SYZYGIES OF TORSION BUNDLES AND THE GEOMETRY OF THE LEVEL ℓ MODULAR VARIETY OVER $\overline{\mathcal{M}}_g$

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ABSTRACT. We formulate, and in some cases prove, three statements concerning the purity or, more generally, the *naturality* of the resolution of various modules one can attach to a generic curve of genus g and a torsion point of ℓ in its Jacobian. These statements can be viewed as analogues of Green's Conjecture and we verify them computationally for bounded genus. We then compute the cohomology class of the corresponding non-vanishing locus in the moduli space $\mathcal{R}_{g,\ell}$ of twisted level ℓ curves of genus g and use this to derive results about the birational geometry of $\mathcal{R}_{g,\ell}$. For instance, we prove that $\mathcal{R}_{g,3}$ is a variety of general type when $g > 11$ and the Kodaira dimension of $\mathcal{R}_{11,3}$ is greater than or equal to 19. In the last section we explain probabilistically the unexpected failure of the Prym-Green conjecture in genus 8 and level 2.

One way of proving that the moduli space of curves of odd genus $g \gg 0$ has general type is via the divisor of curves whose canonical ring has "extra" syzygies. In this paper we apply the same philosophy to prove that, in certain cases, the moduli spaces of curves with torsion bundles—the modular varieties of the title—are also of general type. These modular varieties are natural generalizations, in higher genus, of the much studied modular curves $X_1(\ell) := \mathcal{H}/\Gamma_1(\ell)$ classifying elliptic curves together with an ℓ-torsion point in their Jacobian.

To explain what we mean by "extra" syzygies, consider a finitely generated graded module M over a polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$. Such a module has a minimal free resolution of the form

$$0 \leftarrow M \leftarrow F_0 \leftarrow \cdots \leftarrow F_i \leftarrow \cdots ,$$

where $F_i = \sum_j S(-i-j)^{b_{i,j}}$. The numbers $b_{i,j} = b_{i,j}(M)$, called the graded Betti numbers of M, are uniquely defined; in fact $b_{i,j}$ is the dimension of the degree $i+j$ component of $\mathrm{Tor}_i^S(M, \mathbb{C})$. We say that the resolution of M is *natural* if, for each j, the number $b_{i,j-i}(M)$ is nonzero for at most one value of i, that is, at most one Betti number on each diagonal of the Betti diagram of M is non-zero. In an irreducible flat family of modules M_λ the $b_{i,j}(M_\lambda)$ are semicontinuous, and simultaneously take on minimum values on an open set. We say that M_λ has "extra" syzygies when one of the values $b_{i,j}(M_\lambda)$ is larger than this minimum. If the resolution of some M_λ is natural in the sense above, then its Betti numbers have the minimum value.

For example, given a curve C, a line bundle $L \in \mathrm{Pic}(C)$ and a sheaf $\mathcal{F}$ on C, we consider the $S := \mathrm{Sym}\, H^0(C, L)$-module

$$\Gamma_C(\mathcal{F}, L) := \bigoplus_{q \in \mathbb{Z}} H^0(C, \mathcal{F} \otimes L^{\otimes q}).$$

Following notation introduced by Mark Green, the vector space $\mathrm{Tor}_i^S(\Gamma_C(\mathcal{F}, L), \mathbb{C})_{i+j}$ is often written $K_{i,j}(C; \mathcal{F}, L)$ and one sets $K_{i,j}(C, L) := K_{i,j}(C; \mathcal{O}_C, L)$. Green's Conjecture [G] for generic curves of genus g (proved in [V1] and [V2]), asserts that the resolution of the canonical ring $\Gamma_C(\mathcal{O}_C, K_C)$, is natural. For odd genus, it follows that the resolution is pure, which in Green's notation says

$$K_{\frac{g-1}{2},1}(C, K_C) = 0. \tag{1}$$

The locus of curves whose canonical ring has extra syzygies is a divisor supported on the Hurwitz locus $\mathcal{M}^1_{g, \frac{g+1}{2}}$ used in [HM] to prove that these moduli spaces have general type.

Generalizing the case above, we study modules of the form $\Gamma_C(\xi, K_C \otimes \eta)$ where η and ξ are line bundles of degree 0 on a smooth curve C. As a first step, we prove:

Theorem 0.1. *Let C be a general curve of genus g, $\eta \in \mathrm{Pic}^0(C)[\ell]$ a torsion bundle of order $\ell \geq 2$ and $\xi \in \mathrm{Pic}^0(C)$ a general line bundle of degree degree 0. Then the module $\Gamma_C(\xi, K_C \otimes \eta)$ has natural resolution.*

We shall use a refinement of this result to compute the Kodaira dimension of the moduli spaces $\mathcal{R}_{g,\ell}$ parametrizing level ℓ curves $[C, \eta]$, where C is a smooth curve of genus g and $\eta \in \mathrm{Pic}^0(C)$ is a torsion line bundle of order ℓ. In particular, we shall prove:

Theorem 0.2. *$\mathcal{R}_{g,3}$ is a variety of general type for $g \geq 12$. Furthermore, the Kodaira dimension of $\mathcal{R}_{11,3}$ is at least 19.*

It is known that $\mathcal{R}_{g,3}$ is rational for $g \leq 4$, see [BC], [BV]. The level ℓ modular variety that has received most attention so far is the moduli space $\mathcal{R}_{g,2}$ classifying Prym varieties of dimension $g-1$. It is shown in [FL] that $\mathcal{R}_{g,2}$ is a variety of general type for $g > 13$, whereas $\mathcal{R}_{g,2}$ is unirational for $g \leq 7$, see [FV2] and references therein.

We now explain the statements concerning the naturality of the resolution of certain modules one associates to a level curve $[C, \eta] \in \mathcal{R}_{g,\ell}$. Such predictions can be made for any g and, just like for Green's Conjecture, in about half of the cases they amount to saying that the resolution is actually *pure*. In such case, the locus of points $[C, \eta]$ where the corresponding resolution is not pure is a *virtual* divisor on $\mathcal{R}_{g,\ell}$, that is, the degeneracy locus of a morphism between vector bundles of the same rank over the stack $\mathsf{R}_{g,\ell}$ which is coarsely represented by $\mathcal{R}_{g,\ell}$. Proving the corresponding syzygy conjecture amounts to showing that the respective degeneracy locus is a *genuine* divisor on $\mathcal{R}_{g,\ell}$.

A. Prym-Green Conjecture. *For a general level ℓ curve $[C, \eta] \in \mathcal{R}_{g,\ell}$ of genus $g \geq 6$, the homogeneous coordinate ring of the paracanonical curve $\phi_{K_C \otimes \eta} : C \hookrightarrow \boldsymbol{P}^{g-2}$ has a natural resolution. Equivalently, in even genus, the resolution is pure and the paracanonical curve satisfies property $(N_{\frac{g}{2}-3})$, that is,*

$$K_{\frac{g}{2}-2,1}\big(C, K_C \otimes \eta\big) = 0, \tag{2}$$

whereas in odd genus

$$K_{\frac{g-3}{2},1}\big(C, K_C \otimes \eta\big) = 0 \ \ \textit{and} \ \ K_{\frac{g-7}{2},2}(C, K_C \otimes \eta) = 0. \tag{3}$$

The name of the conjecture is justified by the analogy with Green's Conjecture. In Section 4, we verify the Prym-Green Conjecture for all values of $g \leq 18$ and small ℓ, with the exception of 2-torsion in genus $g = 8$ and $g = 16$. Our findings suggest that very likely, for 2-torsion and g a power of 2 (or perhaps multiple of 8), the Prym-Green Conjecture is actually false. (With our methods, an experiment with $g = 24$ would take about 6500 years.) Our verification of Conjecture A is computational via the use of *Macaulay2*. We verify condition (2) for g-nodal rational curves over a finite field (see Section 4 for details). For $g := 2i + 6$, we denote by

$$\mathcal{Z}_{g,\ell} := \Big\{[C, \eta] \in \mathcal{R}_{g,\ell} : K_{i+1,1}(C, K_C \otimes \eta) \neq 0\Big\},$$

the failure locus of Conjecture A. This is a virtual divisor on $\mathcal{R}_{g,\ell}$.

The second conjecture we address concerns the resolution of torsion bundles.

B. Torsion Bundle Conjecture. *Let $[C, \eta] \in \mathcal{R}_{g,\ell}$ be a general level ℓ curve of* even *genus $g \geq 4$. For each $1 \leq k \leq \ell - 2$, the module $\Gamma_C(\eta^{\otimes k}, K_C \otimes \eta)$ has natural resolution, unless*

$$\eta^{\otimes(2k+1)} = \mathcal{O}_C \ \textit{and}\ g \equiv 2 \mod 4 \ \textit{and}\ \binom{g-3}{\frac{g}{2}-1} \equiv 1 \bmod 2.$$

Equivalently, one has the vanishing statement

$$K_{\frac{g}{2}-1,1}(C; \eta^{\otimes k}, K_C \otimes \eta) = 0. \tag{4}$$

In the exceptional cases there is precisely one extra syzygy.

Theorem 0.1 can be viewed as a weak form of Conjecture B. The first exceptional genera in the Conjecture B are $g = 6, 10, 18, 34$ or 66. For levels $\ell \geq 3$, we set $g := 2i + 2$ and denote the corresponding virtual divisor by

$$\mathcal{D}_{g,\ell} := \Big\{[C, \eta] \in \mathcal{R}_{g,\ell} : K_{i,1}(C; \eta^{\otimes(\ell-2)}, K_C \otimes \eta) \neq 0\Big\}.$$

Conjecture B can be reformulated in the spirit of the *Minimal Resolution Conjecture* [FMP] for points on paracanonical curves as follows. We view a divisor $Z \in |K_C \otimes \eta^{\otimes(1-k)}|$ as a 0-dimensional subscheme of $\phi_L(C) \subset \mathbf{P}^{g-2}$. Then using [FMP] Proposition 1.6, Conjecture B is equivalent to the statement $b_{\frac{g}{2},1}(Z) = b_{\frac{g}{2}-1,2}(Z) = 0$, which amounts to the minimality of the number of syzygies of $Z \subset \mathbf{P}^{g-2}$.

The exceptions to Conjecture B can be explained by (surprising) symmetries in the Koszul differentials (see Section 4). We verify Conjecture B for genus $g \leq 16$ and small level. In any of these cases $\mathcal{D}_{g,\ell}$ is a divisor on $\mathcal{R}_{g,\ell}$.

A prediction about the naturality of the resolution of the module $\Gamma_C(\eta, K_C)$ can also be made. This time we expect no exceptions and this turns out to be the case:

Theorem 0.3. *Let $[C, \eta] \in \mathcal{R}_{g,\ell}$ be a general level ℓ curve of genus $g \geq 3$. Then the resolution of $\Gamma_C(\eta, K_C)$ is natural (respectively pure for odd g), that is,*

$$K_{\lfloor \frac{g-1}{2} \rfloor,1}(C; \eta, K_C) = 0. \tag{5}$$

The proof of Theorem 0.3 is by specialization to hyperelliptic curves. Via the following equality of cycles in the Jacobian of any curve of *odd* genus, see [FMP],

$$C_{\frac{g-1}{2}} - C_{\frac{g-1}{2}} = \Bigl\{\xi \in \mathrm{Pic}^0(C) : K_{\frac{g-1}{2},1}(C;\xi, K_C) \neq 0\Bigr\},$$

where the left hand side denotes the top difference variety of C, Theorem 0.3 admits the following geometrically transparent reformulation:

Corollary 0.4. *For a general curve C of odd genus, the top difference variety $C_{\frac{g-1}{2}} - C_{\frac{g-1}{2}}$ contains no non-trivial ℓ-torsion points, for any $\ell \geq 2$.*

To pass from syzygy statements to the birational structure of $\mathcal{R}_{g,\ell}$, one calculates the cohomology classes of the failure loci of the conditions (2), (4) and (5) respectively on a compact moduli of level ℓ curves. The space $\mathcal{R}_{g,\ell}$ admits a compactification $\overline{\mathcal{R}}_{g,\ell}$, which is the coarse moduli space associated to the smooth proper Deligne-Mumford stack $\overline{\mathsf{R}}_{g,\ell}$ of *level twisted curves*, that is, triples $[\mathsf{C},\eta,\phi]$, where C is a genus g twisted curve, η is a faithful line bundle on the stack C and $\phi : \eta^{\otimes \ell} \to \mathcal{O}_{\mathsf{C}}$ is an isomorphism, see [CF] and Section 1 of this paper. We denote by $f : \overline{\mathcal{R}}_{g,\ell} \to \overline{\mathcal{M}}_g$ the forgetful map.

An essential ingredient in the proof of Theorem 0.2 is one of the main results of [CF]. It is shown that the non log-canonical singularities of $\overline{\mathcal{R}}_{g,3}$ do not impose adjunction conditions, that is, if $\epsilon : \widehat{\mathcal{R}}_{g,3} \to \overline{\mathcal{R}}_{g,3}$ denotes a resolution of singularities, then for each $n \geq 1$, there exists an isomorphism at the level of spaces of global sections

$$\epsilon^* : H^0\Bigl(\overline{\mathcal{R}}_{g,3}^{\mathrm{reg}}, K_{\overline{\mathcal{R}}_{g,3}}^{\otimes n}\Bigr) \xrightarrow{\cong} H^0\Bigl(\widehat{\mathcal{R}}_{g,3}, K_{\widehat{\mathcal{R}}_{g,3}}^{\otimes n}\Bigr).$$

A similar extension result has been proved in [HM] for level one curves, that is, for the moduli space $\overline{\mathcal{M}}_g$ itself, and in [FL] in the case of $\overline{\mathcal{R}}_{g,2}$. We refer to [CF] for partial generalizations of this extension result for higher levels. Using [CF], we thus conclude that the Kodaira dimension of $\overline{\mathcal{R}}_{g,3}$ is equal to the Kodaira-Iitaka dimension of the canonical bundle $K_{\overline{\mathcal{R}}_{g,3}}$. In order to prove that for a given g, the moduli space $\overline{\mathcal{R}}_{g,3}$ is of general type, it suffices to express $K_{\overline{\mathcal{R}}_{g,3}} \in \mathrm{Pic}(\overline{\mathcal{R}}_{g,3})$ as a positive combination of the Hodge class $\lambda \in \mathrm{Pic}(\overline{\mathcal{R}}_{g,3})$, the class $[\overline{\mathfrak{D}}]$ of the closure of a certain effective divisor $\mathfrak{D}$ on $\mathcal{R}_{g,3}$ and boundary divisor classes corresponding to singular level curves. In our case, the divisor $\mathfrak{D}$ is a jumping locus for Koszul cohomology groups of paracanonically embedded curves, defines by one of the conditions (2), (4) or (5).

We denote by $\widetilde{\mathcal{M}}_g$ the open subvariety of $\overline{\mathcal{M}}_g$ classifying irreducible stable curves of genus g and set $\widetilde{\mathcal{R}}_{g,\ell} := f^{-1}(\widetilde{\mathcal{M}}_g)$. The boundary $\widetilde{\mathcal{R}}_{g,\ell} - f^{-1}(\mathcal{M}_g)$ is the union of three proper and closed subloci $\Delta_0^{'} \cup \Delta_0^{''} \cup \Delta_0^{\mathrm{ram}}$, where $\Delta_0^{''}$ and Δ_0^{ram} can be characterized as follows (see Section 1, §1.4 for further details). The locus Δ_0^{ram} is the ramification divisor of f. The locus $\Delta_0^{''}$ is the locus of order-ℓ analogues of Wirtinger covers. These boundary subloci are not irreducible in general and we refer to [CF, §1.4.3-4] for a decomposition into irreducible components. When $\ell = 2$ or 3 however, the loci $\Delta_0^{'}$, $\Delta_0^{''}$ and Δ_0^{ram} are irreducible, whereas for a prime level $\ell > 3$, both $\Delta_0^{''}$ and Δ_0^{ram} are reducible and decompose into $\lfloor \frac{\ell}{2} \rfloor$ irreducible components. For Δ_0^{ram}, we have a decomposition $\Delta_0^{\mathrm{ram}} = \bigcup_{a=1}^{\lfloor \ell/2 \rfloor} \Delta_0^{(a)}$, where the irreducible components are determined by a local index a at the node of the generic level ℓ curve (see Definitions 1.1 and 1.2) and yielding

$\mathbb{Q}$-divisors $\delta_0^{(a)} = [\Delta_0^{(a)}]_{\mathbb{Q}}$ at the level of the moduli stack, fitting in the formula

$$f^*(\delta_0) = \delta_0' + \delta_0'' + \ell \sum_{a=1}^{\lfloor \ell/2 \rfloor} \delta_0^{(a)}.$$

Here δ_0 is the $\mathbb{Q}$-divisor attached to $\Delta_0 = \widetilde{\mathcal{M}}_g - \mathcal{M}_g$. As shown in Section 1, §1.3 and §1.4, the above identity extends word for word for possibly composite levels $\ell \geq 2$.

