# RATIONAL MAPS BETWEEN MODULI SPACES OF CURVES AND GIESEKER-PETRI DIVISORS

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For a a general smooth projective curve $[C] \in \mathcal{M}_g$ and an arbitrary line bundle $L \in \mathrm{Pic}(C)$, the *Gieseker-Petri* theorem states that the multiplication map

$$\mu_0(L) : H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee) \to H^0(C, K_C)$$

is injective. The theorem, conjectured by Petri and proved by Gieseker [G] (see [EH3] for a much simplified proof), lies at the cornerstone of the theory of algebraic curves. It implies that the variety $G^r_d(C) = \{(L, V) : L \in \mathrm{Pic}^d(C), V \in G(r+1, H^0(L))\}$ of linear series of degree $d$ and dimension $r$ is smooth and of expected dimension $\rho(g, r, d) := g - (r+1)(g-d+r)$ and that the forgetful map $G^r_d(C) \to W^r_d(C)$ is a rational resolution of singularities (see [ACGH] for many other applications). It is an old open problem to describe the locus $\mathcal{GP}_g \subset \mathcal{M}_g$ consisting of curves $[C] \in \mathcal{M}_g$ such that there exists a line bundle $L$ on $C$ for which the Gieseker-Petri theorem fails. Obviously $\mathcal{GP}_g$ breaks up into irreducible components depending on the numerical types of linear series. For fixed integers $d, r \geq 1$ such that $g - d + r \geq 2$, we define the locus $\mathcal{GP}^r_{g,d}$ consisting of curves $[C] \in \mathcal{M}_g$ such that there exist a pair of linear series $(L, V) \in G^r_d(C)$ and $(K_C \otimes L^\vee, W) \in G^{g-d+r-1}_{2g-2-d}(C)$ for which the multiplication map

$$\mu_0(V, W) : V \otimes W \to H^0(C, K_C)$$

is not injective. Even though certain components of $\mathcal{GP}_g$ are well-understood, its global geometry seems exceedingly complicated. If $\rho(g, r, d) = -1$, then $\mathcal{GP}^r_{g,d}$ coincides with the Brill-Noether divisor $\mathcal{M}^r_{g,d}$ of curves $[C] \in \mathcal{M}_g$ with $G^r_d(C) \neq \emptyset$ which has been studied by Eisenbud and Harris in [EH2] and used to prove that $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$. The locus $\mathcal{GP}^1_{g,g-1}$ can be identified with the divisor of curves carrying a vanishing theta-null and this has been studied by Teixidor (cf. [T]). We proved in [F2] that for $r = 1$ and $(g+2)/2 \leq d \leq g-1$, the locus $\mathcal{GP}^1_{g,d}$ always carries a divisorial component. It is conjectured that the locus $\mathcal{GP}_g$ is pure of codimension 1 in $\mathcal{M}_g$ and we go some way towards proving this conjecture. Precisely, we show that $\mathcal{GP}_g$ is supported in codimension 1 for every possible numerical type of a linear series:

**Theorem 0.1.** *For any positive integers $g, d$ and $r$ such that $\rho(g, r, d) \geq 0$ and $g - d + r \geq 2$, the locus $\mathcal{GP}^r_{g,d}$ has a divisorial component in $\mathcal{M}_g$.*

The main issue we address in this paper is a detailed intersection theoretic study of a rational map between two different moduli spaces of curves. We fix $g := 2s+1 \geq 3$.

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Since $\rho(2s+1,1,s+2)=1$ we can define a rational map between moduli spaces of curves

$$\phi:\overline{\mathcal{M}}_{2s+1}--> \overline{\mathcal{M}}_{1+\frac{s}{s+1}\binom{2s+2}{s}},\quad \phi([C]):=[W^1_{s+2}(C)].$$

The fact that $\phi$ is well-defined, as well as a justification for the formula of the genus $g':=g(W^1_{s+2}(C))$ of the curve of special divisors of type $\mathfrak{g}^1_{s+2}$, is given in Section 3. It is known that $\phi$ is generically injective (cf. [PT], [CHT]). Since $\phi$ is the only-known rational map between two moduli spaces of curves and one of the very few natural examples of a rational map admitted by $\overline{\mathcal{M}}_g$, its study is clearly of independent interest. In this paper we carry out a detailed enumerative study of $\phi$ and among other things, we determine the pull-back map $\phi^*:\mathrm{Pic}(\overline{\mathcal{M}}_{g'})\rightarrow \mathrm{Pic}(\overline{\mathcal{M}}_g)$ (see Theorem 3.4 for a precise statement). In particular we have the following formula concerning slopes of divisor classes pulled back from $\overline{\mathcal{M}}_{g'}$ (For the definition of the slope function $s:\mathrm{Eff}(\overline{\mathcal{M}}_g)\rightarrow \mathbb{R}\cup\{\infty\}$ on the cone of effective divisors we refer to [HMo] or [FP]):

**Theorem 0.2.** *We set $g:=2s+1$ and $g':=1+\frac{s}{s+1}\binom{2s+2}{s}$. For any divisor class $D\in \mathrm{Pic}(\overline{\mathcal{M}}_{g'})$ having slope $s(D)=c$, we have the following formula for the slope of $\phi^*(D)\in \mathrm{Pic}(\overline{\mathcal{M}}_g)$:*

$$s(\phi^*(D))=6+\frac{8s^3(c-4)+5cs^2-30s^2+20s-8cs-2c+24}{s(s+2)(cs^2-4s^2-c-s+6)}\ .$$

We use this formula to describe the cone $\mathrm{Mov}(\overline{\mathcal{M}}_g)$ of *moving divisors*[1] inside the cone $\mathrm{Eff}(\overline{\mathcal{M}}_g)$ of effective divisors. The cone $\mathrm{Mov}(\overline{\mathcal{M}}_g)$ parameterizes rational maps from $\overline{\mathcal{M}}_g$ in the projective category while the cone $\mathrm{Nef}(\overline{\mathcal{M}}_g)$ of numerically effective divisors, parameterizes regular maps from $\overline{\mathcal{M}}_g$ (see [HK] for details on this perspective). A fundamental question is to estimate the following slope invariants associated to $\overline{\mathcal{M}}_g$:

$$s(\overline{\mathcal{M}}_g):=\inf_{D\in \mathrm{Eff}(\overline{\mathcal{M}}_g)} s(D)\ \text{ and }\ s'(\overline{\mathcal{M}}_g):=\inf_{D\in \mathrm{Mov}(\overline{\mathcal{M}}_g)} s(D).$$

The formula of the class of Brill-Noether divisors $\overline{\mathcal{M}}^r_{g,d}$ when $\rho(g,r,d)=-1$ shows that $\lim_{g\rightarrow\infty} s(\overline{\mathcal{M}}_g)\leq 6$ (cf. [EH2]). In [F1] we provided an infinite sequence of genera of the form $g=a(2a+1)$ with $a\geq 2$ for which $s(\overline{\mathcal{M}}_g)<6+12/(g+1)$, thus contradicting the Slope Conjecture [HMo]. There is no known example of a genus $g$ such that $s(\overline{\mathcal{M}}_g)<6$.

Understanding the difference between $s(\overline{\mathcal{M}}_g)$ and $s'(\overline{\mathcal{M}}_g)$ is a subtle question even for low $g$. There is a strict inequality $s(\overline{\mathcal{M}}_g)<s'(\overline{\mathcal{M}}_g)$ whenever one can find an effective divisor $D\in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ with $s(D)=s(\overline{\mathcal{M}}_g)$, such that there exists a covering curve $R\subset D$ for which $R\cdot D<0$. For $g<12$ the divisors minimizing the slope function have a strong geometric characterization in terms of Brill-Noether theory. Thus computing $s'(\overline{\mathcal{M}}_g)$ becomes a problem in understanding the geometry of Brill-Noether and Gieseker-Petri divisors on $\overline{\mathcal{M}}_g$. To illustrate this point we give two examples (see Section 5 for details): It is known that $s(\overline{\mathcal{M}}_3)=9$ and the minimum slope is realized by the locus of hyperelliptic curves $\overline{\mathcal{M}}^1_{3,2}\equiv 9\lambda-\delta_0-3\delta_1$. However $[\overline{\mathcal{M}}^1_{3,2}]\notin \mathrm{Mov}(\overline{\mathcal{M}}_3)$, because $\overline{\mathcal{M}}^1_{3,2}$ is swept out by pencils $R\subset\overline{\mathcal{M}}_3$ with $R\cdot\delta/R\cdot\lambda=28/3>s(\overline{\mathcal{M}}^1_{3,2})$.

[1] Recall that an effective $\mathbb{Q}$-Cartier divisor $D$ on a normal projective variety $X$ is said to be moving, if the stable base locus $\bigcap_{n\geq 1}\mathrm{Bs}|\mathcal{O}_X(nD)|$ has codimension at least 2 in $X$.

In fact, one has equality $s'(\overline{\mathcal{M}}_3) = 28/3$ and the moving divisor on $\overline{\mathcal{M}}_3$ attaining this bound corresponds to the pull-back of an ample class under the rational map

$$\overline{\mathcal{M}}_3 - - > \mathcal{Q}_4 := |\mathcal{O}_{\mathbf{P}^2}(4)| \, //SL(3)$$

to the GIT quotient of plane quartics which contracts $\overline{\mathcal{M}}_{3,2}^1$ to a point (see [HL] for details on the role of this map in carrying out the Minimal Model Program for $\overline{\mathcal{M}}_3$).

For $g = 10$, it is known that $s(\overline{\mathcal{M}}_{10}) = 7$ and this bound is attained by the divisor $\overline{\mathcal{K}}_{10}$ of curves lying on $K3$ surfaces (cf. [FP] Theorems 1.6 and 1.7 for details):

$$\overline{\mathcal{K}}_{10} \equiv 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - 15\delta_5.$$

Furthermore, $\overline{\mathcal{K}}_{10}$ is swept out by pencils $R \subset \overline{\mathcal{M}}_{10}$ with $R \cdot \delta / R \cdot \lambda = 78/11 > s(\overline{\mathcal{K}}_{10})$ (cf. [FP] Proposition 2.2). Therefore $[\overline{\mathcal{K}}_{10}] \notin \mathrm{Mov}(\overline{\mathcal{M}}_{10})$ and $s'(\overline{\mathcal{M}}_{10}) \geq 78/11$.

For $g = 2s+1$ using the elementary observation that $\phi^*(\mathrm{Ample}(\overline{\mathcal{M}}_{g'})) \subset \mathrm{Mov}(\overline{\mathcal{M}}_g)$, Theorem 0.2 provides a uniform upper bound on slopes of moving divisors on $\overline{\mathcal{M}}_g$:

**Corollary 0.3.** *We set $g := 2s+1$ as and $g' := 1 + \frac{s}{s+1}\binom{2s+2}{s}$ as above. Then*

$$s(\phi^*(D)) < 6 + \frac{16}{g-1} \text{ for every divisor } D \in \mathrm{Ample}(\overline{\mathcal{M}}_{g'}).$$

*In particular one has the estimate $s'(\overline{\mathcal{M}}_g) < 6 + 16/(g-1)$, for every odd integer $g \geq 3$.*

Since we also know that $\lim_{g\to\infty} s(\overline{\mathcal{M}}_g) \leq 6$, Corollary 0.3 indicates that (at least asymptotically, for large $g$) we cannot distinguish between effective and moving divisors on $\overline{\mathcal{M}}_g$ . We ask whether it is true that $\lim_{g\to\infty} s(\overline{\mathcal{M}}_g) = \lim_{g\to\infty} s'(\overline{\mathcal{M}}_g)$?

At the heart of the description in codimension 1 of the map $\phi : \overline{\mathcal{M}}_g - - > \overline{\mathcal{M}}_{g'}$ lies the computation of the cohomology class of the compactified Gieseker-Petri divisor $\overline{\mathcal{GP}}^r_{g,d} \subset \overline{\mathcal{M}}_g$ in the case when $\rho(g,r,d) = 1$. Since this calculation is of independent interest we discuss it in some detail. We denote by $\mathfrak{G}^r_d$ the stack parameterizing pairs $[C, l]$ with $[C] \in \mathcal{M}_g$ and $l = (L, V) \in G^r_d(C)$ and denote by $\sigma : \mathfrak{G}^r_d \to \mathcal{M}_g$ the natural projection. In [F1] we computed the class of $\overline{\mathcal{GP}}^r_{g,d}$ in the case $\rho(g,r,d) = 0$, when $\overline{\mathcal{GP}}^r_{g,d}$ can be realized as the push-forward of a determinantal divisor on $\mathfrak{G}^r_d$ under the generically finite map $\sigma$. In particular, we showed that if we write $g = rs + s$ and $d = rs + r$ where $r \geq 1$ and $s \geq 2$ (hence $\rho(g,r,d) = 0$), then we have the following formula for the slope of $\overline{\mathcal{GP}}^r_{g,d}$ (cf. [F1], Theorem 1.6):

$$s(\overline{\mathcal{GP}}^r_{g,d}) = 6 + \frac{12}{g+1} + \frac{6(s+r+1)(rs+s-2)(rs+s-1)}{s(s+1)(r+1)(r+2)(rs+s+4)(rs+s+1)}.$$

The number $6 + 12/(g+1)$ is the slope of all Brill-Noether divisors on $\overline{\mathcal{M}}_g$, that is $s(\overline{\mathcal{GP}}^r_{g,d}) = 6 + 12/(g+1)$ whenever $\rho(g,r,d) = -1$ (cf. [EH2], or [F1] Corollary 1.2 for a different proof, making use of M. Green's Conjecture on syzygies of canonical curves).

In the technically much-more intricate case $\rho(g,r,d) = 1$, we can realize $\mathcal{GP}^r_{g,d}$ as the push-forward of a codimension 2 determinantal subvariety of $\mathfrak{G}^r_d$ and most of Section 2 is devoted to extending this structure over a partial compactification of $\overline{\mathcal{M}}_g$ corresponding to tree-like curves. If $\sigma : \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathcal{M}}_g$ denotes the stack of limit linear series

$\mathfrak{g}^r_d$, we construct two *locally free* sheaves $\mathcal{F}$ and $\mathcal{N}$ over $\widetilde{\mathfrak{G}}^r_d$ such that $\text{rank}(\mathcal{F}) = r+1$, $\text{rank}(\mathcal{N}) = g-d+r =: s$ respectively, together with a vector bundle morphism

$$\mu : \mathcal{F} \otimes \mathcal{N} \to \sigma^*\big(\mathbb{E} \otimes \mathcal{O}_{\overline{\mathcal{M}}_g}(\sum_{j=1}^{[g/2]} (2j-1) \cdot \delta_j)\big)$$

such that $\overline{\mathcal{GP}}^r_{g,d}$ is the push-forward of the first degeneration locus of $\mu$:

**Theorem 0.4.** *We fix integers $r, s \geq 1$ and we set $g := rs+s+1, d := rs+r+1$ so that $\rho(g,r,d) = 1$. Then the class of the compactified Gieseker-Petri divisor $\overline{\mathcal{GP}}^r_{g,d}$ in $\overline{\mathcal{M}}_g$ is given by the formula:*

$$\overline{\mathcal{GP}}^r_{g,d} \equiv \frac{C_{r+1}\ (s-1)r}{2(r+s+1)(s+r)(r+s+2)(rs+s-1)}\big(a\lambda - b_0\delta_0 - b_1\delta_1 - \sum_{j=2}^{[g/2]} b_j\delta_j\big),$$

*where*

$$C_{r+1} := \frac{(rs+s)!\ r!\ (r-1)! \cdots 2!\ 1!}{(s+r)!\ (s+r-1)! \cdots (s+1)!\ s!}$$

$$a = 2s^3(s+1)r^5 + s^2(2s^3+14s^2+33s+25)r^4 + s(10s^4+59s^3+162s^2+179s+54)r^3 + \\ +(18s^5+138s^4+387s^3+491s^2+244s+24)r^2 + (14s^5+145s^4+464s^3+627s^2+378s+72)r + \\ 4s^5+54s^4+208s^3+314s^2+212s+48$$

$$b_0 := \frac{(r+2)(s+1)(s+r+1)(2rs+2s+1)(rs+s+2)(rs+s+6)}{6}$$

$$b_1 := (r+1)s\Big(2s^2(s+1)r^4 + s(2s^3+12s^2+23s+9s)r^3 + (8s^4+39s^3+75s^2+46s+10)r^2 + \\ +(10s^4+59s^3+108s^2+89s+26)r + 4s^4+30s^3+64s^2+58s+12\Big),$$

*and $b_j \geq b_1$ for $j \geq 2$ are explicitly determined constants.*

Even though the coefficients $a$ and $b_1$ look rather unwieldy, the expression for the slope of $\overline{\mathcal{GP}}^r_{g,d}$ has a simpler and much more suggestive expression which we record:

**Corollary 0.5.** *For $\rho(g,r,d) = 1$, the slope of the Gieseker-Petri divisor $\overline{\mathcal{GP}}^r_{g,d}$ has the following expression:*

$$s(\overline{\mathcal{GP}}^r_{g,d}) = 6 + \frac{12}{g+1} + \frac{24\ s(r+1)(r+s)(s+r+2)(rs+s-1)}{(r+2)(s+1)(s+r+1)(2rs+2s+1)(rs+s+2)(rs+s+6)}.$$

Next we specialize to the case $r = 1$, thus $g = 2s+1$. Using the base point free pencil trick one can see that the divisor $\mathcal{GP}^1_{2s+1,s+2}$ splits into two irreducible components according to whether the pencil for which the Gieseker-Petri theorem fails has a base point or not. Precisely we have the following equality of codimension 1 cycles

$$\overline{\mathcal{GP}}^1_{2s+1,s+2} = (2s-2) \cdot \overline{\mathcal{M}}^1_{2s+1,s+1} + \overline{\mathcal{GP}}^{1,0}_{2s+1,s+2},$$

where $\overline{\mathcal{GP}}^{1,0}_{2s+1,s+2}$ is the closure of the locus of curves $[C] \in \mathcal{M}_g$ carrying a base point free pencil $L \in W^1_{s+2}(C)$ such that $\mu_0(L)$ is not injective. Since we also have the well-known formula for the class of the Hurwitz divisor (cf. [EH2], Theorem 1)

$$\overline{\mathcal{M}}^1_{2s+1,s+1} \equiv \frac{(2s-2)!}{(s+1)!\ (s-1)!}\Big(6(s+2)\lambda - (s+1)\delta_0 - 6s\delta_1 - \cdots\Big),$$

we find the following expression for the slope of $\overline{\mathcal{GP}}^{1,0}_{2s+1,s+2}$:

**Corollary 0.6.** *For $g = 2s+1$, the slope of the divisor $\overline{\mathcal{GP}}^{1,0}_{2s+1,s+2}$ of curves carrying a base point free pencil $L \in W^1_{s+2}(C)$ such that $\mu_0(L)$ is not injective, is given by the formula*

$$s(\overline{\mathcal{GP}}^{1,0}_{2s+1,s+2}) = 6 + \frac{12}{g+1} + \frac{2s-1}{(s+1)(s+2)}.$$

We note that for $s = 2$ and $g = 5$, the divisor $\overline{\mathcal{GP}}^{1,0}_{5,4}$ is equal to Teixidor's divisor of curves $[C] \in \mathcal{M}_5$ having a vanishing theta-null, that is, a theta-characteristic $L^{\otimes 2} = K_C$ with $h^0(C, L) \geq 2$. In this case Corollary 0.6 specializes to her formula [T] Theorem 3.1:

$$\overline{\mathcal{GP}}^{1,0}_{5,4} \equiv 4 \cdot (33\lambda - 4\delta_0 - 15\delta_1 - 21\delta_2) \in \mathrm{Pic}(\overline{\mathcal{M}}_5).$$

To give another example we specialize to the case $r = 1$, $s = 3$ when $g = 7$. Using the base point free pencil trick, the divisor $\overline{\mathcal{GP}}^1_{7,5}$ can be identified with the closure of the locus of curves $[C] \in \mathcal{M}_7$ possessing a linear series $l \in G^2_7(C)$ such that the plane model $C \xrightarrow{l} \mathbf{P}^2$ has 8 nodes, of which 7 lie on a conic. Its class is given by the formula:

$$\overline{\mathcal{GP}}^1_{7,5} \equiv 4 \cdot (201\lambda - 26\delta_0 - 111\delta_1 - 177\delta_2 - 198\delta_3) \in \mathrm{Pic}(\overline{\mathcal{M}}_7).$$

In Section 5 we shall need a characterization of the $k$-gonal loci $\overline{\mathcal{M}}^1_{g,k}$ in terms of effective divisors of $\overline{\mathcal{M}}_g$ containing them. For instance, it is known that if $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ is a divisor such that $s(D) < 8 + 4/g$, then $D$ contains the hyperelliptic locus $\overline{\mathcal{M}}^1_{g,2}$ (see e.g. [HMo], Corollary 3.30). Similar bounds exist for the trigonal locus: if $s(D) < 7+6/g$ then $D \supset \overline{\mathcal{M}}^1_{g,3}$. We have the following extension of this type of result:

**Theorem 0.7.** *1) Every effective divisor $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ having slope $s(D) < \frac{1}{g}\left[\frac{13g+16}{2}\right]$ contains the locus $\overline{\mathcal{M}}^1_{g,4}$ of 4-gonal curves.*

*2) Every effective divisor $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ having slope $s(D) < \frac{1}{g}\Big(5g + 9 + 2[\frac{g+1}{2}]\Big)$ contains the locus $\overline{\mathcal{M}}^1_{g,5}$ of 5-gonal curves.*

The proof uses an explicit unirational parametrization of $\overline{\mathcal{M}}^1_{g,k}$ that is available only when $k \leq 5$. It is natural to ask whether the subvariety $\overline{\mathcal{M}}^1_{g,k} \subset \overline{\mathcal{M}}_g$ is cut out by divisors $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ of slope less than the bound given in Theorem 0.7. Very little seems to be known about this question even in the hyperelliptic case.

