedge^{a-1} M_{\omega_C} \otimes \omega_C\big) \arrow[r] & K_{a-1,1}(C,\omega_C) \arrow[r] & 0
\end{tikzcd}$$

Since $C \subseteq X$ is linearly normal, it follows that res_C is injective, therefore α_f is injective as well. On the other hand, by the snake lemma the surjectivity of α_f is equivalent to the surjectivity of res_C. From the kernel bundle description of Koszul cohomology, we write

$$K_{a-2,2}(X,H) = \mathrm{Ker}\Big\{H^1\big(X, \bigwedge^{a-1} M_H \otimes H\big) \to \bigwedge^{a-1} H^0(X,H) \otimes H^1(X,H)\Big\}.$$

Since $H^1(X,H) = 0$ and $K_{a-2,2}(X,H) = 0$, it follows $H^1\big(X, \bigwedge^{a-1} M_H \otimes H\big) = 0$. We write the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^0\big(X, \bigwedge^{a-1} M_H \otimes H\big) \arrow[r] \arrow[d, "\mathrm{res}_C"] & \bigwedge^{a-1} H^0(X,H) \otimes H^0(X,H) \arrow[r] \arrow[d, "\cong"] & H^0\big(X, \bigwedge^{a-2} M_H \otimes H^{\otimes 2}\big) \arrow[r] \arrow[d, "\beta_f"] & 0\ , \\
0 \arrow[r] & H^0\big(C, \bigwedge^{a-1} M_{\omega_C} \otimes \omega_C\big) \arrow[r] & \bigwedge^{a-1} H^0(C,\omega_C) \otimes H^0(C,\omega_C) \arrow[r] & H^0\big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\big) &
\end{tikzcd}$$

By the snake lemma, the surjectivity of res_C is equivalent to the injectivity of β_f. □

Koszul duality gives an isomorphism $K_{a-2,2}(C,\omega_C) \cong K_{a-1,1}(C,\omega_C)^\vee$, therefore we have a surjection

$$H^0\big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\big) \longrightarrow H^0\big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\big) / \bigwedge^{a-1} H^0(C,\omega_C) \otimes H^0(C,\omega_C) \cong K_{a-1,1}(C,\omega_C)^\vee.$$

The composition of this map with $\alpha_f^\vee$ gives rise to a surjection

$$\psi_f : H^0\big(C, \bigwedge^{a-2} M_{\omega_C} \otimes \omega_C^{\otimes 2}\big) \to K_{a-1,1}(X,H)^\vee.$$

Because $K_{a-2,2}(X,H) = 0$, from the second diagram in the proof of Lemma 4.4, it follows $\psi_f \circ \beta_f = 0$.

Lemma 4.5. *We have a natural isomorphism* $\mathrm{Ker}(\psi_f) \cong H^2\big(X, \bigwedge^a M_H \otimes I_{C/X}\big)^\vee$.

Proof. Since $H^1(X,\mathcal{O}_X)=0$, the description of Koszul cohomology via kernel bundles yields the identification $K_{a-1,1}(X,H)^\vee \cong H^1\big(X,\bigwedge^a M_H\big)^\vee$. Using that $\bigwedge^{a-2} M_{\omega_C}\otimes\omega_C \cong \bigwedge^a M^\vee_{\omega_C}$, Serre-Duality gives the isomorphism

$$H^0\big(C,\bigwedge^{a-2} M_{\omega_C}\otimes\omega_C^{\otimes 2}\big)^\vee \cong H^1\big(C,\bigwedge^{a} M_{\omega_C}\big),$$

which enables us to identify the dual map $\psi_f^\vee$ with the restriction

$$H^1\big(X,\bigwedge^{a} M_H\big)\to H^1\big(C,\bigwedge^{a} M_{\omega_C}\big).$$

Then $\mathrm{Ker}(\psi_f)\cong \mathrm{Coker}\big(\psi_f^\vee)\big)^\vee \cong H^2\big(X,\bigwedge^a M_H\otimes I_{C/X}\big)^\vee$, using also Lemma 4.3. □

Putting the above pieces together, we have constructed a natural map

$$\beta_f:\ H^0\big(X,\bigwedge^{a-2} M_H\otimes H^{\otimes 2}\big)\to H^2\big(X,\bigwedge^{a} M_H\otimes I_{C/X}\big)^\vee$$

such that $b_{a-1,1}(C,\omega_C)>a-1$ if and only if β_f fails to be injective. We shall see that both sides of this map have the same dimension. This allows us to construct $\widetilde{\mathcal{EN}}$ as the degeneracy locus of a morphism between vector bundles of the same rank on the space of stable maps.

Lemma 4.6. *We have:*

$$h^0\big(X,\bigwedge^{a-2} M_H\otimes H^{\otimes 2}\big)=h^2\big(X,\bigwedge^{a} M_H\otimes I_{C/X}\big)=(2a-2)\binom{2a-1}{a}-a+1.$$

Proof. As already pointed out $H^1\big(X,\bigwedge^{a-1} M_H\otimes H\big)=0$. Therefore

$$h^0\big(X,\bigwedge^{a-2} M_H\otimes H^{\otimes 2}\big)=(2a-1)\binom{2a-1}{a}-h^0\big(X,\bigwedge^{a-1} M_H\otimes H\big),$$

by the short exact sequence (5). We further have a short exact sequence

$$0\longrightarrow \bigwedge^{a} H^0(X,H)\longrightarrow H^0\big(X,\bigwedge^{a-1} M_H\otimes H\big)\longrightarrow K_{a-1,1}(X,H)\longrightarrow 0,$$

thus using that $b_{a-1,1}(X,H)=a-1$, we find $h^0\big(X,\bigwedge^{a-1} M_H\otimes H\big)=a-1+\binom{2a-1}{a}$, which leads to the claimed formula for $h^0\big(X,\bigwedge^{a-2} M_H\otimes H^{\otimes 2}\big)$.

Using Lemma 4.5, we compute:

$$h^2\big(X,\bigwedge^{a} M_H\otimes I_{C/X}\big)=\dim(\mathrm{Ker}\ \psi_f)=h^0\big(C,\bigwedge^{a-2} M_{\omega_C}\otimes\omega_C^{\otimes 2}\big)-b_{a-1,1}(X,H).$$

Recall that $b_{a-1,1}(X,H)=a-1$. The Riemann-Roch theorem (still valid for a nodal curve C with no disconnecting nodes) gives

$$h^0\big(C,\bigwedge^{a-2} M_{\omega_C}\otimes\omega_C^{\otimes 2}\big)=\chi\big(C,\bigwedge^{a-2} M_{\omega_C}\otimes\omega_C^{\otimes 2}\big)=(4a-2)\binom{2a-2}{a},$$

which finishes the proof. □

We now explain how the above considerations can be carried out in a relative setting. Let

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[dr, "\nu"'] & \mathcal{P} \arrow[d, "\mu"] \\ & \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a} \end{tikzcd}$$

be the universal degree a cover, where $\mathcal{P}=\widetilde{\mathcal{G}}^{ns}_{2a-1,a}\times \mathbf{P}^1$. The universal Tschirnhausen bundle $\mathcal{E}_f$ on $\mathcal{P}$ fits into an exact sequence:

$$0\longrightarrow \mathcal{E}_f\longrightarrow f_*\omega_f\longrightarrow \mathcal{O}_{\mathcal{P}}\longrightarrow 0.$$