For any $\ell \geq 2$, following the established practice of passing to lower case symbols for the divisor classes on the moduli stack, we have the following results:

Theorem 0.5. *Write $g = 2i+1$ and $\ell \geq 2$. The class of the closure in $\widetilde{\mathcal{R}}_{g,\ell}$ of the effective divisor $\mathcal{U}_{g,\ell} := \big\{[C,\eta] \in \mathcal{R}_{g,\ell} : K_{i,1}(C;\eta,K_C) \neq 0\big\}$ is equal to*

$$[\overline{\mathcal{U}}_{g,\ell}] = \frac{1}{2i-1}\binom{2i}{i}\Big((3i+1)\lambda - \frac{i}{2}(\delta_0' + \delta_0'') - \sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} \frac{1}{2\ell}(i\ell^2 + 2a^2 i - 2a\ell i - a^2 + a\ell)\, \delta_0^{(a)}\Big) \in \mathsf{Pic}(\widetilde{\mathsf{R}}_{g,\ell}).$$

Keeping $g = 2i+1$ and setting $\ell = 3$, in order to form an effective representative of $K_{\overline{\mathcal{R}}_{g,3}}$, we use Theorem 0.5 together with the formula of the slope $s(\overline{\mathcal{M}}^1_{g,i+1}) = \frac{6(i+2)}{i+1}$ of the class of the closure in $\overline{\mathcal{M}}_g$ of the Hurwitz divisor [HM], [EH]

$$\mathcal{M}^1_{g,i+1} := \{[C] \in \mathcal{M}_g : W^1_{i+1}(C) \neq \emptyset\},$$

in order to obtain that, for suitable rational constants $\alpha, \beta > 0$, the $\mathbb{Q}$-divisor class

$$\alpha \cdot [\overline{\mathcal{U}}_{g,3}] + \beta \cdot [f^*(\overline{\mathcal{M}}^1_{g,i+1})] = \frac{6(2i+3)}{i+1}\lambda - 2(\delta_0' + \delta_0'') - 4\delta_0^{(1)} \in \mathsf{Pic}(\widetilde{\mathsf{R}}_{g,3}) \tag{6}$$

is effective. Comparing this formula against that of the canonical class $K_{\overline{\mathcal{R}}_{g,3}}$ (see Section 1), we note that whenever the following inequality

$$\frac{6(2i+3)}{i+1} < 13 \Leftrightarrow i > 5,$$

holds, the canonical class $K_{\overline{\mathcal{R}}_{g,3}}$ is big. Using the extension result of [CF], we conclude that $\overline{\mathcal{R}}_{g,3}$ is of general type for odd genus $g \geq 13$. This argument also shows that when $g = 11$ the class given in (6) is an effective representative for the canonical class; furthermore, one has the following inequalities

$$\kappa(\overline{\mathcal{R}}_{11,3}, K_{\overline{\mathcal{R}}_{11,3}}) \geq \kappa\big(\overline{\mathcal{R}}_{11,3}, f^*(\overline{\mathcal{M}}^1_{11,6})\big) \geq \kappa(\overline{\mathcal{M}}_{11}, \overline{\mathcal{M}}^1_{11,6}) = 19,$$

where the last equality has been proved in [FP]. It is an interesting open question whether the equality $\kappa(\overline{\mathcal{R}}_{11,3}) = 19$ holds. It is known [FV1] that the universal Picard variety over the moduli space $\overline{\mathcal{M}}_{11}$ has Kodaira dimension equal to 19 as well.

We compute the (virtual) class of the failure loci given by Conjectures A and B.

Theorem 0.6. *Set $g := 2i+6$ with $i \geq 0$ and $\ell \geq 2$. The virtual class of the closure in $\widetilde{\mathcal{R}}_{g,\ell}$ of the locus $\mathcal{Z}_{g,\ell}$ of level ℓ curves $[C,\eta] \in \mathcal{R}_{g,\ell}$ with $K_{i+1,1}(C, K_C \otimes \eta) \neq 0$ is equal to*

$$[\overline{\mathcal{Z}}_{g,\ell}]^{\mathrm{virt}} = \binom{2i+2}{i}\Big(\frac{3(2i+7)}{i+3}\lambda - (\delta_0' + \delta_0'') - \sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} \frac{a^2 - a\ell + \ell^2}{2}\delta_0^{(a)}\Big) \in \mathsf{Pic}(\widetilde{\mathsf{R}}_{g,\ell}).$$

We explain the meaning of this result. In Section 3 we construct tautological vector bundles $\mathcal{A}$ and $\mathcal{B}$ over the stack $\widetilde{\mathsf{R}}_{g,\ell}$ with $\mathrm{rk}(\mathcal{A}) = \mathrm{rk}(\mathcal{B})$, as well as a vector bundle morphism $\varphi : \mathcal{A} \to \mathcal{B}$, such that over the open part $\mathcal{R}_{g,\ell} \subset \overline{\mathcal{R}}_{g,\ell}$, the degeneracy locus of φ equals the scheme $\mathcal{Z}_{g,\ell}$. Accordingly, we define $[\overline{\mathcal{Z}}_{g,\ell}]^{\mathrm{virt}} := c_1(\mathcal{B} - \mathcal{A})$. Whenever the Prym-Green Conjecture holds, that is, φ is generically non-degenerate and $\overline{\mathcal{Z}}_{g,\ell}$ is a divisor, we have that $[\overline{\mathcal{Z}}_{g,\ell}]^{\mathrm{virt}} - [\overline{\mathcal{Z}}_{g,\ell}]$ is a (possibly empty) effective class entirely supported on the boundary of $\widetilde{\mathcal{R}}_{g,\ell}$. In particular, the class computed in Theorem 0.6 is effective. Next we describe the universal failure locus of Conjecture B.

Theorem 0.7. *Set $g := 2i + 2 \geq 4$ and $\ell \geq 3$ such that $i \equiv 1 \bmod 2$ or $\binom{2i-1}{i} \equiv 0 \bmod 2$. The virtual class of the closure in $\widetilde{\mathcal{R}}_{g,\ell}$ of the locus $\mathcal{D}_{g,\ell}$ of level ℓ curves $[C, \eta] \in \mathcal{R}_{g,\ell}$ such that $K_{i,1}(C; \eta^{\otimes(\ell-2)}, K_C \otimes \eta) \neq 0$ is equal to*

$$[\overline{\mathcal{D}}_{g,\ell}]^{\mathrm{virt}} = \frac{1}{i-1}\binom{2i-2}{i}\Big((6i+1)\lambda - i(\delta_0^{'}+\delta_0^{''}) - \frac{1}{\ell}\sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor}(i\ell^2+5a^2i-5ai\ell-2a^2+2a\ell)\delta_0^{(a)}\Big).$$

To establish Theorem 0.2 in even genus, when $g \geq 14$ but $g \neq 16$, we can use the class $[\overline{\mathcal{Z}}_{g,3}]$ to show that $K_{\overline{\mathcal{R}}_{g,3}}$ is big. In genus $g = 16$, when the Prym-Green Conjecture appears to fail, we use the class $[\overline{\mathcal{D}}_{g,3}]$ instead. Interestingly, for $g = 12$, the Prym-Green divisor $\overline{\mathcal{Z}}_{12,3} \equiv 13\lambda - 2(\delta_0^{'} + \delta_0^{''}) - \frac{14}{3}\delta_0^{(1)}$ has slope equal to that of the canonical divisor $K_{\overline{\mathcal{R}}_{12,3}}$. However, one can form the effective linear combination

$$\frac{31}{36}[\overline{\mathcal{Z}}_{12,3}] + \frac{1}{36 \cdot 7}[\overline{\mathcal{D}}_{12,3}] = \big(13 - \frac{1}{12}\big)\lambda - 2(\delta_0^{'} + \delta_0^{''}) - 4\delta_0^{(1)} = K_{\widetilde{\mathsf{R}}_{12,3}} - \frac{1}{12}\lambda \in \mathrm{Eff}(\widetilde{\mathsf{R}}_{12,3}),$$

thus showing that $\overline{\mathcal{R}}_{12,3}$ is of general type as well.

Section 5 is devoted to the highly surprising failure of the Prym-Green Conjecture for $g = 8$ and $\ell = 2$. We give a probabilistic proof of the fact that for a general genus 8 Prym canonical curve $\phi_{K_C \otimes \eta} : C \hookrightarrow \mathbf{P}^6$, the multiplication map

$$I_2(C, K_C \otimes \eta) \otimes H^0(C, K_C \otimes \eta) \to I_3(C, K_C \otimes \eta),$$

has a non-trivial 1-dimensional kernel, corresponding to a syzygy of rank 6.

A curve $\phi_L : C \hookrightarrow \mathbf{P}^6$ corresponding to a general element $[C, L]$ of the universal Jacobian variety $\mathfrak{Pic}_8^{14} \to \mathcal{M}_8$ is linked via five quadrics to a genus 14 curve with general moduli $C' \subset \mathbf{P}^6$, embedded such that $K_{C'}(-1) \in W_8^1(C')$ is a pencil of minimal degree.

The universal Koszul locus $\mathfrak{Kosz} := \big\{[C, L] \in \mathfrak{Pic}_8^{14} : K_{1,2}(C, L) \neq 0\big\}$ is a divisor that has at least two components $\mathfrak{Kosz}_6$ and $\mathfrak{Kosz}_7$, distinguished by whether the extra syzygy has rank 6 or 7. The generic curve in $\mathfrak{Kosz}_6$ corresponds via linkage to a curve in the Petri divisor $\mathcal{GP}_{14,8}^1$ on $\mathcal{M}_{14}$, while the generic curve in $\mathfrak{Kosz}_7$ links to a 7-gonal curve of genus 14 such that $K_{C'}(-1)$ has a base point. Using a structure theorem for rank 6 syzygies in $\mathbf{P}^6$, our findings show that $\mathcal{R}_{8,2}$ lies in the component $\mathfrak{Kosz}_6$, that is, the extra syzygy of a Prym-canonical curve is never of maximal rank.

Structure of the paper. The first section is dedicated to the geometry of the stack of twisted level ℓ curves. In Section 2 we describe the syzygy formalism needed to formulate Conjectures A and B and then we prove Theorems 0.1 and 0.3. In Section 3 proofs of Theorems 0.5, 0.6 and 0.7 are provided, whereas in Section 4 we explain how to verify with *Macaulay2* the syzygy conjectures formulated in the Introduction. The last section is devoted entirely to genus 8 and we explain the unexpected failure of the Prym-Green Conjecture on $\mathcal{R}_{8,2}$.

Disclaimer. This paper is a collaborative effort marrying techniques ranging from computer algebra to stacks. The work of the second author is mainly reflected in sections 4 and 5, while the work of the first author is mainly reflected in sections 1–3.

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1. LEVEL ℓ CURVES

This section contains background material concerning the moduli space $\overline{\mathcal{R}}_{g,\ell}$ and complements the paper [CF]. We begin by setting terminology. For a Deligne-Mumford stack X be denote by X its associated coarse moduli space. The morphism $\pi_{\mathsf{X}} : \mathsf{X} \to X$ is universal with respect to morphisms from X to algebraic spaces. The *coarsening* of a morphism $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ between Deligne-Mumford stacks is the induced morphism $f : X \to Y$ between coarse moduli spaces. For a Deligne-Mumford stack X we denote by $\mathsf{Pic}(\mathsf{X})$ the Picard group of the stack with rational coefficients.

We fix two integers g and $\ell \geq 2$, the genus and the level. Throughout §1.2 we assume that ℓ is *prime*, but we generalize all statements to any, possibly composite, level $\ell \geq 2$ in §1.3.

1.1. The geometric points of the moduli space of level ℓ curves.

The geometric points of the moduli space of level ℓ curves can be interpreted in two relatively simple ways, as quasi-stable ℓth roots and twisted ℓth roots respectively. We are recall their definitions.

A *quasi-stable* curve is a nodal curve X such that (i) for each smooth rational component $E \subset X$ the inequality $k_E := |E \cap \overline{X - E}| \geq 2$ holds, and (ii) if E, E' are rational components with $k_E = k_{E'} = 2$, then $E \cap E' = \emptyset$. Rational components of X meeting the rest of the curve in two points are called *exceptional*. If X is a quasi-stable curve, there exists a *stabilization* morphism $\mathrm{st} : X \to C$, obtained by collapsing all rational curves $E \subset X$ with $k_E = 2$. Abusing terminology, we say that X is a *blow-up* of the curve C.

Definition 1.1. A *quasi-stable level ℓ curve* consists of a triple (X, η, ϕ), where X is a quasi-stable curve, $\eta \in \mathrm{Pic}^0(X)$ is a locally free sheaf of total degree 0 and $\phi : \eta^{\otimes \ell} \to \mathcal{O}_X$ is a sheaf homomorphism satisfying the following properties:

(i) $\eta_E = \mathcal{O}_E(1)$, for every *exceptional* component $E \subset X$;
(ii) ϕ is an isomorphism along each non-exceptional component of X;

(iii) if E is an exceptional component and $\{p, q\} := E \cap \overline{X - E}$, then

$$\mathrm{ord}_p(\phi) + \mathrm{ord}_q(\phi) = \ell.$$

The last condition refers to the vanishing orders of the section $0 \neq \phi \in H^0(X, \eta^{\otimes(-\ell)})$.

We recall that a *balanced twisted curve* is a Deligne-Mumford stack C whose coarse moduli space C is a stable curve, and such that locally at a node, the stack C comes endowed with an action of $\mathbb{Z}_\ell$ of determinant 1 (this is equivalent to impose the condition that C be smoothable).

Definition 1.2. A *twisted ℓth root* $(\mathsf{C} \to T, \eta, \phi)$, is a balanced twisted curve C of genus g over a base T, a faithful line bundle η on C (i.e. a representable morphism $\eta : \mathsf{C} \to \mathsf{B}\mathbb{Z}_\ell$) and an isomorphism $\phi : \eta^{\otimes \ell} \to \mathcal{O}_{\mathsf{C}}$. If η has order ℓ in $\mathsf{Pic}(\mathsf{C})$, then $(\mathsf{C} \to T, \eta, \phi)$ is a *level ℓ curve*.

There exist two $(3g-3)$-dimensional projective schemes $\mathrm{Root}_{g,\ell}$ (see [J1, CCC]) and $\overline{\mathcal{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ (see [AV, ACV]) whose (geometric) points represent isomorphism classes of quasi-stable ℓth roots and twisted ℓth roots, respectively. The moduli space $\overline{\mathcal{R}}_{g,\ell}$ of level ℓ curves arises as a connected component of $\overline{\mathcal{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$. As we illustrate in detail below, $\mathrm{Root}_{g,\ell}$ and $\overline{\mathcal{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ are not isomorphic unless $\ell = 2$ or 3, but there exists a natural morphism $\mathrm{nor} : \overline{\mathcal{M}}_g(\mathsf{B}\mathbb{Z}_\ell) \to \mathrm{Root}_{g,\ell}$, which sets a bijection on the sets of geometric points, and may be regarded, scheme-theoretically, as a normalization morphism (for $\ell > 3$ the singularities of $\mathrm{Root}_{g,\ell}$ are not normal). The aim of §1.2 is to describe the twisted ℓth root $(\mathsf{C} \to T, \eta, \phi)$ and the quasi-stable ℓth root (X, η, ϕ) that correspond to each other under the map nor. In fact, following [C2], we lift this correspondence to a correspondence between a universal twisted ℓth root and a universal quasi-stable ℓth root both defined on the moduli stack of level ℓ curves. This correspondence allows us to study the enumerative geometry of the moduli of level curves both in scheme-theoretic and stack-theoretic terms (see Remark 1.4).

1.2. **The moduli stack of level ℓ curves (when ℓ is prime).** We consider the categories of quasi-stable ℓth roots and of twisted ℓ roots. For the sake of clarity, in this section, we assume that ℓ is prime.

A family of quasi-stable ℓth roots consists of a triple (f, η, ϕ), where $f : X \to T$ is a flat family of quasi-stable curves, η is a line bundle on X and $\phi : \eta^{\otimes \ell} \to \mathcal{O}_X$ is a morphism of sheaves such that for each geometric point $t \in T$, the restriction

$$\big(X_t := f^{-1}(t),\ \eta_t := \eta_{|X_t},\ \phi_{|X_t} : \eta_t^{\otimes \ell} \to \mathcal{O}_{X_t}\big)$$

is a quasi-stable ℓth root as defined above. The category of level ℓ curves gives rise to a proper Deligne-Mumford stack $\mathsf{Root}_{g,\ell}$ with associated coarse moduli space $\mathrm{Root}_{g,\ell}$, see [CCC, J1]. Unfortunately, as already mentioned, this stack is singular as soon as $\ell > 3$ (see (8)). To obtain a smooth stack whose coarse moduli space is the normalization of $\mathrm{Root}_{g,\ell}$ one can use twisted ℓ roots [AV, ACV].

Indeed, the category $\overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ of twisted ℓth roots forms a smooth and proper Deligne-Mumford stack, whose coarse moduli space is the normalization of $\mathrm{Root}_{g,\ell}$. The connected component parametrizing order-ℓ line bundles is the Deligne-Mumford stack $\overline{\mathsf{R}}_{g,\ell}$; the coarse space $\overline{\mathcal{R}}_{g,\ell}$ is the $(3g-3)$-dimensional projective variety studied in this paper. The following diagram is commutative

$$\begin{array}{ccc} \overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell) & \xrightarrow{\mathsf{nor}} & \mathsf{Root}_{g,\ell} \\ & {\scriptstyle \mathsf{f}}\searrow \quad \swarrow{\scriptstyle \mathsf{h}} & \\ & \overline{\mathsf{M}}_g & \end{array} \tag{7}$$

and endows $\overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ with a universal twisted ℓth root and a universal quasi-stable ℓth root. The universal twisted ℓth root

$$\left(\mathsf{u}^{\mathsf{C}}_{g,\ell} : \mathsf{C}_{g,\ell} \to \overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell),\ \eta^{\mathsf{C}}_{g,\ell} \in \mathsf{Pic}(\mathsf{C}_{g,\ell}),\ \phi^{\mathsf{C}}_{g,\ell} : (\eta^{\mathsf{C}}_{g,\ell})^{\otimes \ell} \xrightarrow{\sim} \mathcal{O}\right)$$

consists of a universal balanced twisted curve, a line bundle and an isomorphism. The universal quasi-stable level ℓ curve pulled back via nor^*

$$\left(\mathsf{u}^{X}_{g,\ell} : X_{g,\ell} \to \overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell),\ \eta^{X}_{g,\ell} \in \mathsf{Pic}(X_{g,\ell}),\ \phi^{X}_{g,\ell} : (\eta^{X}_{g,\ell})^{\otimes \ell} \to \mathcal{O}\right)$$

consists of a family of quasi-stable curves, endowed with a line bundle and a homomorphism of line bundles. For any morphisms $T \to \overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ from a scheme T, we consider the pullbacks $(X_{g,\ell})_T$ and $(\mathsf{C}_{g,\ell})_T$; the stabilization of the quasi-stable curve $(X_{g,\ell})_T$ and the coarsening of the balanced twisted curve $(\mathsf{C}_{g,\ell})_T$ coincide and yield a representable morphism $C_{g,\ell} \to \overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ which may be also regarded as the universal stable curve pulled back from $\overline{\mathsf{M}}_g$ via f. On $C_{g,\ell}$, we have the following identity of sheaves (see [J2] and, in particular, (1-3) after Lemma 3.3.8, the proof of Theorem 3.3.9, and Figure 1; see also [C2, Lem. 2.2.5] for a more complete statement).