We close by summarizing the structure of the paper. In Section 1 we introduce a certain stack of pairs of complementary limit linear series which we then use to prove Theorem 0.1 by induction on the genus. The class of the compactified Gieseker-Petri divisor is computed in Section 2. This calculation is used in Section 3 to describe maps between moduli spaces of curves. We then study the geometry of $\phi$ in low genus (Section 4) with applications to Prym varieties and we finish the paper by computing the invariant $s'(\overline{\mathcal{M}}_g)$ for $g \leq 11$ (Section 5).

## 1. Divisorial components of the Gieseker-Petri locus

Let us fix positive integers $g, r$ and $g$ such that $\rho(g,r,d)\geq 0$ and set $s:=g-d+r\geq 2$, hence $g=rs+s+j$ and $d=rs+r+j$, with $j\geq 0$. The case $j=0$ corresponds to the situation $\rho(g,r,d)=0$ when we already know that $\mathcal{GP}^r_{g,d}$ has a divisorial component in $\mathcal{M}_g$ whose class has been computed (see [F1], Theorem 1.6). We present an inductive method on $j$ which produces a divisorial component of $\mathcal{GP}^r_{g,d}\subset \mathcal{M}_g$ provided one knows that $\mathcal{GP}^r_{g-1,d-1}$ has a divisorial component in $\mathcal{M}_{g-1}$. The method is based on degeneration to the boundary divisor $\Delta_1\subset \overline{\mathcal{M}}_g$ and is somewhat similar to the one used in [F2] for the case $r=1$.

We briefly recall a few facts about (degeneration of) multiplication maps on curves. If $L$ and $M$ are line bundles on a smooth curve $C$, we denote by

$$\mu_0(L,M): H^0(L)\otimes H^0(M)\rightarrow H^0(L\otimes M)$$

the usual multiplication map and by

$$\mu_1(L,M): \mathrm{Ker}\ \mu_0(L,M)\rightarrow H^0(K_C\otimes L\otimes M),\quad \mu_1(\sum_i \sigma_i\otimes \tau_i):=\sum_i (d\sigma_i)\cdot \tau_i,$$

the first Gaussian map associated to $L$ and $M$ (see [W]). For any $\rho\in H^0(L)\otimes H^0(M)$ and a point $p\in C$, we write that $\mathrm{ord}_p(\rho)\geq k$, if $\rho$ lies in the span of elements of the form $\sigma\otimes\tau$, where $\sigma\in H^0(L)$ and $\tau\in H^0(M)$ are such that $\mathrm{ord}_p(\sigma)+\mathrm{ord}_p(\tau)\geq k$. When $i=0,1$, the condition $\mathrm{ord}_p(\rho)\geq i+1$ for a generic point $p\in C$, is clearly equivalent to $\rho\in \mathrm{Ker}\ \mu_i(L,M)$.

If $X$ is a tree-like curve and $l$ is a limit $\mathfrak{g}^r_d$ on $X$, for an irreducible component $Y\subset X$ we denote by $l_Y=(L_Y, V_Y\subset H^0(L_Y))$ the $Y$-aspect of $l$. For $p\in Y$ we denote by $\{a_i^{l_Y}(p)\}_{i=0\ldots r}$ the *vanishing sequence* of $l$ at $p$ and by $\rho(l_Y,p):=\rho(g(Y),r,d)-\sum_{i=0}^r(a_i^{l_Y}(p)-i)$ the *adjusted Brill-Noether number* with respect to the point $p$ (see [EH1] for a general reference on limit linear series).

We shall repeatedly use the following elementary observation already made in [EH3] and used in [F2]: Suppose $\{\sigma_i\}\subset H^0(L)$ and $\{\tau_j\}\subset H^0(M)$ are bases of global sections with the property that $\mathrm{ord}_p(\sigma_i)=a_i^L(p)$ and $\mathrm{ord}_p(\tau_j)=a_j^M(p)$ for all $i$ and $j$. Then if $\rho\in \mathrm{Ker}\ \mu_0(L,M))$, there must exist two pairs of integers $(i_1,j_1)\neq (i_2,j_2)$ such that $\mathrm{ord}_p(\rho)=\mathrm{ord}_p(\sigma_{i_1})+\mathrm{ord}_p(\tau_{j_1})=\mathrm{ord}_p(\sigma_{i_2})+\mathrm{ord}_p(\tau_{j_2})$.

A technical tool in the paper is the stack $\nu:\widetilde{\mathcal{U}}^r_{g,d}\rightarrow \widetilde{\mathcal{M}}_g$ of pairs of complementary limit linear series defined over a partial compactification of $\mathcal{M}_g$ which will be defined below. Then $\mathcal{GP}^r_{g,d}$ is the push-forward under $\nu_{|\nu^{-1}(\mathcal{M}_g)}$ of a degeneration locus inside $\widetilde{\mathcal{U}}^r_{g,d}$. We denote by $\mathfrak{Pic}^d$ the degree $d$ Picard stack over $\mathcal{M}_g$, that is, the étale sheafification of the Picard functor, and by $\mathbb{E}$ the Hodge bundle over $\overline{\mathcal{M}}_g$. We consider $\mathfrak{G}^r_d\subset \mathfrak{Pic}^d$ to be the stack parameterizing pairs $[C,l]$ with $l=(L,V)\in G^r_d(C)$ and the projection $\sigma:\mathfrak{G}^r_d\rightarrow \mathcal{M}_g$.

We set $\Delta_0^0\subset\Delta_0\subset\overline{\mathcal{M}}_g$ to be the locus of curves $[C/y\sim q]$, where $[C,q]\in\mathcal{M}_{g-1,1}$ is Brill-Noether general and $y\in C$ is an arbitrary point, as well as their degenerations $[C\cup_q E_\infty]$, where $E_\infty$ is a rational nodal curve, that is, $j(E_\infty)=\infty$. For $1\leq i\leq [g/2]$, we denote by $\Delta_i^0 \subset \Delta_i$ the open subset consisting of unions $[C \cup_y D]$, where $[C] \in \mathcal{M}_i$ and $[D, y] \in \mathcal{M}_{g-i,1}$ are Brill-Noether general curves but the point $y \in C$ is arbitrary. Then if we denote by $\widetilde{\mathcal{M}}_g := \mathcal{M}_g \cup \big(\cup_{i=0}^{[g/2]} \Delta_i^0\big)$, one can extend the covering $\sigma : \mathfrak{G}_d^r \to \mathcal{M}_g$ to a proper map $\sigma : \widetilde{\mathfrak{G}}_d^r \to \widetilde{\mathcal{M}}_g$ from the stack $\widetilde{\mathfrak{G}}_d^r$ of limit linear series $\mathfrak{g}_d^r$.

We now introduce the stack $\nu : \widetilde{\mathcal{U}}_{g,d}^r \to \widetilde{\mathcal{M}}_g$ of complementary linear series: For $[C] \in \mathcal{M}_g$, the fibre $\nu^{-1}[C]$ parameterizes pairs $(l, m)$ where $l = (L, V) \in G_d^r(C)$ and $m = (K_C \otimes L^\vee, W) \in G_{2g-2-d}^{g-d+r-1}(C)$. If $[C = C_1 \cup_y C_2] \in \widetilde{\mathcal{M}}_g$, where $[C_1, y] \in \mathcal{M}_{i,1}$ and $[C_2, y] \in \mathcal{M}_{g-i,1}$, the fibre $\nu^{-1}[C]$ consists of pairs of limit linear series $(l, m)$, where $l = \{(L_{C_1}, V_{C_1}), (L_{C_2}, V_{C_2})\}$ is a limit $\mathfrak{g}_d^r$ on $C$ and

$$m = \{\big(K_{C_1} \otimes \mathcal{O}_{C_1}(2(g-i)\cdot p) \otimes L_{C_1}^{-1}, W_{C_1}\big), \big(K_{C_2} \otimes \mathcal{O}_{C_2}(2i \cdot p) \otimes L_{C_2}^{-1}, W_{C_2}\big)\}$$

is a limit $\mathfrak{g}_{2g-2-d}^{g-d+r-1}$ on $C$ which is complementary to $l$. There is a morphism of stacks $\epsilon : \widetilde{\mathcal{U}}_{g,d}^r \to \widetilde{\mathfrak{G}}_{g,d}^r$ which forgets the limit $\mathfrak{g}_{2g-2-d}^{g-d+r-1}$ on each curve. Clearly $\sigma \circ \epsilon = \nu$.

**Definition 1.1.** For a smooth curve $C$ of genus $g$, a Gieseker-Petri $(\mathfrak{gp})_d^r$*-relation* consists of a pair of linear series $(L, V) \in G_d^r(C)$ and $(K_C \otimes L^\vee, W) \in G_{2g-2-d}^{g-d+r-1}(C)$, together with an element $\rho \in \mathbf{P}\mathrm{Ker}\{\mu_0(V, W) : V \otimes W \to H^0(K_C)\}$.

If $C = C_1 \cup_p C_2$ is a curve of compact type with $C_1$ and $C_2$ being smooth curves with $g(C_1) = i$ and $g(C_2) = g - i$ respectively, a $(\mathfrak{gp})_d^r$*-relation* on $C$ is a collection $(l, m, \rho_1, \rho_2)$, where $[C, l, m] \in \widetilde{\mathcal{U}}_{g,d}^r$, and elements

$\rho_1 \in \mathbf{P}\mathrm{Ker}\{V_{C_1} \otimes W_{C_1} \to H^0\big(K_{C_1}(2(g-i)p)\big)\}, \rho_2 \in \mathbf{P}\mathrm{Ker}\{V_{C_2} \otimes W_{C_2} \to H^0\big(K_{C_2}(2ip)\big)\}$

satisfying the compatibility relation $\mathrm{ord}_p(\rho_1) + \mathrm{ord}_p(\rho_2) \geq 2g - 2$.

For every curve $C$ of compact type, the variety $\mathcal{Q}_d^r(C)$ of $(\mathfrak{gp})_d^r$-relations has an obvious determinantal scheme structure. One can construct a moduli stack of $(\mathfrak{gp})_d^r$-relations which has a natural determinantal structure over the moduli stack of curves of compact type. In particular one has a lower bound on the dimension of each irreducible component of this space and we shall use this feature in order to smooth $(\mathfrak{gp})_d^r$-relations constructed over curves from the divisor $\Delta_1$ to nearby smooth curves from $\mathcal{M}_g$. The proof of the following theorem is very similar to the proof of Theorem 4.3 in [F2] which dealt with the case $r = 1$. We omit the details.

**Theorem 1.2.** *We fix integers $g, r, d$ such that $\rho(g, r, d) \geq 0$ and a curve $[C := C_1 \cup_y C_2] \in \overline{\mathcal{M}}_g$ of compact type. We denote by $\pi : \mathcal{C} \to B$ the versal deformation space of $C = \pi^{-1}(0)$, with $0 \in B$. Then there exists a quasi-projective variety $\nu : \mathcal{Q}_d^r \to B$, compatible with base change, such that the fibre over each point $b \in B$ parameterizes $(\mathfrak{gp})_d^r$-relations over $C_b$. Moreover, each irreducible component of $\mathcal{Q}_d^r$ has dimension at least $\dim(B) - 1 = 3g - 4$.*

The dimensional estimate on $\mathcal{Q}_d^r$ comes from its construction as a determinantal variety over $B$. Just like in the case of $\widetilde{\mathcal{U}}_{g,d}^r$, we denote by $\epsilon : \mathcal{Q}_d^r \to \widetilde{\mathfrak{G}}_d^r$ the forgetful map such that $\sigma \circ \epsilon = \nu$. We use the existence of $\mathcal{Q}_d^r$ to prove the following inductive result:

**Theorem 1.3.** *Fix integers $g, r, d$ such that $\rho(g, r, d) \geq 2$ and let us assume that $\mathcal{GP}_{g,d}^r$ has a divisorial component $\mathcal{D}$ in $\mathcal{M}_g$ such that if $[C] \in \mathcal{D}$ is a general point, then the variety $\mathcal{Q}_d^r(C)$*

*has at least one* $0$*-dimensional component corresponding to two complementary base point free linear series* $(l,m)\in G^r_d(C)\times G^{g-d+r-1}_{2g-2-d}(C)$*, such that* $[C,l]\in\widetilde{\mathfrak{G}}^r_d$ *is a smooth point. Then* $\mathcal{GP}^r_{g+1,d+1}$ *has a divisorial component* $\mathcal{D}'$ *in* $\mathcal{M}_{g+1}$ *such that a general point* $[C']\in\mathcal{D}'$ *enjoys the same properties, namely that* $\mathcal{Q}^r_{d+1}(C')$ *possesses a* $0$*-dimensional component corresponding to a pair of base point free complementary linear series* $(l',m')\in G^r_{d+1}(C')\times G^{g-d+r-1}_{2g-1-d}(C')$ *such that* $[C',l']\in\widetilde{\mathfrak{G}}^r_{d+1}$ *is a smooth point.*

*Proof.* We choose a general curve $[C]\in\mathcal{D}\subset\mathcal{GP}^r_{g,d}$, a general point $p\in C$ and we set $[C_0:=C\cup_p E]\in\overline{\mathcal{M}}_{g+1}$, where $E$ is an elliptic curve. By assumption, there exist base point free linear series $l_0=(L,V)\in G^r_d(C)$ and $m_0=(K_C\otimes L^\vee,W)\in G^{s-1}_{2g-2-d}(C)$, together with an element $\rho\in\mathbf{P}\mathrm{Ker}\big(\mu_0(V,W)\big)$ such that $\dim_{(l_0,m_0,\rho)}\mathcal{Q}^r_d(C)=0$. In particular, then Ker $\mu_0(V,W)$ is 1-dimensional. Let $\pi:\mathcal{C}\to B$ be the versal deformation space of $C_0=\pi^{-1}(0)$ and $\Delta\subset B$ the boundary divisor corresponding to singular curves. We consider the scheme $\nu:\mathcal{Q}^r_{d+1}\to B$ parameterizing $(\mathfrak{gp})^r_{d+1}$-relations (cf. Theorem 1.2). Since $[C,l_0]\in\mathfrak{G}^r_d$ is a smooth point and $l_0$ is base point free, Lemma 2.5 from [AC] implies that $\mu_1(V,W)$ : Ker $\mu_0(V,W)\to H^0(K_C^{\otimes 2})$ is injective, in particular $\mu_1(V,W)(\rho)\neq 0$. (Here $\sigma_0:\mathfrak{G}^r_d\to\mathcal{M}_g$ denotes the stack of $\mathfrak{g}^r_d$'s over the moduli space of curves of genus $g$). Thus we can assume that $\mathrm{ord}_p(\rho)=1$ for a generic choice of $p$.

We construct a $(\mathfrak{gp})^r_{d+1}$-relation $z=(l,m,\rho_C,\rho_E)\in\mathcal{Q}^r_{d+1}(C_0)$ as follows: the $C$-aspect of the limit $\mathfrak{g}^r_{d+1}$ denoted by $l$ is obtained by adding $p$ as a base point to $(L,V)$, that is $l_C=\big(L_C:=L\otimes\mathcal{O}_C(p),V_C:=V\subset H^0(L_C)\big)$. The aspect $l_E$ is constructed by adding $(d-r)\cdot p$ as a base locus to $|L^0_E|$, where $L^0_E\in\mathrm{Pic}^{r+1}(E)$ is such that $L^0_E\neq\mathcal{O}_E((r+1)\cdot p)$ and $(L^0_E)^{\otimes 2}=\mathcal{O}_E((2r+2)\cdot p)$, and where $|V_E|=(d-r)\cdot p+|L^0_E|$. Since $p\in C$ is general, we may assume that $p$ is not a ramification point of $l_0$, which implies that $a^{l_C}(p)=(1,2,\ldots,r+1)$. Clearly, $a^{l_E}(p)=(d-r,d-r+1,\cdots,d)$, hence $l=\{l_C,l_E\}$ is a refined limit $\mathfrak{g}^r_{d+1}$ on $C_0$. The $C$-aspect of the limit $\mathfrak{g}^{s-1}_{2g-2-d}$ we denote by $m$, is given by $m_C:=\big(K_C\otimes L^\vee\otimes\mathcal{O}_C(p),W_C:=W\subset H^0(K_C\otimes L^\vee\otimes\mathcal{O}_C(p))\big)$. The aspect $m_E$ is constructed by adding $(g-r-1)\cdot p$ to the complete linear series $|\mathcal{O}_E((r+1+s)\cdot p)\otimes(L^0_E)^\vee|$. Since we may also assume that $p$ is not a ramification point of $m_0$, we find that $a^{m_C}(p)=(1,2,\ldots,s)$ and $a^{m_E}(p)=(g-r-1,g-r,\ldots,2g-2-d)$, that is, $m=\{m_C,m_E\}$ is a refined limit $\mathfrak{g}^{s-1}_{2g-1-d}$ on $C_0$. Next we construct the elements $\rho_C$ and $\rho_E$. We choose

$$\rho_C=\rho\in\mathbf{P}\mathrm{Ker}\{\mu_0(V,W):V\otimes W\to H^0(K_C\otimes\mathcal{O}_C(2p))\},$$

that is, $\rho_C$ equals $\rho$ except that we add $p$ as a simple base point to both linear series $l_C$ and $m_C$ whose sections get multiplied. Clearly $\mathrm{ord}_p(\rho_C)=\mathrm{ord}_p(\rho)+2=3$. Then we construct an element $\rho_E\in\mathbf{P}\mathrm{Ker}\{V_E\otimes W_E\to H^0(\mathcal{O}_E(2g\cdot p))\}$ with the property that $\mathrm{ord}_p(\rho_E)=2g-3\ \big(=d-1+(2g-2-d)=d+(2g-3-d)\big)$. Such an element lies necessarily in the kernel of the map

$$H^0\big(L^0_E\otimes\mathcal{O}_E(-(r-1)\cdot p)\big)\otimes H^0\big(\mathcal{O}_E((r+3)\cdot p)\otimes(L^0_E)^\vee\big)\to H^0(\mathcal{O}_E(4\cdot p)),$$

which by the base point free pencil trick is isomorphic to the 1-dimensional space $H^0\big(E,\mathcal{O}_E((2r+2)\cdot p)\otimes(L^0_E)^{\otimes(-2)}\big)$, that is, $\rho_E$ is uniquely determined by the property that $\mathrm{ord}_p(\rho_E)\geq 2g-3$.