We further have the projective bundle $\varphi:\ \mathcal{X}:=\mathbf{P}(\mathcal{E}_f\otimes \omega_\mu)\to \mathcal{P}$ and a closed immersion $j:\mathcal{C}\hookrightarrow \mathcal{X}$. Set $h:=\mu\circ\varphi:\ \mathbf{P}(\mathcal{E}_f\otimes\omega_\mu)\to \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$. By Grauert's Theorem, $h_*\big(\mathcal{O}_{\mathcal{X}}(1)\big)$ is a vector bundle of rank $2a-1$. Define the determinant $\xi:=\det h_*\big(\mathcal{O}_{\mathcal{X}}(1)\big)$. The evaluation map $h^*h_*\mathcal{O}_{\mathcal{X}}(1)\to \mathcal{O}_{\mathcal{X}}(1)$ is furthermore surjective, thus we can define the kernel bundle $\mathcal{M}$ by

$$0\longrightarrow \mathcal{M}\longrightarrow h^*h_*(\mathcal{O}_{\mathcal{X}}(1)\to \mathcal{O}_{\mathcal{X}}(1)\longrightarrow 0.$$

Then $\mathcal{M}$ restricts to the kernel bundle M_H for each scroll induced by an element $[C\to\mathbf{P}^1]$. Note that j is defined by the surjection $f^*(\mathcal{E}_f\otimes\omega_\mu)\to \omega_f\otimes f^*\omega_\mu\cong\omega_\nu$, hence $\mathcal{O}_{\mathcal{C}}(1)\cong\omega_\nu$. Set

$$\mathcal{F}_1:=h_*\Big(\bigwedge^{a-2}\mathcal{M}\otimes\mathcal{O}_{\mathcal{X}}(2)\Big)\otimes\xi^\vee,$$

which is a vector bundle of rank $(2a-2)\binom{2a-1}{a}-a+1$, by Lemma 4.6. Set

$$\mathcal{F}_2:=h_*\Big(\bigwedge^{a-2}\mathcal{M}\otimes\mathcal{O}_{\mathcal{C}}(2)\Big)\otimes\xi^\vee,$$

which is a vector bundle of rank $(2a-2)\binom{2a-1}{a}$. Restriction to $\mathcal{C}$ induces a morphism

$$\beta:\ \mathcal{F}_1\to\mathcal{F}_2.$$

Relative duality gives the isomorphism

$$R^1\nu_*\big(\bigwedge^a\mathcal{M}_{|\mathcal{C}}\big)\cong\Big(\nu_*\big(\bigwedge^a\mathcal{M}^\vee_{|\mathcal{C}}\otimes\omega_\nu\big)\Big)^\vee\cong\mathcal{F}_2^\vee,$$

using $\det(\mathcal{M})\cong h^*\xi\otimes\mathcal{O}_{\mathcal{X}}(-1)$. Define the rank $a-1$ vector bundle by $\mathcal{F}_3:=R^1h_*\Big(\bigwedge^a\mathcal{M}\Big)^\vee$, The dual of the restriction morphism $\psi^\vee:\ R^1h_*\big(\bigwedge^a\mathcal{M}\big)\to R^1\nu_*\big(\bigwedge^a\mathcal{M}_{|\mathcal{C}}\big)$ gives a morphism

$$\psi:\ \mathcal{F}_2\to\mathcal{F}_3$$

with fibre over a moduli point $[f:C\to\mathbf{P}^1]$ equal to ψ_f. As already explained, $\psi\circ\beta=0$.

We get a short exact sequence of vector bundles over $\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$:

$$0\longrightarrow R^1h_*\big(\bigwedge^a\mathcal{M}\big)\longrightarrow R^1\nu_*\big(\bigwedge^a\mathcal{M}\otimes\mathcal{O}_{\mathcal{C}}\big)\longrightarrow R^2h_*\big(\bigwedge^a\mathcal{M}\otimes I_{\mathcal{C}/\mathcal{X}}\big)\longrightarrow 0,$$

where $\mathcal{F}_4:=R^2h_*\big(\bigwedge^a\mathcal{M}\otimes I_{\mathcal{C}/\mathcal{X}}\big)$ is a vector bundle of rank $(2a-2)\binom{2a-1}{a}-a+1$ by Lemma 4.6. Thus we may canonically identify

$$\mathrm{Ker}(\psi)\cong\mathcal{F}_4^\vee$$

and we have an induced morphism between vector bundles $\beta:\mathcal{F}_1\to\mathcal{F}_4^\vee$ globalizing the morphisms β_f as the moduli point $[f]\in\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$ varies. Since $\mathrm{rk}(\mathcal{F}_1)=\mathrm{rk}(\mathcal{F}_4)$, we define the extended Eagon-Northcott divisor

$$\widetilde{\mathcal{EN}}\subseteq\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a}$$

as the degeneracy locus of β. By the results of the previous chapter, this is a genuine divisor.

Define $\mathcal{EN}^{\mathrm{sm}}$ as the union of all components of $\widetilde{\mathcal{EN}}$ containing an element $[f : C \to \mathbf{P}^1]$, with C being a smooth curve and all ramification simple. The following lemma is a direct consequence of Theorem 3.8.

Lemma 4.7. *Let C be a curve of genus g and gonality $k \leq \frac{g+1}{2}$ satisfying bpf-linear growth. For $i = 1, \ldots, g-2k+1$, choose pairs (x_i, y_i) of general points on C and let B be the semistable curve given as the union of C with $g-2k+1$ smooth rational curves R_i meeting the rest of B precisely at x_i, y_i. Let*

$$[f : B \to \boldsymbol{P}^1] \in \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2g-2k+1, g-k+1}$$

be a morphism with $\deg(f_C) = k$ and f_{R_i} an isomorphism. Assume that f_C is the unique minimal pencil on C and, further, $h^0(C, f^*\mathcal{O}_{\boldsymbol{P}^1}(2)) = 3$. Then $[f] \notin \mathcal{EN}^{\mathrm{sm}}$.

Proof. Consider the closure $\overline{\mathcal{EN}}^{\mathrm{sm}} \subseteq \overline{\mathcal{M}}_{2g-2k+1}(\mathbf{P}^1, g-k+1)$ in the moduli space of stable maps. We have the projections $\pi' : \overline{\mathcal{M}}_{2g-2k+1}(\mathbf{P}^1, g-k+1) \to \overline{\mathcal{M}}_{2g-2k+1}$, as well as the projection π from the space of admissible covers. There is an equality of closed sets $\pi'(\overline{\mathcal{EN}}^{\mathrm{sm}}) = \pi(\widetilde{\mathcal{EN}})$, since $\overline{\mathcal{M}}_{2g-2k+1}(\mathbf{P}^1, g-k+1)$ is a $PGL(2)$-cover of $\overline{\mathcal{H}}_{2g-2k+1, g-k+1}$ over the open set of morphisms with smooth source and simple ramification. By Theorem 3.8, the point $[D] \in \overline{\mathcal{M}}_{2g-2k+1}$ defined by the stabilization of B does not lie in $\pi'(\overline{\mathcal{EN}}^{\mathrm{sm}})$, therefore, $[f] \notin \overline{\mathcal{EN}}^{\mathrm{sm}}$. ☐

To complete the proof of Theorem 0.5 (assuming Theorem 3.7), we need to show that, in the situation of Lemma 4.7, the point $[f]$ does not lie in the extended Eagon-Northcott divisor $\widetilde{\mathcal{EN}}$. Note that $[f]$ lies in precisely one boundary divisor of $\widetilde{\mathcal{G}}^{\mathrm{ns}}_{2g-2k+1, g-k+1}$, namely the divisor Δ whose general point corresponds to maps $h : C \to \mathbf{P}^1$, where C is a union of two curves C_1 and C_2 of genera $g-1$ and 0 respectively, meeting at two points, and such that $\deg(h_{C_1}) = g-k$ and $\deg(h_{C_2}) = 1$. Since $\widetilde{\mathcal{EN}}$ is pure of codimension one, we need to show that $\widetilde{\mathcal{EN}}$ does not contain Δ. We carry this out in the next section, using $K3$ surfaces.

5. K3 Surfaces and Schreyer's Conjecture

We start by considering a $K3$ surface $X = X_d$ with Picard group generated by two classes L and E with self intersections given by $(L)^2 = 4d-4$, $(E)^2 = 0$ and $(L \cdot E) = d$, for $d \geq 3$. By performing Picard-Lefschetz transformations and a reflection if necessary, we may assume that L is big and nef.