Proposition 1.3. *Over $\overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$, consider the stabilization morphism $\mathrm{st}_{g,\ell} : X_{g,\ell} \to C_{g,\ell}$ and the coarsening morphism $\pi_{g,\ell} : \mathsf{C}_{g,\ell} \to C_{g,\ell}$. On the universal stable curve $C_{g,\ell}$, we have the following identity between coherent sheaves $(\pi_{g,\ell})_*(\eta^{\mathsf{C}}_{g,\ell}) = (\mathrm{st}_{g,\ell})_*(\eta^{X}_{g,\ell})$.* □

Let us fix a closed point $\overline{\tau} := [\mathsf{C}, \eta, \phi]$ representing a twisted ℓth root and its image $\overline{t} := [X, \eta, \phi]$ representing a quasi-stable ℓth root. By the above proposition, the coarsening C of C equals the stabilization of X. Consider the exceptional components $E^1, \dots, E^k \subset X$ and write $\{p_i, q_i\} := E^i \cap \overline{X - E^i}$ and $a_i := \mathrm{ord}_{p_i}(\phi)$ and $b_i := \mathrm{ord}_{q_i}(\phi)$, for $i = 1, \dots, k$. Then $a_i + b_i = \ell$ and we have $\gcd(a_i, b_i) = 1$ (because ℓ is prime). The only nontrivial stabilizers of C occur at the nodes $\mathsf{n}_1,\dots,\mathsf{n}_k$ of C mapping to the nodes $\mathrm{st}(E_1), \dots, \mathrm{st}(E_1)$ of C. The local picture at n_i is the spectrum of $\mathbb{C}[\widetilde{x}_i, \widetilde{y}_i]/(\widetilde{x}_i\widetilde{y}_i = 0)$ and the local coordinates x_i and y_i of $\overline{X - E^i}$ at p_i and at q_i are related to $\widetilde{x}_i$ and to $\widetilde{y}_i$ by $\widetilde{x}_i^\ell = x_i$ and $\widetilde{y}_i^\ell = y_i$. The action of $\mathbb{Z}_\ell$ on the local picture $\operatorname{Spec} R[t] \to \operatorname{Spec} R$ of $\eta \to \mathsf{C}$ at n_i is given by $(\widetilde{x}_i, \widetilde{y}_i) \mapsto (\xi_\ell \widetilde{x}_i, \xi_\ell^{-1} \widetilde{y}_i)$ and $t_i \mapsto \xi_\ell^{a_i} t_i$. Following [CF], we refer to the pair (a_i, b_i) as the *multiplicity* of the quasi-stable ℓ-root $[X, \eta, \phi]$ along the exceptional component E^i and, equivalently, of the twisted ℓth root at n_i.

We further describe the local picture of the universal quasi-stable ℓth root over $\mathsf{Root}_{g,\ell}$ along one exceptional component $E = E^i$ meeting the rest of X at p and q and having multiplicity $(a, b) := (a_i, b_i)$. It is proved in [CCC, §3.1], that if one denotes by $C_{a,b} \subset \mathbb{A}^2_{w,z}$ the affine plane curve given by the equation $w^a = z^b$ and the following surface by

$$S_{a,b} := \Big\{ \big((x, y, z, w), [s_0 : s_1]\big) \in \mathbb{A}^4_{x,y,z,w} \times \mathbf{P}^1 : xs_0 = ws_1,\ ys_1 = zs_0,\ w^a = z^b \Big\}, \tag{8}$$

then the covering $f_{a,b} : S_{a,b} \to C_{a,b}$ given by $f_{a,b} : ((x,y,w,z),[s_0 : s_1]) \mapsto (w,z)$ is a local model for the simultaneous smoothing of the nodes p and q within the universal quasi-stable ℓth root over $\mathsf{Root}_{g,\ell}$. In particular, as soon as $a,b > 1$, the space $C_{a,b}$ and hence the moduli space $\mathrm{Root}_{g,\ell}$ are not normal. The normalization $\widetilde{f}_{a,b} : \widetilde{S}_{a,b} \to \mathbb{A}^1_\tau$ of $f_{a,b}$ is constructed by setting

$$\widetilde{S}_{a,b} := \Big\{ \big((x,y,\tau),[s_0 : s_1]\big) \in \mathbb{A}^3_{x,y,\tau} \times \mathbf{P}^1 : xs_0 = \tau^b s_1, ys_1 = \tau^a s_0 \Big\}, \tag{9}$$

and mapping $\mathbb{A}^1_\tau \to C_{a,b}$ via $\tau \mapsto (\tau^b, \tau^a)$. At the level of the normalization, the point $q \in X$ corresponds to the A_{a-1}-singularity $\big((0,0,0),[1 : 0]\big) \in \widetilde{S}_{a,b}$, whereas the A_{b-1}-singularity $\big((0,0,0),[0 : 1]\big) \in \widetilde{S}_{a,b}$ corresponds to the point $p \in X$. Globalizing this description, one obtains a local picture of the diagram (7) at the points $\overline{\tau}$ and $\overline{t}$. There exist local coordinates such that

$$\hat{\mathcal{O}}_{\mathsf{Root}_{g,\ell},\, \overline{t}} = \mathbb{C}[[w_1, z_1, \dots, w_k, z_k, t_{k+1}, \dots, t_{3g-3}]]/(w_i^{a_i} = z_i^{b_i}), \text{ for } i = 1, \dots, k, \tag{10}$$

$$\hat{\mathcal{O}}_{\overline{\mathsf{M}}_g, \mathrm{st}(X)} = \mathbb{C}[[t_1, \dots, t_{3g-3}]], \tag{11}$$

$$\hat{\mathcal{O}}_{\overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell),\, \overline{\tau}} = \mathbb{C}[[\tau_1, \dots, \tau_k, t_{k+1}, \dots, t_{3g-3}]]. \tag{12}$$

Locally, the morphisms f, nor and h are given by

$$\mathsf{f} : t_i \mapsto \begin{cases} \tau_i^\ell & i \le k \\ t_i & i > k \end{cases} \qquad \mathsf{nor} : \begin{cases} w_i \mapsto \tau_i^{b_i} & i \le k \\ z_i \mapsto \tau_i^{a_i} & i \le k \\ t_i \mapsto t_i & i > k \end{cases} \qquad \mathsf{h} : t_i \mapsto \begin{cases} w_i z_i & i \le k \\ t_i & i > k. \end{cases} \tag{13}$$

We regard X as a fiber of a universal quasi-stable curve on $\overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ over $\overline{\tau}$. Set $\tau := \tau_i$, $(a,b) := (a_i, b_i)$. Then, at p, we obtain an A_{b-1}-singularity of equation $x(s_0/s_1) = \tau^b$ along the locus where the node p persists. At q, we obtain an A_{a-1}-singularity of equation $y(s_1/s_0) = \tau^a$ along the locus where the node q persists. In view of intersection theory computations, following [C2], we desingularize such singularities. For any étale morphism $T \to \overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ the desingularization of $(X_{g,\ell})_T$ yields a semi-stable curve, globally on $\overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$,

$$X'_{g,\ell} \to X_{g,\ell} \to \overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$$

equipped with a line bundle $\mathcal{P} := \eta^{X'}_{g,\ell}$ and a homomorphism $\Phi = \phi^{X'}_{g,\ell}$ from the ℓth tensor power of $\mathcal{P}$ to $\mathcal{O}$: the pull-backs of $\eta^X_{g,\ell}$ and $\phi^X_{g,\ell}$. The curve X' over $\overline{\tau} \in \overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ is obtained by iterated blow-ups; *i.e.* for a fibre of $X_{g,\ell}$ of the form $C' \cup E$ with E exceptional and $E \cup C' = \{p, q\}$. One inserts a chain of $\ell - a - 1$ rational curves $E_1, \dots, E_{\ell-a-1}$ at p and a chain of $a - 1$ rational curves $E_{\ell-a+1}, \dots, E_{\ell-1}$ at q. Setting $E_{\ell-a} := E$, we obtain the nodal semi-stable curve $C' \cup E_1 \cup \dots \cup E_{\ell-1}$ of Figure 1, where $C' = \overline{(X - E)}$.

By iterating this procedure at all exceptional curves we get X', whose stable model is $C = \mathrm{st}(X)$, obtained by contracting the chains of the form $E_1 \cup \dots \cup E_{\ell-1}$: this explains that the singular locus $C_{g,\ell} \to \overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ is formed by $A_{\ell-1}$-singularities (since $C_{g,\ell}$ is the pullback via f of the universal stable curve of $\overline{\mathsf{M}}_g$, this may be regarded as a consequence of the local description of f of (13)). The restriction of $\mathcal{P} = \eta^{X'}_{g,\ell}$ to X' is a line bundle η' satisfying $\eta'_{E_i} = \mathcal{O}_{E_i}$ for $i \neq \ell - a$ and $\eta'_{E_{\ell-a}} = \mathcal{O}_{E_{\ell-a}}(1)$.

FIGURE 1. The curve X'.

Remark 1.4. There is an isomorphism $H^0(X', \omega_{X'} \otimes \eta') \cong H^0(X, \omega_X \otimes \eta)$ and we often identify the two spaces. This may be regarded as a consequence of the fact that $X'_{g,\ell} \to X_{g,\ell}$ is crepant (the relative canonical bundles match under pullback) and of Proposition 1.3. In practice this means that any cohomological question involving kernel bundles on a twisted ℓth root can be settled by working, equivalently, over $\overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$, with the twisted ℓth root $(\mathsf{C}_{g,\ell}, \eta^{\mathsf{C}}_{g,\ell}, \phi^{\mathsf{C}}_{g,\ell})$, with the quasi-stable level ℓ curve $(X_{g,\ell}, \eta^X_{g,\ell}, \phi^X_{g,\ell})$, or with the pullback $\mathcal{P} := \eta^{X'}_{g,\ell}$ of $\eta^X_{g,\ell}$ on the semi-stable curve $X'_{g,\ell}$.

1.3. **The moduli stack of level ℓ curves (for any level ℓ).** Let us assume only $\ell \in \mathbb{Z}_{\geq 2}$, without any condition on ℓ being prime. The local picture of $\mathsf{Root}_{g,\ell}$ (10) is still valid; in particular the local model for the curve smoothing the nodes p and q of an exceptional component E is still $S_{a,b}$ where (a, b) are the multiplicites of the quasistable ℓth root at the nodes. However, since a and b are no longer necessarily coprime, $S_{a,b}$ has $d := \gcd(a, b)$ irreducible components, each one isomorphic to $S_{\frac{a}{d}, \frac{b}{d}}$. Then the local picture of f, nor and h of (7) is

(14)

$$\mathsf{f} : t_i \mapsto \begin{cases} \tau_i^{\ell/d_i} & i \leq m \\ t_i & i > m \end{cases} \qquad \widetilde{\mathsf{nor}} : \begin{cases} w_i \mapsto \tau_i^{\ell - a_i/d_i} & i \leq k \\ z_i \mapsto \tau_i^{a_i/d_i} & i \leq k \\ t_i \mapsto t_i & i > k \end{cases} \qquad \mathsf{h} : t_i \mapsto \begin{cases} w_i z_i & i \leq k \\ t_i & i > k, \end{cases}$$

for $d_i = \gcd(a_i, b_i)$. This happens because the stabilizers of a node of C attached to an exceptional component whose multiplicities satisfy $d_i = \gcd(a_i, b_i)$ have order ℓ/d_i. Notice that the normalization morphism nor induces a surjection at the level of closed points, but does not induce an injection as soon as ℓ is composite. Furthermore the desingularization of the pullback of the universal quasi-stable curve on $\overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ is a semistable curve with chain of rational curves of length $\ell/d_i - 1$ over the node corresponding to the the exceptional component E^i. Fortunately, there is a simple way to avoid these exceptions.

In [C1] the first author introduced a series of variants of smooth modifications $\widetilde{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ of $\overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ allowing us to reproduce, even when ℓ is composite, the same configuration with chains of $\ell - 1$ rational curves illustrated in Figure 1. The stack $\widetilde{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ parametrizes triples $(\mathsf{C}, \eta \to \mathsf{C}, \phi : \eta^{\otimes \ell} \to \mathcal{O})$, where the stabilizer G_{n} at a node n is trivial if and only if the restriction $\eta_{|\mathsf{n}}$ is trivial and, otherwise, satisfies $G_{\mathsf{n}} \cong \mathbb{Z}_\ell$. In this way, by removing the condition of faithfulness and by imposing stabilizers of order ℓ on n whenever $\eta_{|\mathsf{n}}$ is nontrivial, the local picture of C at each node $\mathsf{n}_1, \ldots, \mathsf{n}_m$

where η has nontrivial restriction is given by the spectrum of $R = \mathbb{C}[\widetilde{x}_i, \widetilde{y}_i]/(\widetilde{x}_i\widetilde{y}_i = 0)$ with $\mathbb{Z}_\ell$ operating as $(\widetilde{x}_i, \widetilde{y}_i) \mapsto (\xi_\ell \widetilde{x}_i, \xi_\ell^{-1}\widetilde{y}_i)$. There, the line bundle $\eta \to \mathsf{C}$ has local picture $\operatorname{Spec} R[t_i] \to \operatorname{Spec} R$ with $\mathbb{Z}_\ell$ operating as $t_i \mapsto \xi_\ell^{a_i} t_i$ with $a_i \in \{1, \ldots, \ell-1\}$, not necessarily prime to ℓ, for $i = 1, \ldots, m$. As a consequence, there exist local coordinates at the closed point $\widetilde{\tau} \in \widetilde{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ representing (C, η, ϕ)

$$\hat{\mathcal{O}}_{\widetilde{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell),\, \widetilde{\tau}} = \mathbb{C}[[\widetilde{\tau}_1, \ldots, \widetilde{\tau}_m, t_{m+1}, \ldots, t_{3g-3}]].$$

and a morphism $\mathsf{e} : \widetilde{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell) \to \overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ fitting in

$$\begin{array}{ccccc} \widetilde{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell) & \xrightarrow{\ \mathsf{e}\ } & \overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell) & \xrightarrow{\ \mathsf{nor}\ } & \mathsf{Root}_{g,\ell} \\ & \searrow^{\widetilde{\mathsf{f}}} & \downarrow^{\mathsf{f}} & \swarrow_{\mathsf{h}} & \\ & & \overline{\mathsf{M}}_g & & \end{array} \tag{15}$$

given (at the level of local rings) by $\mathsf{e} : \tau_i \mapsto \widetilde{\tau}_i^{d_i}$ for $i \le m$ and by $\widetilde{\mathsf{f}} : t_i \mapsto \widetilde{\tau}_i^{\ell}$ for $i \le m$.

The morphism e yields an isomorphism at the level of coarse spaces. Therefore, as in [C2], when ℓ is not prime, we can work throughout the rest of the paper with the substack $\overline{\mathsf{R}}_{g,\ell}$ arising as the connected component of $\widetilde{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ of triples (C, η, ϕ) where η has order ℓ. This is a smooth Deligne-Mumford stack whose coarse space is a projective $(3g-3)$-dimensional variety $\overline{\mathcal{R}}_{g,\ell}$. With a slight abuse of notation, let us denote by $X_{g,\ell}$ the pullback of the universal quasi-stable curve on $\overline{\mathsf{R}}_{g,\ell}$ via $\mathsf{nor} \circ \mathsf{e}$. By construction, the singularities arising at the nodes p_i and q_i of an exceptional curve E_i are of type $A_{\ell-a_i-1}$ and A_{a_i-1} and the desingularization $X'_{g,\ell}$ is fibred in semi-stable curves obtained by inserting chains of $\ell - a_i - 1$ rational curves at p_i and chains of $a_i - 1$ rational curves at q_i as in Figure 1. Remark 1.4 generalizes word for word. From now on we shall work with $\overline{\mathsf{R}}_{g,\ell} \subset \widetilde{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ equipped with the universal semistable curve $X'_{g,\ell}$ and with the universal line bundle $\mathcal{P} := \eta_{g,\ell}^{X'}$ pulled back from $\mathsf{Root}_{g,\ell}$.

1.4. **The boundary divisors of $\overline{\mathcal{R}}_{g,\ell}$.** We briefly discuss the geometry of the boundary divisors of $\overline{\mathcal{R}}_{g,\ell}$ by describing the structure of the fibre $f^{-1}([C])$ corresponding to a general point of each boundary of the boundary divisors $\Delta_0, \ldots, \Delta_{\lfloor \frac{g}{2} \rfloor}$ of $\overline{\mathcal{M}}_g$. Let us assume first that $C := C_1 \cup_p C_2$ is a transverse union of two smooth curves C_1 and C_2 of genus i and $g-i$ respectively. If $[X, \eta, \phi] \in f^{-1}([C])$, then necessarily $X = C$ and η is uniquely determined by the data of two line bundles $\eta_{C_1} \in \operatorname{Pic}^0(C_1)[\ell]$ and $\eta_{C_2} \in \operatorname{Pic}^0(C_2)[\ell]$. Depending on which of these line bundles is trivial, we define the boundary divisors Δ_i, Δ_{g-i} and $\Delta_{i:g-i}$ respectively. For $1 \le i \le g-1$, the general point of Δ_i corresponds to a level curve of compact type

$$\left[C_1 \cup_p C_2, \eta_{C_1} \text{ of order } \ell,\ \eta_{C_2} \cong \mathcal{O}_{C_2}\right] \in \overline{\mathcal{R}}_{g,\ell}.$$

Finally, we denote by $\Delta_{i:g-i}$ the closure in $\overline{\mathcal{R}}_{g,\ell}$ of the locus of twisted level curves on $C_1 \cup C_2$ such that $\eta_{C_1} \ncong \mathcal{O}_{C_1}$ and $\eta_{C_2} \ncong \mathcal{O}_{C_2}$. Denoting by $\delta_i := [\Delta_i]_{\mathbb{Q}}, \delta_{g-i} := [\Delta_{g-i}]_{\mathbb{Q}}$, $\delta_{i:g-i} := [\Delta_{i:g-i}]_{\mathbb{Q}}$ the corresponding classes in $\operatorname{Pic}(\overline{\mathsf{R}}_{g,\ell})$, we have the following relation, showing that the morphism of stacks $\mathsf{f} : \overline{\mathsf{R}}_{g,\ell} \to \overline{\mathsf{M}}_g$ is étale over Δ_i where $i \ge 1$.

$$\mathsf{f}^*(\delta_i) = \delta_i + \delta_{g-i} + \delta_{i:g-i}. \tag{16}$$

For $i = 1$ and $\ell \geq 3$, observe that at the level of coarse moduli spaces the formula $f^*(\Delta_1) = 2(\Delta_1 + \Delta_{1:g-1}) + \Delta_{g-1}$ holds, in particular f is ramified along Δ_1 and $\Delta_{1:g-1}$.