Since $\mathrm{ord}_p(\rho_C) + \mathrm{ord}_p(\rho_E) = 2g$, we find that $z = (l, m, \rho_C, \rho_E) \in \mathcal{Q}^r_{d+1}$. Theorem 1.2 guarantees that any component of $\mathcal{Q}^r_{d+1}$ passing through $z$ has dimension at least $3g-1$. To prove the existence of a component of $\mathcal{Q}^r_{d+1}$ mapping rationally onto a divisor $\mathcal{D}' \subset \mathcal{M}_{g+1}$, it suffices to show that $z$ is an isolated point in $\nu^{-1}([C_0])$. Suppose that $z' = (l', m', \rho'_C, \rho'_E) \in \mathcal{Q}^r_{d+1}$ is another point lying in the same component of $\nu^{-1}([C_0])$ as $z$. Since the scheme $\mathcal{Q}^r_{d+1}$ is constructed as a disjoint union over the possibilities of the vanishing sequences of the limit linear series $\mathfrak{g}^r_{d+1}$ and $\mathfrak{g}^{s-1}_{2g-1-d}$, we may assume that $a^{l'_C}(p) = a^{l_C}(p) = (1, 2, \ldots, r+1)$, $a^{m'_C}(p) = a^{m_C}(p) = (1, 2, \ldots, s)$. Similarly for the $E$-aspects, we assume that $a^{l'_E}(p) = a^{l_E}(p)$ and $a^{m'_E}(p) = a^{m_E}(p)$. Then necessarily, $\mathrm{ord}_p(\rho'_C) = 3 (= 1+2 = 2+1)$, otherwise we would contradict the assumption $\mu_1(V, W)(\rho) = 0$. Moreover, $l_C = l_0$ and $m_C = m_0$ because of the inductive assumption on $[C]$. Using the compatibility relation between $\rho'_C$ and $\rho'_E$ we then get that $\mathrm{ord}_p(\rho'_E) \geq 2g-3$. The only way this can be satisfied is when the underlying line bundle $L'_E$ of the linear series $l'_E(-(d-r)\cdot p)$ satisfies the relation $(L'_E)^{\otimes 2} = \mathcal{O}_E((2r+2)\cdot p)$, which gives a finite number of choices for $l'_E$ and then for $m'_E$. Once $l'_E$ is fixed, then as pointed out before, $\rho'_E$ is uniquely determined by the condition $\mathrm{ord}_p(\rho'_E) \geq 2g-3$ (and in fact one must have equality). This shows that $z \in \nu^{-1}([C_0])$ is an isolated point, thus $z$ must smooth to $(\mathfrak{gp})^r_{d+1}$ relations on smooth curves filling-up a divisor $\mathcal{D}'$ in $\mathcal{M}_{g+1}$.

We now prove that $[C_0, l] \in \widetilde{\mathfrak{G}}^r_{d+1}$ is a smooth point (Recall that $\sigma : \widetilde{\mathfrak{G}}^r_{d+1} \to B$ denotes the stack of limit $\mathfrak{g}^r_{d+1}$'s on the fibres of $\pi$). This follows once we show that $[C_0, l]$ is a smooth point of $\sigma^*(\Delta)$ and then observe that $\widetilde{\mathfrak{G}}^r_{d+1}$ commutes with base change. By explicit description, a neighbourhood of $[C_0, l] \in \sigma^*(\Delta)$ is locally isomorphic to an étale neighbourhood of $(\mathfrak{G}^r_d \times_{\mathcal{M}_g} \mathcal{M}_{g,1}) \times \mathcal{M}_{1,1}$ around the point $\big([C, l_0], [C, y], [E, y]\big)$ and we can use our inductive assumption that $\mathfrak{G}^r_d$ is smooth at the point $[C, l_0]$.

Finally, we prove that a generic point $[C'] \in \mathcal{D}'$ corresponds to a pair of base point free linear series $(l', m') \in G^r_{d+1}(C') \times G^{s-1}_{2g-1-d}(C')$. Suppose this is not the case and assume that, say, $l' \in G^r_{d+1}(C')$ has a base point. As $[C', l'] \in \widetilde{\mathfrak{G}}^r_{d+1}$ specializes to $[C_0, l_0]$ the base point of $l'$ specializes to a point $y \in (C_0)_{\mathrm{reg}}$ (If the base point specialized to the $p \in C \cap E$, then necessarily $l$ would be a non-refined limit $\mathfrak{g}^r_{d+1}$). If $y \in C - \{p\}$ then it follows that $l_0 = l_C(-p) \in G^r_d(C)$ has a base point at $y$, which is a contradiction. If $y \in E - \{p\}$, then $L^0_E$ must have a base point at $y$ which is manifestly false. $\square$

## 2. The class of the Gieseker-Petri divisors.

In this section we determine the class of the Gieseker-Petri divisor $\overline{\mathcal{GP}}^r_{g,d}$. We start by setting some notation. We fix integers $r, s \geq 1$ and set $g := rs + s + 1$ and $d := rs + r + 1$, hence $\rho(g, r, d) = 1$. We denote by $\mathcal{M}^0_g$ the open substack of $\mathcal{M}_g$ consisting of curves $[C] \in \mathcal{M}_g$ such that $W^{r+1}_d(C) = \emptyset$. Since $\rho(g, r+1, d) = -r-s-1$, it follows that $\mathrm{codim}(\mathcal{M}_g - \mathcal{M}^0_g, \mathcal{M}_g) \geq 3$. In this section we denote by $\mathfrak{G}^r_d \subset \mathfrak{Pic}^d$ the stack parameterizing pairs $[C, l]$ with $[C] \in \mathcal{M}^0_g$ and $l \in G^r_d(C)$ and $\widetilde{\mathcal{M}}_g := \mathcal{M}^0_g \cup (\cup_{i=0}^{[g/2]} \Delta^0_i)$. We have a natural projection $\sigma : \mathfrak{G}^r_d \to \mathcal{M}^0_g$. Furthermore, we denote by $\pi : \mathcal{M}^0_{g,1} \to \mathcal{M}^0_g$ the universal curve and by $f : \mathcal{M}^0_{g,1} \times_{\mathcal{M}^0_g} \mathfrak{G}^r_d \to \mathfrak{G}^r_d$ the second projection. Note that the forgetful map $\epsilon : \mathcal{U}^r_{g,d} \to \mathfrak{G}^r_d$ is an isomorphism over $\mathcal{M}^0_g$, and we make the identification between $\mathcal{U}^r_{g,d}$ and $\mathfrak{G}^r_d$ (This identification obviously no longer holds over $\widetilde{\mathcal{M}}_g - \mathcal{M}^0_g$).

From general Brill-Noether theory it follows that there exists a unique component of $\mathfrak{G}^r_d$ which maps onto $\mathcal{M}^0_g$. Moreover, any irreducible component $\mathcal{Z}$ of $\mathfrak{G}^r_d$ of dimension $> 3g-3+\rho(g,r,d)$ has the property that $\text{codim}\big(\sigma(\mathcal{Z}), \mathcal{M}^0_g\big) \geq 2$ (see [F1], Corollary 2.5 for a similar statement when $\rho(g,r,d)=0$, the proof remains essentially the same in the case $\rho(g,r,d)=1$).

If $\mathcal{L}$ is a Poincaré bundle over $\mathcal{M}^0_{g,1} \times_{\mathcal{M}^0_g} \mathfrak{G}^r_d$ (one may have to make an étale base change $\Sigma \to \mathfrak{G}^r_d$ to ensure the existence of $\mathcal{L}$, see [Est]), we set $\mathcal{F} := f_*(\mathcal{L})$ and $\mathcal{N} := R^1 f_*(\mathcal{L})$. By Grauert's theorem, both $\mathcal{F}$ and $\mathcal{N}$ are vector bundles over $\mathfrak{G}^r_d = \mathcal{U}^r_{g,d}$ with $\text{rank}(\mathcal{F}) = r+1$ and $\text{rank}(\mathcal{N}) = s$ respectively, and there exists a bundle morphism $\mu : \mathcal{F} \otimes \mathcal{N} \to \sigma^*(\mathbb{E})$, which over each point $[C, L] \in \mathfrak{G}^r_d$ restricts to the Petri map $\mu_0(L)$. If $\mathcal{U} := Z_{rs+s-1}(\mu)$ is the first degeneration locus of $\mu$, then clearly $\mathcal{GP}^r_{g,d} = \sigma_*(\mathcal{U})$. Each irreducible component of $\mathcal{U}$ has codimension at most 2 inside $\mathfrak{G}^r_d$. We shall prove that every such component mapping onto a divisor in $\mathcal{M}_g$ is in fact of codimension 2 (see Proposition 2.3), which will enable us to use Porteous' formula to compute its class. While the construction of $\mathcal{F}$ and $\mathcal{N}$ clearly depends on the choice of the Poincaré bundle $\mathcal{L}$ (and of $\Sigma$), it is easy to check that the degeneracy class $Z_{rs+s-1}(\mu) \in A^2(\mathfrak{G}^r_d)$ is independent of such choices.

Like in [F1], our technique for determining the class of the divisor $\overline{\mathcal{GP}}^r_{g,d}$ is to intersect $\mathcal{U}$ with pull-backs of test curves sitting in the boundary of $\overline{\mathcal{M}}_g$: We fix a general pointed curve $[C, q] \in \mathcal{M}_{g-1,1}$ and a general elliptic curve $[E, y] \in \mathcal{M}_{1,1}$. Then we define the families

$$C^0 := \{C/y \sim q : y \in C\} \subset \Delta_0 \subset \overline{\mathcal{M}}_g \text{ and } C^1 := \{C \cup_y E : y \in C\} \subset \Delta_1 \subset \overline{\mathcal{M}}_g.$$

These curves intersect the generators of $\text{Pic}(\overline{\mathcal{M}}_g)$ as follows:

$$C^0 \cdot \lambda = 0,\ C^0 \cdot \delta_0 = -2g+2,\ C^0 \cdot \delta_1 = 1 \text{ and } C^0 \cdot \delta_j = 0 \text{ for } 2 \leq j \leq [g/2], \text{ and}$$

$$C^1 \cdot \lambda = 0,\ C^1 \cdot \delta_0 = 0,\ C^1 \cdot \delta_1 = -2g+4 \text{ and } C^1 \cdot \delta_j = 0 \text{ for } 2 \leq j \leq [g/2].$$

Next we fix a genus $[g/2] \leq j \leq g-2$ and general curves $[C] \in \mathcal{M}_j, [D, y] \in \mathcal{M}_{g-j,1}$. We define the 1-parameter family $C^j := \{C^j_y = C \cup_y D\}_{y \in C} \subset \Delta_j \subset \overline{\mathcal{M}}_g$. We have the formulas

$$C^j \cdot \lambda = 0,\ \ C^j \cdot \delta_j = -2j+2 \text{ and } C^j \cdot \delta_i = 0 \ \text{ for } i \neq j.$$

To understand the intersections $C^j \cdot \overline{\mathcal{GP}}^r_{g,d}$ for $0 \leq j \leq [g/2]$, we shall extend the vector bundles $\mathcal{F}$ and $\mathcal{N}$ over the partial compactification $\widetilde{\mathcal{U}}^r_{g,d}$ constructed in Section 1.

The next propositions describe the pull-back surfaces $\sigma^*(C^j)$ inside $\widetilde{\mathfrak{G}}^r_d$:

**Proposition 2.1.** *We set $g := rs+s+1$ and fix general curves $[C] \in \mathcal{M}_{rs+s}$ and $[E, y] \in \mathcal{M}_{1,1}$ and consider the associated test curve $C^1 \subset \Delta_1 \subset \overline{\mathcal{M}}_g$. Then we have the following equality of 2-cycles in $\widetilde{\mathfrak{G}}^r_d$:*

$$\sigma^*(C^1) = X + X_1 \times X_2 + \Gamma_0 \times Z_0 + n_1 \cdot Z_1 + n_2 \cdot Z_2 + n_3 \cdot Z_3,$$

*where*

$$X := \{(y, L) \in C \times W^r_d(C) : h^0(C, L \otimes \mathcal{O}_C(-2y)) = r\}$$

$$X_1 := \{(y,L)\in C\times W^r_d(C) : h^0(L\otimes\mathcal{O}_C(-2\cdot y)) = r,\ h^0(L\otimes\mathcal{O}_C(-(r+2)\cdot y)) = 1\}$$

$$X_2 := \{(y,l)\in G^r_{r+2}(E) : a_1^l(y)\geq 2,\ a_r^l(y)\geq r+2\}\cong \boldsymbol{P}\Big(\frac{H^0(\mathcal{O}_E((r+2)\cdot y))}{H^0(\mathcal{O}_E(r\cdot y))}\Big)$$

$$\Gamma_0 := \{(y, A\otimes\mathcal{O}_C(y)) : y\in C, A\in W^r_{d-1}(C)\}\ ,\ Z_0 = G^r_{r+1}(E) = \mathrm{Pic}^{r+1}(E)$$

$$Z_1 := \{l\in G^r_{r+3}(E) : a_1^l(y)\geq 3,\ a_r^l(y)\geq r+3\}\cong \boldsymbol{P}\Big(\frac{H^0(\mathcal{O}_E((r+3)\cdot y))}{H^0(\mathcal{O}_E(r\cdot y))}\Big)$$

$$Z_2 := \{l\in G^r_{r+2}(E) : a_2^l(y)\geq 3,\ a_r^l(y)\geq r+2\}\cong \boldsymbol{P}\Big(\frac{H^0(\mathcal{O}_E((r+2)\cdot y))}{H^0(\mathcal{O}_E((r-1)\cdot y))}\Big)$$

$$Z_3 := \{l\in G^r_{r+2}(E) : a_1^l(y)\geq 2\} = \bigcup_{z\in E}\boldsymbol{P}\Big(\frac{H^0(\mathcal{O}_E((r+1)\cdot y+z))}{H^0(\mathcal{O}_E((r-1)\cdot y+z))}\Big),$$

*where the constants $n_1, n_2, n_3$ are explicitly known positive integers.*

*Proof.* Every point in $\sigma^*(C^1)$ corresponds to a limit $\mathfrak{g}^r_d$, say $l = \{l_C, l_E\}$, on some curve $[C^1_y := C\cup_y E]\in C^1$. By investigating the possible ways of distributing the Brill-Noether numbers $\rho(l_C,y)$ and $\rho(l_E,y)$ in a way such that the inequality $1=\rho(g,r,d)\geq \rho(l_C,y)+\rho(l_E,y)$ is satisfied, we arrive to the six components in the statement (We always use the elementary inequality $\rho(l_E,y)\geq 0$, hence $\rho(l_C,y)\leq 1$). We mention that $X$ corresponds to the case when $\rho(l_C,y)=1, \rho(l_E,y)=0$, the surfaces $X_1\times X_2$ and $\Gamma_0\times Z_0$ correspond to the case $\rho(l_C,y)=0,\rho(l_E,y)=0$, while $Z_1,Z_2,Z_3$ appear in the cases when $\rho(l_C,y)=-1,\rho(l_E,y)=1$. The constants $n_i$ for $1\leq i\leq 3$ have a clear enumerative meaning: First, $n_1$ is the number of points $y\in C$ for which there exists $L\in W^r_d(C)$ such that $a^L(y)=(0,2,3,\ldots,r,r+3)$. Then $n_2$ is the number of points $y\in C$ for which there exists $L\in W^r_d(C)$ such that $a^L(y)=(0,2,3,\ldots,r-1,r+1,r+2)$. Finally, $n_3$ is the number of points $y\in C$ which appear as ramification points for one of the finitely many linear series $A\in W^r_{d-1}(C)$. □

Next we describe $\sigma^*(C^0)$ and we start by fixing more notation. We choose a general pointed curve $[C,q]\in\mathcal{M}_{rs+s,1}$ and denote by $Y$ the following surface:

$$Y := \{(y,L)\in C\times W^r_d(C) : h^0(C, L\otimes\mathcal{O}_C(-y-q)) = r\}.$$

Let $\pi_1 : Y\to C$ denote the first projection. Inside $Y$ we consider two curves corresponding to $\mathfrak{g}^r_d$'s with a base point at $q$:

$$\Gamma_1 := \{(y, A\otimes\mathcal{O}_C(y)) : y\in C, A\in W^r_{d-1}(C)\}\ \text{ and}$$

$$\Gamma_2 := \{(y, A\otimes\mathcal{O}_C(q)) : y\in C, A\in W^r_{d-1}(C)\},$$

intersecting transversally in $n_0 := \#(W^r_{d-1}(C))$ points. Note that $\rho(g,r-1,d)=0$ and $W^r_{d-1}(C)$ is a reduced 0-dimensional cycle. We denote by $Y'$ the blow-up of $Y$ at these $n_0$ points and at the points $(q,B)\in Y$ where $B\in W^r_d(C)$ is a linear series with the property that $h^0(C, B\otimes\mathcal{O}_C(-(r+2)\cdot q))\geq 1$. We denote by $E_A, E_B\subset Y'$ the exceptional divisors corresponding to $(q, A\otimes\mathcal{O}_C(q))$ and $(q,B)$ respectively, by $\epsilon : Y'\to Y$ the projection and by $\widetilde{\Gamma}_1,\widetilde{\Gamma}_2\subset Y'$ the strict transforms of $\Gamma_1$ and $\Gamma_2$.

**Proposition 2.2.** *Fix a general curve $[C, q] \in \mathcal{M}_{rs+s,1}$ and consider the associated test curve $C^0 \subset \Delta_0 \subset \overline{\mathcal{M}}_{rs+s+1}$. Then we have the following equality of 2-cycles in $\widetilde{\mathfrak{G}}^d_r$:*

$$\sigma^*(C^0) = Y'/\widetilde{\Gamma}_1 \cong \widetilde{\Gamma}_2,$$

*that is, $\sigma^*(C^0)$ can be naturally identified with the surface obtained from $Y'$ by identifying the disjoint curves $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ over each pair $(y, A) \in C \times W^r_{d-1}(C)$.*

*Proof.* We fix a point $y \in C - \{q\}$, denote by $[C^0_y := C/y \sim q] \in \overline{\mathcal{M}}_g$, $\nu : C \to C^y_0$ the normalization map, and we investigate the variety $\overline{W}^r_d(C^0_y) \subset \overline{\mathrm{Pic}}^d(C^0_y)$ of torsion-free sheaves $L$ on $C^0_y$ with $\deg(L) = d$ and $h^0(C^0_y, L) \geq r + 1$. If $L \in W^r_d(C^0_y)$, that is, $L$ is locally free, then $L$ is determined by $\nu^*(L) \in W^r_d(C)$ which has the property that $h^0(C, \nu^* L \otimes \mathcal{O}_C(-y - q)) = r$. However, the line bundles of type $A \otimes \mathcal{O}_C(y)$ or $A \otimes \mathcal{O}_C(q)$ with $A \in W^r_{d-1}(C)$, do not appear in this association even though they have this property. They correspond to the situation when $L \in \overline{W}^r_d(C^y_0)$ is not locally free, in which case necessarily $L = \nu_*(A)$ for some $A \in W^r_{d-1}(C)$. Thus $Y \cap \pi_1^{-1}(y)$ is the partial normalization of $\overline{W}^r_d(C^0_y)$ at the $n_0$ points of the form $\nu_*(A)$ with $A \in W^r_{d-1}(C)$. A special analysis is required when $y = q$, that is, when $C^0_y$ degenerates to $C \cup_q E_\infty$, where $E_\infty$ is a rational nodal cubic. If $\{l_C, l_{E_\infty}\} \in \sigma^{-1}([C \cup_q E_\infty])$, then an analysis along the lines of Theorem 2.1 shows that $\rho(l_C, q) \geq 0$ and $\rho(l_{E_\infty}, q) \leq 1$. Then either $l_C$ has a base point at $q$ and then the underlying line bundle of $l_C$ is of type $A \otimes \mathcal{O}_C(q)$ while $l_{E_\infty}(-(d - r - 1) \cdot q) \in \overline{W}^r_{r+1}(E_\infty)$, or else, $a^{l_C}(q) = (0, 2, 3, \ldots, r, r + 2)$ and then $l_{E_\infty}(-(d-r-2)\cdot q) \in \mathbf{P}\big(H^0(\mathcal{O}_{E_\infty}((r+2)\cdot q))/H^0(\mathcal{O}_{E_\infty}(r\cdot q))\big) \cong E_B$, where $B \in W^r_d(C)$ is the underlying line bundle of $l_C$. □

We now show that every irreducible component of $\mathcal{U}$ has the expected dimension:

**Proposition 2.3.** *Every irreducible component $\mathcal{X}$ of $\mathcal{U}$ having the property that $\sigma(\mathcal{X})$ is a divisor in $\mathcal{M}_g$ has $\mathrm{codim}(\mathcal{X}, \mathfrak{G}^r_d) = 2$.*

*Proof.* Suppose that $\mathcal{X}$ is an irreducible component of $\mathcal{U}$ satisfying (1) $\mathrm{codim}(\mathcal{X}, \mathfrak{G}^r_d) \leq 1$ and (2) $\mathrm{codim}(\sigma(\mathcal{X}), \mathcal{M}_g) = 1$. We write $D := \overline{\sigma(\mathcal{X})} \subset \overline{\mathcal{M}}_g$ for the closure of this divisor in $\overline{\mathcal{M}}_g$, and we express its class as $D \equiv a\lambda - b_0\delta_0 - b_1\delta_1 - \cdots - b_{[g/2]}\delta_{[g/2]} \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$. To reach a contradiction, it suffices to show that $a = 0$.