Lemma 5.1. *For X as above, the class L is base point free and E is the class of a smooth elliptic curve.*

Proof. We firstly show that L is base point free. As L is big and nef, it suffices to show there is no smooth elliptic curve F with $(L \cdot F) = 1$, see [M, Proposition 8]. Assume such F exists, and write $F = aL + bE$ for $a, b \in \mathbb{Z}$. As F is smooth and elliptic, $(F)^2 = 0$, implying $0 = (aL + bE) \cdot F = a + b(E \cdot F) = a(1 + db)$. If $a = 0$, then $(L \cdot F) = bd \neq 1$, since $d \geq 2$, so $db = -1$, which is again impossible. Thus L is base point free.

We next show that E is the class of a smooth elliptic curve. As $(E)^2 = 0$ and E is primitive, it suffices to show that E is nef. Since $(E \cdot L) > 0$, and L is big and nef, E is effective. Suppose E is not nef. Then there exists a smooth, rational curve R with $(R \cdot E) < 0$. Write $R = aL + bE$ for $a, b \in \mathbb{Z}$. Then $(R \cdot E) < 0$ implies $a < 0$. As $(R)^2 = -2$ and R is effective, we must have $b > 0$. We have $-2 = (R)^2 = R \cdot (aL + bE) = a(R \cdot L) + b(R \cdot E) = a\big((R \cdot L) + bd\big)$, which is impossible for $d \geq 3$. ☐

We now discuss the Brill-Noether theory of a smooth curve $C \in |L|$. To that end, we follow [K1, §2] which works in the situation of a higher rank Picard lattice containing the lattice $\mathrm{Pic}(X_d)$.

Lemma 5.2. *Let $D \in \mathrm{Pic}(X_d)$ be effective with $(D)^2 \geq 0$. Assume in addition $L-D$ is effective and $(L-D)^2 > 0$. Then $D = cE$, for some integer c.*

Proof. This is a slight modification of [K1, Lemma 2.5]. Write $D = aL + bE$. As $L - D$ is effective and E nef, $(L-D) \cdot E = (1-a)(L \cdot E) \geq 0$, so $a \leq 1$. From $(D \cdot E) \geq 0$, we obtain $a \geq 0$. If $a = 1$, then $(L-D)^2 = b^2(E)^2 = 0$, so we must have $a = 0$ as required. $\square$

The next lemma describes the Brill-Noether behaviour of curves in the linear system $|L|$.

Lemma 5.3. *Let $C \in |L|$ be a smooth curve. Then* $\mathrm{Cliff}(C) = d-2$ *and $W^1_d(C)$ is reduced and consists of the single point $\mathcal{O}_C(E)$.*

Proof. The proof that $\mathrm{Cliff}(C) = d-2$ is essentially the same as [K1, Lemma 2.6]. Arguing as in [K1, Lemmas 2.7, 2.8], we see that $W^1_d(C)$ is set-theoretically a single point, namely $\mathcal{O}_C(E)$.

It remains to establish that $W^1_d(C)$ is reduced, which amounts to showing that $h^0(\mathcal{O}_C(2E)) = 3$. From the exact sequence

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_X(2E) \longrightarrow \mathcal{O}_E(2E) \cong \mathcal{O}_E \longrightarrow 0,$$

we deduce $h^1(X, 2E) = 1$ and then $h^0(X, 2E) = 3$ by Riemann–Roch. By the exact sequence

$$0 \longrightarrow \mathcal{O}_X(2E - C) \longrightarrow \mathcal{O}_X(2E) \longrightarrow \mathcal{O}_C(2E) \longrightarrow 0,$$

it suffices to show $h^0(X, 2E-C) = h^1(X, 2E-C) = 0$. As $(C-2E)^2 = -4$, by Riemann-Roch, it suffices to show that neither $2E - C$ nor $C - 2E$ are effective. As $(E \cdot 2E - C) < 0$ and E is nef, $2E - C$ is not effective. Now suppose $C - 2E$ is effective with integral components $R_1, \ldots, R_\ell$, for $\ell \geq 1$. We write $R_i = a_i L + b_i E$, for integers a_i, b_i, with $\sum_{i=1}^{\ell} a_i = 1$ and $\sum_{i=1}^{\ell} b_i = -2$. As $(E \cdot R_i) \geq 0$, we find $a_i \geq 0$ for all i. Without loss of generality, we may assume $a_1 = 1$ and $a_i = 0$ for $2 \leq i \leq \ell$. As R_i is integral, we must then have $b_i = 1$ for $i > 1$. Thus $R_1 = L - (\ell+1)E$, which implies $(R_1)^2 = 4d - 4 - 2d(\ell+1) \leq -4$, contradicting that R_1 is integral. $\square$

We can now prove Theorem 0.5, that is, establish the Schreyer Conjecture.

Proof of Theorem 0.5. Let $[f : B \to \mathbf{P}^1]$ be as in the statement of Lemma 4.7. By an argument along the lines of [V1, Corollary 1], we have an injection $K_{g-k,1}(C, \omega_C) \hookrightarrow K_{g-k,1}(B, \omega_B)$. For the sake of completeness we recall the proof.

The Mayer-Vietoris sequence induces an injection $H^0(C, \omega_C) \hookrightarrow H^0(B, \omega_B)$, as well as the composition of injections $H^0(C, \omega_C^{\otimes 2}) \hookrightarrow H^0\big(C, \omega_C^{\otimes 2}(\sum_{i=1}^{g-2k+1}(x_i + y_i))\big) \hookrightarrow H^0(B, \omega_B^{\otimes 2})$. We then get a commutative diagram

$$\begin{array}{ccccc}
\bigwedge^{g-k+1} H^0(\omega_C) & \xrightarrow{\delta_0} & \bigwedge^{g-k} H^0(\omega_C) \otimes H^0(\omega_C) & \longrightarrow & \bigwedge^{g-k-1} H^0(\omega_C) \otimes H^0(\omega_C^{\otimes 2}) \\
\downarrow & & \downarrow & & \downarrow \\
\bigwedge^{g-k+1} H^0(\omega_B) & \xrightarrow{\delta_0'} & \bigwedge^{g-k} H^0(\omega_B) \otimes H^0(\omega_B) & \longrightarrow & \bigwedge^{g-k-1} H^0(\omega_C) \otimes H^0(\omega_C^{\otimes 2})
\end{array}.$$

The conclusion now follows from the existence of maps, see also [AN, Lemma 7.1]

$$\wedge : \wedge^{g-k} H^0(\omega_C) \otimes H^0(\omega_C) \to \wedge^{g-k+1} H^0(\omega_C) \ \text{ and } \wedge' : \wedge^{g-k} H^0(\omega_B) \otimes H^0(\omega_B) \to \wedge^{g-k+1} H^0(\omega_B),$$

with $\wedge \circ \delta_0 = \pm(g-k)\mathrm{Id}$ and $\wedge' \circ \delta_0' = \pm(g-k)\mathrm{Id}$.

We secondly claim that $[f]$ does not lie in the extended Koszul divisor $\widetilde{\mathcal{EN}}$. In light of the injective map above, this will complete the proof. As $[f]$ lies in exactly one boundary divisor, namely Δ, all that remains is to show that the divisor $\widetilde{\mathcal{EN}}$ does not contain Δ. By Lemma 5.3, we know that any smooth curve $C \in |L|$ on the K3 surface $X = X_{g-k+1}$ satisfies $b_{g-k,1}(C, \omega_C) = g - k$. By the Lefschetz Theorem for Koszul cohomology [G], the same holds for any integral nodal curve $C_0 \in |L|$. As any integral, nodal curve C_0 (with at least one node) defines a point in Δ, it suffices to show that such curves exist for the general X_{g-k+1}.