Suppose now that $[C] \in \Delta_0$ is a general irreducible 1-nodal curve of genus g with normalization $\mathrm{nor} : C' \to C$, and let $p, q \in C'$ be such that $\mathrm{nor}(p) = \mathrm{nor}(q) \in \mathrm{Sing}(C)$. Assume that $[X, \eta, \phi] \in \pi^{-1}([C])$. We write down the exact sequence

$$1 \longrightarrow \mathbb{Z}_\ell \longrightarrow \mathrm{Pic}^0(C)[\ell] \stackrel{\mathrm{nor}^*}{\longrightarrow} \mathrm{Pic}^0(C')[\ell] \longrightarrow 0. \tag{17}$$

If $X = C$, we set $\eta_{C'} := \mathrm{nor}^*(\eta) \in \mathrm{Pic}^0(C')[\ell]$. We denote by $\Delta_0^{'}$ the closure in $\overline{\mathcal{R}}_{g,\ell}$ of the locus of level curves $[C, \eta, \phi]$ as above, where $\eta_{C'} \ncong \mathcal{O}_{C'}$. An ℓ-torsion line bundle on C is determined by the choice of $\eta_{C'}$ and the choice of a $\mathbb{Z}_\ell$-gluing of the fibres $\eta_{C'}(p)$ and $\eta_{C'}(q)$. We observe that, when ℓ is prime, $\deg(\Delta_0^{'}/\Delta_0) = \ell(\ell^{2g-2} - 1)$ (see [CF] for the general statement).

We denote by $\Delta_0^{''}$ the closure in $\overline{\mathcal{R}}_{g,\ell}$ of the locus of level curves $[C, \eta, \phi]$ such that $\eta_{C'} \cong \mathcal{O}_{C'}$ (we referred to these curves as Wirtinger covers in the introduction; these arise as the preimage of a constant section via the morphism $\eta \to \mathcal{O}$ between total spaces). Using (17), an order-ℓ line bundle $\eta \in \mathrm{Pic}^0(C)[\ell]$ with $\mathrm{nor}^*(\eta) = \mathcal{O}_{C'}$ is determined by a $\mathbb{Z}_\ell^*$-gluing of the fibres $\eta_{C'}(p)$ and $\eta_{C'}(q)$, that is, a root of unity $\xi_\ell^a \in \mathbb{Z}_\ell^*$, with $1 \leq a \leq \ell - 1$ prime to ℓ, such that sections $\sigma \in \eta_{C'}$ that descend to C are characterized by the equation $\sigma(p) = \xi_\ell^a \sigma(q)$. Since by reversing the role of p and q we interchange a and $\ell - a$, we obtain the a decomposition into irreducible $\lfloor l/2 \rfloor$ components, each of them of order 2 over Δ_0. For any $\ell \in \mathbb{Z}_{\geq 2}$ we get $\deg(\Delta_0^{''}/\Delta_0) = \ell - 1$.

We now consider the case $X = C' \cup_{\{p,q\}} E$, where E is an exceptional component. Then by definition $\eta_E = \mathcal{O}_E(1)$, therefore $\deg(\eta_{C'}) = -1$. Furthermore, there exists an integer $1 \leq a \leq \ell - 1$ such that $\eta_{C'}^{\otimes(-\ell)} = \mathcal{O}_{C'}(a \cdot p + (\ell - a) \cdot q)$. Let us denote by $\Delta_0^{(a)}$ the closure in $\overline{\mathcal{R}}_{g,\ell}$ of the locus of such points. By switching the role of p and q we can obviously restrict ourselves to the case $1 \leq a \leq \lfloor \frac{\ell}{2} \rfloor$. Since the choice of $\eta_{C'} \in \mathrm{Pic}^{-1}(C')$ as above uniquely determines the level curve $[C' \cup E, \eta, \phi]$, it follows that, when ℓ is prime, $\deg(\Delta_0^{(a)}/\Delta_0) = 2\ell^{2g-2}$ (the factor 2 accounts for the possibility of interchanging p and q). We refer to [CF] for the treatment of the general case $\ell \in \mathbb{Z}_{\geq 2}$ and the decomposition into irreducible components.

We follow again the convention of passing to lower case symbols for the divisor classes in the moduli stack $\overline{\mathsf{R}}_{g,\ell}$: $\delta_0' = [\Delta_0']_{\mathbb{Q}}$, $\delta_0'' = [\Delta_0'']_{\mathbb{Q}}$ and $\delta_0^{(a)} = [\Delta_0^{(a)}]_{\mathbb{Q}}$. As a direct consequence of the local description (13) the morphism f is étale over Δ_0' and Δ_0'' and ramified with order ℓ at $\Delta_0^{(a)}$. (When ℓ is not prime the order of ramification at $\Delta_0^{(a)}$ is still ℓ because $\overline{\mathsf{R}}_{g,\ell}$ as been defined as a substack of the covering $\widetilde{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$ of $\overline{\mathsf{M}}_g(\mathsf{B}\mathbb{Z}_\ell)$). Summarizing we have the following relation in $\mathrm{Pic}(\overline{\mathsf{R}}_{g,\ell})$

$$\mathsf{f}^*(\delta_0) = \delta_0^{'} + \delta_0^{''} + \ell \sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} \delta_0^{(a)}.$$

Proposition 1.5. *For $\ell \geq 3$, the canonical class of the coarse moduli space $\overline{\mathcal{R}}_{g,\ell}$ is equal to*

$$K_{\overline{\mathcal{R}}_{g,\ell}} = 13\lambda - 2(\delta_0^{'} + \delta_0^{''}) - (\ell+1)\sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} \delta_0^{(a)} - 2\sum_{i=1}^{\lfloor \frac{g}{2} \rfloor}(\delta_i + \delta_{g-i} + \delta_{i:g-i}) - \delta_{g-1} \in \mathsf{Pic}(\overline{\mathsf{R}}_{g,\ell}).$$

Proof. It follows from the Hurwitz formula $K_{\overline{\mathcal{R}}_{g,\ell}} = f^* K_{\overline{\mathcal{M}}_g} + \delta_1 + \delta_{1:g-1} + (\ell-1)\sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} \delta_0^{(a)}$, coupled with the expression for $K_{\overline{\mathcal{M}}_g}$, cf. [HM] Theorem 2 and formula (16). □

1.5. **The geometry of the universal semi-stable level ℓ curve.** Consider the universal semi-stable curve $\mathsf{u} : X'_{g,\ell} \to \overline{\mathsf{R}}_{g,\ell}$ introduced in §1.2, endowed with the tautological bundle $\mathcal{P} := \eta_{g,\ell}^{X'}$ and the homomorphism of line bundles

$$\Phi : \mathcal{P}^{\otimes \ell} \to \mathcal{O}_{X'_{g,\ell}}.$$

For sake of clarity and in preparation of the enumerative geometry studied in this paper, we detail the proof of a preliminary result on the self-intersection of $c_1(\mathcal{P})$. First, let us fix an integer $1 \leq a \leq \lfloor \frac{\ell}{2} \rfloor$ and observe that the the pull-back of the boundary divisor $\Delta_0^{(a)}$ in $\overline{\mathsf{R}}_{g,\ell}$ splits into irreducible components

$$\mathsf{u}^*(\Delta_0^{(a)}) = \mathcal{E}_1^{(a)} + \ldots + \mathcal{E}_{\ell-1}^{(a)},$$

where $\mathcal{E}_i^{(a)}$ parametrizes the closure of the locus of points that lie on the ith rational component of the chain $E_1 \cup \cdots \cup E_{\ell-1}$ of $\ell - 1$ rational lines over a general point of $\Delta_0^{(a)}$.

Proposition 1.6. *The following relations hold in* $\mathsf{Pic}(\overline{\mathsf{R}}_{g,\ell})$*:*
(1) $\mathsf{u}_*\big([\mathcal{E}_i^{(a)}] \cdot [\mathcal{E}_{i+1}^{(a)}]\big) = \delta_0^{(a)}$ *and* $\mathsf{u}_*\big([\mathcal{E}_i^{(a)}]^2\big) = -2\delta_0^{(a)}$*, for each* $1 \leq i \leq \ell - 1$.
(2) $\mathsf{u}_*\big(c_1(\mathcal{P}) \cdot c_1(\omega_{\mathsf{u}})\big) = 0$.
(3) $\mathsf{u}_*(c_1^2(\mathcal{P})) = -\sum_{1 \leq a \leq \lfloor \frac{\ell}{2} \rfloor} \frac{a(\ell-a)}{\ell} \delta_0^{(a)}$.

Proof. The first two statements being immediate, we proceed to the last one (see also [C2, Lem. 3.1.4]). For each $1 \leq a \leq \lfloor \frac{\ell}{2} \rfloor$, we fix a general quasi-stable ℓth root (X^a, η, ϕ) over $\Delta_0^{(a)}$. Here $X^a := C' \cup_{\{p,q\}} E$ is a quasi-stable curve and $\eta_{C'}^{\otimes \ell} = \mathcal{O}_{C'}(-a \cdot p - (\ell - a) \cdot q)$. The corresponding semi-stable fibre of $X'_{g,\ell}$ mapping to X^a is a semi-stable curve X' obtained by setting $E_{\ell-a}^{(a)} := E$, and gluing to the points $p, q \in C$ smooth rational curves $E_1^{(a)}, \ldots, E_{\ell-a-1}^{(a)}, E_{\ell-a}^{(a)}, E_{\ell-a+1}^{(a)}, \ldots, E_{\ell-1}^{(a)}$ forming a chain. The line bundle η' on X' is the restriction $\mathcal{P}_{|X'}$ isomorphic to $\eta_{C'}$ on C', to $\mathcal{O}$ on $E_i^{(a)}$ for $i \neq \ell - a$, and to $\mathcal{O}(1)$ on $E_{\ell-a}^{(a)}$. By construction we have that $X' \cap \mathcal{E}_i^{(a)} = E_i^{(a)}$.

We claim that the vanishing locus of the section $\Phi \in H^0(X'_{g,\ell}, \mathcal{P}^{\otimes(-\ell)})$ is precisely the divisor

$$\sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor}\Big(a\sum_{i=1}^{\ell-a} i\, \mathcal{E}_i^{(a)} + (\ell-a)\sum_{i=1}^{a-1}(a-i)\mathcal{E}_{\ell-a+i}^{(a)}\Big). \tag{18}$$

Indeed, we already know that $\mathcal{D} := \text{supp}(Z(\Phi))$ can be expressed as a linear combination of exceptional divisors $\sum_{a,i} c_i^{(a)}\mathcal{E}_i^{(a)}$, where $1 \leq i \leq \ell-1$ and $1 \leq a \leq \lfloor \frac{\ell}{2} \rfloor$, which is furthermore characterized by two sets of conditions:

(i) For each $1 \leq a \leq \lfloor \frac{\ell}{2} \rfloor$, one has that $c_1^{(a)} = a$ and $c_{\ell-1}^{(a)} = \ell - a$.
(ii) For each $i \neq \ell - a$, one has $\deg(\mathcal{D}_{|E_i^{(a)}}) = 0$.

The first condition expressed the fact that the vanishing order of $\Phi_{|C}$ at p (respectively q) is equal to a (respectively $\ell - a$). The second condition expresses the fact that the restriction of Φ on $E_i^{(a)}$ is the trivial morphism along each exceptional divisor $E_i^{(a)}$ with $i \neq \ell - a$, inserted when passing from $X_{g,\ell}$ to $X'_{g,\ell}$. Taking into account that $\mathcal{O}(\mathcal{E}_i^{(a)}) = \mathcal{O}(-2)$ on $E_i^{(a)}$, the two conditions lead to a linear system of equation in the coefficients $c_i^{(a)}$, which we then solve to obtain the claimed formula (18), which also computed $-\ell c_1(\mathcal{P})$. Squaring this formula, pushing it forward while using formulas (1), (2), we obtain

$$\ell^2 \mathsf{u}_*(c_1^2(\mathcal{P})) = -2\Big(\sum_{i=1}^{\ell-a} a^2 i^2 + \sum_{i=1}^{a-1}(\ell-a)^2 i^2 + \sum_{i=1}^{\ell-a-1} a^2 i(i+1) + \sum_{i=1}^{a-1}(\ell-a)^2 i(i+1)\Big)\delta_0^{(a)},$$

which after manipulations leads to the claimed formula. □

2. SYZYGY JUMPING LOCI OVER $\overline{\mathcal{R}}_{g,\ell}$

Following a principle already explained in [F1], one can construct tautological divisors on moduli spaces of curves defined in terms of syzygies of the parametrized objects. Such loci have a determinantal structure over an open subset of $\mathcal{R}_{g,\ell}$, hence one can speak of their expected (co)dimension. When the expected codimension is 1, one has virtual divisors on $\mathcal{R}_{g,\ell}$.

We begin by setting notation. For a smooth curve C, a line bundle L and a sheaf $\mathcal{F}$ on C, we define the Koszul cohomology group $K_{p,q}(C;\mathcal{F},L)$ as the cohomology of the complex

$$\bigwedge^{p+1} H^0(C,L)\otimes H^0(C,\mathcal{F}\otimes L^{\otimes(q-1)}) \stackrel{d_{p+1,q-1}}{\longrightarrow} \bigwedge^{p} H^0(C,L)\otimes H^0(C,\mathcal{F}\otimes L^{\otimes q}) \stackrel{d_{p,q}}{\longrightarrow}$$
$$\stackrel{d_{p,q}}{\longrightarrow} \bigwedge^{p-1} H^0(C,L)\otimes H^0(C,\mathcal{F}\otimes L^{\otimes(q+1)}).$$

When $\mathcal{F} = \mathcal{O}_C$, one writes $K_{p,q}(C,L) := K_{p,q}(C;\mathcal{O}_C,L)$. When $\mathcal{F}$ is a line bundle, these Koszul cohomology groups can be interpreted in terms of syzygies of certain 0-dimensional subschemes of C.

Example 2.1. Suppose $[C,\eta] \in \mathcal{R}_{g,\ell}$ and $\eta \notin C_2 - C_2$. Let $L := K_C \otimes \eta \in \text{Pic}^{2g-2}(C)$ be the very ample paracanonical line bundle inducing an embedding $\phi_L : C \to \mathbf{P}^{g-2}$. For an integer $1 \leq k \leq \ell - 1$, since $H^0(C,\eta^{\otimes k}) = 0$ one has that

$$K_{p,1}(C;\eta^{\otimes k},L) = \text{Ker}\Big\{\bigwedge^{p} H^0(L)\otimes H^0(L\otimes\eta^{\otimes k}) \to \bigwedge^{p-1} H^0(L)\otimes H^0(L^{\otimes 2}\otimes\eta^{\otimes k})\Big\}.$$

This group can be viewed as parametrizing the syzygies of a 0-dimensional scheme $Z \subset C \subset \mathbf{P}^{g-2}$, where $Z \in |K_C \otimes \eta^{\otimes(1-k)}|$, see [FMP] Proposition 1.6. Precisely,

$$b_{p+1,1}(Z) = \dim K_{p,1}(C; \eta^{\otimes k}, L). \tag{19}$$

We explain Conjecture B and Theorems 0.1 and 0.3 formulated in the introduction. We fix line bundles $\eta, \xi \in \mathrm{Pic}^0(C)$, with $\xi \neq \mathcal{O}_C$ and $\eta \otimes \xi \neq \mathcal{O}_C$. To the paracanonical linear series $L := K_C \otimes \eta$ we associate the kernel bundle defined via the exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(C, L) \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0.$$

Using standard arguments, see e.g. [FMP] Proposition 1.6, one has the identifications

$$K_{i,1}(C; \xi, K_C \otimes \eta) = H^0\big(C, \bigwedge^i M_L \otimes L \otimes \xi\big), \tag{20}$$

$$K_{i-1,2}(C; \xi, K_C \otimes \eta) = H^1\big(C, \bigwedge^i M_L \otimes L \otimes \xi\big).$$

The difference of the dimensions of the two cohomology groups is the Euler-Poincaré characteristic of a vector bundle on C, which in the case $\eta \neq \mathcal{O}_C$ is equal to

$$\dim K_{i,1}(C; \xi, K_C \otimes \eta) - \dim K_{i-1,2}(C; \xi, K_C \otimes \eta) = (g-1)\binom{g-2}{i}\Big(1 - \frac{2i}{g-2}\Big). \tag{21}$$

The naturality of the resolution of $\Gamma_C(\xi, L)$ as a Sym $H^0(C, L)$-module is equivalent to the vanishing for all i of one of the groups $K_{i,1}(C; \xi, L)$ or $K_{i-1,2}(C; \xi, L)$, depending on the sign computed by (21). The resolution being minimal, it suffices to verify this for the values of i when the expression (21), viewed as a function of i, changes sign. The resolution is pure, if the corresponding cohomology group vanishes for i such that the expression in (21) is equal to zero.