Keeping the notation from Propositions 2.1 and 2.2, we are going to show that $C^0 \cap D = C^1 \cap D = \emptyset$ which implies that $b_0 = b_1 = 0$. Then we shall show that if $R \subset \overline{\mathcal{M}}_g$ denotes the pencil obtained by attaching to a general pointed curve $[C, q] \in \mathcal{M}_{rs+s,1}$ at the fixed point $q$, a pencil of plane cubics (i.e. an elliptic pencil of degree 12), then $R \cap D = \emptyset$. This implies the relation $a - 12b_0 + b_1 = 0$ which of course yields that $a = 0$.

We assume by contradiction that $C^1 \cap D \neq \emptyset$. Then there exists a point $y \in C$ and a limit $\mathfrak{g}^r_d$ on $C^1_y := C \cup_y E$, say $l = \{l_C, l_E\}$, such that if $L_C \in W^r_d(C)$ denotes the underlying line bundle of $l_C$, then the multiplication map

$$\mu_0(L_C, y) : H^0(L_C) \otimes H^0(K_C \otimes L_C^\vee \otimes \mathcal{O}_C(2y)) \to H^0(K_C \otimes \mathcal{O}_C(2y))$$

is not injective. We claim that this can happen only when $\rho(l_C, y) = 1$ and $\rho(l_E, y) = 0$, that is, when $[C^1_y, l] \in X$ (we are still using the notation from Proposition 2.1). Indeed, assuming that $\rho(l_C, y) \leq 0$, there are two cases to consider. Either $L_C$ has a base point at $y$ and then we can write $L_C = A \otimes \mathcal{O}_C(y)$ for $A \in W^r_{d-1}(C)$ and then we find that $\mu_0(A)$ is not injective which contradicts the assumption that $[C] \in \mathcal{M}_{rs+s}$ is Petri general. Or $y \notin \mathrm{Bs}|L_C|$ and then $a^{L_C}(y) \geq (0, 2, 3, \ldots, r, r+2)$. A degeneration argument along the lines of [F1] Proposition 3.2 shows that $[C]$ can be chosen general enough such that every $L_C$ with this property has $\mu_0(L_C, y)$ injective. Thus we may assume that $\rho(l_C, y) = 1$ and then $\mu_0(L_C, y)$ is not injective for *every* point $(y, L_C)$ belonging to an irreducible component of the fibre $\pi_1^{-1}(y) \subset X$.

On the other hand, whenever one has an irreducible projective variety $A \subset G^r_d(C)$ with $\dim(A) \geq 1$ and a Schubert index $\overline{\alpha} := (0 \leq \alpha_0 \leq \ldots \leq \alpha_r \leq d-r)$ such that $\alpha^l(y) \geq \overline{\alpha}$ for all $l \in A$, there exists a Schubert index of the same type $\overline{\beta} > \overline{\alpha}$, such that $\alpha^{l_0}(y) \geq \overline{\beta}$ for a certain $l_0 \in A$. In our case, this implies that $\mu_0(L_C, y)$ is not injective for a linear series $L_C \in W^r_d(C)$ such that either $a^{L_C}(y) \geq (0, 2, \ldots, r, r+2)$ (and this case has been dealt with before), or $a^{L_C}(y) \geq (1, 2, \ldots, r+1)$. Then $L_C = A \otimes \mathcal{O}_C(y)$ for $A \in W^r_{d-1}(C)$ and $\mu_0(A)$ is not injective. This violates the assumption that $[C] \in \mathcal{M}_{rs+s}$ is Petri general. To prove that $C^0 \cap D = \emptyset$ we use the same principle in the context of the explicit description of $\sigma^*(C^0)$ provided by Proposition 2.2. Finally, to show that $R \cap D = \emptyset$ it suffices to show that if $[C, q] \in \mathcal{M}_{rs+s,1}$ is sufficiently general, then $\mu_0(L_C, q)$ is injective for every $(q, L_C) \in \pi_1^{-1}(q)$. This is the statement of Theorem 2.13. □

We extend $\mathcal{F}$ and $\mathcal{N}$ as vector bundles over the stack $\widetilde{\mathcal{U}}^r_{g,d}$ of pairs of limit linear series. Note that every irreducible component of $\widetilde{\mathcal{U}}^r_{g,d}$ which meets one of the test surfaces $\nu^*(C^j)$ has dimension $3g-2$. This follows from an explicit description of $\nu^*(C^j)$ similar to the one for $j = 0, 1$ given in Propositions 2.1 and 2.2. Such a description, although straightforward, is combinatorially involved (see [F1] Proposition 2.4, for the answer in the case $\rho(g, r, d) = 0$). Since we are not going to make direct use of it in this paper, we skip such details. Recall that we denote by $\epsilon : \widetilde{\mathcal{U}}^r_{g,d} \to \widetilde{\mathfrak{G}}^r_d$ the forgetful map and $\nu = \sigma \circ \epsilon$.

**Proposition 2.4.** *There exist two vector bundles $\mathcal{F}$ and $\mathcal{N}$ over $\widetilde{\mathcal{U}}^r_{g,d}$ with $\mathrm{rank}(\mathcal{F}) = r+1$ and $\mathrm{rank}(\mathcal{N}) = s$, together with a vector bundle morphism $\mu : \mathcal{F} \otimes \mathcal{N} \to \nu^*\big(\mathbb{E} \otimes \sum_{j=1}^{[g/2]} (2j-1) \cdot \delta_j\big)$, such that the following statements hold:*

- *For a point $[C, L] \in \mathfrak{G}^r_d = \mathcal{U}^r_{g,d}$ we have that $\mathcal{F}(C, L) = H^0(C, L)$, $\mathcal{N}(C, L) = H^0(C, K_C \otimes L^\vee)$ and $\mu_0(C, L) : H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee) \to H^0(K_C)$ is the Petri map.*
- *For $t = \big[C \cup_y D, (l_C, l_D), (m_C, m_D)\big] \in \sigma^{-1}(\Delta_j^0)$, with $[g/2] \leq j \leq g-1$, $[C, y] \in \mathcal{M}_{j,1}$, $[D, y] \in \mathcal{M}_{g-j,1}$ and*

$$l_C = (L_C, V_C) \in G^r_d(C), \quad m_C = (K_C \otimes L_C^\vee \otimes \mathcal{O}_C(2(g-j) \cdot y), W_C) \in G^{s-1}_{2g-2-d}(C),$$

*we have that $\mathcal{F}(t) = V_C$, $\mathcal{N}(t) = W_C$ and*

$$\mu(t) = \mu_0(V_C, W_C) : V_C \otimes W_C \to H^0\big(K_C \otimes \mathcal{O}_C(2(g-j) \cdot y)\big).$$

- *Fix $t = [C^0_y := C/y \sim q, L] \in \sigma^{-1}(\Delta_0^0)$, with $q, y \in C$ and $L \in \overline{W}^r_d(C^0_y)$ such that $h^0(C, \nu^* L \otimes \mathcal{O}_C(-y-q)) = r$. Here $\nu : C \to C^0_y$ is the normalization map.*

*When $L$ is locally free, $\mathcal{F}(t)=H^0(C,\nu^*L)$, $\mathcal{N}(t)=H^0(C,K_C\otimes\nu^*L^{\vee}\otimes\mathcal{O}_C(y+q))$ and $\phi(t)$ is the multiplication map*

$$H^0(\nu^*L)\otimes H^0\big(K_C\otimes\nu^*L^{\vee}\otimes\mathcal{O}_C(y+q)\big)\rightarrow H^0(K_C\otimes\mathcal{O}_C(y+q))=H^0(C_y^0,\omega_{C_y^0}).$$

*In the case when $L$ is not locally free, that is, $L\in\overline{W}^r_d(C_0^y)-W^r_d(C_0^y)$, then $L=\nu_*(A)$, where $A\in W^r_{d-1}(C)$, and*

$$\mathcal{F}(t)=H^0(A)=H^0(\nu_*A)\ \textit{and}\ \mathcal{N}(t)=H^0(K_C\otimes A^{\vee}\otimes\mathcal{O}_C(y+q))=H^0(\omega_{C_y^0}\otimes\nu_*A^{\vee}).$$

Briefly stated, over each curve of compact type, the vector bundle $\mathcal{F}$ (resp. $\mathcal{N}$) retains the sections of the limit $\mathfrak{g}^r_d$ (resp. $\mathfrak{g}^{s-1}_{2g-2-d}$) coming from the component having the largest genus. The Gieseker-Petri theorem ensures that the vector bundle morphism $\mu:\mathcal{F}\otimes\mathcal{N}\rightarrow\nu^*\big(\mathbb{E}\otimes\sum_{j=1}^{[g/2]}(2j-1)\cdot\delta_j\big)$ is generically non-degenerate. Moreover, $\nu_{|\nu^{-1}(\Delta_0^0)}$ and $\nu_{|\nu^{-1}(\Delta_1^0)}$ are also generically-nondegenerate along each irreducible component (see Theorem 2.13), hence one can write that

$$\nu_*\, c_1\big(\nu^*(\mathbb{E}\otimes\sum_{j=1}^{[g/2]}(2j-1)\cdot\delta_j)-\mathcal{F}\otimes\mathcal{N}\big)=[\overline{\mathcal{GP}}^r_{g,d}]+\sum_{j=2}^{[g/2]}e_j\cdot\delta_j,$$

where $e_j\geq 0$. We can compute explicitly the left-hand-side of this formula and show that the smallest boundary coefficient of $\nu_*c_1\big(\nu^*(\mathbb{E}\otimes\sum_{j=1}^{[g/2]}(2j-1)\cdot\delta_j)-\mathcal{F}\otimes\mathcal{N}\big)$ is that corresponding to $\delta_0$. Thus $s([\overline{\mathcal{GP}}^r_{g,d}])=s\big(\nu_*\,c_1(\nu^*(\mathbb{E}\otimes\sum_{j=1}^{[g/2]}(2j-1)\cdot\delta_j)-\mathcal{F}\otimes\mathcal{N})\big)$.

Throughout the paper we use a few facts about intersection theory on Jacobians which we briefly recall (see [ACGH] for a general reference). We fix integers $r,s\geq 1$ and set $g:=rs+s$ and $d:=rs+r+1$. If $[C]\in\mathcal{M}_g$ is a Brill-Noether general curve, we denote by $\mathcal{P}$ a Poincaré bundle on $C\times\mathrm{Pic}^d(C)$ and by $\pi_1:C\times\mathrm{Pic}^d(C)\rightarrow C$ and $\pi_2:C\times\mathrm{Pic}^d(C)\rightarrow\mathrm{Pic}^d(C)$ the projections. We define the cohomology class $\eta=\pi_1^*([point])\in H^2(C\times\mathrm{Pic}^d(C))$, and if $\delta_1,\ldots,\delta_{2g}\in H^1(C,\mathbb{Z})\cong H^1(\mathrm{Pic}^d(C),\mathbb{Z})$ is a symplectic basis, then we set

$$\gamma:=-\sum_{\alpha=1}^{g}\Big(\pi_1^*(\delta_\alpha)\pi_2^*(\delta_{g+\alpha})-\pi_1^*(\delta_{g+\alpha})\pi_2^*(\delta_\alpha)\Big).$$

We have the formula $c_1(\mathcal{P})=d\cdot\eta+\gamma$, corresponding to the Hodge decomposition of $c_1(\mathcal{P})$. We also record that $\gamma^3=\gamma\eta=0$, $\eta^2=0$ and $\gamma^2=-2\eta\pi_2^*(\theta)$. Since $W^{r+1}_d(C)=\emptyset$, it follows that $W^r_d(C)$ is smooth of dimension $\rho(g,r,d)=r+1$. Over $W^r_d(C)$ there is a tautological rank $r+1$ vector bundle $\mathcal{M}:=(\pi_2)_*(\mathcal{P}_{|C\times W^r_d(C)})$. The Chern numbers of $\mathcal{M}$ can be computed using the Harris-Tu formula (cf. [HT]) as follows: We write $\sum_{i=0}^{r+1}c_i(\mathcal{M}^{\vee})=(1+x_1)\cdots(1+x_{r+1})$ and then for every class $\zeta\in H^*(\mathrm{Pic}^d(C),\mathbb{Z})$ one has the following formula:

$$x_1^{i_1}\cdots x_{r+1}^{i_{r+1}}\,\zeta=\det\Big(\frac{\theta^{g+r-d+i_j-j+l}}{(g+r-d+i_j-j+l)!}\Big)_{1\leq j,l\leq r+1}\,\zeta.$$

If we use the expression of the Vandermonde determinant, we get the identity

$$\det\Big(\frac{1}{(a_j+l-1)!}\Big)_{1\leq j,l\leq r+1}=\frac{\Pi_{j>l}\,(a_l-a_j)}{\Pi_{j=1}^{r+1}\,(a_j+r)!},$$

which quickly leads to the following formula in $H^{2r+2}(W^r_d(C),\mathbb{Z})$:

(1) $$x_1^{i_1}\cdots x_{r+1}^{i_{r+1}}\cdot\theta^{r+1-i_1-\cdots-i_{r+1}}=\frac{\Pi_{j>l}(i_l-i_j+j-l)}{\Pi_{j=1}^{r+1}(s+r+i_j-j)!}.$$

By repeatedly applying (1), we get all intersection numbers on $W^r_d(C)$ we shall need. We define the integer

$$n_0=C_{r+1}:=\frac{(rs+s)!\ r!\ (r-1)!\cdots 2!\ 1!}{(s+r)!\ (s+r-1)!\cdots(s+1)!\ s!}\ =\#(W^r_{d-1}(C))$$

and we have the following formulas:

**Proposition 2.5.** *Let $C$ be a general curve of genus $rs+s$ and we set $d:=rs+r+1$. We denote by $c_i:=c_i(\mathcal{M}^\vee)\in H^{2i}(W^r_d(C),\mathbb{Z})$ the Chern classes of the dual of the tautological bundle on $W^r_d(C)$. Then one has the following identities in $H^*(W^r_d(C),\mathbb{Z})$:*

$$c_{r+1}=x_1x_2\dots x_{r+1}=C_{r+1},\ c_r\cdot c_1=x_1x_2\dots x_{r+1}+x_1^2x_2\dots x_r$$
$$c_{r-1}\cdot c_2=x_1x_2\dots x_{r+1}+x_1^2x_2\dots x_r+x_1^2x_2^2x_3\dots x_{r-1}$$
$$c_{r-1}\cdot c_1^2=x_1x_2\dots x_{r+1}+2x_1^2x_2\dots x_r+x_1^2x_2^2x_3\dots x_{r-1}+x_1^3x_2x_3\dots x_{r-1}$$
$$c_r\cdot\theta=x_1x_2\dots x_r\cdot\theta=(r+1)s\ C_{r+1},\quad c_{r-1}\cdot c_1\cdot\theta=x_1x_2\dots x_r\cdot\theta+x_1^2x_2+\dots x_{r-1}\cdot\theta$$
$$c_{r-2}\cdot c_2\cdot\theta=x_1x_2\dots x_r\cdot\theta+x_1^2x_2\dots x_{r-1}\cdot\theta+x_1^2x_2^2x_3\dots x_{r-2}\cdot\theta$$
$$c_{r-2}\cdot c_1^2\cdot\theta=x_1x_2\dots x_r\cdot\theta+2x_1^2x_2\dots x_{r-1}\cdot\theta+x_1^2x_2^2x_3\dots x_{r-2}\cdot\theta+x_1^3x_2x_3\dots x_{r-2}\cdot\theta$$
$$c_{r-1}\cdot\theta^2=x_1x_2\dots x_{r-1}\cdot\theta^2,\quad c_{r-2}\cdot c_1\cdot\theta^2=x_1x_2\dots x_{r-1}\cdot\theta^2+x_1^2x_2\dots x_{r-2}\cdot\theta^2$$

Next we record the values of the monomials in the $x_i$'s and $\theta$ that appeared in Proposition 2.5. The proof amounts to a systematic application of formula (1):

**Proposition 2.6.** *We set $d:=rs+r+1$ and write $c_t(\mathcal{M}^\vee)=(1+x_1)\cdots(1+x_{r+1})$ as above. Then one has the following identities in $H^{2r+2}(W^r_d(C),\mathbb{Z})$:*

$$x_1x_2\dots x_{r+1}=C_{r+1},\quad x_1^2x_2^2x_3\dots x_{r-1}=\frac{s(s+1)(r+1)^2(r-2)(r+2)}{4(s+r)(s+r+1)}C_{r+1}$$
$$x_1^2x_2\dots x_r=\frac{r(r+2)s}{s+r+1}C_{r+1},\quad x_1^3x_2x_3\dots x_{r-1}=\frac{r(r-1)(r+2)(r+3)s(s+1)}{4(s+r+1)(s+r+2)}C_{r+1}$$
$$x_1x_2\dots x_r\cdot\theta=(r+1)sC_{r+1},\quad x_1^2x_2\dots x_{r-1}\cdot\theta=\frac{(s+1)(r-1)(r+2)}{2(s+r+1)}x_1x_2\dots x_r\cdot\theta,$$
$$x_1^2x_2^2x_3\dots x_{r-2}=\frac{(r-3)(r+1)(r+2)r(s+1)(s+2)}{12(s+r+1)(s+r)}x_1x_2\dots x_r\cdot\theta$$
$$x_1^3x_2x_3\dots x_{r-2}\cdot\theta=\frac{(r+2)(r+3)(r-1)(r-2)(s+1)(s+2)}{12(s+r+1)(s+r+2)}x_1x_2\dots x_r\cdot\theta$$
$$x_1x_2\dots x_{r-1}\cdot\theta^2=\frac{r(r+1)s(s+1)}{s}C_{r+1}$$
$$x_1^2x_2\dots x_{r-2}\cdot\theta^2=\frac{(r+2)(r-2)(s+2)}{3(s+r+1)}x_1x_2\dots x_{r-1}\cdot\theta^2$$
$$x_1x_2\dots x_{r-2}\cdot\theta^3=\frac{(r+1)r(r-1)(s+2)(s+1)s}{6}C_{r+1}.$$

**Proposition 2.7.** *Let $[C, q] \in \mathcal{M}_{rs+s,1}$ be a general pointed curve. If $\mathcal{M}$ denotes the tautological vector bundle over $W^r_d(C)$ and $c_i := c_i(\mathcal{M}^\vee)$, then one has the following relations:*

(1) $[X] = \pi_2^*(c_r) - 6\pi_2^*(c_{r-2})\eta\theta + \big((4rs+2r+2s)\eta + 2\gamma\big)\pi_2^*(c_{r-1}) \in H^{2r}(C \times W^r_d(C))$.
(2) $[Y] = \pi_2^*(c_r) - 2\pi_2^*(c_{r-2})\eta\theta + \big((rs+r)\eta + \gamma\big)\pi_2^*(c_{r-1}) \in H^{2r}(C \times W^r_d(C))$.