In order to do this, it suffices to take $2g-2k+1 \geq 8$, as the conclusion of the Theorem is well-known for $g \leq 8$ by [Sch1]. Indeed, if $g-k \leq 3$, then, since we are assuming $g \geq 2k-1$, we must have $k \leq 4$ and $g \leq 7$. The class $L-E$ is very ample for a general $K3$ surface X_{g-k+1} general with the given Picard lattice, by degenerating to the $K3$ surface $Y_{\Omega_{2g-2k+1}}$ from [K1, Lemma 2.3]. Choose a curve $C_1 \in |L-E|$ meeting a smooth elliptic curve $E_0 \in |E|$ transversally, and consider the nodal curve $C_1 \cup E_0$. Pick any node $p_1 \in C_1 \cup E_0$. Then, by [Ta, Theorem 3.8], the moduli space $\bar{\mathcal{V}}_1(X_d)$ parametrising deformation of $C_1 \cup E_0$ preserving the assigned node p_1 is smooth near $(C_1 \cup E_0, p_1)$ of dimension $2g-2k$. As $\dim |L-E| + \dim |E| = g-k+1 < g-1$ for $k \geq 3$, there exist integral 1-nodal curves $C_0 \in |L|$, completing the proof. □

6. Twistings and pencils on singular curves

In this section we prove Theorem 3.7. We need to construct twistings of line bundles on reducible curves. Suppose we have a family $\mathcal{B} \to \Delta$ of nodal curves over a smooth base Δ, with general fibre smooth and with special fibre $C \cup R$, where $R \cong \mathbf{P}^1$ and C is smooth meeting R transversally in two points. If the total family $\mathcal{B}$ is smooth, then $\mathcal{O}_{\mathcal{B}}(R)$ is a line bundle which is trivial outside of the special fibre. We will generalize the definition of this twist to the case where the general fibre is only integral and $\mathcal{B}$ is not necessarily smooth, so R may not be Cartier.

We first introduce some convenient notation.

Definition 6.1. *Let X be a connected, nodal curve and $p \in X$. The "blow-up" $c_p : \widetilde{X} \to X$ of X at p is defined as such:*

1. *If $p \in X$ is a node, let $\nu : X' \to X$ denote the partial normalisation of X at p and let $\nu^{-1}(p) = \{a, b\}$. Then $\widetilde{X}$ is defined to be $X' \cup E$, where $E \cong \boldsymbol{P}^1$ and $E \cap X' = \{a, b\}$.*
2. *If p is not a node, then define $\widetilde{X} = X \cup E$, where $E \cong \boldsymbol{P}^1$ with $X \cap E = \{p\}$. We further define the "strict transform" X' of X to be the closure $\overline{\widetilde{X} \setminus E}$.*

In both cases, $c_p : \widetilde{X} \to X$ is given by contracting the unstable component E to the point p.

An *abelian differential* of type $(1)^{2g-2}$ on a smooth curve C of genus g is an ordered marking α of degree $2g-2$ such that there exists a nonzero section $s \in H^0(C, \omega_C)$ with $s(\alpha) = 0$. Such (C, α) define elements of $\overline{\mathcal{M}}_{g,2g-2}$. Let $\overline{\mathcal{M}}_g\big((1)^{2g-2}\big) \subseteq \overline{\mathcal{M}}_{g,2g-2}$ be the space of twisted abelian differentials of type $(1)^{2g-2}$, i.e. the closure of the space of abelian differentials of this type, [FP].

The first twisting construction we will use is described below and is rather well-known. We attach a proof due to lack of a suitable reference.

Proposition 6.2. *Let $(\Delta, 0)$ be an irreducible, pointed, variety and $\mathcal{B} \to \Delta$ be a flat family of nodal curves of genus $g \geq 2$ such that the fibre B_t is irreducible for a general $t \in \Delta$ and*

$$B_0 \cong C \cup R, \;\; R \cong \boldsymbol{P}^1, \;\; R \cap C = \{u, v\},$$

and C is irreducible. After a base change, there is a birational morphism $\nu : \widetilde{\mathcal{B}} \to \mathcal{B}$ of families of nodal curves over Δ and a line bundle $\tau \in \mathrm{Pic}(\widetilde{\mathcal{B}})$, such that one of the following cases occur:

1. *$\widetilde{B}_0 \cong B_0$. Furthermore $\tau_C \cong \mathcal{O}_C(u+v)$ and $\deg(\tau_R) = -2$.*
2. *$\widetilde{B}_0$ is a blow-up of B_0 at a node $p \in \{u, v\}$ with exceptional component E. Identifying R and C with their strict transforms, $\tau_C \cong \mathcal{O}_C(u+v)$ and $\deg(\tau_R) = \deg(\tau_E) = -1$.*

Informally, case (1) corresponds to twisting by a line bundle, whereas case (2) corresponds to twisting by a torsion-free sheaf.

Proof. Performing a base change if necessary, we choose markings $p_i : \Delta \to \mathcal{B}$, for $i = 1, \ldots, g$, with $p_g(0) \in R \subseteq B_0$ and $p_1(0), \ldots, p_{g-1}(0)$ being general points of C. For all $t \in \Delta$, up to scaling, there exists a unique form $0 \neq s_t \in H^0(B_t, \omega_{B_t})$ vanishing along $p_1(t)+\cdots+p_{g-1}(t)$. Note that s_0 vanishes identically on R. By the generality of the points $p_i(0)$ for $i=1,\ldots,g-1$, the restriction $s_{0|C}$ vanishes on an abelian differential of type $(1)^{2g-4}$. Hence, after possibly shrinking Δ, those components of the vanishing set of s_t which limit to points on C provide additional sections $p_i : \Delta \to \mathcal{B}$, for $i = g+1,\ldots,2g-3$, with $\sum_{i\neq g} p_i(0)$ defining an abelian differential on C, such that $[B_t, p_1(t),\ldots,p_{2g-3}(t)]$ lies in the image of the morphism $\overline{\mathcal{M}}_g\big((1)^{2g-2}\big) \to \overline{\mathcal{M}}_{g,2g-3}$ obtained by forgetting the last marking. After a finite base change, we obtain a map $\nu : \widetilde{\mathcal{B}} \to \mathcal{B}$ between families of nodal curves, together with sections $q_i : \Delta \to \widetilde{\mathcal{B}}$ for $i = 1,\ldots,2g-2$, such that $[\widetilde{B}_t, q_i(t)] \in \overline{\mathcal{M}}_g\big((1)^{2g-2}\big)$ and, further, after forgetting the last marking $q_{2g-2}(t)$, these curves stabilize to $[B_t, p_1(t),\ldots,p_{2g-3}(t)]$.

Notice that $\nu\big(q_{2g-2}(0)\big) \in R$. We now distinguish three cases. If $\nu\big(q_{2g-2}(0)\big) \notin \{u, v, p_g(0)\}$, then ν is an isomorphism, and we may take

$$\tau := \omega_{\mathcal{B}/\Delta}\Big(-\sum_{i=1}^{2g-2}(\nu\circ q_i)(\Delta)\Big).$$

If $\nu(q_{2g-2}(0)) = p_g(0)$, then $\nu\circ q_{2g-2}$ defines a section in the smooth locus of each fibre B_t, and we define τ by the same formula. Lastly, suppose $\nu(q_{2g-2}(0)) \in R\cap C$. We first observe that $\nu(q_{2g-2}(t))$ is not a node of B_t for a general t. Indeed, otherwise $\widetilde{B}_t$ would be a blow-up of B_t at a node. Further, there must be a nonzero section of $\omega_{\widetilde{B}_t}$ vanishing at the exceptional component E, as well as on smooth points $q_1(t),\ldots,q_{2g-3}(t)$, which is impossible. Hence, in the last case, $\widetilde{\mathcal{B}}$ is a blow-up of $\mathcal{B}$ at a node in the intersection $C\cap R$, and we may now take $\tau := \omega_{\widetilde{\mathcal{B}}/\Delta}\Big(-\sum_{i=1}^{2g-2} q_i(\Delta)\Big)$. $\square$

The next twisting result is more sophisticated. Under further hypotheses, it ensures that we can put ourselves in the more degenerate case (2) of Proposition 6.2.