Lemma 2.2. *Let C be a curve of genus g and $\eta, \xi \in \mathrm{Pic}^0(C) - \{\mathcal{O}_C\}$ such that $\eta \otimes \xi \neq \mathcal{O}_C$.*

(1) The resolution of $\Gamma_C(\xi, K_C \otimes \eta)$ as a $\mathrm{Sym} H^0(C, K_C \otimes \eta)$-module is natural if and only if

$$K_{\lfloor \frac{g-1}{2} \rfloor,1}(C; \xi, K_C \otimes \eta) = 0 \quad \textit{and} \quad K_{\lfloor \frac{g-1}{2} \rfloor,1}(C; (\eta \otimes \xi)^\vee, K_C \otimes \eta) = 0.$$

(2) The resolution of $\Gamma_C(\xi, K_C)$ as a $\mathrm{Sym} H^0(C, K_C)$-module is natural if and only if

$$K_{\lfloor \frac{g}{2} \rfloor,1}(C; \xi, K_C) = 0 \quad \textit{and} \quad K_{\lfloor \frac{g}{2} \rfloor,1}(C; \xi^\vee, K_C) = 0.$$

Proof. Assume that g is odd and write $g = 2i+1$. From (21), the resolution of the module $\Gamma_C(\xi, K_C \otimes \eta)$ is natural if and only if $K_{i,1}(C; \xi, K_C \otimes \eta) = K_{i-2,2}(C; \xi, K_C \otimes \eta) = 0$. By the duality theorem [G], the dual of the last group is canonically isomorphic to

$$K_{i,0}(C; K_C \otimes \xi^\vee, K_C \otimes \eta) = K_{i,1}(C; (\eta \otimes \xi)^\vee, K_C \otimes \eta).$$

The case of the resolution of $\Gamma_C(\xi, K_C)$ (that is, $\eta = \mathcal{O}_C$) is similar and we skip details. $\square$

We now describe the structure of the locus $\mathcal{U}_{g,\ell}$ mentioned in Theorem 0.3. For $g = 2i+1$, a result from [FMP] provides an identification of cycles valid for each curve:

$$\Big\{\eta \in \mathrm{Pic}^0(C) : h^0(C, \bigwedge^i M_{K_C} \otimes K_C \otimes \eta) \geq 1\Big\} = C_i - C_i \subset \mathrm{Pic}^0(C).$$

The right hand side denotes the i-th difference variety consisting of line bundles of the form $\mathcal{O}_C(D-E)$, where $D, E \in C_i$. This establishes the following equivalence

$$K_{i,1}(C;\eta,K_C) \neq 0 \Leftrightarrow \eta \in C_i - C_i.$$

After tensoring with $K_C \otimes \eta$ and taking cohomology in the short exact sequence

$$0 \longrightarrow \bigwedge^i M_{K_C} \longrightarrow \bigwedge^i H^0(C,K_C)\otimes \mathcal{O}_C \longrightarrow \bigwedge^{i-1} M_{K_C}\otimes K_C \longrightarrow 0,$$

we reformulate the last condition as follows: $[C,\eta] \in \mathcal{U}_{g,\ell}$ if and only if the map

$$\chi([C,\eta]) : \bigwedge^i H^0(K_C)\otimes H^0(K_C\otimes \eta) \to H^0\big(C, \bigwedge^{i-1} M_{K_C}\otimes K_C^{\otimes 2}\otimes \eta\big) \tag{22}$$

is an isomorphism. Since $h^0(\bigwedge^{i-1} M_{K_C}\otimes K_C^{\otimes 2}\otimes \eta) = \chi(\bigwedge^{i-1} M_{K_C}\otimes K_C^{\otimes 2}\otimes \eta) = \binom{2i}{i-1}(4i+2)$, note that the locus $\mathcal{U}_{g,\ell}$ can be defined as the degeneracy locus of a morphism between vector bundles of the same rank over $\mathsf{R}_{g,\ell}$, whose fibre over a point $[C,\eta]$ is precisely the map $\chi([C,\eta]$ defined above. Thus $\mathcal{U}_{g,\ell}$ is a virtual divisor over $\mathcal{R}_{g,\ell}$.

We show that that $\chi([C,\eta])$ is an isomorphism for a general level curve, thus establishing Theorem 0.3. In fact, we prove a more precise result valid for all genera:

Theorem 2.3. *Let $[C,p] \in \mathcal{M}_{g,1}$ be a general hyperelliptic curve of genus $g \geq 2$ with a Weierstrass point and $\eta \in \mathrm{Pic}^0(C)[\ell] - \{\mathcal{O}_C\}$. One has that $H^0\big(C, \eta((g-1)p)\big) = 0$.*

First we explain how Theorem 2.3 implies Theorem 0.3. Let C be hyperelliptic, $A = \mathcal{O}_C(2p)$ the hyperelliptic line bundle and $i := \lfloor \frac{g}{2} \rfloor$. Then $M_{K_C}^{\vee} = A^{\oplus(g-1)}$, therefore

$$\bigwedge^i M_{K_C}^{\vee} = \big(A^{\otimes i}\big)^{\oplus \binom{g-1}{i}}.$$

It follows from Lemma (2.2) that $K_{i,1}(C;\eta,K_C) = 0$ if and only if $H^0(C,\eta((g-1)p)) = 0$ when g is odd, respectively $H^0(C,\eta((g-2)p)) = 0$, when g is even. Clearly, the statement of Theorem 2.3 implies the naturality of the resolution $\Gamma_C(\eta, K_C)$ for all genera.

Proof of Theorem 2.3. Inductively we assume that $[C',\eta'] \in \mathcal{R}_{g-1,\ell}$ is a hyperelliptic level ℓ curve with a point $p' \in C'$ such that $h^0(C',\mathcal{O}_{C'}(2p')) = 2$ and $H^0\big(C',\eta'((g-2)p')\big) = 0$. We consider a pointed elliptic curve $[E,p'] \in \mathcal{M}_{1,1}$, which we attach to C' at the point p', then choose $p \in E - \{p'\}$ such that $2(p-p') \equiv 0$. Then $[C := C \cup_{p'} E, p] \in \overline{\mathcal{M}}_{g,1}$ is a degenerate hyperelliptic curve and $p \in C$ is a hyperelliptic Weierstrass point. This follows by exhibiting an admissible double covering

$$\varphi : C \to (\mathbf{P}^1)_1 \cup_t (\mathbf{P}^1)_2,$$

where $\varphi_{C'} : C' \to (\mathbf{P}^1)_1$ is the hyperelliptic cover and $\varphi_E : E \to (\mathbf{P}^1)_2$ is the double cover ramified at p and p'. Note that $\varphi(p') = t \in (\mathbf{P}^1)_1 \cap (\mathbf{P}^1)_2$.

We choose an ℓ-torsion point $\eta_E \in \mathrm{Pic}^0(E)[\ell]$ such that $\eta_E \neq \mathcal{O}_C\big((g-1)(p'-p)\big)$. Such a choice is possible since there at least two points of order ℓ inside $\mathrm{Pic}^0(E)$. Then

$$[C' \cup_{p'} E,\ \eta_{C'},\ \eta_E] \in \overline{\mathcal{R}}_{g,\ell},$$

and we claim that this curve does not lie in the closure of the locus of hyperelliptic level ℓ curves $[X_t,\eta_t] \in \mathcal{R}_{g,\ell}$ with a point $p_t \in X_t$ such that $h^0(X_t,\mathcal{O}_{X_t}(2p_t)) = 2$ and

$h^0(X_t, \eta_t((g-1)p_t)) \geq 1$. Indeed, assuming this not to be true, applying a limit linear series argument we find non-zero sections

$$\sigma_{C'} \in H^0\big(C', \eta_{C'} \otimes \mathcal{O}_{C'}((g-1)p')\big) \text{ and } \sigma_E \in H^0\big(E, \eta_E \otimes \mathcal{O}_E((g-1)p)\big),$$

such that $\mathrm{ord}_{p'}(\sigma_{C'}) + \mathrm{ord}_{p'}(\sigma_E) \geq g-1$. Our assumption implies $\mathrm{ord}_{p'}(\sigma_E) \leq g-2$, therefore $\mathrm{ord}_{p'}(\sigma_{C'}) \geq 1$, that is, $H^0\big(C', \eta_{C'} \otimes \mathcal{O}_{C'}((g-2)p')\big) \neq 0$. This is a contradiction and completes the proof. $\square$

2.1. **The syzygies of the twisted paracanonical module.** We now show that for a general point $[C, \eta] \in \mathcal{R}_{g,\ell}$, the module $\Gamma_C(\xi, K_C \otimes \eta)$ has a pure resolution for a general choice of $\xi \in \mathrm{Pic}^0(C)$. The argument relies on specialization to bielliptic curves.

Suppose E is an elliptic curve, C a curve of genus g and $f : C \to E$ a double cover ramified along the divisor $R \in C_{2g-2}$ and branched along the divisor $B \in E_{2g-2}$. Let $\delta \in \mathrm{Pic}^{g-1}(E)$ denote the line bundle determining f, that is, $\delta^{\otimes 2} = \mathcal{O}_E(B)$. In particular,

$$f_*\mathcal{O}_C = \mathcal{O}_E \oplus \delta^\vee, \ \ f^*(\delta) = \mathcal{O}_C(R).$$

Proposition 2.4. *Let $\epsilon \in \mathrm{Pic}^0(E)[\ell]$ an ℓ-torsion point and $\eta := f^*(\epsilon) \in \mathrm{Pic}^0(C)[\ell]$. Then*

$$M_{K_C \otimes \eta} = f^* M_{\delta \otimes \epsilon}.$$

Proof. From adjunction $K_C = f^*(\delta)$, therefore from the push-pull formula

$$H^0(C, K_C \otimes \eta) = f^*H^0(E, \delta \otimes \epsilon) \oplus f^*H^0(E, \epsilon) \cong H^0(E, \delta \otimes \epsilon).$$

In particular, pulling-back via f the exact sequence defining the kernel bundle on E

$$0 \longrightarrow M_{\delta \otimes \epsilon} \longrightarrow H^0(E, \delta \otimes \epsilon) \longrightarrow \delta \otimes \epsilon \longrightarrow 0,$$

we retrieve the sequence defining the kernel bundle on C, that is, $M_{K_C \otimes \eta} = f^*(M_{\delta \otimes \epsilon})$. $\square$

Lemma 2.5. *The push-forward $f_*\xi$ of a general line bundle $\xi \in \mathrm{Pic}^0(C)$ is stable (respectively semistable) if g is even (respectively odd).*

Proof. We treat only the case of even genus $g = 2i+2$, the odd genus case being similar. The push-forward $f_*\xi$ is a rank two vector bundle on E with $\det f_*(\xi) = \mathrm{Nm}_f(\xi) \otimes \delta^\vee$, in particular $\deg(f_*\xi) = 1-g$. Assume that $f_*\xi$ is not semistable. Then there exists a line subbundle $M \hookrightarrow f_*\xi$ with $\deg(M) \geq \deg(f_*\xi)/2$, that is, $\deg(M) \geq -i$. Then $H^0(C, \xi \otimes f^*M^\vee) \neq 0$, hence we can write $\xi = f^*(M)(D)$, where D is an effective divisor on C, with $\deg(D) \leq 2i$. Counting parameters, line bundles on C having this type depend on at most $2i+1 = g-1$ parameters, hence they do not fill-up $\mathrm{Pic}^0(C)$. $\square$

Thus one obtains a rational map $f_* : \mathrm{Pic}^0(C) \dashrightarrow \mathcal{U}_E(2, 1-g)$. By describing the differential of this map, it is easy to show that this map is dominant, see also [B].

We complete the proof of Theorem 0.1. More precisely we prove the following:

Theorem 2.6. *Let $f : C \to E$ be a bielliptic curve and $\eta \in \mathrm{Pic}^0(C)[\ell]$ as above. Then*

$$K_{\lfloor \frac{g-1}{2} \rfloor, 1}(C; \xi, K_C \otimes \eta) = 0,$$

for a general $\xi \in \mathrm{Pic}^0(C)$. In particular, the resolution of $\Gamma_C(\xi, K_C \otimes \eta)$ is natural.

Proof. We treat only the case $g = 2i+2$, the odd genus case being quite similar. Using identification (20) we write

$$K_{i,1}(C;\xi, K_C\otimes\eta) = H^0\big(C, \bigwedge^i M_{K_C\otimes\eta}\otimes K_C\otimes\eta\otimes\xi\big) = H^0\big(E, \bigwedge^i M_{\delta\otimes\epsilon}\otimes\delta\otimes\epsilon\otimes f_*\xi\big).$$

It is well-known that the vector bundle $M_{\delta\otimes\epsilon}$ is stable, hence using also Lemma 2.5, the vector bundle $\mathcal{F} := \bigwedge^i M_{\delta\otimes\epsilon}\otimes\epsilon\otimes\delta\otimes f_*\xi$ is semistable. Futhermore, the choice of a general $\xi\in \mathrm{Pic}^0(C)$ corresponds to the choice of a general $\mathcal{F}\in\mathcal{U}_E\big(\binom{g-2}{i}, 0\big)$. Since on an elliptic curve, there exists precisely one semistable bundle of prescribed rank, degree zero and with a section, we find that $H^0(E,\mathcal{F}) = 0$, which finishes the proof. □

Theorem 2.6 admits the following reformulation in the case of even genus.

Corollary 2.7. *For a general curve $[C,\eta]\in\mathcal{R}_{g,\ell}$ of even genus, the vector bundle $\bigwedge^{\frac{g-2}{2}} M_{K_C\otimes\eta}$ admits a theta divisor.*

Since $\mu\big(\bigwedge^{\frac{g-2}{2}} M_{K_C\otimes\eta}\big) = g-1\in\mathbb{Z}$ it makes sense to ask whether the vector bundle in question admits a theta divisor. Conjecture B is a more refined statement, predicting which line bundles $K_C\otimes\eta^{\otimes k}$ belong to the theta divisor of $\bigwedge^{\frac{g-2}{2}} M_{K_C\otimes\eta}$. Note that it is proved in [FMP] that all powers $\bigwedge^i M_{K_C}$ admit a theta divisor for *every* smooth curve C.

3. Intersection theory on $\overline{\mathsf{R}}_{g,\ell}$

The aim of this section is to describe the characteristic classes of tautological bundles on $\overline{\mathsf{R}}_{g,\ell}$ that are used to calculate the classes $[\overline{\mathcal{Z}}_{g,\ell}]^{\mathrm{virt}}$, $[\overline{\mathcal{D}}_{g,\ell}]^{\mathrm{virt}}$ and $[\overline{\mathcal{U}}_{g,\ell}]$ respectively. For our purposes it will suffice to consider only level ℓ curves whose underlying stable model is irreducible. Let $\widetilde{\mathsf{R}}_{g,\ell}$ be the open substack of level ℓ curves whose underlying stable model is a 1-nodal irreducible curves of arithmetic genus g. We denote by $\mathsf{u}:\widetilde{\mathsf{X}}_{g,\ell}\to\widetilde{\mathsf{R}}_{g,\ell}$ the restriction of the universal level ℓ curve, by $\mathcal{P}\in\mathsf{Pic}(\widetilde{\mathsf{X}}_{g,\ell})$ the tautological ℓth root bundle and by $\Phi:\mathcal{P}^{\otimes\ell}\to\mathcal{O}_{\widetilde{\mathsf{X}}_{g,\ell}}$ the universal sheaf homomorphism. We shall use the Hodge bundle $\mathbb{E} := \mathsf{u}_*(\omega_{\mathsf{u}})$ respectively the Prym-Hodge bundle $\mathbb{E}' := \mathsf{u}_*(\omega_{\mathsf{u}}\otimes\mathcal{P})$. These are locally free sheaves over $\widetilde{\mathsf{R}}_{g,\ell}$ of ranks g and $g-1$ respectively.

The following technical statement will be used to show that various tautological sheaves on $\widetilde{\mathsf{R}}_{g,\ell}$ are locally free.

Proposition 3.1. *For a level curve $[X,\eta,\phi]\in\widetilde{\mathcal{R}}_{g,\ell}$ and integers $b\geq 2$ and $0\leq j\leq\frac{g}{2}$, the following vanishing statements hold:*

(i) $H^1\big(X,\bigwedge^j M_{\omega_X}\otimes\omega_X^{\otimes b}\otimes\eta\big) = 0.$
(ii) $H^1\big(X,\bigwedge^j M_{\omega_X\otimes\eta}\otimes\omega_X^{\otimes b}\otimes\eta^{\otimes(b-2)}\big) = 0.$

Proof. As pointed out in Section 1, such a question can be studied at the level of root curves. To ease notation we identify $\mathrm{nor}([X,\eta,\phi])\in\mathrm{Root}_{g,\ell}$ and $[X,\eta,\phi]$. If X is smooth, since $\mu(\wedge^j M_{\omega_X}\otimes \omega_X^{\otimes b}\otimes \eta)=b(2g-2)-2j\geq 2g-1$, the statements are a consequence of the semistability of M_{ω_X} and that of $M_{\omega_X\otimes \eta}$ respectively.

Assume now that X is 1-nodal and $\eta\in \mathrm{Pic}(X)$ is locally free, that is, $[X,\eta,\phi]\in \Delta_0^{'}\cup\Delta_0^{''}$. Let $\mathrm{nor}:C\to X$ be the normalization map, with $\mathrm{nor}^{-1}(X_{\mathrm{sing}})=\{p,q\}$ and set $\eta_C:=\mathrm{nor}^*(\eta)$. Via the exact sequence $0\to \mathcal{O}_X\to \mathrm{nor}_*(\mathcal{O}_C)\to \mathrm{nor}_*(\mathcal{O}_C)/\mathcal{O}_X\to 0$, to show that (ii) holds it suffices to show that

$$H^1\Big(C,\bigwedge^j M_{K_C(p+q)\otimes \eta_C}\otimes K_C^{\otimes b}\otimes \mathcal{O}_C(bp+bq)\otimes \eta_C\Big)=0,$$

and

$$H^1\Big(C,\bigwedge^j M_{K_C(p+q)\otimes \eta_C}\otimes K_C^{\otimes b}\otimes \mathcal{O}_C((b-1)p+(b-1)q)\otimes \eta_C\Big)=0.$$

This again is a consequence of the stability of the vector bundle $M_{K_C(p+q)\otimes \eta_C}$, which in turn follows from [FL] Proposition 2.4.