*Proof.* We realize the surface $X$ as the degeneracy locus of a vector bundle map over $C \times W^r_d(C)$. For each pair $(y, L) \in C \times W^r_d(C)$ there is a natural map

$$H^0(C, L \otimes \mathcal{O}_{2y})^\vee \to H^0(C, L)^\vee$$

which globalizes to a vector bundle morphism $\zeta : J_1(\mathcal{P})^\vee \to \pi_2^*(\mathcal{M})^\vee$ over $C \times W^r_d(C)$ (Recall that $W^r_d(C)$ is a smooth $(r+1)$-fold). Then we have the identification $X = Z_1(\zeta)$ and the Thom-Porteous formula gives that $[X] = c_r\big(\pi_2^*(\mathcal{M}) - J_1(\mathcal{P}^\vee)\big)$. From the usual exact sequence over $C \times \text{Pic}^d(C)$

$$0 \longrightarrow \pi_1^*(K_C) \otimes \mathcal{P} \longrightarrow J_1(\mathcal{P}) \longrightarrow \mathcal{P} \longrightarrow 0,$$

we can compute the total Chern class of the jet bundle

$$c_t(J_1(\mathcal{P})^\vee) = \Big(\sum_{j \geq 0}(d\eta+\gamma)^j\Big)\cdot\Big(\sum_{j\geq 0}((2g(C)-2+d)\eta+\gamma)^j\Big) = 1 - 6\eta\theta + (2d+2g(C)-2)\eta + 2\gamma,$$

which quickly leads to the formula for $[X]$. To compute $[Y]$ we proceed in a similar way. We denote by $p_1, p_2 : C \times C \times \text{Pic}^d(C) \to C \times \text{Pic}^d(C)$ the two projections, by $\Delta \subset C \times C \times \text{Pic}^d(C)$ the diagonal and we set $\Gamma_q := \{q\} \times \text{Pic}^d(C)$. We introduce the rank 2 vector bundle $\mathcal{B} := (p_1)_*\big(p_2^*(\mathcal{P}) \otimes \mathcal{O}_{\Delta + p_2^*(\Gamma_q)}\big)$ defined over $C \times W^r_d(C)$ and we note that there is a bundle morphism $\chi : \mathcal{B}^\vee \to (\pi_2)^*(\mathcal{M})^\vee$ such that $Y = Z_1(\chi)$. Since we also have that

$$c_t(\mathcal{B}^\vee)^{-1} = \big(1 + (d\eta + \gamma) + (d\eta+\gamma)^2 + \cdots\big)(1 - \eta),$$

we immediately obtained the desired expression for $[Y]$. □

**Remark 2.8.** For future reference we also record the following formulas:

(2) $c_{r+1}(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee) = \pi_2^*(c_{r+1}) - 6\pi_2^*(c_{r-1})\eta\theta + \big((4rs+2r+2s)\eta + 2\gamma\big)\pi_2^*(c_r)$

(3) $$c_{r+2}(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee) = \pi_2^*(c_{r+1})\big((4rs+2r+2s)\eta + 2\gamma\big) - 6\pi_2^*(c_r)\eta\theta.$$

(4) $$c_{r+1}(\pi_2^*(\mathcal{M})^\vee - \mathcal{B}^\vee) = \pi_2^*(c_{r+1}) - 2\pi_2^*(c_{r-1})\eta\theta + \big((rs+r)\eta + \gamma\big)\pi_2^*(c_r)$$

and

(5) $$c_{r+2}(\pi_2^*(\mathcal{M})^\vee - \mathcal{B}^\vee) = \pi_2^*(c_{r+1})\big((rs+r)\eta + \gamma\big) - 2\pi_2^*(c_r)\eta\theta.$$

**Proposition 2.9.** *Let $[C] \in \mathcal{M}_{rs+s}$ be a Brill-Noether general curve and denote by $\mathcal{P}$ the Poincaré bundle on $C \times \text{Pic}^d(C)$. We have the following identities in $H^*(\text{Pic}^d(C), \mathbb{Z})$:*

$$c_1\big(R^1\pi_{2*}(\mathcal{P}_{|C\times W^r_d(C)})\big) = \theta - c_1 \ \textit{and} \ c_2\big(R^1\pi_{2*}(\mathcal{P}_{|C\times W^r_d(C)})\big) = \frac{\theta^2}{2} - \theta c_1 + c_2.$$

*Proof.* We recall that in order to obtain a determinantal structure on $W^r_d(C)$ one fixes a divisor $D \in C_e$ of degree $e >> 0$ and considers the morphism

$$(\pi_2)_*\big(\mathcal{P} \otimes \mathcal{O}(\pi_1^* D)\big) \to (\pi_2)_*\big(\mathcal{P} \otimes \mathcal{O}(\pi_1^* D_{|\pi_1^* D})\big).$$

Then $W^r_d(C)$ is the degeneration locus of rank $d-g-r+e$ of this map and there is an exact sequence of vector bundles over $W^r_d(C)$:

$$0 \to \mathcal{M} \longrightarrow (\pi_2)_*\big(\mathcal{P}\otimes\mathcal{O}(\pi_1{}^* D)\big) \longrightarrow (\pi_2)_*\big(\mathcal{P}\otimes\mathcal{O}(\pi_1^* D)_{|\pi_1^* D}\big) \to R^1\pi_{2*}\big(\mathcal{P}_{|C\times W^r_d(C)}\big) \longrightarrow 0.$$

From this sequence our claim follows if we take into account that $(\pi_2)_*\big(\mathcal{P}\otimes\mathcal{O}(\pi_1^* D)_{|\pi_1^* D}\big)$ is numerically trivial and $c_t\big((\pi_2)_*(\mathcal{P} \otimes \mathcal{O}(\pi_1^* D))\big) = e^{-\theta}$. □

**Remark 2.10.** For future reference we note that Proposition 2.9 provides a quick way to compute the canonical class $K_{W^r_d(C)}$. Indeed, since $T_{\mathrm{Pic}^d(C)}$ is trivial, we have that $K_{W^r_d(C)} = c_1(N_{W^r_d(C)/\mathrm{Pic}^d(C)})$. From the realization of $W^r_d(C)$ as a determinantal variety, we obtain that $N_{W^r_d(C)} = Hom\big(\mathcal{M}, R^1\pi_{2*}\big(\mathcal{P}_{|C\times W^r_d(C)}\big)\big)$, which leads to the expression:

$$K_{W^r_d(C)} \equiv (r+1)\theta + (s-r-2)c_1. \tag{6}$$

We shall also need in Section 3 the expressions for $K_X$ and $K_Y$. To start with the surface $X$, we have that $K_X \equiv (2rs+2s-2)\eta + K_{W^r_d(C)} + c_1(N_{X/C\times W^r_d(C)})$. Next we use Proposition 2.7, to express the normal bundle of the determinantal subvariety $X \subset C \times W^r_d(C)$ as $N_{X/C\times W^r_d(C)} = Hom(\mathrm{Ker}(\zeta), \mathrm{Coker}(\zeta))$ which leads to the formula:

$$K_X \equiv (r+1)\theta + (r-1)c_1(\mathrm{Ker}(\zeta)^\vee) + (s-r-1)\pi_2 * (c_1) + 2\gamma + (6rs+2r+4s-2)\eta. \tag{7}$$

In a similar manner, using the vector bundle map $\chi$, we find the canonical class of $Y$:

$$K_Y \equiv (r+1)\theta + (r-1)c_1(\mathrm{Ker}(\chi)^\vee) + (s-r-1)\pi_2^*(c_1) + \gamma + (3rs+r+2s-2)\eta. \tag{8}$$

As a first step towards computing $[\overline{\mathcal{GP}}^r_{g,d}]$ we determine the $\delta_1$ coefficient in its expression. For simplicity we set $\tilde{\mathbb{E}} := \mathbb{E} \otimes \mathcal{O}_{\overline{\mathcal{M}}_g}\big(\sum_{j=1}^{[g/2]}(2j-1)\cdot\delta_j\big)$ for the twist of the Hodge bundle.

**Theorem 2.11.** *Let $[C] \in \mathcal{M}_{rs+s}$ be a Brill-Noether general curve and denote by $C^1 \subset \Delta_1$ the associated test curve. Then the coefficient of $\delta_1$ in the expansion of $\overline{\mathcal{GP}}^r_{g,d}$ in terms of the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ is equal to*

$$b_1 = \frac{rs(r+1)(s-1)\,C_{r+1}}{2(s+r+1)(s+r)(s+r+2)(rs+s-1)}\Big((2s^2+2s^3)r^4+(2s^4+12s^3+23s^2+9s)r^3+$$
$$+$$
$$(8s^4+39s^3+75s^2+46s+10)r^2+(10s^4+59s^3+108s^2+89s+26)r+4s^4+30s^3+64s^2+58s+12)\Big).$$

*Proof.* We intersect the degeneracy locus of the map $\mathcal{F} \otimes \mathcal{N} \to \sigma^*(\tilde{\mathbb{E}})$ with the surface $\sigma^*(C^1)$ and use that the vector bundles $\mathcal{F}$ and $\mathcal{N}$ were defined by retaining the sections of the genus $g-1$ aspect of each limit linear series and dropping the information coming from the elliptic curve. It follows that $Z_i \cdot c_2(\sigma^*(\tilde{\mathbb{E}}) - \mathcal{F} \otimes \mathcal{N}) = 0$ for $1 \le i \le 3$ because both $\sigma^*\tilde{\mathbb{E}}$ and $\mathcal{F} \otimes \mathcal{N}$ are trivial along the surfaces $Z_i$. Furthermore, we also have that $[X_1 \times X_2] \cdot c_2(\sigma^*(\tilde{\mathbb{E}}) - \mathcal{F} \otimes \mathcal{N}) = 0$, because $c_2(\sigma^*(\tilde{\mathbb{E}}) - \mathcal{F} \otimes \mathcal{N})_{|X_1\times X_2}$ is in fact the pull-back of a codimension 2 class from the 1-dimensional cycle $X_1$, therefore the intersection number is 0 for dimensional reasons. We are left with estimating the contribution coming from $X$ and we write

$$\sigma^*(C^1)\cdot c_2(\sigma^*\tilde{\mathbb{E}}-\mathcal{F}\otimes\mathcal{N}) = c_2(\sigma^*\tilde{\mathbb{E}}_{|X})-c_1(\sigma^*\tilde{\mathbb{E}}_{|X})\cdot c_1(\mathcal{F}\otimes\mathcal{N}_{|X})+c_1^2(\mathcal{F}\otimes\mathcal{N}_{|X})-c_2(\mathcal{F}\otimes\mathcal{N}_{|X})$$

and we are going to compute each term in the right-hand-side of this expression.

Since we have a canonical identification $\tilde{\mathbb{E}}_{|C^1}[C^1_y] = H^0(C, K_C\otimes \mathcal{O}_C(2y))$ for each $y\in C$, we obtain that $c_2(\sigma^*\tilde{\mathbb{E}}_{|X}) = 0$ and $c_1(\sigma^*\tilde{\mathbb{E}}_{|X}) = -(2g-4)\eta$. Recall also that we have set $c_i(\mathcal{F}^\vee_{|X}) = \pi_2^*(c_i)$ for $0\le i\le r+1$, where $c_i\in H^{2i}(W^r_d(C),\mathbb{Z})$.

In Proposition 2.7 we introduced a vector bundle morphism $\zeta : J_1(\mathcal{P})^\vee \to \pi_2^*(\mathcal{M})$ over $C\times W^r_d(C)$. We denote by $U := \mathrm{Ker}(\zeta)$ and we view $U$ as a line bundle over $X$ with fibre over a point $(y,L)\in X$ being the space

$$U(y,L) = \frac{H^1(C, L\otimes\mathcal{O}_C(-2y))^\vee}{H^1(C,L)^\vee}.$$

The Chern numbers of $U^\vee$ can be computed from the Harris-Tu formula and we find that for any class $\xi\in H^2(C\times W^r_d(C))$ we have the following (cf. (2)): $c_1(U^\vee)\cdot\xi_{|X} =$

$$= c_{r+1}\big(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee\big)\cdot\xi_{|X} = \big(\pi_2^*(c_{r+1})-6\eta\theta\pi_2^*(c_{r-1})+((4rs+2r+2s)\eta+2\gamma)\pi_2^*(c_r)\big)\cdot\xi_{|X},$$

and

$$c_1^2(U^\vee) = c_{r+2}\big(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee\big) = \pi_2^*(c_{r+1})((4rs+2r+2s)\eta+2\gamma) - 6\pi_2^*(c_r)\eta\theta.$$

The line bundle $U$ is used to evaluate the Chern numbers of $\mathcal{N}_{|X}$ via the exact sequence:

$$0\longrightarrow \pi_2^*R^1\pi_{2*}\big(\mathcal{P}_{|C\times W^r_d(C)}\big)^\vee \longrightarrow \mathcal{N}_{|X}\longrightarrow U\longrightarrow 0, \tag{9}$$

from which we obtain (by also using Proposition 2.9), that $c_1(\mathcal{N}_{|X}) = -\theta + c_1 - c_1(U^\vee)$ and

$$c_2(\mathcal{N}_{|X}) = c_2 - \theta\cdot c_1 + \frac{\theta^2}{2} + (\theta - c_1)\cdot c_1(U^\vee).$$

Therefore we can write that

$$\sigma^*(C^1)\cdot c_2(\sigma^*\tilde{\mathbb{E}}-\mathcal{F}\otimes\mathcal{N}) = (2g-4)\eta\cdot c_1(\mathcal{F}\otimes\mathcal{N}_{|X}) + c_1^2(\mathcal{F}\otimes\mathcal{N}_{|X}) - c_2(\mathcal{F}\otimes\mathcal{N}_{|X}) =$$

$$= \binom{r+2}{2}c_1^2(U^\vee)+\Big((r+1)^2\cdot\theta+2(r+1)(1-s(r+1))\cdot\eta+((r+1)(s-r-1)+1)\cdot c_1\Big)c_1(U^\vee)+\cdots,$$

where the term we omitted is a quadratic polynomial in $\theta,\eta$ and $\gamma$ which will be multiplied by the class $[X]$. Since we have already computed $c_1(U^\vee)$ and $c_1^2(U^\vee)$, we can write $\sigma^*(C^1)\cdot c_2(\sigma^*(\tilde{\mathbb{E}})-\mathcal{F}\otimes\mathcal{N})$ as a polynomial in the classes $\pi_2^*(c_i)$, $\eta$, $\theta$ and $\gamma$ and the only non-zero terms will be those which contain $\eta$. Then we apply Propositions 2.5 and 2.6 and finally compute the coefficient $b_1 := \sigma^*(C^1)\cdot c_2(\sigma^*(\tilde{\mathbb{E}})-\mathcal{F}\otimes\mathcal{N})/(2g-4)$ which finishes the proof.

□

**Theorem 2.12.** *Let $[C,q]\in\mathcal{M}_{rs+s,1}$ be a Brill-Noether general pointed curve and denote by $C^0\subset\Delta_0$ the associated test curve. Then the $\delta_0$-coefficient of $[\overline{\mathcal{GP}}^r_{g,d}]$ is given by the formula:*

$$b_0 = \frac{r(r+2)(s-1)(s+1)(2rs+2s+1)(rs+s+2)(rs+s+6)}{12(rs+s-1)(s+r+2)(s+r)}C_{r+1}.$$

*Proof.* We look at the virtual degeneracy locus of the morphism $\mathcal{F}\otimes\mathcal{N}\to\sigma^*(\tilde{\mathbb{E}})$ along the surface $\sigma^*(C^0)$. The first thing to note is that the vector bundles $\mathcal{F}_{|\sigma^*(C^0)}$ and $\mathcal{N}_{|\sigma^*(C^0)}$ are both pull-backs of vector bundles on $Y$. For convenience we denote this vector bundles also by $\mathcal{F}$ and $\mathcal{N}$, hence to use the notation of Proposition 2.2, $\mathcal{F}_{|\sigma^*(C^0)}) = \epsilon^*(\mathcal{F}_{|Y})$ and $\mathcal{N}_{|\sigma^*(C^0)} = \epsilon^*(\mathcal{N}_{|Y})$. We find that

$$\sigma^*(C^0)\cdot c_2(\sigma^*\tilde{\mathbb{E}}-\mathcal{F}\otimes\mathcal{N}) = c_2(\sigma^*\tilde{\mathbb{E}}_{|Y})-c_1(\sigma^*\tilde{\mathbb{E}}_{|Y})\cdot c_1(\mathcal{F}\otimes\mathcal{N}_{|Y})+c_1^2(\mathcal{F}\otimes\mathcal{N}_{|Y})-c_2(\mathcal{F}\otimes\mathcal{N}_{|Y}),$$

and like in the proof of Theorem 2.11, we are going to compute each term in this expression. We denote by $V := \mathrm{Ker}(\chi)$, where $\chi : \mathcal{B}^\vee \to \pi_2^*(\mathcal{M})^\vee$ is the bundle morphism coming from Proposition 2.7. Thus $V$ is a line bundle on $Y$ with fibre

$$V(y,L) = \frac{H^1(C, L\otimes\mathcal{O}_C(-y-q))^\vee}{H^1(C,L)^\vee}$$

over each point $(y,L)\in Y$. By using again the Harris-Tu Theorem, we find the following formulas for the Chern numbers of $V^\vee$ (cf. (4) and (5)):

$$c_1(V^\vee)\cdot\xi_{|Y} = c_{r+1}\big(\pi_2^*(\mathcal{M})^\vee-\mathcal{B}^\vee\big)\cdot\xi_{|Y} = \big(\pi_2^*(c_{r+1})+\pi_2^*(c_r)((d-1)\eta+\gamma)-2\pi_2^*(c_{r-1})\eta\theta\big)\cdot\xi_{|Y},$$

for any class $\xi\in H^2(C\times W^r_d(C))$, and

$$c_1^2(V^\vee) = c_{r+2}\big(\pi_2^*(\mathcal{M})^\vee-\mathcal{B}^\vee\big) = \pi_2^*(c_{r+1})((d-1)\eta+\gamma)-2\pi_2^*(c_r)\eta\theta.$$

To evaluate the Chern numbers of $\mathcal{N}_{|Y}$ we fit the line bundle $V$ in the following exact sequence:

$$0\longrightarrow \pi_2^*R^1\pi_{2*}\big(\mathcal{P}_{|C\times W^r_d(C)}\big)^\vee\longrightarrow\mathcal{N}_{|Y}\longrightarrow V\longrightarrow 0. \tag{10}$$

This allows us to compute $c_1(V^\vee)$ and $c_1^2(V^\vee)$ and then we can write that

$$\sigma^*(C^0)\cdot c_2(\sigma^*\tilde{\mathbb{E}}-\mathcal{F}\otimes\mathcal{N}) = \eta\cdot c_1(\mathcal{F}\otimes\mathcal{N}_{|Y})+c_1^2(\mathcal{F}\otimes\mathcal{N}_{|Y})-c_2(\mathcal{F}\otimes\mathcal{N}_{|Y}) =$$

$$= \binom{r+2}{2}c_1^2(V^\vee)+\Big((r+1)^2\cdot\theta+2(r+1)(r+1)\cdot\eta+((r+1)(s-r-1)+1)\cdot c_1\Big)c_1(V^\vee)+\cdots,$$

where the term we omitted is a quadratic polynomial in $\theta,\eta$ and $\gamma$ which will be multiplied by the class $[Y]$. Using repeatedly Propositions 2.5 and 2.6, we finally evaluate all the terms and obtain the stated expression for $b_0$ using the relation $(2g-2)b_0-b_1 = \sigma^*(C^0)\cdot c_2(\sigma^*\tilde{\mathbb{E}}-\mathcal{F}\otimes\mathcal{N})$. ☐

We finish the calculation of $s(\overline{\mathcal{GP}}^r_{g,d})$ by proving the following result:

**Theorem 2.13.** *Let $[C,q]\in\mathcal{M}_{rs+s,1}$ be a suitably general pointed curve and $L\in W^r_d(C)$ any linear series with a cusp at $q$. Then the multiplication map*

$$\mu_0(L,q): H^0(C,L)\otimes H^0(K_C\otimes L^\vee\otimes\mathcal{O}_C(2q))\to H^0(C,K_C\otimes\mathcal{O}_C(2q))$$

*is injective. If $\overline{\mathcal{GP}}^r_{g,d}\equiv a\lambda-\sum_{j=0}^{[g/2]}b_j\delta_j\in\mathrm{Pic}(\overline{\mathcal{M}}_g)$, we have the relation $a-12b_0+b_1=0$.*

*Proof.* We consider again the pencil $R\subset\overline{\mathcal{M}}_g$ obtained by attaching to $C$ at the point $q$ a pencil of plane cubics. It is well-known that $R\cdot\lambda=1$, $R\cdot\delta_0=12$ and $R\cdot\delta_1=-1$, thus the relation $a-12b_0+b_1=0$ would be immediate once we show that $R\cap\overline{\mathcal{GP}}^r_{g,d}=\emptyset$. Assume by contradiction that $R\cap\overline{\mathcal{GP}}^r_{g,d}\neq\emptyset$ and then according to Proposition 2.1 there exists $L \in W^r_d(C)$ with $h^0(L \otimes \mathcal{O}_C(-2q)) = r$ such that the multiplication map $\mu_0(L, q)$ is not injective.