Proposition 6.3. *Keeping the notation of Proposition 6.2, assume additionally we have a family*

$$f : \mathcal{B} \to \boldsymbol{P}^1 \times \Delta$$

of stable maps of degree k with $\deg(f_{0|R}) = 1$, $h^0(B_0, f_0^*\mathcal{O}_{\boldsymbol{P}^1}(1)) = 2$, and $\omega_C \otimes f_C^*\mathcal{O}_{\boldsymbol{P}^1}(-1)$ base point free. Assume the locus of points $t \in \Delta$ for which B_t is reducible has codimension two. Then, after a base change, we have $\nu : \widetilde{\mathcal{B}} \to \mathcal{B}$ and $\tau \in \mathrm{Pic}(\widetilde{\mathcal{B}})$ as in case (2) of Proposition 6.2.

Proof. Let $\overline{\mathcal{M}}^{\dagger} \subseteq \overline{\mathcal{M}}_g(\mathbf{P}^1, k; 2g-2-k)$ denote the substack of the moduli space of degree k stable maps $h : B \to \mathbf{P}^1$ with markings $p_1,\ldots,p_{2g-2-k}$ defined by the following conditions (i) $h^0(B, h^*\mathcal{O}_{\mathbf{P}^1}(1)) = 2$ and (ii) $H^0\big(B, \omega_B \otimes h^*\mathcal{O}_{\mathbf{P}^1}(-1)(-p_1-\cdots-p_{2g-2-k})\big) \neq 0$.

The moduli space $\overline{\mathcal{M}}^{\dagger}$ may be constructed from an incidence variety in the obvious fashion, see also [BCGGM, §2]. We have a forgetful morphism $\overline{\mathcal{M}}^{\dagger} \to \overline{\mathcal{M}}_g(\mathbf{P}^1, k)$. Let

$$r : \Delta^{\dagger} := \Delta \times_{\overline{\mathcal{M}}_g(\mathbf{P}^1,k)} \overline{\mathcal{M}}^{\dagger} \to \Delta,$$

where $\Delta \to \overline{\mathcal{M}}_g(\mathbf{P}^1, k)$ is induced by the family of stable maps $f : \mathcal{B} \to \mathbf{P}^1 \times \Delta$.

We denote by $\widetilde{B}_0$ the blow-up of $B_0 = C \cup_{\{u,v\}} R$ at u and set $\widetilde{f}_0 := f_0 \circ c_u : \widetilde{B}_0 \to \mathbf{P}^1$. Choose distinct points $p_1,\ldots,p_{2g-3-k}$ in the smooth locus of B_0, with $\sum_{i=1}^{2g-3-k} p_i \in |\omega_C \otimes f_C^*\mathcal{O}_{\mathbf{P}^1}(-1)|$ and pick a general point $p_{2g-2-k} \in E$. Then

$$t := [\widetilde{f}_0, p_1,\ldots,p_{2g-2-k}] \in \Delta^{\dagger}.$$

We claim that any component $I \subseteq \Delta^{\dagger}$ containing the point t dominates Δ under the forgetful morphism r. We are then done, by replacing Δ with I, setting $\widetilde{f} := f\circ\nu : \widetilde{\mathcal{B}} \to \mathbf{P}^1 \times I$ to be the universal family and choosing

$$\tau := \widetilde{f}^* \mathcal{O}_{\mathbf{P}^1 \times I}(-1) \otimes \omega_{\widetilde{\mathcal{B}}/I}\Big(-\sum_{i=1}^{2g-2-k} \widetilde{p}_i\Big),$$

where $\widetilde{p}_i : I \to \widetilde{\mathcal{B}}$ are the markings.

From the construction of $\Delta^\dagger$ as an incidence variety it follows that

$$\dim I \geq \dim \Delta + \dim |\omega_{B_0} \otimes f_0^* \mathcal{O}_{\mathbf{P}^1}(-1)|.$$

Since $\dim |\omega_C \otimes f_C^* \mathcal{O}_{\mathbf{P}^1}(-1)| = \dim |\omega_{B_0} \otimes f_0^* \mathcal{O}_{\mathbf{P}^1}(-1)|$, it follows that $r(I)$ has codimension at most one in Δ. By assumption, the general point of $r(I)$ corresponds to a stable map with *irreducible* source B. But, in an open subset about $t \in I$, the fibre of r over such a stable map with irreducible base has expected dimension $\dim |\omega_{B_0} \otimes f_0^* \mathcal{O}_{\mathbf{P}^1}(-1)|$. Thus I dominates Δ. $\Box$

6.1. **Induction Step.** Our task is now to prove the induction step of Theorem 3.7. We first prove a weakening of the induction step, using the more basic Proposition 6.2.

Proposition 6.4. *Assume Theorem 3.7 holds for $n = m$. Let C be a general curve of genus $2a-2-m$ and gonality $a-m-1$ with pencil $f : C \to \boldsymbol{P}^1$ of degree $a-m-1$. Let (x_i, y_i) be general pairs of points in C for $i = 1, \ldots, m+1$. Then $[f, x_1, y_1, \ldots, x_{m+1}, y_{m+1}] \notin Z_{m+1}$. In particular, Z_{m+1} has codimension one in $\widetilde{\mathcal{M}}^{\mathrm{ns}}_{2a-1-(m+1)}\big(\boldsymbol{P}^1, a-(m+1); 2(m+1)\big)$.*

The conclusion of Proposition 6.4 make it possible to later apply Proposition 6.3 and thus finish the proof of Theorem 3.7.

Proof. Suppose $[f, x_1, y_1, \ldots, x_{m+1}, y_{m+1}] \in Z_{m+1}$. Let $B_{m+1} := C \cup R_{m+1}$ with $R_{m+1} \cong \mathbf{P}^1$ and $R_{m+1} \cap C = \{x_{m+1}, y_{m+1}\}$ and let $f_{m+1} : B_{m+1} \to \mathbf{P}^1$ be the stable map with $f_{m+1|C} = f$ and $\deg(f_{m+1|R_{m+1}}) = 1$. Our hypothesis implies

$$t = [f_{m+1}, x_1, y_1, \ldots, x_m, y_m] \in Z_m.$$

Let $J \subseteq Z_m$ be any irreducible component containing t. As $\dim J \geq \dim \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a} - 2m - 1$ but $\dim Z_{m+1} \leq \dim \widetilde{\mathcal{M}}^{\mathrm{ns}}_{2a-2-m}(\mathbf{P}^1, a-m-1; 2(m+1)) = \dim \widetilde{\mathcal{G}}^{\mathrm{ns}}_{2a-1,a} - 2m - 2$, the general point

$$[g : T \to \mathbf{P}^1, x'_1, y'_1, \ldots, x'_m, y'_m]$$

of J is a marked stable map with *irreducible* source curve T. Let $\widetilde{B}_{m+1}$ be the curve obtained from B_{m+1} by glueing x_i to y_i for $i = 1, \ldots, m$ and contracting the unstable component R_{m+1}. Then $\widetilde{B}_{m+1}$ has gonality a by Proposition 3.4. It follows that the curve $\widetilde{T}$ obtained from T by glueing x'_i to y'_i for $i = 1, \ldots, m$ has gonality at least a, in particular $\mathrm{gon}(T) = a - m$.

Let $S \subseteq \{x'_i, y'_i\}_{i=1}^m$ be a set of cardinality at most m. As explained previously, a Riemann–Roch calculation gives $h^0\big(C, f^* \mathcal{O}_{\mathbf{P}^1}(2)(\sum_{s \in S'} s)\big) = 3$ for any $S' \subseteq \{x_i, y_i\}_{i=1}^{m+1}$ of cardinality $|S'| \leq m+2$. Further, there is a unique admissible cover in $\pi^{-1}([\widetilde{B}_{m+1}])$, by Proposition 3.5. In the following two lemmas, we establish the following statements:

- $h^0\big(T, g^*(\mathcal{O}_{\mathbf{P}^1}(2))(\sum_{s \in S} s)\big) = 3$.
- $\pi^{-1}([\widetilde{T}])$ consists of a unique admissible cover.