Assume now that $[X,\eta,\phi]\in \Delta_0^{\mathrm{ram}}$ and $X=C\cup_{\{p,q\}}E$, where $E\cong \mathbf{P}^1$ and $[C,p,q]\in \mathcal{M}_{g-1,2}$. Furthermore $\eta_E=\mathcal{O}_E(1)$ and $\deg(\eta_C)=-1$. The kernel bundles M_{ω_X} and $M_{\omega_X\otimes\eta}$ have the following restrictions to the components of X:

$$M_{\omega_X|C}=M_{K_C(p+q)} \text{ and } M_{\omega_X|E}=\mathcal{O}_E^{\oplus(g-1)},$$

respectively

$$M_{\omega_X\otimes\eta|C}=M_{K_C(p+q)\otimes\eta_C} \text{ and } M_{\omega_X\otimes \eta|E}=\mathcal{O}_E(-1)\oplus\mathcal{O}_E^{\oplus(g-3)}.$$

Via the Mayer-Vietoris sequence on X, a sufficient condition for (i) to hold is given by the following vanishing statements:

$$H^1\Big(\bigwedge^j M_{\omega_X|C}\otimes K_C^{\otimes b}\big(b(p+q)\big)\otimes\eta_C\Big)=0,\ \ H^1\Big(\bigwedge^j M_{\omega_X|C}\otimes K_C^{\otimes b}\big((b-1)(p+q)\big)\otimes\eta_C\Big)=0,$$

as well as $H^1\big(E,\bigwedge^j M_{\omega_X|E}\otimes\eta_E\big)=0$ (note that $\omega_{X|E}=\mathcal{O}_E$). The vanishing on E is immediate, whereas that on C follows again by using that $M_{K_C(p+q)}$ and $M_{K_C(p+q)\otimes\eta_C}$ are semistable and computing the slopes of the corresponding vector bundles whose first cohomology group is supposed to vanish. Statement (ii) is entirely similar and we skip details. □

To be able to define tautological sheaves over $\widetilde{\mathsf{R}}_{g,\ell}$, we consider the global *kernel bundle* defined via the exact sequence over the universal curve $\widetilde{\mathsf{X}}_{g,\ell}$

$$0\longrightarrow \mathcal{M}_{\mathsf{u}}\longrightarrow \mathsf{u}^*(\mathbb{E})\longrightarrow \omega_{\mathsf{u}}\longrightarrow 0,$$

respectively the global *Prym kernel bundle*

$$0\longrightarrow \mathcal{M}'_{\mathsf{u}}\longrightarrow \mathsf{u}^*(\mathbb{E}')\longrightarrow \omega_{\mathsf{u}}\otimes\mathcal{P}\longrightarrow 0.$$

3.1. **Tautological sheaves.** We introduce tautological vector bundles over $\widetilde{\mathsf{R}}_{g,\ell}$ whose fibres are various Koszul cohomology groups. For integers $0 \leq j \leq \frac{g}{2}$ and $b \geq 2$, as well as for $(j, b) = (0, 1)$, we define the sheaves

$$\mathbb{E}_{j,b} := \mathsf{u}_*\big(\bigwedge^j \mathcal{M}_{\mathsf{u}} \otimes \omega_{\mathsf{u}}^{\otimes b} \otimes \mathcal{P}\big) \text{ and } \mathbb{F}_{j,b} := \mathsf{u}_*\big(\bigwedge^j \mathcal{M}'_{\mathsf{u}} \otimes \omega_{\mathsf{u}}^{\otimes b} \otimes \mathcal{P}^{\otimes(b-2)}\big).$$

Grauert's theorem used via Proposition 3.1 imply that $\mathbb{E}_{j,b}$ and $\mathbb{F}_{j,b}$ are locally free. Note that $\mathbb{E}_{0,1} = \mathbb{E}'$. One can now carry a Grothendieck-Riemann-Roch calculation over the universal level ℓ curve $\mathsf{u} : \widetilde{\mathsf{X}}_{g,\ell} \to \widetilde{\mathsf{R}}_{g,\ell}$ and prove the following:

Proposition 3.2. *For each integer $b \geq 1$, the following formula holds in $\mathsf{Pic}(\widetilde{\mathsf{R}}_{g,\ell})$:*

(i) $c_1(\mathbb{E}_{0,b}) = \lambda + \binom{b}{2}\kappa_1 - \frac{1}{2\ell}\sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} a(\ell - a)\delta_0^{(a)}$.

(ii) $c_1(\mathbb{F}_{0,b}) = \lambda + \binom{b}{2}\kappa_1 - \frac{(b-2)^2}{2\ell}\sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} a(\ell - a)\delta_0^{(a)}$.

3.2. **The divisor $\overline{\mathcal{U}}_{g,\ell}$.** The divisor $\mathcal{U}_{g,\ell}$ has a scheme theoretic characterization via condition (22). We extend this description over the boundary of $\widetilde{\mathcal{R}}_{g,\ell}$ and compute the class of the resulting degeneracy locus, thus completing the proof of Theorem 0.5.

By taking exterior powers in the sequence defining the sheaf $\mathcal{M}_{\mathsf{u}}$ and then using Proposition 3.1, we find that for each $1 \leq j \leq i - 1$, one has the following exact sequences, which can be used to compute inductively the Chern classes of the vector bundles $\mathbb{E}_{j,b}$, starting from level $j = 0$:

$$0 \longrightarrow \mathbb{E}_{j,i+1-j} \longrightarrow \bigwedge^j \mathbb{E} \otimes \mathbb{E}_{0,i+1-j} \longrightarrow \mathbb{E}_{j-1,i+2-j} \longrightarrow 0. \tag{23}$$

With these ingredients, we can prove Theorem 0.5 and calculate $[\overline{\mathcal{U}}_{g,\ell}]$:

Proof of Theorem 0.5. We consider the morphism $\chi : \bigwedge^i \mathbb{E} \otimes \mathbb{E}_{0,1} \to \mathbb{E}_{i-1,2}$ of vector bundles over the stack $\widetilde{\mathsf{R}}_{g,\ell}$, which at the level of fibres is given by the map

$$\chi([X, \eta, \phi]) : \bigwedge^i H^0(X, \omega_X) \otimes H^0(X, \omega_X \otimes \eta) \to H^0\big(X, \bigwedge^{i-1} M_{\omega_X} \otimes \omega_X^{\otimes 2} \otimes \eta\big).$$

The intersection of the degeneracy locus of χ with $\mathcal{R}_{g,\ell}$ is precisely the divisor $\mathcal{U}_{g,\ell}$, therefore the difference $c_1(\mathbb{E}_{i-1,2} - \bigwedge^i \mathbb{E} \otimes \mathbb{E}_{0,1}) - [\overline{\mathcal{U}}_{g,\ell}] \in \mathrm{Pic}(\widetilde{\mathsf{R}}_{g,\ell})$ is a (possibly empty) effective class supported only on the boundary classes of $\widetilde{\mathcal{R}}_{g,\ell}$. In particular, the class $c_1(\mathbb{E}_{g-1,2} - \bigwedge^i \mathbb{E} \otimes \mathbb{E}_{0,1})$ is effective. In order to compute it, we use (23) and write that:

$$c_1\big(\mathbb{E}_{i-1,2} - \bigwedge^i \mathbb{E} \otimes \mathbb{E}_{0,1}\big) = \sum_{b=0}^{i} (-1)^{b+1} c_1\big(\bigwedge^{i-b} \mathbb{E} \otimes \mathbb{E}_{0,b+1}\big) =$$

$$\sum_{b=0}^{i} (-1)^{b+1}\Big[\binom{g}{i-b}\Big(\lambda + \binom{b+1}{2}\kappa_1 - \sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} \frac{a(\ell-a)}{2\ell}\delta_0^{(a)}\Big) + (2b+1)(g-1)\binom{g-1}{i-b-1}\lambda\Big],$$

which after routine calculations leads to Theorem 0.5. □

3.3. **The virtual divisor $\overline{\mathcal{D}}_{g,\ell}$.** We prove Theorem 0.7. Since the proof resembles the previous calculation, we succintly explain the main points. We set $g := 2i+2$ and recall that $[C,\eta] \in \mathcal{D}_{g,\ell}$ if and only if the map

$$\bigwedge^i H^0(K_C \otimes \eta) \otimes H^0(C, K_C \otimes \eta^\vee) \to H^0\big(C, \bigwedge^{i-1} M_{K_C \otimes \eta} \otimes K_C^{\otimes 2}\big)$$

is not an isomorphism. Equivalently, $\mathcal{D}_{g,\ell}$ is the restriction to $\mathcal{R}_{g,\ell}$ of the degeneracy locus of the vector bundle morphism $\bigwedge^i \mathbb{E}' \otimes \mathbb{F}_{0,1} \to \mathbb{F}_{i-1,2}$.

Proof of Theorem 0.7. For each integer $1 \le j \le i-1$ we have the exact sequence on $\widetilde{\mathsf{R}}_{g,\ell}$

$$0 \longrightarrow \mathbb{F}_{i-j,j+1} \longrightarrow \bigwedge^{i-j} \mathbb{E}' \otimes \mathbb{F}_{0,j+1} \longrightarrow \mathbb{F}_{i-j-1,j+2} \longrightarrow 0,$$

thus one writes that

$$[\overline{\mathcal{D}}_{g,\ell}]^{\mathrm{virt}} = \sum_{j=0}^{i} (-1)^{j+1} c_1\big(\bigwedge^{i-j} \mathbb{E}' \otimes \mathbb{F}_{0,j+1}\big) =$$

$$\sum_{j=0}^{i} (-1)^{j+1} \Big[(g-1)(2j+1) \binom{g-2}{i-j-1} c_1(\mathbb{E}') + \binom{g-1}{i-j} c_1(\mathbb{F}_{0,j+1})\Big].$$

Using Proposition 3.2 and that $\mathbb{E}^{'} = \mathcal{G}_{0,1}$, we finish the proof after some calculations. $\Box$

3.4. **The Prym-Green divisor.** Recall that $\mathbb{E}'$ is the Prym-Hodge bundle with fibres $\mathbb{E}'([X,\eta,\phi]) = H^0(X, \omega_X \otimes \eta)$. For $b \ge 1$, we set $\mathbb{H}_{0,b} := \mathrm{Sym}^b \mathbb{E}'$. Then for $a \ge 1$, we define inductively the sheaves $\mathbb{H}_{a,b}$ via the following exact sequences over $\widetilde{\mathsf{R}}_{g,\ell}$:

$$0 \longrightarrow \mathbb{H}_{a,b} \longrightarrow \bigwedge^{a} \mathbb{E}' \otimes \mathbb{H}_{0,b} \longrightarrow \mathbb{H}_{a-1,b+1} \longrightarrow 0. \tag{24}$$

Note that $\mathbb{H}_{a,b}$ is locally free and its fibre over a point $[X,\eta,\phi]$ inducing a paracanonical map $\phi_L : X \to \mathbf{P}^{g-2}$, where $L := \omega_X \otimes \eta$ is canonically identified with the space $H^0(\mathbf{P}^{g-2}, \Omega^1_{\mathbf{P}^{g-2}}(a+b))$.

For each $b \ge 1$, we also define the sheaf $\mathbb{G}_{0,b} := \mathsf{u}_*(\omega_{\mathsf{u}}^{\otimes b} \otimes \mathcal{P}^{\otimes b})$. Note that $\mathbb{G}_{0,1} = \mathbb{E}'$ and there exist sheaf homomorphisms $\varphi_{0,b} : \mathbb{H}_{0,b} \to \mathbb{G}_{0,b}$, which fibrewise over $[X,\eta,\phi] \in \overline{\mathcal{R}}_{g,\ell}$ correspond to the multiplication maps of global sections

$$\mathrm{Sym}^b H^0(X, \omega_X \otimes \eta) \to H^0(X, \omega_X^{\otimes b} \otimes \eta^{\otimes b}).$$

Inductively, for each $a \ge 1$, we define sheaves $\mathbb{G}_{a,b}$ over $\widetilde{\mathsf{R}}_{g,\ell}$ via the exact sequences

$$0 \longrightarrow \mathbb{G}_{a,b} \to \bigwedge^{a} \mathbb{E}' \otimes \mathbb{G}_{0,b} \to \mathbb{G}_{a-1,b+1} \longrightarrow 0, \tag{25}$$

where the right exactness of the sequence (25) is to be soon justified. Inductively, one also constructs sheaf homomorphisms

$$\varphi_{a,b} : \mathbb{H}_{a,b} \to \mathbb{G}_{a,b},$$

fibrewise given by restriction of twisted forms on projective space

$$\varphi_{a,b}([X,\eta,\phi]) : H^0(\mathbf{P}^{g-2}, \bigwedge^{a} M_{\mathbf{P}^{g-2}}(b)) \to H^0(X, \bigwedge^{a} M_L \otimes L^{\otimes b}).$$

The right exactness of the sequence (25) is established via Grauert's theorem by the following:

Proposition 3.3. *Let $[X, \eta, \beta] \in \widetilde{\mathcal{R}}_{g,\ell}$ a twisted level curve. Then for all integers $a \geq 0$, $b \geq 2$, the vanishing $H^1\big(X, \bigwedge^a M_L \otimes L^{\otimes b}\big) = 0$ holds.*

Proof. It follows closely [FL] Proposition 3.2, where the corresponding statement when $\ell = 2$ is established. □

We now compute the Chern classes of the above defined tautological bundles (this matches the main theorem of [C2] for $s = b\ell$ via $\frac{1}{2}B_2(\frac{s}{\ell}) = \binom{b}{2} + 1/6$; we refer to Section 3.2, (44) for the explicit example in degree 1).

Proposition 3.4. *For $b \geq 1$ we have the following formula in $\mathrm{Pic}(\widetilde{\mathsf{R}}_g)$:*

$$c_1(\mathbb{G}_{0,b}) = \lambda + \binom{b}{2}\kappa_1 - \frac{b^2}{2}\sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} \frac{a(\ell-a)}{\ell}\delta_0^{(a)}.$$

Proof. We apply Grothendieck-Riemann-Roch for the universal curve $\mathsf{u} : \widetilde{\mathsf{X}}_{g,\ell} \to \widetilde{\mathsf{M}}_g$ and the sheaf $\mathbb{G}_{0,b}$. Noting that $R^1\mathsf{u}_*(\omega_{\mathsf{u}}^{\otimes b} \otimes \mathcal{P}^{\otimes b}) = 0$, we write:

$$\mathrm{ch}(\mathbb{G}_{0,b}) = \mathsf{u}_*\Big[\Big(1 + b\, c_1(\omega_{\mathsf{u}} \otimes \mathcal{P}) + \frac{b^2}{2}c_1^2(\omega_{\mathsf{u}} \otimes \mathcal{P}) + \cdots\Big)\cdot\Big(1 - \frac{c_1(\omega_{\mathsf{u}})}{2} + \frac{c_1^2(\omega_{\mathsf{u}}) + [\mathrm{Sing}(\mathsf{u})]}{12}\Big)\Big],$$

where $\mathrm{Sing}(\mathsf{u}) \subset \widetilde{X}_{g,\ell}$ denotes the codimension 2 singular locus of the u, and clearly $\mathsf{u}_*([\mathrm{Sing}(\mathsf{u})]) = \mathsf{f}^*(\delta_0)$, where $\delta_0 \in \mathrm{Pic}(\widetilde{\mathsf{M}}_g)$. Using Mumford's formula [HM] $\mathsf{u}_*(c_1^2(\omega_{\mathsf{u}})) = 12\lambda - \delta$ as well as Proposition 1.6, we obtained the claimed formula by evaluating the push-forward under u of the quadratic terms. □

We now compute the virtual class of the Prym-Green divisor.