We degenerate $[C, q]$ to $[C_0 := E_0 \cup_{p_1} E_1 \cup_{p_2} E_2 \cup \ldots \cup_{p_{g-2}} E_{g-2}, p_0] \in \overline{\mathcal{M}}_{g-1,1}$, consisting of a string of elliptic curves such that $p_0 \in E_0$ and the differences $p_{i+1} - p_i \in \mathrm{Pic}^0(E_i)$ for $0 \leq i \leq g-3 = rs + s - 2$, are not torsion classes. For each $0 \leq i \leq g-2$ we denote by $L^i \in \mathrm{Pic}^d(C_0)$ the unique limit of the line bundles $L_t \in \mathrm{Pic}^d(C_t)$ having the property that $\deg(L^i_{|E_j}) = 0$ for $i \neq j$. Here $[C_t, p_t] \in \mathcal{M}_{g-1,1}$ is a 1-dimensional family of smooth pointed curves with the property $\lim_{t\to 0}[C_t, p_t] = [C_0, p_0] \in \overline{\mathcal{M}}_{g-1,1}$ and where we also assume that Ker $\mu_0(L_t, p_t) \neq 0$ for all $t \neq 0$.

Similarly, we define $M^i \in \mathrm{Pic}^{2g-2-d}(C_0)$ to be the unique limit of the line bundles $K_{C_t} \otimes L_t^\vee \otimes \mathcal{O}_{C_t}(2p_t)$ characterized by the property $\deg(M^i_{|E_j}) = 0$ for $i \neq j$. We denote by $\{(L^i_{|E_i}, V_i)\}_{i=0}^{g-2}$ and by $\{(M^i_{|E_i}, W_i)\}_{i=0}^{g-2}$ the limit linear series on $C_0$ corresponding to $L_t$ and $K_{C_t} \otimes L_t^\vee \otimes \mathcal{O}_{C_t}(2p_t)$ respectively as $t \to 0$. Reasoning along the lines of [EH3] or [F1] Proposition 3.2, for each $0 \leq i \leq g-2$ we find elements

$$0 \neq \rho_i \in \mathrm{Ker}\{V_i \otimes W_i \to H^0(E_i, L^i \otimes M^i_{|E_i})\}$$

satisfying $\mathrm{ord}_{p_{i+1}}(\rho_{i+1}) \geq \mathrm{ord}_{p_i}(\rho_i) + 2$ for $0 \leq i \leq g-3$. Since $\mathrm{ord}_{p_0}(\rho) \geq 2$, we find that $\mathrm{ord}_{p_{g-2}}(\rho_{g-2}) \geq 2g-2 = \deg(L^{g-2} \otimes M^{g-2}_{|E_{g-2}})$, which is impossible. □

## 3. MAPS BETWEEN MODULI SPACES OF CURVES

We begin the study of the map $\phi : \overline{\mathcal{M}}_g - - > \overline{\mathcal{M}}_{g'}$ given by $\phi([C]) := [W^r_d(C)]$ in the case $\rho(g, r, d) = 1$, so that $g = rs + s + 1$ and $d = rs + r + 1$. The genus of $W^r_d(C)$ for a general $[C] \in \mathcal{M}_g$ has been computed in [EH2] Theorem 4, and we have the formula:

$$g' = g(W^r_d(C)) = 1 + g!\, \frac{s(r+1)}{s+r+1} \prod_{i=0}^{r} \frac{i!}{(s+i)!}. \tag{11}$$

We shall describe the pull-back map $\phi^* : \mathrm{Pic}(\overline{\mathcal{M}}_{g'}) \to \mathrm{Pic}(\overline{\mathcal{M}}_g)$ and to avoid confusion we denote, as usual, by $\lambda, \delta_0, \ldots, \delta_{[g/2]}$ the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$, and by $\lambda', \delta'_0, \ldots, \delta'_{[g'/2]}$ the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{g'})$. We start by describing the map $\phi$ over a generic point of each boundary divisors $\Delta_j$ for $0 \leq j \leq [g/2]$. If $[C^j_y := C \cup_y D] \in \Delta_j$ is a general point, then $\phi([C^j_y])$ is the stable reduction of the variety $\overline{G}^r_d(C^j_y)$ of limit linear series $\mathfrak{g}^r_d$. Our analysis shows that $\overline{G}^r_d(C^j_y)$ is always a semi-stable curve and this observation completely determines $\phi$ in codimension 1.

Suppose that $[C] \in \mathcal{M}_{rs+s}$ is a Brill-Noether-Petri general curve and that $[E, y] \in \mathcal{M}_{1,1}$ is a pointed elliptic curve. We recall that we have introduced the smooth surface $X = \{(y, L) \in C \times W^r_d(C) : h^0(C, L \otimes \mathcal{O}_C(-2y)) = r\}$. For $y \in C$ we denote by $X_y := \pi_1^{-1}(y)$ the fibre of the first projection $\pi_1 : X \to C$. For each of the $n_0$ linear series $A \in W^r_{d-1}(C)$ there exists a section $\sigma_A : C \to X$ given by $\sigma_A(y) = (y, A \otimes \mathcal{O}_C(y))$ and we set $\Sigma_A := \mathrm{Im}(\sigma_A)$. From the description given in Proposition 2.1, it follows that $\phi([C \cup_y E])$ is the stable curve of genus $g'$ obtained by attaching to the spine $X_y$ copies of $E \cong \mathrm{Pic}^{r+1}(E)$ at the points $\sigma_A(y)$ for each $A \in W^r_{d-1}(C)$.

Similarly, having fixed a general pointed curve $[C, q] \in \mathcal{M}_{rs+s,1}$, we recall that we have introduced the surface $Y = \{(y, L) \in C \times W^r_d(C) : h^0(L \otimes \mathcal{O}_C(-y-q)) = r\}$. (cf. Proposition 2.2). For each $y \in Y$ we set $Y_y := \pi_1^{-1}(y)$. To every $A \in W^r_{d-1}(C)$ correspond two sections $u_A : C \to Y$, $v_A : C \to Y$ given by $u_A(y) = (y, A \otimes \mathcal{O}_C(y))$ and $v_A(y) = (y, A \otimes \mathcal{O}_C(q))$ respectively. If as before, we denote by $[C^0_y] := [C/y \sim q] \in \Delta_0$, then $\phi([C^0_y])$ is the stable curve obtained from $Y_y$ by identifying the points $v_A(y)$ and $u_A(y)$ for all linear series $A \in W^r_{d-1}(C)$.

For $2 \leq j \leq [g/2]$, the irreducible components of $\phi([C^j_y])$ are indexed by Schubert indices $\overline{\alpha} := (\alpha_0 \leq \ldots \leq \alpha_r)$ such that there exists limit linear series $l = \{l_C, l_D\} \in \overline{G}^r_d(C^j_y)$ with $\alpha^{l_C}(y) = \overline{\alpha}$, $\rho(l_C, y) \in \{0, 1\}$ and $\rho(l_C, y) + \rho(l_D, y) = 1$ (a precise list of such $\overline{\alpha}$'s is given in the proof of Theorem 3.4). To describe the pull-backs of the tautological classes under $\phi$ we need a description of the numerical properties of the push-forwards under $\phi$ of the standard test curves $R$ and $C^j$ where $0 \leq j \leq [g/2]$. We carry this out in detail only for $j = 0, 1$ which is sufficient to compute the slopes of pull-backs $\phi^*(D')$ where $[D'] \in \mathrm{Pic}(\overline{\mathcal{M}}_{g'})$. The case $j \geq 2$ is quite similar and again we skip these details. To keep our formulas relatively simple we only deal with the case $r = 1$, when $g = 2s + 1$ and

$$\phi : \overline{\mathcal{M}}_{2s+1} - - > \overline{\mathcal{M}}_{1+\frac{s}{s+1}\binom{2s+2}{s}}.$$

**Proposition 3.1.** *We fix a general pointed curve $[C, q] \in \mathcal{M}_{2s,1}$ and we consider the test curve $R \subset \overline{\mathcal{M}}_{2s+1}$ obtained by attaching a pencil of plane cubics to $C$ at the fixed point $q$. If $n_0 := \#(W^1_{s+2}(C))$, then we have the following relations:*

$$\phi_*(R) \cdot \lambda' = n_0, \ \ \phi_*(R) \cdot \delta'_0 = 12n_0, \ \ \phi_*(R) \cdot \delta'_1 = -n_0, \textit{ and } \phi_*(R) \cdot \delta'_j = 0 \textit{ for } j \geq 2.$$

*Proof.* We denote by $f : \widetilde{\mathbf{P}}^2 := \mathrm{Bl}_9(\mathbf{P}^2) \to \mathbf{P}^1$ the fibration induced by a pencil of plane cubics after blowing-up the 9 base points of the pencil. Since $f$ has 9 sections, there is an isomorphism between $f$ and its Picard fibration $\mathrm{Pic}^2(f) \to \mathbf{P}^1$. The curve $\phi_*(R) \subset \overline{\mathcal{M}}_{g'}$ is induced by a fibration of stable curves $\pi : T \to \mathbf{P}^1$, where

$$\pi^{-1}(t) = X_q \bigcup \{\cup_{\sigma_A(q)} \mathrm{Pic}^2(f^{-1}(t)) : A \in W^r_{d-1}(C)\}, \ \text{ for each } t \in \mathbf{P}^1.$$

In other words, $\pi$ is obtained by attaching to the fixed curve $X_q$, $n_0$ copies of the elliptic curve $f^{-1}(t)$ at each of the points $\sigma_A(q)$. The claimed formulas are now immediate. □

**Proposition 3.2.** *We fix general curves $[C] \in \mathcal{M}_{2s}$ and $[E, y] \in \mathcal{M}_{1,1}$ and consider the associated test curve $C^1 \subset \Delta_1 \subset \overline{\mathcal{M}}_{2s+1}$. Then we have the formulas*

$$\phi_*(C^1) \cdot \lambda' = n_0 \frac{2s(s-1)(6s^2 + 10s + 1)}{s+2}, \ \ \phi_*(C^1) \cdot \delta'_0 = C^1 \cdot \overline{\mathcal{GP}}^1_{2s+1,s+2},$$

$$\phi_*(C^1) \cdot \delta'_1 = -n_0(4s-2) \textit{ and } \phi_*(C^1) \cdot \delta'_j = 0 \textit{ for } j \geq 2.$$

*Proof.* The 1-cycle $\phi_*(C^1)$ corresponds to a family of curves constructed as follows: We start with $\pi_1 : X \to C$ and consider the sections $\{\sigma_A : C \to X\}_{A \in W^1_{s+1}(C)}$. We also consider $n_0$ disjoint copies of the trivial family $C \times E \to C$ which we glue to $\pi_1$ along each of the sections $\sigma_A$. From this description it follows that $\phi_*(C^1) \cdot \delta'_0 = C^1 \cdot \overline{\mathcal{GP}}^1_{2s+1,s+2}$ and this equals the number of points $y \in C$ (counted with the appropriate multiplicities) such that $X_y$ is singular at some point $(y, L)$. This translates into saying that the Petri map $\mu_0(L, y) : H^0(C, L) \otimes H^0(K_C \otimes L^\vee \otimes \mathcal{O}_C(2y)) \to H^0(C, K_C \otimes \mathcal{O}_C(2y))$ is not injective. Also, $\phi_*(C^1) \cdot \delta_j' = 0$ for $j \geq 2$. Using the description of $N_{\Delta_1'/\overline{\mathcal{M}}_{g'}}$ we have that

$$\phi_*(C^1) \cdot \delta_1' = \sum_{A \in W^1_{s+1}(C)} (\Sigma_A)^2 = \sum_{A \in W^1_{s+1}(C)} \Big( 2g(C) - 2 - \Sigma_A \cdot K_X \Big).$$

To estimate this sum, we recall that we have computed the canonical class of $X$ (cf. (7)):

$$K_X \equiv 2\theta + (s-2) \cdot \pi_2^*(c_1) + 2\gamma + 10s \cdot \eta.$$

By direct computation we obtain that $\Sigma_A \cdot \theta = \sigma_A^*(\theta) = 2s, \Sigma_A \cdot \eta = 1$ and $\Sigma_A \cdot \gamma = -4s$ (all these intersection numbers are being computed on the smooth surface $X$).

We now compute $\Sigma_A \cdot \pi_2^*(c_1)$ and note that $\sigma_A^* \pi_2^*(\mathcal{M}) = (p_2)_*\big(\mu^*(\mathcal{P}_{|C \times W^r_d(C)})\big)$, where $\mu : C \times C \to C \times \mathrm{Pic}^d(C)$ is defined as $\mu(x, y) = (x, A \otimes \mathcal{O}_C(y))$ and $p_1, p_2 : C \times C \to C$ are the two projections. The key observation here is that if the Poincaré bundle $\mathcal{P}$ is chosen in such a way that $\mathcal{P}_{|\{q\} \times \mathrm{Pic}^d(C)}$ is trivial for a point $q \in C$, then

$$\mu^*(\mathcal{P}) = p_1^*(A) \otimes \mathcal{O}_{C \times C}(\Delta) \otimes p_2^*(\mathcal{O}_C(-q)),$$

hence $(p_2)_* \mu^*(\mathcal{P}) = (p_2)_*\big(p_1^* A \otimes \mathcal{O}_{C \times C}(\Delta)\big) \otimes \mathcal{O}_C(-q)$. Then we note that the vector bundle $(p_2)_*\big(p_1^*(A) \otimes \mathcal{O}_{C \times C}(\Delta)\big)$ is trivial, thus it follows that $\deg\big(\sigma_A^* \pi_2^*(\mathcal{M}^\vee)\big) = \mathrm{rank}(\mathcal{M}) = h^0(A) = 2$. Putting these calculations together, we obtain that $\Sigma_A \cdot K_X = 8s - 4$ and then $(\Sigma_A^2) = -4s + 2$, that is, $\phi_*(C^1) \cdot \delta_1' = -n_0(4s - 2)$.

We are left with the computation of $\phi_*(C^1) \cdot \lambda'$, which equals the degree of the Hodge bundle over the family $\pi_1 : X \to C$. From the Mumford relation $\kappa_1 = 12\lambda - \delta$ we find that

$$\phi_*(C^1) \cdot \lambda' = \frac{K^2_{X/C} + \delta(\pi_1)}{12} = \frac{K^2_{X/C} + C^1 \cdot \overline{\mathcal{GP}}^1_{2s+1,s+2}}{12},$$

where $K_{X/B} = K_X - \pi_1^*(K_C)$ is the relative canonical class. A direct calculation involving Propositions 2.5 and 2.6 shows that

$$K^2_{X/C} = (6s^3 - 10s^2 - 4s)\pi_2^*(c_1^2) + (24s^2 - 32s + 16)\pi_2^*(c_1) \cdot \theta + 24s^2\theta^2.$$

The calculation of $[\overline{\mathcal{GP}}^1_{2s+1,s+2}]$ (precisely Theorem 2.11), yields that $C^1 \cdot \overline{\mathcal{GP}}^1_{2s+1,s+2} = 4n_0 s(s-1)(12s^2 + 23s + 8)/(s+2)$, which leads to the stated formula for $\phi_*(C^1) \cdot \lambda'$. □

**Proposition 3.3.** *We fix a general pointed curve* $[C, q] \in \mathcal{M}_{2s,1}$ *and consider the test curve* $C^0 \subset \Delta_0 \subset \overline{\mathcal{M}}_{2s+1}$. *Then we have the following formulas:*

$$\phi_*(C^0) \cdot \delta_1' = n_0, \ \phi_*(C^0) \cdot \delta_0' = C^0 \cdot \overline{\mathcal{GP}}^1_{2s+1,s+2} - 4n_0 s,$$

$$\phi_*(C^0) \cdot \lambda' = n_0 \frac{s(s-1)(2s^2 + 4s + 1)}{s+2} \ \ \textit{and} \ \phi_*(C^0) \cdot \delta_j' = 0 \ \textit{for} \ j \geq 2.$$

*Proof.* We describe the family of stable curves inducing $\phi_*(C^0)$. We start with the family $\pi_1 : Y \to C$ and consider the sections $\{u_A, v_A : C \to Y\}_{A \in W^1_{s+1}(C)}$ with images $U_A := u_A(C)$ and $V_A := v_A(C)$ respectively. We denote by $Y'$ the blow-up of $Y$ at the $n_0$ points of intersections $\{U_A \cap V_A = (q, A \otimes \mathcal{O}_C(q))\}_{A \in W^1_{s+1}(C)}$ (see also Proposition 2.2), and we denote by $\widetilde{U}_A$ and $\widetilde{V}_A$ the strict transforms of $U_A$ and $V_A$ respectively. Then $\phi_*(C^1) \subset \overline{\mathcal{M}}_{g'}$ is induced by the fibration $\pi : \widetilde{Y} \to C$, where $\widetilde{Y} = \sigma^*(C^0)$ is the surface obtained from $Y'$ by identifying the sections $\widetilde{U}_A$ and $\widetilde{V}_A$ for each $A \in W^1_{s+1}(C)$. The numerical characters of $\phi_*(C^0)$ are now easily describable. We have that $\phi_*(C^0) \cdot \delta'_j = 0$ for $j \geq 2$, $\phi_*(C^0) \cdot \delta'_1 = n_0$, and

$$\phi_*(C^0) \cdot \delta'_0 = C^0 \cdot \overline{\mathcal{GP}}^1_{2s+1,s+2} + \sum_{A \in W^1_{s+1}(C)} \Big((U_A)^2 + (V_A)^2 - 2\Big).$$

We recall that we have computed the canonical class of $Y$ (cf. (8)):

$$K_Y \equiv 2\theta + (s-2)\pi_2^*(c_1) + \gamma + (5s-1) \cdot \eta.$$

Since $u_A^*(\theta) = g(C) = 2s,\ u_A^*(\gamma) = -4s,\ u_A^*(\eta) = 1$ and $u_A^*\pi_2^*(c_1) = 2$ (the proof of this last equality follows from the calculation in Proposition 3.2), we find that $K_Y \cdot U_A = 7s - 5$, hence by the adjunction formula $(U_A)^2 = -3(s-1)$. Similarly, $(V_A)^2 = -s-1$, therefore $(U_A)^2 + (V_A)^2 - 2 = -4s$, for every $A \in W^1_{s+1}(C)$, which determines $\phi_*(C^0) \cdot \delta'_0$.