All these facts yield a contradiction by the assumption that Theorem 3.7 holds for $n = m$. $\Box$

For the next two lemmas we fix the following notation. We let C be an integral, nodal curve of genus $2a-2-m$ and gonality $a-m-1$, together with a pencil $f : C \to \mathbf{P}^1$ of degree $a-m-1$. For $i = 1, \ldots, m+1$, let $x_i, y_i \in C_{\mathrm{reg}}$ be distinct points such that $f(x_i) \neq f(y_i)$. Let $B_{m+1} := C \cup R_{m+1}$ with $R_{m+1} \cong \mathbf{P}^1$ and $R_{m+1} \cap C = \{x_{m+1}, y_{m+1}\}$ and we denote by

$$f_{m+1} : B_{m+1} \to \mathbf{P}^1 \tag{7}$$

the map with $f_{m+1|C} = f$ and $\deg_{R_{m+1}}(f_{m+1}) = 1$. Finally, $\widetilde{B}_{m+1}$ is the curve obtained from B_{m+1} by glueing x_i to y_i for $i = 1, \ldots, m$ and contracting the unstable component R_{m+1}.

Lemma 6.5. *Let $(\Delta, 0)$ be an irreducible, pointed variety and let*

$$\phi : \mathcal{B} \to \boldsymbol{P}^1 \times \Delta, \ \ \sigma_i, \tau_i : \ \Delta \to \mathcal{B} \text{ for } \ i = 1, \ldots, m$$

be a family of marked stable maps over $(\Delta, 0)$ such that $\phi_0 = f_{m+1} : B_{m+1} \to \boldsymbol{P}^1$, $\sigma_i(0) = x_i$ and $\tau_i(0) = y_i$ for $i = 1, \ldots, m$. For a general point $t \in \Delta$, assume the fibre B_t is irreducible and let $S_t \subseteq \{\sigma_i(t), \tau_i(t)\}_{i=1}^m$ be a set of cardinality at most m.

(I) *If $h^0\big(C, f^*\mathcal{O}_{\boldsymbol{P}^1}(2)(\sum_{s \in S'} s)\big) = 3$ for any subset $S' \subseteq \{x_i, y_i\}_{i=1}^{m+1}$ of cardinality at most $m+2$, then $h^0\big(B_t, \phi_t^*\mathcal{O}_{\boldsymbol{P}^1}(2)(\sum_{s_t \in S_t} s_t)\big) = 3$.*

(II) *Assume instead $h^0\big(C, f^*\mathcal{O}_{\boldsymbol{P}^1}(2)(\sum_{s \in S'} s)\big) = 3$ for any subset $S' \subseteq \{x_i, y_i\}_{i=1}^{m+1}$ of cardinality at most $m+1$. If furthermore, B_t is irreducible for $t \in \Delta$ outside a set of codimension two and $\omega_C \otimes f^*\mathcal{O}_{\boldsymbol{P}^1}(-1)$ is base point free, then $h^0\big(B_t, \phi_t^*\mathcal{O}_{\boldsymbol{P}^1}(2)(\sum_{s_t \in S_t} s_t)\big) = 3$.*

Proof. We choose a subset of sections $\mathcal{S} \subseteq \{\sigma_i, \tau_i\}_{i=1}^m$ of cardinality at most m and set

$$\mathcal{M} := \phi^*\mathcal{O}_{\mathbf{P}^1 \times \Delta}(2)\Big(\sum_{s \in \mathcal{S}} s\Big) \in \mathrm{Pic}(\mathcal{B}).$$

We shall prove that $h^0(B_t, \mathcal{M}_t) = 3$, for a general $t \in \Delta$. There exists a birational map $\nu : \widetilde{\mathcal{B}} \to \mathcal{B}$ together with a line bundle $\tau \in \mathrm{Pic}(\widetilde{\mathcal{B}})$ enjoying the properties listed in Proposition 6.2. Furthermore, when Assumption (II) applies, then, by Proposition 6.3, we may ensure that τ is as in case (2) of Proposition 6.2. Set $\mathcal{L} := \nu^*\mathcal{M} \otimes \tau \in \mathrm{Pic}(\widetilde{\mathcal{B}})$. It suffices to show $h^0(\widetilde{B}, \mathcal{L}_0) = 3$, where $\widetilde{B}$ denotes the fibre of $\widetilde{\mathcal{B}}$ over the point $0 \in \Delta$.

Assume first we are in case (1) of Proposition 6.2, thus $\widetilde{B} = B_{m+1}$. Then $C \subseteq \widetilde{B}$ and $\mathcal{L}_C \cong f^*\mathcal{O}_{\mathbf{P}^1}(2)\big(\sum_{s \in S} s + x_{m+1} + y_{m+1}\big)$, where $S = \{s(0) : s \in \mathcal{S}\} \subseteq \{x_1, y_1, \ldots, x_m, y_m\}$. Furthermore, $\mathcal{L}_{R_{m+1}} \cong \mathcal{O}_{R_{m+1}}$ and the claim follows by twisting the exact sequence

$$0 \longrightarrow \mathcal{O}_{R_{m+1}}(-2) \longrightarrow \mathcal{O}_{B_{m+1}} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

by $\mathcal{L}$ and using our assumptions.

Assume now $\widetilde{B}$ is as in case (2) of Proposition 6.2. Then

$$\big(\deg \mathcal{L}_{R_{m+1}}, \deg \mathcal{L}_E\big) = (1, -1) \text{ and } \quad \mathcal{L}_C \cong f^*\mathcal{O}_{\mathbf{P}^1}(2)\Big(\sum_{s \in S} s + x_{m+1} + y_{m+1}\Big).$$

Set $B' := C \cup R_{m+1} \subseteq \widetilde{B}$. Via the exact sequence $0 \to \mathcal{O}_{B'}(-z-w) \to \mathcal{O}_{\widetilde{B}} \to \mathcal{O}_E \to 0$, where $\{w\} = E \cap R_{m+1}$ and $\{z\} = C \cap E$, it suffices to show $h^0(B', \mathcal{L}_{B'}(-z-w)) = 3$, following from

$$h^0(C, f^*\mathcal{O}_{\mathbf{P}^1}(2)(\sum_{s \in S} s + z')) = 3,$$

where $\{z'\} = C \cap R_{m+1} \subseteq \{x_{m+1}, y_{m+1}\}$, which is a consequence of our assumptions. $\square$

To finish the proof of Theorem 3.7, it suffices to establish the following.

Proposition 6.6. *Let $\big(\phi : \mathcal{B} \to \boldsymbol{P}^1 \times \Delta, \ \ \sigma_i, \tau_i : \ \Delta \to \boldsymbol{P}^1\big)$ be as in Lemma 6.5, with irreducible general fibre B_t. Let X_t be the curve obtained from B_t by identifying the points $\sigma_i(t)$ and $\tau_i(t)$ for $i = 1, \ldots, m$. Assume either assumption (I) or (II) holds as in Lemma 6.5 and that $\pi^{-1}([\widetilde{B}_{m+1}])$ consists of a single point. Then $\pi^{-1}([X_t])$ consists of a single point, for a general $t \in \Delta$.*

Proof. We describe the unique element $[f' : B' \to T] \in \pi^{-1}([\widetilde{B}_{m+1}])$ using [HM, Theorem 5]. Firstly, B' has a component $\widetilde{C}$ isomorphic to the normalization of C. All other components are isomorphic to $\mathbf{P}^1$. The restriction $f'_{\widetilde{C}}$ is the composition $\widetilde{f} = f \circ \mu$, of f with the normalization morphism $\mu : \widetilde{C} \to C$. The fact that $\pi^{-1}([\widetilde{B}_{m+1}])$ consists of a single element implies that the pencil $f : C \to \mathbf{P}^1$ is unramified at the points x_i, y_i for $i = 1, \ldots, m+1$. Moreover, $\widetilde{f}$ is unramified at all points above nodes of C. Set $\mathbf{P}^1_0 := f'(\widetilde{C})$. For each pair (x_i, y_i) the curve B' contains a rational chain with three components which meets $\widetilde{C}$ in x_i, y_i, with precisely one component dominating $\mathbf{P}^1_0$. The map f' has degree one on this component dominating $\mathbf{P}^1_0$ and degree two on the other two components (cf. Proposition 3.5). For each node $p \in C$ with points $q_1, q_2 \in \widetilde{C}$ over p, there is a single rational component $R_p \cong \mathbf{P}^1$ of B' which meets $\widetilde{C}$ at q_1, q_2, and which does *not* map to $\mathbf{P}^1_0$. The admissible cover f' has degree two on this component R_p. There are further rational tails attached to $\widetilde{C}$, none of which dominate $\mathbf{P}^1_0$.