Proof of Theorem 0.6. We set $g = 2i + 6$ and observe that the locus $\mathcal{Z}_{g,\ell}$ of smooth level curves $[C, \eta] \in \mathcal{R}_{g,\ell}$ is the degeneracy locus of the morphism $\varphi_{i,2} : \mathbb{H}_{i,2|\mathsf{R}_{g,\ell}} \to \mathbb{G}_{i,2|\mathsf{R}_{g,\ell}}$. Whenever Conjecture A is true, the class $[\overline{\mathcal{Z}}_{g,\ell}]^{\mathrm{virt}}$ is effective, and differs from the class of the closure $\overline{\mathcal{Z}}_{g,\ell}$ by a (possibly empty) effective combination of boundary divisors. We now compute $[\overline{\mathcal{Z}}_{g,\ell}]^{\mathrm{virt}} := c_1(\mathbb{G}_{i,2} - \mathbb{H}_{i,2})$ as follows:

$$c_1(\mathbb{H}_{i,2}) = \sum_{j=0}^{i}(-1)^j c_1\Big(\bigwedge^{i-j}\mathbb{H}_{0,1} \otimes \mathrm{Sym}^{j+2}(\mathbb{H}_{0,1})\Big),$$

and

$$c_1(\mathbb{G}_{i,2}) = \sum_{j=0}^{i}(-1)^j\binom{g-1}{i-j}c_1(\mathbb{G}_{0,i+2}) + \sum_{j=0}^{i}(-1)^j(g-1)(2j+3)\binom{g-2}{i-j-1}c_1(\mathbb{E}').$$

Using Proposition 3.4 we can finish the proof. □

Remark 3.5. In the course of the proof of Theorem 0.2 it is important to note that for any $\ell \geq 2$ and $g \leq 23$, in order to establish that the class $K_{\overline{\mathcal{R}}_{g,\ell}}$ is effective (respectively big), it suffices to prove the seemingly weaker statement that the restriction $K_{\widetilde{\mathcal{R}}_{g,\ell}}$ is effective (respectively big). Precisely, if $\mathcal{D}$ is an effective divisor on $\mathcal{R}_{g,\ell}$ such that

$$[\overline{\mathcal{D}}] = s\lambda - 2(\delta_0^{'} + \delta_0^{''}) - \sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} b_0^{(a)} \delta_0^{(a)} - \sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} (b_i \, \delta_i + b_{g-i} \, \delta_{g-i} + b_{i:g-i} \delta_{i:g-i}) \in \mathsf{Pic}(\overline{\mathsf{R}}_{g,\ell}),$$

with $s \leq 13$ and $b_0^{(a)} \geq \ell + 1$ for $a = 1, \ldots, \lfloor \frac{\ell}{2} \rfloor$, then $K_{\overline{\mathcal{R}}_{g,\ell}} - [\overline{\mathcal{D}}] \in \mathsf{Eff}(\overline{\mathsf{R}}_{g,\ell})$ is an effective class, that is $b_{g-1} \geq 3$ and $b_i, b_{i:g-i} \geq 2$, for all $i = 1, \ldots, g-1$. The proof uses pencils of level curves on $K3$ surfaces and is similar to [FL] Proposition 1.2. For $1 \leq i \leq \min\{\frac{g}{2}, 11\}$, we fix a general curve $[C_2, p] \in \mathcal{M}_{g-i,1}$, an ℓ-torsion point $\eta_{C_2} \in \mathrm{Pic}^0(C_2)$ and a moduli curve $B_i \subset \overline{\mathcal{M}}_{i,1}$, induced by a Lefschetz pencil $\{(C_t, p)\}_{t \in \mathbf{P}^1}$ of pointed curves on genus i on a fixed $K3$ surface $S \subset \mathbf{P}^i$, the marked point p being one of the base points of the pencil. We construct three 1-dimensional families filling-up the divisors Δ_{g-i}, Δ_i and $\Delta_{i:g-i}$ as follows. Firstly, $A_{g-i} \subset \Delta_{g-i}$ consists of level curves $\big\{[C_t \cup_p C_2, \eta_{C_t} = \mathcal{O}_{C_t}, \eta_{C_2}]\big\}_{t \in \mathbf{P}^1} \subset \overline{\mathcal{R}}_{g,\ell}$. Then $A_i \subset \Delta_i$ parametrizes level curves $\big\{[C_t \cup_p C_2, \ \eta_{C_t} \in \overline{\mathrm{Pic}}^0(C_t)[\ell], \ \mathcal{O}_{C_2}]\big\}_{t \in \mathbf{P}^1}$. Finally, $A_{i:g-i} \subset \Delta_{i:g-i}$ consists of level curves $\big\{[C_t \cup_p C_2, \ \eta_{C_t} \in \overline{\mathrm{Pic}}^0(C_t)[\ell], \ \eta_{C_2}]\big\}_{t \in \mathbf{P}^1}$. For $i \neq 10$, the curves A_i, A_{g-i} and $A_{i:g-i}$ fill-up the respective boundary divisors in $\overline{\mathcal{R}}_{g,\ell}$. By imposing the conditions $A_i \cdot \overline{\mathcal{D}} \geq 0$, $A_{g-i} \cdot \overline{\mathcal{D}} \geq 0$ and $A_{i:g-i} \cdot \overline{\mathcal{D}} \geq 0$ and computing the actual intersection numbers via [FP], we obtain the desired bounds on the boundary coefficients of $[\overline{\mathcal{D}}]$. The case $g = 10$ is a little special but can be handled in a similar way, cf. [FP] Theorem 1.1 (b).

4. Syzygy computations with nodal curves

In this section we explain how to verify computationally Conjectures A and B for small g and bounded level ℓ using nodal curves.

Let C be a rational g-nodal curve with normalization $\mathrm{nor} : \mathbf{P}^1 \to C$ and denote by $\{P_j, Q_j\}_{j=1}^{g}$ the preimages of the nodes of C. A line bundle $L \in \mathrm{Pic}^d(C)$ is given by an isomorphism $\mathrm{nor}^*(L) \cong \mathcal{O}_{\mathbf{P}^1}(d)$ and gluing data between residue class fields

$$a_j : \mathcal{O}_{\mathbf{P}^1}(d) \otimes \kappa(P_j) \cong \mathcal{O}_{\mathbf{P}^1}(d) \otimes \kappa(Q_j).$$

In particular, $\mathrm{Pic}^0(C) \cong \mathbf{G}_m \times \ldots \times \mathbf{G}_m$ is a g-dimensional torus.

Let $K[x_0, x_1]$ be the homogeneous coordinate ring of $\mathbf{P}^1$, and let $z = \frac{x_1}{x_0}$ be the affine coordinate on the chart $U_0 \cong \mathbb{A}^1$. We assume that all preimage points of the nodes are contained in U_0, say $P_j = (1 : p_j)$ and $Q_j = (1 : q_j)$. Then

$$H^0(C, L) \cong \{f \in K[z]_{\leq d} : a_j f(p_j) = f(q_j) \text{ for } \ i = 1, \ldots, g\}$$

can be identified with the space of polynomials f of degree at most d whose values in p_j and q_j differ by the factor $a_j \in K^*$. In terms of coefficients of polynomials, the space $H^0(C, L) \subset H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))$ is the solution space of a homogeneous system of equations.

Example 4.1. A basis of sections of the dualizing sheaf ω_C is given by

$$\omega_j = \frac{dz}{(z-p_j)(z-q_j)}, \text{ for } j=1,\ldots,g,$$

and the canonical map is induced by

$$\phi_{\omega_C} : \mathbf{P}^1 \to \mathbf{P}^{g-1}, \ z \mapsto \Big(\prod_{i\neq 1}(z-p_i)(p-q_i) : \ldots : \prod_{i\neq g}(z-p_i)(z-q_i)\Big).$$

Thus $\mathrm{nor}^*(\omega_C)=\mathcal{O}_{\mathbf{P}^1}(2g-2)$ and the the canonical multipliers are $a_j^{\mathrm{can}} = \prod_{i\neq j} \frac{(q_j-p_i)(q_j-q_i)}{(p_j-p_i)(p_j-q_i)}$.

Example 4.2. Suppose $A, B \in \mathrm{Pic}(C)$ correspond to the pairs $\big(\mathcal{O}_{\mathbf{P}^1}(d), (a_1,\ldots,a_g)\big)$ and $\big(\mathcal{O}_{\mathbf{P}^1}(e), (b_1,\ldots,b_g)\big)$ respectively, then $A\otimes B$ is given by $\big(\mathcal{O}_{\mathbf{P}^1}(d+e), (a_1b_1,\ldots,a_gb_g)\big)$.

Example 4.3. Let $\eta \in \mathrm{Pic}^0(C)$ be a line bundle corresponding to $(\mathcal{O}_{\mathbf{P}^1}, (\zeta_1,\ldots,\zeta_g))$. Then η is a torsion bundle, if and only if all multipliers ζ_j are roots of unity. In particular, one obtains a level ℓ paracanonical curve by altering some or all of the multipliers a_j^{can} by a primitive ℓth root of unity.

We verify the Prym-Green Conjecture following the following steps:

(i) If $\ell = 2$, we take $r = 1$ and choose a prime p of moderate size, for instance $10^4 < p < 3\cdot 10^4$. In case $\ell > 2$ we choose an integer r and a prime p such that r represents a primitive ℓth root of unity in $K = \mathbb{F}_p$. So p is one of prime factors of $r^\ell - 1$.
(ii) Randomly pick $2g$ points $P_1, Q_1, \ldots, P_g, Q_g \in \mathbf{P}^1(K)$ and compute the canonical multipliers a_j^{can} for $j = 1,\ldots,g$.
(iii) Alter all (or some) multiplier $a_j^\eta = r a_j^{\mathrm{can}}$ and compute a basis $f_0,\ldots,f_{g-2}$ of the Prym canonical system $H^0(C,\omega_C\otimes\eta) \subset H^0(\mathbf{P}^1,\mathcal{O}_{\mathbf{P}^1}(2g-2))$. By Riemann-Roch, this space is $(g-1)$-dimensional.
(iv) Compute the kernel I_C of the map $K[y_0,\ldots,y_{g-2}] \to K[x_0,x_1]$ given by $y_j \mapsto f_j$ and its free resolution up to the appropriate step.

A *Macaulay2* code which does this job can be found in the package *NodalCurves.m2* available at *http://www.math.uni-sb.de/ag/schreyer/home/computeralgebra.htm*.

Proposition 4.4. *The Prym-Green Conjecture holds for all even genera with $g \le 18$ and $\ell \le 5$ over a field K of characteristic 0 with the possible exception of the cases $(g,\ell) = (8,2)$ or $(16,2)$.*

Proof. By the *Macaulay2* computation documented above, the statement holds true for some g-nodal rational curve defined over a finite field $\mathbb{F}_p$. This computation can be viewed as the reduction mod p of the same computation of an example defined over $\mathbb{Z}$. By semicontinuity of Betti numbers in families, then there exists a g-nodal rational curve defined over $\mathbb{Q}$ with a pure resolution. Applying again semicontinuity, a general pair $[C,\eta] \in \mathcal{R}_{g,\ell}$ has a pure resolution of its paracanonical embedding. □

In more concrete terms, Proposition 4.4 says that a general level ℓ paracanonical curve of genus 10 has the following syzygies

```
        0  1  2   3   4   5  6  7
total: 1 18 42 126 210 162 63 10
    0: 1  .  .   .   .   .  .  .
    1: . 18 42   .   .   .  .  .
    2: .  .  . 126 210 162 63 10
```

for all $2 \leq \ell \leq 5$. The relevant computation in this example takes about 1.80 seconds. Using 2 minutes of cpu we can extend this result for $g = 10$ to all levels $\ell \leq 30$.

For larger genus the approach has to to modified to be still computationally feasible. Recall [E] that for a graded Cohen-Macaulay module M of dimension d over a standard graded polynomial ring S, if $x_1, \ldots, x_d \in S_1$ is an M-sequence, the minimal free resolution of the artinian quotient $A = M/\langle x_1, \ldots, x_d\rangle M$ over the polynomial ring $T = S/\langle x_1, \ldots, x_d\rangle$ is obtained from the minimal free resolution of M as an S-module by tensoring with T. In particular the graded Betti numbers

$$b_{ij}(M) = \dim_K \operatorname{Tor}_i^S(M, K)_{i+j} = \dim_K \operatorname{Tor}_i^T(A, K)_{i+j} = b_{ij}(A)$$

coincide. We also can use duality [E, Theorem 21.15] : Let $\omega_S \cong S(-\dim S)$ denote the dualizing module of S, and $\omega_M := \operatorname{Ext}_S^{\operatorname{codim} M}(M, \omega_S)$. If M is Cohen-Macaulay, then the minimal free resolution of ω_M is obtained by applying $\operatorname{Hom}_S(-, \omega_S)[-\operatorname{codim} M]$ to the minimal free resolution of M. In particular,

$$\dim_K \operatorname{Tor}_i^S(M, K)_j = \dim_K \operatorname{Tor}^S_{\operatorname{codim} M - i}(\omega_M, K)_{\dim S - j}.$$

Finally, we can use the Koszul complex, that is, the minimal free resolution of K as an S-module in order to compute $\operatorname{Tor}^S(M, K)$. We use all three options.

(i) Compute a basis $s_0, \ldots, s_{g-2}$ of $H^0(C, K_C \otimes \eta) \subset K[x_0, x_1]_{2g-2}$ and $\omega_1, \ldots, \omega_g$ of $H^0(C, K_C) \subset K[x_0, x_1]_{2g-2}$.

(ii) Check that s_{g-3}, s_{g-2} have no common zero in $\mathbf{P}^1$, i.e. they correspond to a R-sequence for $R = K[y_0, \ldots, y_{g-2}]/I_C \cong \Gamma_C(\mathcal{O}_C, K_C \otimes \eta)$.

(iii) Compute representatives in $K[x_0, x_1]$ of a K-basis for the artinian reduction $A = \omega_R/\langle (y_{g-3}, y_{g-2}\rangle \omega_R$. Note that A is a graded artinian $T = K[y_0, \ldots y_{g-4}]$-module with Hilbert series $H_A(t) = \sum_d \dim A_d t^d = g + (g-3)t + t^2$.

(iv) Set $m = \frac{g}{2}$. Substitute $s_0, \ldots s_{g-4}$ into the m^{th} Koszul matrix on $y_0, \ldots, y_{g-4}$ and compute the tensor product with the $1 \times g$ matrix $(\omega_1, \ldots, \omega_g)^t$. The result is a $\binom{g-3}{\frac{g}{2}-1} \times g\binom{g-3}{\frac{g}{2}}$ matrix with entries in $H^0(C, \omega_C^2 \otimes \eta) \subset K[x_0, x_1]_{4g-4}$.

(v) Reduce the entries module the ideal $\langle s_{g-3}, s_{g-2}\rangle\langle \omega_1, \ldots, \omega_g\rangle \subset K[x_0, x_1]$. The result is now a matrix M_{pol} of polynomials with entries in the $(g-3)$-dimensional space of representatives of A_2.

(vi) Compute the $(g-3)\binom{g-3}{\frac{g}{2}-1} \times g\binom{g-3}{\frac{g}{2}}$ coefficient matrix of M_{field} with values in K. The kernel of M_{field} is isomorphic to

$$\operatorname{Tor}_S^m(\omega_R, K)_m \cong \operatorname{Tor}_S^{g-3-m}(R, K)_{g-1-m} \cong K_{g-3-m,2}(C, K_C \otimes \eta),$$

whose dimension equals $\dim K_{\frac{g}{2}-2,1}(C, K_C \otimes \eta)$ since $g - 3 - m + 1 = \frac{g}{2} - 2$.

(vii) Use the finite field linear algebra subroutines/packages *FFLAS* [DGP] to compute the rank of M_{field} over the finite field $K = \mathbb{F}_p$.

Our approach to the torsion bundle conjecture (Conjecture B) for bounded g and ℓ is similar. Again we work with a g-nodal curve C and carry out the following steps:

(i) Compute bases of $s_0, \ldots, s_{g-2}$ of $H^0(C, K_C \otimes \eta) \subset K[x_0, x_1]_{2g-2}$ and $t_0, \ldots, t_{g-3}$ of $H^0(C, K_C \otimes \eta^k) \subset K[x_0, x_1]_{2g-2}$ and the kernel of the multiplication map

$$\mu : H^0(C, K_C \otimes \eta) \otimes H^0(C, K_C \otimes \eta^{\otimes k}) \to H^0(C, K_C^2 \otimes \eta^{\otimes (k+1)}).$$

Since the map is surjective, $\mathrm{Ker}(\mu)$ is $(g-1)^2 - (3g-3) = (g-1)(g-4)$ dimensional.

(ii) $\mathrm{Ker}(\mu)$ can be reinterpreted as the linear presentation matrix

$$\phi : S^{g^2-5g+4}(-2) \to S^{g-1}(-1)$$

of $M = \Gamma_C(\eta^{\otimes k}, K_C \otimes \eta)$ as an $S = K[y_0, \ldots, y_{g-2}]$ module. We wish to compute $\mathrm{Tor}^S_{\frac{g}{2}-1}(M, K)_{\frac{g}{2}} \cong K_{\frac{g}{2}-1,1}(\eta^{\otimes k}, K_C \otimes \eta)$.

(iii) Check that s_{g-3}, s_{g-2} have no common zero in $\mathbf{P}^1$ and compute basis for the artinian reduction $B = M/\langle y_{g-3}, y_{g-2}\rangle M$. This time B is a graded artinian $T = K[y_0, \ldots y_{g-4}]$-module with Hilbert series $H_B(t) = (g-1)t + (g-1)t^2$.

(iv) Reinterpret the tensor product of the $(\frac{g}{2}-1)^{\mathrm{th}}$ Koszul matrix on $y_0, \ldots, y_{g-4}$ with B as a $(g-1)\binom{g-3}{\frac{g}{2}}$ square matrix with entries in the ground field, and compute its rank. The dimension of the kernel is the desired Betti number.

Proposition 4.5. *Conjecture B hold over a field of characteristic zero for all* $6 \le g \le 18$ *and all primitive* ℓ*-torsion bundle with* $3 \le \ell \le 6$.

In more concrete terms the result says for $g = 10$ that a general pair $[C, \eta] \in \overline{\mathcal{R}}_{g,\ell}$ the module $\Gamma_C(C; \eta^{\otimes k}, K_C \otimes \eta)$ for $1 \le k \le \ell - 2$ has syzygies

```
       0  1   2   3   4   5  6 7
total: 9 54 126 126 126 126 54 9
    0: 9 54 126 126   .   .  . .
    1: .  .   .   . 126 126 54 9
```

unless $\ell = 2k + 1$, in which case we have instead

```
       0  1   2   3   4   5  6 7
total: 9 54 126 126 126 126 54 9
    0: 9 54 126 126   1   .  . .
    1: .  .   .   1 126 126 54 9
```

Proof. By our computation, Conjecture B holds for an g-nodal example over a finite field. For documentation of the computation based on the *Macaulay2* package *NodalCurves.m2* see *http://www.math.uni-sb.de/ag/schreyer/home/computeralgebra.htm*. To complete the proof with a semi-continuity argument, we have to show that in the exceptional cases the Betti number cannot become zero.

The extra syzygy are explained as follows. The resolution of $\eta^{\otimes k}$ is self-dual if and only if

$$\begin{array}{rclcl}\eta^{\otimes k} & \cong & \mathcal{E}\mathrm{xt}^{g-3}_{\mathcal{O}}(\eta^{\otimes k}, \mathcal{O}(-g)) & \cong & \mathcal{E}\mathrm{xt}^{g-3}_{\mathcal{O}}(\eta^{\otimes k}, \omega_{\mathbf{P}^{g-2}})(-1) \\ & \cong & \mathcal{H}\mathrm{om}(\eta^{\otimes k}, K_C) \otimes K_C^{-1} \otimes \eta^{-1} & \cong & \eta^{\otimes(-k-1)}\end{array}$$

that is, $\eta^{\otimes(2k+1)} = \mathcal{O}_C$. In this case the artinian reduction, $B = B_1 \oplus B_2$ is self-dual as well, that is, $B_2 = B_1^\vee$ and the multiplication map $V \otimes B_1 \to B_2$ with $V = K[y_0, \dots, y_{g-4}]_1$ corresponds to a symmetric $(g-1)$ square matrix with entires in $V^\vee$. The relevant Koszul cohomology map for B

$$\kappa_B : \Lambda^{(g-2)/2} V \otimes B_1 \to \Lambda^{(g-4)/2} V \otimes B_2$$

is obtained from the Koszul map

$$\kappa : \Lambda^{(g-2)/2} V \to \Lambda^{(g-4)/2} V \otimes V$$

by tensor product with $\mu' : B_1 \to B_2 \otimes V^\vee$ and contraction $V^\vee \otimes V \to K$. We now apply the following well-known fact:

Lemma 4.6. *The middle Koszul matrix*

$$\kappa' : \Lambda^{m+1} V \otimes T \to \Lambda^m V \otimes T(1)$$

of a polynomial ring $T = \operatorname{Sym} V$ in $2m+1$ variables is symmetric if $m \equiv 0 \mod 2$. Otherwise κ' is skew-symmetric.