We still have to estimate $\phi_*(C^0) \cdot \lambda'$. Like in Proposition 3.2, using Mumford's relation, this number equals the degree of the Hodge bundle on the family $\pi_1 : Y \to C$:

$$\phi_*(C^0) \cdot \lambda' = \frac{K^2_{Y/C} + \delta(\pi_1)}{12} = \frac{K^2_{Y/C} + C^0 \cdot \overline{\mathcal{GP}}^1_{2s+1,s+2}}{12}.$$

From Theorem 2.11 we know that $C^0 \cdot \overline{\mathcal{GP}}^1_{2s+1,s+2} = 2n_0 s(s-1)(4s^2+9s+4)/(s+2)$. By direct computation we also obtain that

$$K^2_{Y/C} = (s^3 - s^2 - 2s)\pi_2^*(c_1^2) + (4s^2 - 4s + 2)\pi_2^*(c_1) \cdot \theta + 4(s-1)\theta^2.$$

Moreover, $\pi_2^*(c_1^2) = n_0(4s+2)/(s+2),\ \pi_2^*(c_1) \cdot \theta = 2n_0 s$ and $\theta^2 = n_0 s(s+1)$ (all these intersection numbers are being computed on $Y$ using Proposition 2.5). This completes the calculation of $\phi_*(C^0) \cdot \lambda'$. □

We are in a position to describe pull-backs of divisors classes under the map $\phi$:

**Theorem 3.4.** *We consider the rational map* $\phi : \overline{\mathcal{M}}_g - - > \overline{\mathcal{M}}_{g'},\ \phi[C] = [W^1_{s+2}(C)]$, *where*

$$g := 2s+1 \ \text{and}\ g' := 1 + \frac{s}{s+1}\binom{2s+2}{s}.$$

*We then have the following description of the map* $\phi^* : \mathrm{Pic}(\overline{\mathcal{M}}_{g'}) \to \mathrm{Pic}(\overline{\mathcal{M}}_g)$:

$$\phi^*(\lambda') = n_0\Big(\frac{6s^4 + 20s^3 - s^2 - 20s - 2}{(s+2)(2s-1)}\lambda - \frac{s(s^2-1)}{2s-1}\delta_0 - \frac{2s(s-1)(6s^2+10s+1)}{(s+2)(4s-2)}\delta_1 - \sum_{i=2}^{[g'/2]} b_i\delta_i\Big),$$

*where* $b_i \geq \frac{s(s^2-1)}{2s-1}$ *for* $2 \leq i \leq [g/2]$,

$$\phi^*(\delta'_0) = n_0 \cdot \delta_0 + [\overline{\mathcal{GP}}^1_{2s+1,s+2}],\ \phi^*(\delta'_1) = n_0 \cdot \delta_1 \ \textit{and}\ \phi^*(\delta'_j) = 0 \ \textit{for}\ 2 \leq j \leq [g'/2].$$

*Proof.* The formulas involving $\phi^*(\lambda'), \phi^*(\delta'_0)$ and $\phi^*(\delta'_1)$ are consequences of Propositions 3.2 and 3.3 via the push-pull formula. To prove that $\phi^*(\delta'_j) = 0$ for $2 \leq j \leq [g'/2]$, we show that $\phi_*(\overline{\mathcal{M}}_g) \cap \Delta'_j = \emptyset$. This follows once we note that (i) the generic point of every component of $\overline{\mathcal{GP}}^1_{2s+1,s+2}$ corresponds to a curve $[C] \in \mathcal{M}_{2s+1}$ for which $W^1_{s+2}(C)$ is irreducible with precisely one node, and that (ii) $\phi_*(\Delta_j) \subset \overline{\mathcal{M}}_{g'} - \bigcup_{i\geq 2}^{[g'/2]} \Delta'_i$ for every $2 \leq j \leq [g/2]$. Indeed, let us fix a general point $[C^j_y := C \cup_y D] \in \Delta_j$ where $[C, y] \in \mathcal{M}_{j,1}$ and $[D, y] \in \mathcal{M}_{g-j,1}$ are Brill-Noether general pointed curves. For a real number $t$ we introduce the notation $t_+ := \max\{t, 0\}$. The irreducible components of the stable curve $\phi([C^j_y])$ are indexed by the set $\mathcal{P}_j$ of Schubert indices

$$\overline{\alpha} := \big(0 \leq \alpha_0 \leq \ldots\ldots \leq \alpha_r \leq rs+1\big)$$

satisfying the conditions (cf. [EH2], Proposition 1.2):

$$\text{(12)} \quad \sum_{i=0}^{r}(\alpha_i + j - rs - 1) \in \{j-1, j\}, \quad \sum_{i=0}^{r}(\alpha_i + j - rs - 1)_+ \leq j \text{ and } \sum_{i=0}^{r}(g - j - \alpha_i)_+ \leq g - j.$$

For $\alpha \in \mathcal{P}_j$ we consider the (non-empty) variety $\overline{G}^r_d(X)_{\overline{\alpha}} := \{l \in \overline{G}^r_d(X) : \alpha^{l_C}(y) \geq \overline{\alpha}\}$ which is a disjoint union of irreducible components of $\phi([X])$. When $j \geq 2$, we claim that the stable curve $\phi([C^j_y])$ is not of compact type. Using (12) one checks that for every $\alpha \in \mathcal{P}_j$ there are at least two partitions $\overline{\beta}_1, \overline{\beta}_2 \in \mathcal{P}_j$ such that $\overline{G}^r_d(X)_{\overline{\alpha}} \cap \overline{G}^r_d(X)_{\overline{\beta}_k} \neq \emptyset$ for $k = 1, 2$. Thus for every component $Z$ of $\phi([C^j_y])$ we have that $\#\big(Z \cap \overline{\phi([C^j_y]) - Z}\big) \geq 2$, which proves our claim. $\square$

Theorem 3.4 contains enough information to encode the slope of the pull-backs $\phi^*(D)$ for all classes $D \in \mathrm{Pic}(\overline{\mathcal{M}}_{g'})$ and thus to prove Theorem 0.2: If $s(D) = c$, then we have the following formula for the slope of $\phi^*(D) \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$:

$$s(\phi^*(D)) = 6 + \frac{8s^3(c-4) + 5cs^2 - 30s^2 + 20s - 8cs - 2c + 24}{s(s+2)(cs^2 - 4s^2 - c - s + 6)}.$$

## 4. The map $\phi$ in small genus and applications to Prym varieties

In this section we denote by $\mathcal{R}_g$ the stack of étale double covers of smooth curves of genus $g$ and by $\overline{\mathcal{R}}_g$ its compactification by means of Beauville admissible double covers, cf. [B]. It is proved in [BCF] that $\overline{\mathcal{R}}_g$ is isomorphic to the stack parameterizing *Prym curves* of genus $g$, that is, data of the form $(X, L, \beta)$, where $X$ is a quasi-stable curve with $p_a(X) = g$, $L \in \mathrm{Pic}^0(X)$ is a line bundle such that $L_{|R} = \mathcal{O}_R(1)$ for every destabilizing rational component $R \subset X$ with $\#(R \cap (\overline{X - R})) = 2$, and $\beta : L^{\otimes 2} \to \mathcal{O}_X$ is a sheaf homomorphism whose restriction to the generic point of each component of $X$ is non-zero. One has a finite branched cover $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ and a regular morphism $\chi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_{2g-1}$ which assigns to an admissible double cover the stable model of its source curve. We set $\lambda := \pi^*(\lambda) \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$ and define the following three irreducible boundary divisors in $\overline{\mathcal{R}}_g$:

- $\Delta_0^{'}$, with generic point being a Prym curve $t := [C^0_y := C/y \sim q, L]$, where $[C, y, q] \in \mathcal{M}_{g-1,2}$ and $L \in \mathrm{Pic}^0(C^0_y)[2]$ is a line bundle such that if $\nu : C^0_y \to C$ denotes the normalization map, then $\nu^*(L) \neq \mathcal{O}_C$. If $\tilde{C} \to C$ is the étale $2:1$ cover induced by $\nu^*(L)$ and $y_i, q_i (i = 1, 2)$ are the inverse images of $y$ and $q$, then $\chi(t) = [\tilde{C}/y_1 \sim q_1, y_2 \sim q_2]$.

• $\Delta_0^{''}$, with generic point corresponding to $t := [C_y^0, L]$ as above, but where $\nu^*(L) = \mathcal{O}_C$. In this case $\chi(t)$ consists of two copies $[C_i, y_1, q_i]$ $(i = 1, 2)$ of $[C, y, q]$, where we identify $y_1$ with $q_2$ and $y_2$ and $q_1$ respectively.

• $\Delta_0^r$, with generic point corresponding to a Prym curve $t := [X := C \cup_{\{y,q\}} \mathbf{P}^1, L]$, with $[C, y, q] \in \mathcal{M}_{g-1,2}$ and $L \in \mathrm{Pic}(C)$ is a line bundle such that $L^{\otimes 2} = \mathcal{O}_C(-y-q)$. In this case, if $\tilde{C} \to C$ is the double cover induced by $L_{|C}$ and branched at $y$ and $q$ and if $\tilde{y}, \tilde{q} \in \tilde{C}$ are the ramification points above $y$ and $q$ respectively, then $\chi(t) = [\tilde{C}/\tilde{y} \sim \tilde{q}]$.

For a straightforward dictionary between Beauville covers and Prym curves we refer to [D]. Note that $\pi^*(\Delta_0) = \Delta_0' + \Delta_0^{''} + 2\Delta_0^r$ and $\Delta_0^r$ is the ramification locus of $\pi$. As usual, we set $\delta_0^{'} := [\Delta_0^{'}], \delta_0^{''} := [\Delta_0^{''}]$ and $\delta_0^r := [\Delta_0^r] \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$. We also denote by $p : \mathcal{C} \to \overline{\mathcal{R}}_g$ the universal curve and by $\mathcal{L}$ the line bundle over $\mathcal{C}$ whose restriction to each fibre of $p$ is the underlying line bundle corresponding to a Prym curve. In [F3], for each $i \geq 1$ we introduce the tautological vector bundles $\mathbb{E}_i := p_*(\omega_p \otimes \mathcal{L}^{\otimes i})$ over $\overline{\mathcal{R}}_g$ and we show that

$$c_1(\mathbb{E}_i) = \binom{i}{2}\pi^*(\kappa_1) + \lambda - \frac{i^2}{4}\delta_0^r. \qquad (13)$$

We discuss the geometry of the rational map $\phi : \overline{\mathcal{M}}_g --> \overline{\mathcal{M}}_{g'}$ for small values of $g = 2s+1$. When $s = 1$, then $g = g' = 3$ and the map $\phi : \overline{\mathcal{M}}_3 \to \overline{\mathcal{M}}_3$ is simply the identity. Indeed, for a smooth curve $[C] \in \mathcal{M}_3$, we have a natural isomorphism $C \cong W_3^1(C)$ given by $C \ni y \mapsto K_C \otimes \mathcal{O}_C(-y)$ (Note that this isomorphism extends over the hyperelliptic locus as well, when $W_3^1(C) = C + W_2^1(C)$).

The first truly interesting case is $s = 2$, when we have a map $\phi : \overline{\mathcal{M}}_5 --> \overline{\mathcal{M}}_{11}$

$$\phi([C]) := [W_4^1(C)] = [\mathrm{Sing}(\Theta_C)].$$

By duality there is an involution $\tau : W_4^1(C) \to W_4^1(C)$ given by $\tau(L) = K_C \otimes L^\vee$. For $[C] \in \mathcal{M}_5 - \mathcal{GP}_{5,4}^1$ (that is, when $C$ is not trigonal and possesses no vanishing theta-nulls), $\tau$ has no fixed points and it induces an étale $2:1$ cover $f : W_4^1(C) \to \Gamma$, where $[\Gamma] \in \mathcal{M}_6$. Therefore $\phi$ factors to give a map $\nu : \overline{\mathcal{M}}_5 --> \overline{\mathcal{R}}_6$. Moreover, there is an isomorphism of principally polarized abelian varieties of dimension 5:

$$\big(\mathrm{Prym}(W_4^1(C)/\Gamma), \Xi\big) \cong \big(\mathrm{Jac}(C), \Theta_C\big)$$

(see [ACGH] pg. 296-301 or [DS] for details on this classically understood situation). The genus 6 curve $\Gamma$ is identified with the locus of rank 4 quadrics containing the canonical curve $C \subset \mathbf{P}^4$, and if $Q \in \Gamma$ is such a quadric, then $f^{-1}(Q)$ consists of the $\mathfrak{g}_4^1$'s determined by the two rulings on $Q$. If $[C] \in \mathcal{M}_5 - \mathcal{GP}_{5,4}^1$ then $\Gamma$ is a smooth plane quintic. When $[C] \in \mathcal{GP}_{5,4}^{1,0}$, the curve $\Gamma$ has nodes at the points corresponding to quadrics of rank 3. We have the following result which completely determines $\phi$ in codimension 1:

**Proposition 4.1.** *The image of the rational map $\phi : \overline{\mathcal{M}}_5 --> \overline{\mathcal{M}}_{11}$ given by $\phi([C]) = [W_4^1(C)]$ equals the closure $\overline{\mathcal{MQ}}^+$ of the locus of genus 11 curves which are even étale double covers of smooth plane quintic curves.*

• *For a trigonal curve $[C] \in \mathcal{M}_{5,3}^1$, if $A \in W_3^1(C)$ denotes the unique $\mathfrak{g}_3^1$, then $\phi([C])$ consists of two copies of $C$ joined together at two points $x, y \in C$ such that $x + y = |K_C \otimes A^{\otimes(-2)}|$.*

• *For a curve* $[C \cup_y E] \in \Delta_1 \subset \overline{\mathcal{M}}_5$ *where* $g(C) = 4$ *and* $g(E) = 1$, $\phi([C \cup_y E])$ *is a stable curve of compact type consisting of a genus* 9 *spine* $\{L \in W^1_4(C) : h^0(L \otimes \mathcal{O}_C(-2y)) \geq 1\}$ *and two elliptic tails isomorphic to* $E$ *attached at the points* $A \otimes \mathcal{O}_C(y)$ *where* $A \in W^1_3(C)$.

• *For a curve* $[C/y \sim q] \in \Delta_0 \subset \overline{\mathcal{M}}_5$ *where* $[C, y, q] \in \mathcal{M}_{4,2}$, $\phi([C/y \sim q])$ *is the irreducible stable curve obtained from the smooth genus* 9 *curve* $\{L \in W^1_4(C) : h^0(L \otimes \mathcal{O}_C(-y-q)) \geq 1\}$ *by identifying the two pairs of points* $A \otimes \mathcal{O}_C(y)$ *and* $A \otimes \mathcal{O}_C(q)$ *for every* $A \in W^1_3(C)$.

• *For a curve* $[C \cup_y D] \in \Delta_2 \subset \overline{\mathcal{M}}_5$ *where* $g(C) = 3$ *and* $g(D) = 2$, $\phi([C \cup_y D])$ *is a stable curve of genus* 11 *consisting of two disjoint copies* $Y_1$ *and* $Y_2$ *of* $C$ *and two disjoint copies* $D_1$ *and* $D_2$ *of* $D$, *such that* $Y_i \cap D_j = \{y_{ij}\}$ *for* $i, j = 1, 2$. *The set* $\{y_{1i}, y_{2i}\} \subset D_i$ *consists of* $y \in D$ *and its hyperelliptic conjugate for each* $i = 1, 2$. *The set* $\{y_{1i}, y_{2i}\} \subset Y_i$ *for* $i = 1, 2$, *consists of the pairs of points lying on the tangent line to the smooth plane quartic model of* $C$ *which passes through the point* $y$.

*Proof.* The only case which requires explanation is that when $[C \cup_y D] \in \Delta_2$, when $\phi([C \cup_y D])$ is the stable reduction of the variety $\overline{G}^1_4(C \cup_y D)$ of limit $\mathfrak{g}^1_4$'s on $C \cup_y D$. Components of $\overline{G}^1_4(C \cup_y D)$ are indexed by numerical possibilities for the ramification sequences of a limit linear series $l$ such that $\rho(l_C, y) + \rho(l_D, y) = 1$ and $\rho(l_C, y), \rho(l_D, y) \geq 0$. When $\rho(l_C, y) = 1$ and $\rho(l_D, y) = 0$, we have two numerical possibilities:
**(1)** $a^{l_D}(y) = (1, 4)$, hence $l_D = l^1_D := y + |\mathcal{O}_D(3y)|$ and $a^{l_C}(y) \geq (0, 3)$. Then the curve $Y_1 := \{l \in G^1_4(C) : a^l(y) \geq (0, 3)\} \times \{l^1_D\}$ is an irreducible component of $\phi([C \cup_y D])$.
**(2)** $a^{l_D}(y) = (2, 3)$, hence $l_D = l^2_D := 2y + \mathfrak{g}^1_2 \in G^1_4(D)$ and $a^{l_C}(y) \geq (1, 2)$. Then $Y_2 := \{l \in G^1_4(C) : a^l(y) \geq (1, 2)\} \times \{l^2_D\}$ is another irreducible component of $\phi([C \cup_y D])$.

Before we deal with the remaining case when $\rho(l_C, y) = 0$ and $\rho(l_D, y) = 1$, we note that for a general $[C, y] \in \mathcal{M}_{3,1}$, there are two linear series $l^1_C, l^2_C \in G^1_4(C)$ such that $a^{l^i_C}(y) \geq (1, 3)$. If $\rho(l_C, y) = 0$, then necessarily $a^{l_C}(y) = (1, 3)$, hence $l_C \in \{l^1_C, l^2_C\}$.

We introduce the curves $D_1 := \{l^1_C\} \times \{l \in G^1_4(D) : a^l(y) \geq (1, 3)\}$ and $D_2 := \{l^2_C\} \times \{l \in G^1_4(D) : a^l(y) \geq (1, 3)\}$ which are the remaining two irreducible components of $\phi([C \cup_y D])$. We single out the points $y_{11} = (l^1_C, l^1_D) \in Y_1 \cap D_1$, $y_{12} = (l^2_C, l^1_D) \in Y_1 \cap D_2$, $y_{21} = (l^1_C, l^2_D) \in Y_2 \cap D_1$ and $y_{22} = (l^2_C, l^2_D) \in Y_2 \cap D_2$ and then $\phi([C \cup_y D])$ is the stable curve of genus 11 having irreducible components $Y_1, Y_2, D_1$ and $D_2$ meeting at the points $y_{11}, y_{12}, y_{21}$ and $y_{22}$. □

Proposition 4.1 coupled with Theorem 3.4 allows us to completely describe the pull-back map of divisor classes $\nu^* : \mathrm{Pic}(\overline{\mathcal{R}}_6) \to \mathrm{Pic}(\overline{\mathcal{M}}_5)$.

**Proposition 4.2.** *For* $\nu : \overline{\mathcal{M}}_5 - - > \overline{\mathcal{R}}_6$ *given by* $[C] \mapsto [W^1_4(C)/\Gamma]$, *we have the formulas:*

$$\nu^*(\lambda) = 34\lambda - 4\delta_0 - 15\delta_1 - (?)\delta_2,\ \nu^*(\delta_0^r) = [\overline{\mathcal{GP}}^{1,0}_{5,4}] = 4(33\lambda - 4\delta_0 - 15\delta_1 - 21\delta_2),$$

$$\nu^*(\delta_0') = \delta_0,\ \ \nu^*(\delta_0'') = [\overline{\mathcal{M}}^1_{5,3}] = 8\lambda - \delta_0 - 4\delta_1 - 6\delta_2.$$

*Proof.* Most of this follows directly by comparing Proposition 4.1 with the description of the classes $\delta_0', \delta_0''$ and $\delta_0^r$. Then we use that the generic point of the Teixidor divisor $\overline{\mathcal{GP}}^{1,0}_{5,4}$ corresponds to a curve $[C] \in \mathcal{M}_5$ having precisely one vanishing theta-null (that is, quadric of rank 3 containing the canonical image of $C \subset \mathbf{P}^4$). In such a case the curve of singular quadrics $[\Gamma \subset |I_{C/\mathbf{P}^4}(2)|] \in \overline{\mathcal{M}}_6$ is 1-nodal, the node corresponding precisely to the vanishing theta-null. This implies that $\nu^*(\delta_0^r) = [\overline{\mathcal{GP}}_{5,4}^{1,0}]$. Showing that $\nu^*(\delta_0') = \delta_0$ and $\nu^*(\delta_0'') = [\overline{\mathcal{M}}_{5,3}^1]$ proceeds along similar lines. Finally, we write that

$$35\lambda - 4\delta_0 - 15\delta_1 - \cdots = \phi^*(\lambda) = \nu^*(\chi^*(\lambda)) = \nu^*\big(2\lambda - \frac{1}{4}\,\delta_0^r\big) = 2\nu^*(\lambda) - \frac{1}{4}[\overline{\mathcal{GP}}_{5,4}^{1,0}],$$

which yields the formula for $\nu^*(\lambda)$. □

The main result of [DS] is that the Prym map $\mathrm{Prym} : \mathcal{R}_6 \to \mathcal{A}_5$ is generically finite, of degree 27. We denote by $\mathcal{D}$ the ramification divisor of $\mathcal{R}_6 \to \mathcal{A}_5$ and by $\overline{\mathcal{D}}$ its closure in $\overline{\mathcal{R}}_6$. It is proved in [B] that the codifferential of the Prym map

$$\mathcal{P}^* : T_{\mathrm{Prym}[C,L]}\big(\mathcal{A}_5\big)^\vee \to T_{[C,L]}\big(\mathcal{R}_6\big)^\vee$$

can be identified with the multiplication map $\mathrm{Sym}^2 H^0(C, K_C \otimes L) \to H^0(C, K_C^{\otimes 2})$ (Note that $L^{\otimes 2} = \mathcal{O}_C$). Thus $[C,L] \in \mathcal{D}$ if and only if $C \stackrel{|K_C\otimes L|}{\longrightarrow} \mathbf{P}^4$ lies on a quadric. An immediate application of Proposition 4.2 gives the following characterization of covers of plane quintics which fail the local Torelli theorem for Pryms:

**Theorem 4.3.** *For the map $\nu : \overline{\mathcal{M}}_5 - - > \overline{\mathcal{R}}_6$ given by $[C] \mapsto [W_4^1(C)/\Gamma]$, we have the scheme theoretic equality $\nu^*(\overline{\mathcal{D}}) = 4 \cdot \overline{\mathcal{M}}_{5,3}^1$. Thus the abelian variety $\mathrm{Prym}(W_4^1(C)/\Gamma)$ fails the local Torelli theorem if and only if the curve $[C] \in \mathcal{M}_5$ is trigonal.*

*Proof.* We use (13) to compute the class of the compactification $\overline{\mathcal{D}}$ in $\overline{\mathcal{R}_6}$ of the ramification locus of $\mathrm{Prym} : \mathcal{R}_6 \to \mathcal{A}_5$ (see [F3] for more details and examples). Precisely, there is a generically non-degenerate morphism between vector bundles of the same rank $\alpha : \mathrm{Sym}^2(\mathbb{E}_1) \to \mathbb{E}_2$ over $\overline{\mathcal{R}}_g$ and $\overline{\mathcal{D}} = Z_1(\alpha) \cap \mathcal{R}_g$. From (13) we find that $c_1(\mathbb{E}_2 - \mathrm{Sym}^2(\mathbb{E}_1)) = 7\lambda - \delta_0' - \delta_0'' - \frac{3}{2}\delta_0^r - \cdots$. By direct computation it follows that $\nu^*(\overline{\mathcal{D}}) \equiv 4 \cdot (8\lambda - \delta_0 - a_1\delta_1 - a_2\delta_2)$, where $a_1, a_2 > 1$, that is, $s(\nu^*(\overline{\mathcal{D}})) = 8$. The only irreducible effective divisor on $\overline{\mathcal{M}}_5$ having slope $\leq 8 = 6 + 12/(g+1)$ is the trigonal locus $\overline{\mathcal{M}}_{5,3}^1$, hence we must have the equality of divisors $\nu^*(\overline{\mathcal{D}}) = 4 \cdot \overline{\mathcal{M}}_{5,3}^1$. □

**Remark 4.4.** Theorem 4.3 is certainly not surprising. Beauville proves using relatively elementary methods that for any smooth curve $[C] \in \mathcal{M}_5 - (\mathcal{M}_{5,3}^1 \cup \mathcal{GP}_{5,4}^{1,0})$, the variety $\mathrm{Prym}(W_4^1(C)/\Gamma)$ satisfies local Torelli (cf. [B], Proposition 6.4).