Suppose $\pi^{-1}([X_t])$ consists of at least two elements. Thus, for general $t \in \Delta$ there exists an admissible cover $[f'_t : B'_t \to T_t]$ over X_t which, (i) specializes to the unique element of $\pi^{-1}([\widetilde{B}_{m+1}])$ as $t \to 0$ and (ii) it is distinct from the admissible cover constructed from $(\mu_t)_*\big(\phi_t^*\mathcal{O}_{\mathbf{P}^1}(1)\big)$ as in [HM, Theorem 5], where $\mu_t : B_t \to X_t$ denotes the partial normalization.

For general $t \in \Delta$, there exists a subcurve $B^n_t \subseteq B'_t$ isomorphic to the normalization of B_t. The limit of B^n_t at $t \to 0$ is a connected subcurve B^n_0 of B' containing $\widetilde{C}$. For each node $p \in C$, the component R_p of B' meeting $\widetilde{C}$ at q_1, q_2 either lies in B^n_0, or else, there is a smooth rational component R_{p_t} of B'_t meeting B^n_t at two distinct points $q_{1,t}$ and $q_{2,t}$ and with $f'_t(q_{1,t}) = f'_t(q_{2,t})$. As $t \to 0$, the curve $R_{p,t}$ approaches R_p and $q_{i,t}$ approaches q_i for $i = 1, 2$.

Let $\overline{B}^n_t$ be the union of B^n_t with all components $R_{p,t}$ as above and let D_t denote its stabilization. Define the finite stable map $\chi_t : D_t \to \mathbf{P}^1$ by first restricting f'_t to $\overline{B}^n_t$, then contracting the target T_t to $\mathbf{P}^1_0 := f'_t(B^n_t)$ and then lastly stabilizing the resulting morphism (this contracts each component $R_{p,t}$). As $t \to 0$, the map χ_t tends to the finite stable map

$$\chi_0 : D_0 \to \mathbf{P}^1,$$

where D_0 is a connected curve which is either:

(1) The blow-up of C at some set $S \subseteq \{x_i, y_i\}_{i=1}^{m+1}$ of cardinality at most $m+1$ containing at most one term from each pair (x_i, y_i) for all i, or
(2) $B_{m+1} = C \cup R_{m+1}$ blown-up at some set $S \subseteq \{x_i, y_i\}_{i=1}^{m}$ of cardinality at most m.

In the first case, D_t is isomorphic to the partial normalization $\mu_{q_t} : D_t \to B_t$ of B_t at one node q_t, with $\sigma_{m+1}(t)$ and $\tau_{m+1}(t)$ lying over this node. In the second case, $D_t \cong B_t$. In both cases we define C_2 to be the curve formed by contracting all exceptional components E_i of D_0. If $E_1, \ldots, E_\ell$ are the exceptional components of D_0, where $\ell \leq m+1$, then $\deg_{E_i}(\chi_0) = 1$ for $i = 1, \ldots, \ell$ and $\chi_{0|C} = f$. If, further, $R_{m+1} \subseteq D_0$ then $\deg_{R_{m+1}}(\chi_0) = 1$.

In the case $C_2 = B_{m+1}$, that is, when D_0 contains R_{m+1}, we must have $D_t = B_t$. In this situation, we set $Y_t := D_t$ and define the finite stable map $\epsilon_t := \chi_t : Y_t \to \mathbf{P}^1$, for all $t \in \Delta$. Notice that the stable map ϵ_t cannot be equal to ϕ_t, or else the admissible cover f'_t would coincide with the admissible cover constructed from $(\mu_t)_*(\phi_t^*\mathcal{O}_{\mathbf{P}^1}(1))$ as in [HM]. In the case $C_2 = C$, that is, when D_0 does not contain R_{m+1}, there exist distinct points $\sigma_{m+1}(t)$ and $\tau_{m+1}(t)$ in the smooth locus of D_t, together with a connected chain of rational curves in B'_t which meets B^n_t at $\sigma_{m+1}(t)$ and $\tau_{m+1}(t)$. There are now two cases which we must consider separately. First of all, assume $x_{m+1}, y_{m+1} \notin S$. Then, $(\sigma_{m+1}(t), \tau_{m+1}(t)) \to (x_{m+1}, y_{m+1})$ as $t \to 0$. Since f does not identify the points $x_{m+1}, y_{m+1} \in C$, the map $\chi_t : D_t \to \mathbf{P}^1$ also cannot identify $\sigma_{m+1}(t)$ and $\tau_{m+1}(t)$. Once again, we define $\epsilon_t := \chi_t : Y_t \to \mathbf{P}^1$ with $Y_t := D_t$. In the second case $C_2 = C$ and $x_{m+1} \in S$ or $y_{m+1} \in S$. In this case, χ_t identifies $\sigma_{m+1}(t)$ with $\tau_{m+1}(t)$. We set $Y_t := B_t$ and define

$$\epsilon_t : Y_t \to \mathbf{P}^1$$

for $t \neq 0$ as the unique map such that $\epsilon_t \circ \mu_{q_t} = \chi_t$. The limit of ϵ_t at $t \to 0$ is the stable map

$$\epsilon_0 : \ Y_0 \to \mathbf{P}^1,$$

where $Y_0 = D_0 \cup R_{m+1}$, $\deg_{R_{m+1}}(\epsilon_0) = 1$ and $\epsilon_{0|_{D_0}} = \chi_0$. Once again, $\epsilon_t \ncong \phi_t$.

We define C_3 as the curve obtained by contracting all exceptional components of Y_0. This curve coincides with B_{m+1} with the exception of the case where D_0 does not contain R_{m+1} and $x_{m+1}, y_{m+1} \notin S$, in which case it coincides with C. Let S_1 be the set of all points of intersection of $C_3 \subseteq Y_0$ with the exceptional divisors. This coincides with S, unless we are in the case where D_0 does not contain R_{m+1} and $x_{m+1} \in S$ or $y_{m+1} \in S$, in which case it is a subset of cardinality one less than S. In all cases, $|S_1| \leq m$ and $x_{m+1}, y_{m+1} \notin S_1$.

After a finite base change, we have a family of maps $\epsilon : \ \mathcal{Y} \to \mathbf{P}^1 \times \Delta$ with fibre $\epsilon_t : Y_t \to \mathbf{P}^1$ for all $t \in \Delta$. Let $\mu' : \mathcal{Y} \to \mathcal{Y}'$ be the morphism which contracts the unstable components $E_1, \ldots, E_\ell$ and is otherwise an isomorphism (this exists after a further base change). By Lemma 6.7 below applied to $\epsilon^* \mathcal{O}_{\mathbf{P}^1 \times \Delta}(1)$, there exists a line bundle

$$\mathcal{N} \in \mathrm{Pic}(\mathcal{Y}'),$$

with $\mathcal{N}_t \cong \epsilon_t^* \mathcal{O}_{\mathbf{P}^1}(1)$ for $t \neq 0$, and $\mathcal{N}_0 \cong f^*_{m+1|C_3} \mathcal{O}_{\mathbf{P}^1}(1)\big(\sum_{s \in S_1} s\big)$.

We now turn our attention back to the first family $\phi : \mathcal{B} \to \mathbf{P}^1 \times \Delta$. If $C_3 \cong C$, then for a general $t \in \Delta$ there exists a node q_t of B_t lying above $\sigma_{m+1}(t)$ and $\tau_{m+1}(t)$. In this case, possibly after base change, $\mathcal{B}$ is singular along a section $q : \Delta \to \mathcal{B}$. We let

$$\mu_q : \ \mathcal{B}' \to \mathcal{B}$$

be the partial normalization at q. In the case $C_3 \cong B_{m+1}$, we define $\mathcal{B}'$ to be $\mathcal{B}$. Let $\mathcal{B}^n$ be the surface obtained from $\mathcal{B}'$ by contracting any exceptional components. We now show that

$$\mathcal{B}^n \cong \mathcal{Y}'.$$

Firstly, after a finite base change $\mathcal{B}^{n*} := \mathcal{B}^n \times_\Delta \Delta^*$ is isomorphic to $\mathcal{Y}^* := \mathcal{Y}' \times_\Delta \Delta^*$ over $\Delta^* = \Delta \setminus \{0\}$, as both surfaces have the same fibres for $t \neq 0$ and these fibres are stable curves.

The claim is clear when $C_3 \cong C$, as both $\mathcal{B}^n$ and $\mathcal{Y}'$ are families of stable curves with stable central fibre. So assume $C_3 \cong B_{m+1}$ in which case $\mathcal{B}^n = \mathcal{B}$. For $\alpha \in \{0, 1, \infty\}$ may assume we have sections $s_\alpha : \Delta \to \mathcal{B}$ which specialize to general points of R_{m+1}. Further, by varying the natural $PGL(2)$ action, we may assume $\phi_t(s_\alpha) = \epsilon_t(s_\alpha) = \alpha \in \mathbf{P}^1$ for $t \neq 0$. Then $(s_\alpha)_{|\Delta^*}$ extends to a section $t_\alpha : \Delta \to \mathcal{Y}$. These three sections specialize to *distinct* points in R_{m+1} on the central fibre (as their images under f_{m+1} are distinct), and hence at least one section t_α must land in the smooth locus. Taking the image under $\mu' : \mathcal{Y} \to \mathcal{Y}'$, we may treat t_α as a section of $\mathcal{Y}'$. Thus the families $(\mathcal{B}, s_\alpha)$ and $(\mathcal{Y}', t_\alpha)$ of marked stable curves are isomorphic. In particular, $\mathcal{N}$ defines a line bundle on $\mathcal{B}^n$ in all cases.

In the case $C_3 \cong C$ the fibred surface $\mathcal{B}'$ has an unstable component. Let

$$\phi' := \phi \circ \mu_q : \mathcal{B}' \to \mathbf{P}^1 \times \Delta.$$

Applying Lemma 6.7, there exists a line bundle $\mathcal{G} \in \mathrm{Pic}(\mathcal{B}^n) \cong \mathrm{Pic}(\mathcal{Y}')$ with $\mathcal{G}_t \cong (\phi'_t)_* \mathcal{O}_{\mathbf{P}^1}(1)$ and with $\mathcal{G}_0 \cong f^* \mathcal{O}_{\mathbf{P}^1}(1)(z)$, where $z \in \{x_{m+1}, y_{m+1}\}$. By the base-point free pencil trick $h^0(\mathcal{N}_t \otimes \mathcal{G}_t) \geq 4$ for general $t \in \Delta$. But our hypotheses give $h^0(\mathcal{N}_0 \otimes \mathcal{G}_0) = 3$, which is a contradiction.

Now consider the case $C_3 \cong B_{m+1}$, so $\mathcal{B}_1^n = \mathcal{B}_1$. In the notation of the Proposition 6.2 (for Assumption (I)), or Proposition 6.3 (for Assumption (II)) consider

$$\mathcal{L} := \nu^* \big(\phi^* \mathcal{O}_{\mathbf{P}^1 \times \Delta}(1) \otimes \mathcal{N}\big) \otimes \tau \in \mathrm{Pic}(\widetilde{\mathcal{B}})$$

on a blow-up $\nu : \widetilde{\mathcal{B}} \to \mathcal{B}$. By the same argument as in Lemma 6.5, we have $h^0(B_0', \mathcal{L}_0) \leq 3$, which again gives a contradiction. $\square$

The following lemma was needed in the proof of Proposition 6.4. The proof is well-known.

Lemma 6.7. *Let $(\Delta, 0)$ be a smooth pointed curve and $\mathcal{C} \to \Delta$ a proper family of nodal curves with smooth fibers C_t for $t \neq 0$ and central fibre $C_0 = T \cup E$, where T is nodal and connected and $E \cong \boldsymbol{P}^1$, with $\{z\} = E \cap T$ a single point. Let $\mathcal{L} \in \mathrm{Pic}(\mathcal{C})$ be a line bundle with $\deg_E(\mathcal{L}) = 1$. If $\mu_E : \mathcal{C} \to \mathcal{C}'$ denotes the contraction morphism of E, then there exists a line bundle $\mathcal{L}'$ on $\mathcal{C}'$ with $\mathcal{L}_t' \cong \mathcal{L}_t$ for $t \neq 0$ and $\mathcal{L}_0' \cong \mathcal{L}_{0|T}(z)$.*

After a finite base change a contraction morphism as in the lemma above always exists. We now bootstrap on Proposition 6.4 in order to prove the induction step for Theorem 3.7.

Proof of Theorem 3.7. It remains to prove the induction step. Let C be an integral nodal curve of genus $2a-2-m$ with a unique pencil $f : C \to \mathbf{P}^1$ of degree $a-m-1$ and points $x_i, y_i \in C$ for $i = 1, \ldots, m+1$ as in the hypotheses of the theorem for $n = m+1$. Set $A := f^*\mathcal{O}_{\mathbf{P}^1}(1)$, hence $h^0(C, A) = 2$ and $\dim W^1_{a-m-1}(C) = 0$.

We claim $\omega_C \otimes A^\vee$ is base point free. Otherwise, there exists a point $p \in C$ such that all sections of $\omega_C \otimes A^\vee$ vanish at p. Let $N := \mathrm{Ker}\{\mathrm{ev}_p : \omega_C \otimes A^\vee \to \mathbb{C}_p\}$. There is a partial normalization $\mu : \widetilde{C} \to C$ at $\delta \geq 0$ nodes and a line bundle $\widetilde{N}$ on $\widetilde{C}$ such that $\mu_*\widetilde{N} = N$. Then $\omega_{\widetilde{C}} \otimes \widetilde{N}^\vee \in W^2_{a-m-\delta}(\widetilde{C})$, implying $\dim W^1_{a-m-1-\delta}(\widetilde{C}) \geq 1$, thus $\dim W^1_{a-m-1}(C) \geq 1$, a contradiction.

We now argue exactly as in Proposition 6.4. Suppose $[f, x_1, y_1, \ldots, x_{m+1}, y_{m+1}] \in Z_{m+1}$ which implies that $t = [f_{m+1}, x_1, y_1, \ldots, x_m, y_m] \in Z_m$. Let $J \subseteq Z_m$ be any component containing t. By the same dimension count as in Proposition 6.4, the general point $[g : T \to \mathbf{P}^1, (x_i', y_i')]$ of J is once again a marked stable map with *irreducible* source T. Moreover, from the conclusion of Proposition 6.4, we have the improved bound

$$\dim Z_{m+1} \leq \dim \widetilde{\mathcal{M}}^{\mathrm{ns}}_{2a-2-m}(\mathbf{P}^1, a-m-1; 2(m+1)) - 1,$$

so that the locus of reducible curves in J has codimension two about t. By Lemma 6.5 and Lemma 6.6, using Assumption (II) and applied to $\Delta = J$, we immediately derive a contradiction to the induction hypothesis. $\square$

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