## 5. THE MOVING SLOPE OF $\overline{\mathcal{M}}_g$

We introduce a fundamental invariant of $\overline{\mathcal{M}}_g$ which carries information about all rational maps from $\overline{\mathcal{M}}_g$ to other projective varieties. If $\mathrm{Mov}(\overline{\mathcal{M}}_g) \subset \mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{R}$ is the cone of moving effective divisors, we define the *moving slope* of $\overline{\mathcal{M}}_g$ by the formula

$$s'(\overline{\mathcal{M}}_g) := \inf_{D \in \mathrm{Mov}(\overline{\mathcal{M}}_g)} s(D) \geq s(\overline{\mathcal{M}}_g).$$

Any non-trivial rational map $f : \overline{\mathcal{M}}_g - - > \mathbf{P}^N$ provides an upper bound for $s'(\overline{\mathcal{M}}_g)$ because one has the obvious inequality $s'(\overline{\mathcal{M}}_g) \leq s\big(f^*(\mathcal{O}_{\mathbf{P}^N}(1))\big)$. This observation is not so useful for large $g$ when there are very few known examples of rational maps admitted by $\overline{\mathcal{M}}_g$. For low $g$, in the range where $\overline{\mathcal{M}}_g$ is unirational, there are several explicit examples of such maps which allows us to determine $s'(\overline{\mathcal{M}}_g)$. Parts of the next theorem are certainly known to experts. The slopes $s(\overline{\mathcal{M}}_g)$ for $g \leq 11$ have been computed in [FP], [HMo], [Ta] and we record them in the following table for comparison purposes.

**Theorem 5.1.** *For integers* $3 \leq g \leq 11$ *we have the following table concerning the slope and the moving slope of* $\overline{\mathcal{M}}_g$ *respectively:*

| $g$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|
| $s(\overline{\mathcal{M}}_g)$ | 9 | $\frac{17}{2}$ | 8 | $\frac{47}{6}$ | $\frac{15}{2}$ | $\frac{22}{3}$ | $\frac{36}{5}$ | 7 | 7 |
| $s'(\overline{\mathcal{M}}_g)$ | $\frac{28}{3}$ | $[\frac{17}{2}, \frac{44}{5}]$ | $[\frac{41}{5}, \frac{33}{4}]$ | $[\frac{47}{6}, \frac{65}{8}]$ | $[\frac{53}{7}, \frac{201}{26}]$ | $[\frac{59}{8}, \frac{149}{20}]$ | $(\frac{36}{5}, \frac{95}{13}]$ | $[\frac{78}{11}, \frac{36}{5}]$ | 7 |

In the proof of Theorem 5.1 we use a result, of independent interest, concerning the slopes of curves in $\overline{\mathcal{M}}_g$ which cover the $k$-gonal loci $\mathcal{M}^1_{g,k}$ for $k \leq 5$. It is a theorem of Tan that if $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ is an effective divisor such that $s(D) < 7 + 6/g$ then $D \supset \overline{\mathcal{M}}^1_{g,3}$ (cf. [T]). It is also well-known that if $s(D) < 8 + 4/g$ then $D \supset \overline{\mathcal{M}}^1_{g,2}$ (use that the family arising from a Lefschetz pencil of curves of type $(2, g+1)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ is a covering curve for $\overline{\mathcal{M}}^1_{g,2}$). Next we prove a similar result for the locus of $4$ and $5$-gonal curves:

*Proof of Theorem 0.7.* We begin by recalling that if $f : X \to \mathbf{P}^1$ is a pencil of semi-stable curves of genus $g$ with $X$ a smooth surface such that there are no $(-1)$-curves in the fibres of $f$, if $m : \mathbf{P}^1 \to \overline{\mathcal{M}}_g$ denotes the corresponding moduli map, then the numerical characters of $f$ are computed as follows:

$$\deg m^*(\lambda) = \chi(\mathcal{O}_X) + g - 1 \ \text{ and } \ \deg m^*(\delta) = c_2(X) + 4(g-1).$$

Of course, these invariants are related by the Noether formula $12\chi(\mathcal{O}_X) = K_X^2 + c_2(X)$.

The idea of the proof is to use Beniamino Segre's theorem [S]: A general $k$-gonal curve $[C] \in \mathcal{M}^1_{g,k}$ has a plane model $\Gamma \subset \mathbf{P}^2$ of degree $n \geq (g+k+2)/2$ having one $(n-k)$-fold point $p$ and

$$\delta = \binom{n-1}{2} - \binom{n-k}{2} - g$$

nodes as the remaining singularities. The pencil $\mathfrak{g}^1_k$ on $C$ is recovered by projecting $\Gamma$ from $p$. We denote by $S := \mathrm{Bl}_{\delta+1}(\mathbf{P}^2)$ the blow-up of the plane at $\delta + 1$ general points $p_0, \ldots, p_\delta \in \mathbf{P}^2$ and consider the linear system on $S$

$$|\mathcal{L}| = |n \cdot h - (n-k) \cdot E_{p_0} - 2 \cdot \sum_{i=1}^{\delta} E_{p_i}|$$

where $h \in \mathrm{Pic}(S)$ is the class of a line. It is known that $|\mathcal{L}|$ is base point free whenever

$$\text{virt-dim}(|\mathcal{L}|) = \frac{n(n+3)}{2} - \binom{n-k+1}{2} - 3\delta \geq 0$$

(cf. [AC2]). This inequality is compatible with the Segre condition precisely when $k \leq 5$, that is, in this range the nodes and the $(n-k)$-fold point of the Segre plane model $\Gamma$ can be chosen to be general points in $\mathbf{P}^2$.

A covering curve for $\overline{\mathcal{M}}^1_{g,k}$ is obtained by blowing-up the $n^2-(n-k)^2-4\delta$ base points of a Lefschetz pencil in the linear system $|\mathcal{L}|$ (see [AC2], Theorem 5.3 for the fact that one can recover the general curve $[C]\in\mathcal{M}^1_{g,k}$ in this way). If $F\subset\overline{\mathcal{M}}^1_{g,k}$ denotes the induced family, then we have the formulas

$$F\cdot\lambda=g,\ \ F\cdot\delta_0=\frac{k(k+3)}{2}+7g+(3-n)k-3\ \text{ and }\ F\cdot\delta_i=0 \text{ for all } i\geq 1$$

(For $3\leq k\leq 5$ one checks that there are no $(-1)$-curves in the fibres of $F$, which is not the case for $k=2$). Choosing $n=[(g+k+3)/2]$ (that is, minimal such that Segre's inequality is satisfied), we find that $F\cdot D<0$ implies the inclusion $\overline{\mathcal{M}}^1_{g,k}\subset D$ which finishes the proof. Note that for $k=3$ we find that $F\cdot\delta=7g+6$ (independent of $n$), hence $F\cdot\delta/F\cdot\lambda=7+6/g$ and this gives a different proof of Tan's result [T]. □

**Corollary 5.2.** *There exists no non-trivial rational map $f:\overline{\mathcal{M}}_g - - > X$ in the projective category such that the indeterminacy locus of $f$ is contained in $\overline{\mathcal{M}}^1_{g,k-1}$ and which contracts the variety $\overline{\mathcal{M}}^1_{g,k}(k=4,5)$ to a point.*

*Proof.* By Theorem 0.7 we can find two different covering curves $F$ and $F'$ for $\overline{\mathcal{M}}^1_{g,k}$ according to different choice of $n\geq(g+k+2)/2$ such that $F\cdot\delta/F\cdot\lambda\neq F'\cdot\delta/F'\cdot\lambda$. □

**Remark 5.3.** This last result is in contrast with the situation in the case of the hyperelliptic locus. For instance, the rational map $f:\overline{\mathcal{M}}_3 - - > \mathcal{Q}_3:=|\mathcal{O}_{\mathbf{P}^2}(4)|//SL(3)$ to the GIT quotient of plane quartics blows down $\overline{\mathcal{M}}^1_{3,2}$ to the point corresponding to the double conic. Moreover, we have that $f^*(A)\equiv 28\lambda-3\delta-8\delta_1$, where $A\in\mathrm{Ample}(\mathcal{Q}_3)$.

*Proof of Theorem 5.1.* **(i)** $g=4$. The Petri divisor $\overline{\mathcal{GP}}^1_{4,3}$ is the closure in $\overline{\mathcal{M}}_4$ of the locus of curves $[C]\in\overline{\mathcal{M}}_4$ for which the canonical model of $C\stackrel{|K_C|}{\longrightarrow}\mathbf{P}^3$ lies on a quadric cone. One knows that $\overline{\mathcal{GP}}^1_{4,3}\equiv 34\lambda-4\delta_0-14\delta_1-18\delta_2$. By taking a Lefschetz pencil $R\subset\overline{\mathcal{M}}_4$ of curves of type $(3,3)$ on a smooth quadric in $\mathbf{P}^3$, we find that $R\cdot\lambda=4, R\cdot\delta_0=34$ which implies that $s(\overline{\mathcal{M}}_4)=34/4$. If $R$ is chosen generically then $R\cap\overline{\mathcal{GP}}^1_{4,3}=\emptyset$. Next we construct a covering curve $F\subset\overline{\mathcal{GP}}^1_{4,3}$ for the Gieseker-Petri divisor. We take the Hirzebruch surface $\mathbb{F}_2$ viewed as the blow-up of the cone $\Lambda\subset\mathbf{P}^3$ over a conic. We denote as usual, $\mathrm{Pic}(\mathbb{F}_2)=\mathbb{Z}\cdot[C_0]\oplus\mathbb{Z}\cdot f$, where $f^2=0$, $C_0^2=-2$ and $C_0\cdot f=1$, and $\mathbb{F}_2\stackrel{|C_0+2f|}{\longrightarrow}\mathbf{P}^3$. Then we consider a Lefschetz pencil in the linear system $|3C_0+6f|$ corresponding to intersections of $\Lambda$ with a pencil of cubic surfaces. We blow-up $\mathbb{F}_2$ in $18=(3C_0+6f)^2$ base points and denote by $f:X=\mathrm{Bl}_{18}(\mathbb{F}_2)\to\mathbf{P}^1$ the resulting family of *semistable* curves. Note that $f$ has precisely one fibre of the form $C_0+D$ with $D\in|2C_0+6f|$, where $C_0\cdot D=2$. By blowing-down the $(-2)$-curve $C_0$ we obtain a map $\nu:X\to X'$ and a family of *stable* genus 4 curves $f':X'\to\mathbf{P}^1$, where $X'$ has one surface double point and $f=f'\circ\nu$. If $F\subset\overline{\mathcal{GP}}^1_{4,3}$ is the curve in the moduli space induced by $f'$, then $F$ is a covering curve for $\overline{\mathcal{GP}}^1_{4,3}$. Since $\omega_f=\nu^*(\omega_{f'})$, the $\lambda$- degree of $F$ can be computed directly on $X$, that is, $F\cdot\lambda=\chi(\mathcal{O}_X)+g-1=4$. Then, we can write $F\cdot\delta=\mathrm{deg}\nu_*([Z])$, where $Z\subset X$ is the 0-cycle of nodes in the fibres of $f$, hence

$$F\cdot\delta=12\chi(\mathcal{O}_X)-K_X^2+4(g-1)=34.$$

Since $F$ and $R$ have the same numerical invariants, it follows that there is no rational contraction $\overline{\mathcal{M}}_4 - - > X$ having indeterminacy locus contained in $\overline{\mathcal{M}}_{4,2}^1$, which blows the divisor $\overline{\mathcal{GP}}_{4,3}^1$ down to a point. The upper bound on $s'(\overline{\mathcal{M}}_4)$ is obtained by considering the irreducible divisor

$$\mathfrak{D}_4 := \{[C] \in \mathcal{M}_4 : \exists p \in C \text{ with } h^0(C, \mathcal{O}_C(3p)) \geq 2\}$$

introduced by S. Diaz. It is known that $s(\overline{\mathfrak{D}}_4) = 44/5$ (cf. [Di]) , hence $s'(\overline{\mathcal{M}}_4) \leq s(\overline{\mathfrak{D}}_4)$.

**(ii)** $g = 5$. We construct a covering curve for $\overline{\mathcal{GP}}_{5,3}^1 = \overline{\mathcal{M}}_{5,3}^1$ as follows: On $\mathbb{F}_1 = \mathrm{Bl}_1(\mathbf{P}^2)$ we denote by $C_0$ and $f$ respectively, the generators of the Picard group where $f^2 = 0, C_0^2 = -1, f \cdot C_0 = 1$. Then we consider the family of genus 5 curves $F \subset \overline{\mathcal{M}}_5$ obtained by blowing-up the base points of a Lefschetz pencil inside the ample linear system $|3C_0 + 5f|$ on $\mathbb{F}_1$. By direct computation we find $F \cdot \lambda = 5, F \cdot \delta = 41$, hence $F \cdot \overline{\mathcal{M}}_{5,3}^1 = -1$. This implies that $[\overline{\mathcal{M}}_{5,3}^1] \notin \mathrm{Mov}(\overline{\mathcal{M}}_5)$ and that $s'(\overline{\mathcal{M}}_5) \geq 41/5$. The upper bound on $s'(\overline{\mathcal{M}}_5)$ uses the Teixidor divisor $\overline{\mathcal{GP}}_{5,4}^{1,0}$ which has slope $s(\overline{\mathcal{GP}}_{5,4}^{1,0}) = 33/4$.

**(iii)** $g = 6$. We use that $s(\overline{\mathcal{GP}}_{6,4}^1) = s(\overline{\mathcal{M}}_6) = 47/6$ and $s(\overline{\mathcal{GP}}_{6,5}^1) = 65/8$, hence $s(\overline{\mathcal{GP}}_{6,4}^1) \leq s'(\overline{\mathcal{M}}_6) \leq s(\overline{\mathcal{GP}}_{6,5}^1)$.

**(iv)** $g = 7$. We consider the tetragonal divisor $\overline{\mathcal{M}}_{7,4}^1 \equiv 15\lambda - 2\delta_0 - 9\delta_1 - 15\delta_2 - 18\delta_3$ and we construct a covering curve for $\overline{\mathcal{M}}_{7,4}^1$ using Theorem 0.7: A general $[C] \in \mathcal{M}_{7,4}^1$ has a septic plane model with one triple point and 5 nodes. A covering curve $F \subset \overline{\mathcal{M}}_{7,4}^1$ is obtained by blowing up $\mathbf{P}^2$ at $26 = 1 + 5 + 20$ points, corresponding to the triple point, the assigned nodes and the unassigned base points of a Lefschetz pencil in the linear system $|7 \cdot h - 3 \cdot E_{p_0} - 2 \cdot \sum_{i=1}^5 E_{p_i}|$. We find that $F \cdot \lambda = 7,\ F \cdot \delta = 53$, hence $F \cdot \overline{\mathcal{M}}_{7,4}^1 < 0$. We obtain that $[\overline{\mathcal{M}}_{7,4}^1] \notin \mathrm{Mov}(\overline{\mathcal{M}}_7)$ and $s'(\overline{\mathcal{M}}_7) \geq F \cdot \delta / F \cdot \lambda = 53/7$.

**(v)** $g = 8$. In this case we consider the Brill-Noether divisor $\overline{\mathcal{M}}_{8,7}^2$ corresponding to septic plane curves with 7 nodes. To obtain a covering curve $F \subset \overline{\mathcal{M}}_{8,7}^2$ one has to blow-up $\mathbf{P}^2$ in the $28 = 21 + 7$ base points of a Lefschetz pencil of 7-nodal septics. It easily follows that $F \cdot \lambda = 8$, $F \cdot \delta = 59$, hence $F \cdot \overline{\mathcal{GP}}_{8,7}^2 < 0$, that is $[\overline{\mathcal{M}}_{8,7}^2] \notin \mathrm{Mov}(\overline{\mathcal{M}}_8)$ and $s'(\overline{\mathcal{M}}_8) \geq 59/8$. Moreover, $s'(\overline{\mathcal{M}}_8) \leq s(\overline{\mathcal{GP}}_{8,5}^1) = 149/20$ (cf. [F1]).

**(vi)** $g = 9$. The smallest known slopes of effective divisors on $\overline{\mathcal{M}}_9$ are $s(\overline{\mathcal{M}}_{9,5}^1) = 36/5$ and $s(\overline{\mathcal{GP}}_{9,8}^2) = 95/13$ respectively (cf. [F1], Theorem 1.5). It follows that a multiple of the linear system $|\overline{\mathcal{GP}}_{9,8}^2|$ contains a moving divisor on $\overline{\mathcal{M}}_9$.

**(vii)** $g = 10$. We use the results from [FP] and denote by $\overline{\mathcal{K}}_{10}$ the closure of the locus of curves $[C] \in \mathcal{M}_{10}$ lying on a $K3$ surface, hence $s(\overline{\mathcal{K}}_{10}) = 7$. If $F \subset \overline{\mathcal{K}}_{10}$ is the 1-dimensional family obtained from a Lefschetz pencil of curves of genus 10 lying on a general $K3$ surface, then $F \cdot \delta / F \cdot \lambda = 78/11$, hence $s'(\overline{\mathcal{M}}_{10}) \geq 78/11 > s(\overline{\mathcal{K}}_{10})$ and moreover $[\overline{\mathcal{K}}_{10}] \notin \mathrm{Mov}(\overline{\mathcal{M}}_{10})$. Since $s(\overline{\mathcal{GP}}_{10,6}^1) = 36/5$ (cf. [F1], Proposition 1.6), we obtain the estimate $78/11 \leq s'(\overline{\mathcal{M}}_{10}) \leq 36/5$.

**(viii)** $g = 11$**.** This is also a consequence of [FP], Proposition 6.2. If $\overline{\mathcal{F}}_g$ is the Baily-Borel compactification of the moduli space of polarized $K3$ surfaces of degree $2g-2$, then there is a rational map $f : \overline{\mathcal{M}}_{11} - - > \overline{\mathcal{F}}_{11}$ given by $f([C]) = [S, C]$, where $S$ is the unique $K3$ surface containing $C$. If $F \subset \overline{\mathcal{M}}_{11}$ is a Lefschetz pencil of curves corresponding to a general choice of $[S, C] \in \mathcal{F}_{11}$, then $F \cdot \lambda = g + 1 = 12$ and $F \cdot \delta = 84$. The map $f$ contracts the pencil $F$, hence for each divisor $A \in \text{Ample}(\overline{\mathcal{F}}_{11})$, we must have that $s(f^*(A)) = 7$, that is, $s'(\overline{\mathcal{M}}_{11}) \leq 7$. Since $F$ is a covering curve for $\overline{\mathcal{M}}_{11}$ one also has that $s(\overline{\mathcal{M}}_{11}) \geq F \cdot \delta / F \cdot \lambda = 7$, hence $s(\overline{\mathcal{M}}_{11}) = s'(\overline{\mathcal{M}}_{11}) = 7$. □

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