Thus in case $\ell = 2k+1$ the matrix κ_B is as tensor product of matrices with symmetry properties, skew-symmetric precisely in case $\frac{g-4}{2} \equiv 1 \mod 2$ i.e. $g \equiv 2 \mod 4$. If in addition $\binom{g-3}{\frac{g}{2}-1}$ is odd, then κ_B is a skew-symmetric matrix of odd size, and κ_B cannot have maximal rank. This completes the argument and justifies the exceptions in the statement of Conjecture B. ☐

Proof of Lemma 4.6. Under the pairing $\Lambda^m V \times \Lambda^{m+1} V \to \Lambda^{2m+1} V \cong K$ the matrix κ' corresponds to the composition

$$\psi : \Lambda^{m+1} V \otimes \Lambda^{m+1} V \to V \otimes \Lambda^m V \otimes \Lambda^{m+1} V \to V \otimes \Lambda^{2m+1} V.$$

Hence for basis elements $y_I = y_{i_0} \wedge \ldots \wedge y_{i_m}, y_J = y_{j_0} \wedge \ldots \wedge y_{j_m} \in \Lambda^{m+1} V$ for $I, J \subset \{0, \dots, 2m\}$ we have that $\psi(y_I \otimes y_J) = 0$ unless $I \cup J = \{0, \dots, 2m\}$, which means that $I \cap J$ consists of precisely one element. Furthermore for disjoint sets $I' = \{i_1, \dots, i_m\}$ and $J' = \{j_1, \dots, j_m\}$ we compute that

$$\psi(y_{i_0} \wedge y_{I'} \otimes y_{i_0} \wedge y_{J'}) = y_{i_0} \otimes y_{I'} \wedge y_{i_0} \wedge y_{J'} = (-1)^m y_{i_0} \otimes y_{i_0} \wedge y_{I'} \wedge y_{J'}$$

$$= y_{i_0} \otimes y_{i_0} \wedge y_{J'} \wedge y_{I'} = (-1)^m y_{i_0} \otimes y_{J'} \wedge y_{i_0} \wedge y_{I'} = (-1)^m \psi(y_{i_0} \wedge y_{J'} \otimes y_{i_0} \wedge y_{I'})$$

as claimed. ☐

Remark 4.7. Notice that for a field of characteristic 2 the matrix ψ is symmetric with zeroes on the diagonal. Thus for $\ell = 2k+1$ and $\binom{g-3}{\frac{g}{2}-1} \equiv 1 \mod 2$ a torsion bundle has an extra syzygy independent of the parity of $\frac{g}{2}$.

5. THE FAILURE OF THE PRYM-GREEN CONJECTURE ON $\mathcal{R}_8$

We argue that the Prym-Green Conjecture A fails on $\mathcal{R}_8$. Our findings suggest that for a general Prym canonical curve $\phi_{K_C\otimes\eta}: C \to \mathbf{P}^6$ of genus 8, the non-vanishing $K_{2,1}(C, K_C \otimes \eta) \neq 0$ holds and the corresponding Betti table is the following:

```
        0 1  2  3  4 5
total:  1 8 36 56 35 8
    0:  1 .  .  .  . .
    1:  . 7  1  .  . .
    2:  . 1 35 56 35 8
```

Moreover, in all cases the extra syzygy in $K_{2,1}(C, K_C \otimes \eta)$ is never of maximal rank 7. We analyze this situation geometrically.

Throughout this section $\ell = 2$ and we set $\mathcal{R}_g := \mathcal{R}_{g,2}$. We recall that $\mathcal{GP}^r_{g,d}$ is the closure in $\mathcal{M}_g$ of the locus of curves with a base point free linear series $L \in W^r_d(C)$ such that the Petri map $\mu_0(L) : H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee) \to H^0(C, K_C)$ is not injective. The syzygy locus $\mathcal{Z}_{8,2} \subset \mathcal{R}_8$ considered in Theorem 0.6 can be extended to the level of the universal genus 8 Jacobian $\mathfrak{Pic}^{14}_8 \to \mathcal{M}_8$ and we introduce the Koszul locus

$$\mathfrak{Kosz} := \big\{[C, L] \in \mathfrak{Pic}^{14}_8 : K_{2,1}(C, L) \neq 0\big\}.$$

Note that dim $K_{2,1}(C, L) =$ dim $K_{1,2}(C, L)$, for each $[C, L] \in \mathfrak{Pic}^{14}_8$. It is proved in [Ve] that $\mathfrak{Kosz}$ is indeed a divisor on $\mathfrak{Pic}^{14}_8$, that is, $\mathfrak{Kosz} \neq \mathfrak{Pic}^{14}_8$. Via the map $\iota : \mathcal{R}_8 \hookrightarrow \mathfrak{Pic}^{14}_8$ given by $\iota([C, \eta]) := [C, K_C \otimes \eta]$, one has that $\iota^*(\mathfrak{Kosz}) = \mathcal{Z}_{8,2}$.

Let us consider the parameter space Σ of triples (C, L, V), where $[C, L] \in \mathfrak{Pic}^{14}_8$ induces a map $\phi_L : C \to \mathbf{P}^6$ and $V \in G\big(5, H^0(\mathbf{P}^6, \mathcal{I}_{C/\mathbf{P}^6}(2))\big)$. The variety Σ is birational to a Grassmann bundle over an open subset of $\mathfrak{Pic}^{14}_8$. Denoting by $X := \mathrm{Bs}|V| \subset \mathbf{P}^6$ the base locus of the system of quadrics in V, one can write that $X = C + C'$, where the linked curve $C' \subset \mathbf{P}^6$ has genus 14, degree $\deg(C') = 18$ and $h^1(C', \mathcal{O}_{C'}(1)) = 2$. For a general $(C, L, V) \in \Sigma$, we have an isomorphism

$$H^0(\mathbf{P}^6, \mathcal{I}_{C/\mathbf{P}^6}(2))/V \xrightarrow{\cong} H^0(C', K_{C'}(-1)). \tag{26}$$

This linkage construction induces a dominant rational map

$$\varphi : \Sigma \dashrightarrow \mathcal{M}_{14}, \ \ (C, L, \Sigma) \mapsto [C'].$$

The image of $\mathfrak{Kosz}$ under this map has a transparent geometric interpretation. It is proved in [Ve] Lemma 4.4 that the Petri map for the genus 14 curve, that is,

$$\mu_0(\mathcal{O}_{C'}(1)) : H^0(C', \mathcal{O}_{C'}(1)) \otimes H^0(C', K_{C'}(-1)) \to H^0(C', K_{C'})$$

is an isomorphism if and only if $K_{2,1}(C, L) = 0$. We summarize the situation as follows:

Proposition 5.1. *A paracanonical genus 8 curve $C \subset \mathbf{P}^6$ belongs to the Koszul divisor if and only if the residual genus 14 curve $C' \subset \mathbf{P}^6$ is Petri special.*

The locus of such Petri special curves $C' \subset \mathbf{P}^6$ splits into two components depending on whether $K_{C'}(-1) \in W^1_8(C')$ is base point free, and then $[C'] \in \mathcal{GP}^1_{14,8}$, or else, it has a base point, in which case C' is 7-gonal.

We fix a point $[C, L] \in \mathfrak{Kosz}$ corresponding to an embedded curve $\phi_L : C \hookrightarrow \mathbf{P}^6$ of degree 14 having Betti diagram as above, where S denote the homogeneous coordinate ring of $\mathbf{P}^6$. Consider the second syzygy matrix $S^7(-2) \oplus S(-3) \leftarrow S(-3) \oplus S^{35}(-4)$ of S/I_C. The entries of the submatrix $S^7(-2) \leftarrow S(-3)$ are given by linear forms $(\ell_0, \ldots, \ell_6)$ corresponding to a syzygy

$$0 \neq \gamma := \sum_{i=0}^{6} \ell_i \otimes q_i \in \operatorname{Ker}\big\{H^0(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(1)) \otimes H^0(\mathbf{P}^6, \mathcal{I}_{C/\mathbf{P}^6}(2)) \to H^0(\mathbf{P}^6, \mathcal{I}_{C/\mathbf{P}^6}(3))\big\}.$$

Definition 5.2. The *syzygy scheme* $\mathrm{Syz}(\gamma)$ of $\gamma \in K_{2,1}(C, L)$ is the largest subscheme $Y \subset \mathbf{P}^6$ such that $\gamma \in H^0(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(1)) \otimes I_Y(2)$. The *rank* of γ is the dimension of the linear subspace $\langle \ell_0, \ldots, \ell_6 \rangle \subset H^0(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(1))$.

We refer to [AN] and [vB] for general background on syzygy schemes. In case of rank 6 first order syzygies among quadrics, say $\ell_0 = 0$, there exists a 6×6 skew-symmetric matrix of linear forms $(a_{ij})_{i,j=1}^6$ such that $q_i = \sum_{j=1}^6 \ell_j a_{ji}$ for $i = 1, \ldots, 6$ by [S, Lemma 4.3]. In this case the syzygy scheme is $C' = \mathbf{P}^6 \cap X_6$, where $X_6 \subset \mathbf{P}^{20} := \mathbf{P}_{\ell_i, a_{ij}}$ is the universal rank 6 syzygy scheme, given by equations $\{q_i = 0\}_{i=1}^6$ and $\mathrm{Pfaff}\big((a_{ij})\big) = 0$.

We consider first the case when $\gamma \in K_{2,1}(C, L)$ is a rank 7 syzygy, which we can view as a section of the vector bundle $\Omega^1_{\mathbf{P}^6}(3)$. By direct calculation using the Euler sequence, one finds $c_6(\Omega^1_{\mathbf{P}^6}(3)) = 43$, which is the number of zeroes of a general section of $\Omega^1_{\mathbf{P}^6}(3)$. Since $\gamma \in H^0(\mathbf{P}^6, \Omega^1_{\mathbf{P}^6}(3) \otimes \mathcal{I}_C)$, to account for the contribution of C we use the excess intersection formula and from $c_6(\Omega^1_{\mathbf{P}^6}(3))$ we subtract

$$c_1\big(\Omega^1_{\mathbf{P}^6}(3)_{|C} - N_{C/\mathbf{P}^6}\big) = \deg(M_L \otimes L^{\otimes 2}) - \deg(N_{C/\mathbf{P}^6}) = 11 \cdot 14 - 8 \cdot 14 = 42.$$

This shows that $Z(\gamma) = C \cup \{p\}$, where $p \in \mathbf{P}^6 - C$. Choose now a 5-dimensional subspace $V \subset H^0(\mathbf{P}^6, \mathcal{I}_{C/\mathbf{P}^6}(2))$ and let $C' \subset \mathbf{P}^6$ be the corresponding linked curve. Since all the quadrics in $H^0(\mathbf{P}^6, \mathcal{I}_{C/\mathbf{P}^6}(2))$ vanish at p, we find via the isomorphism (26) that the pencil $K_{C'}(-1) \in W^1_8(C')$ has a base point at C'. In conclusion, via linkage, to rank 7 syzygies $\gamma \in K_{2,1}(C, L)$ correspond 7-gonal curves C' of genus 14.

Assume now that γ is a rank 6 syzygy. Let $p \in \mathbf{P}^6$ denote the point defined by the 6 forms. The syzygy γ may be interpreted as a section in $H^0(\mathbf{P}^6, \mathcal{F}(3))$, where $\mathcal{F}$ is the first syzygy sheaf of the ideal sheaf $\mathcal{I}_{p/\mathbf{P}^6}$:

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathbf{P}^6}^{\oplus 6}(-1) \to \mathcal{I}_{p/\mathbf{P}^6} \to 0.$$

Since $\mathrm{rk}(\mathcal{F}) = 5$, the scheme $\mathrm{Syz}(\gamma)$ is expected to be 1-dimensional away from p. Indeed, the zero locus of a general global section of $\mathcal{F}(3)$ vanishes in p and a smooth half canonically embedded curve C' of degree 21 and genus 22, see [ELMS, Theorem 4.4]. We refer to [ES] for more details.

In our case C' is reducible, because C is a component of C'. In fact

$$C' = C \cup E,$$

where E is an elliptic normal curve of degree 7, intersecting C in 14 points. Moreover,

$$C \cdot E \in |\mathcal{O}_E(2)| \text{ and } C \cdot E \in |L^{\otimes 2} \otimes K_C^{\vee}|.$$

Observe that $[C, L] \in \mathcal{R}_8$ if and only if $E \cdot C \in |K_C|$. Reversing this construction, we obtain the following result.

Theorem 5.3. *The divisor $\mathfrak{Kosz} \subset \mathfrak{Pic}_8^{14}$ is reducible and has a unirational component $\mathfrak{Kosz}_6$ whose general point is a paracanonical curve with extra syzygies of rank 6, as well as a component $\mathfrak{Kosz}_7$ whose general point is a curve with syzygies of rank 7. The syzygy scheme corresponding to a general point $[C, L] \in \mathfrak{Kosz}_6$ is the disjoint union of a point $p \in \mathbf{P}^6$ and a nodal half-canonical reducible curve $C \cup E$ of genus 22, where E is a smooth elliptic normal curve of degree 7 which intersects C in 14 distinct points.*

Proof. In order to construct $\mathfrak{Kosz}_6$, we reverse the construction and start with an elliptic normal curve $E \subset \mathbf{P}^6$ and a point $p \in \mathbf{P}^6 - E$. The minimal free resolution of E has the following Betti table:

```
        0  1  2  3  4 5
total: 1 14 35 35 14 1
    0: 1  .  .  .  . .
    1: . 14 35 35 14 .
    2: .  .  .  .  . 1
```

The group $K_{1,2}(E, \mathcal{O}_E(1))$ is 35-dimensional and there is a 22-dimensional vector space $H^0\big(\mathbf{P}^6, \Omega^1_{\mathbf{P}^6}(3) \otimes \mathcal{I}_{E \cup \{p\}}\big)$ of second syzygies, whose order ideal vanishes at the point p. The zero locus of such a syzygy consists of p together with a half-canonical curve, having E as one of the components. Its residual curve is the desired point $[C, \mathcal{O}_C(1)] \in \mathfrak{Kosz}$. To show that such curves fill-up a component of $\mathfrak{Kosz}$, we count dimensions. Modulo the action of $PGL(7)$, septic elliptic curves $E \subset \mathbf{P}^6$ depend on one parameter. Indeed, if $H_7 \subset SL_7(\mathbb{C})$ denotes the level 7 Heisenberg group, there is a $1:1$ correspondence between elliptic curves with a level 7 structure and H_7-equivariantly embedded normal elliptic curves $E \subset \mathbf{P}^6$. The choice of $p \in \mathbf{P}^6$ gives 6 dimensions. Having chosen E and p as above, the choice $[\gamma] \in \mathbf{P}\big(H^0(\mathbf{P}^6, \Omega^1_{\mathbf{P}^6}(3) \otimes \mathcal{I}_{E \cup \{p\}})\big)$ gives another 21 dimensions. Since the construction depends on $1 + 6 + 21 = \dim(\mathfrak{Pic}_8^{14}) - 1$ parameters, curves obtained in this way fill-up a component $\mathfrak{Kosz}_6$ of $\mathfrak{Kosz}$. To complete the proof of the unirationality of $\mathfrak{Kosz}_6$, it remains to show that the construction leads to a smooth curve for general choices of parameters. By semi-continuity this can checked with *Macaulay2* over a finite field, for details see the function *unirationaliyOfD1* in our *Macaulay2* package *KoszulDivisorOnPic14M8*.

By running the function *getCurveOnKoszulDivisor* contained in the *Macaulay2* package *KoszulDivisorOnPic14M8*, we observe that there exist points $[C, L] \in \mathfrak{Kosz}$ with syzygies $0 \neq \gamma \in K_{1,2}(C, L)$ having full rank 7. They cannot lie in the closure of $\mathfrak{Kosz}_6$ for the rank of a syzygy is upper semi-continuous, therefore $\mathfrak{Kosz}_6 \subsetneq \mathfrak{Kosz}$. □

Remark 5.4. Since $\mathfrak{Pic}_8^{14}$ is unirational [M], [Ve], over a finite field $\mathbb{F}_p$ one can find points on absolutely irreducible components of $\mathfrak{Kosz}$ with probability approximately equal to $\frac{1}{p}$. By our *experiment1* of the package *KoszulDivisorOnPic14M8*, we observe that the rate of points $[C, L] \in \mathfrak{Kosz}$ corresponding to syzygies of rank 6 respectively 7 is approximately the same. We conclude that most likely, $\mathfrak{Kosz}$ has precisely two components, namely $\mathfrak{Kosz}_6$ and $\mathfrak{Kosz}_7$.

The remaining part of this section is devoted to obtain strong evidence for the inclusion $\mathcal{R}_8 \subset \mathfrak{Kosz}$. Since we do not have a unirational parametrization of $\mathcal{R}_8$ (however see [FV3] for results in this direction), we are only able to perform *Macaulay2* experiments with smooth curves from lower dimensional subvarieties of $\overline{\mathcal{R}}_8$, for instance, we tested the curves constructed by the function *getCanonicalCurveOfGenus8With2Torsion* of our package *PrymCanonicalCurves*. All these calculations lead to curves in the component $\mathfrak{Kosz}_6$, having an extra syzygy of rank 6. We first observe that in order to show the inclusion $\mathcal{R}_8 \subset \mathfrak{Kosz}$, it suffices to show that general 1-nodal curves from the boundary divisor $\Delta_0^{'} \subset \overline{\mathcal{R}}_8$ lie in the locus $\overline{\mathcal{Z}}_{8,2}$.

Proposition 5.5. *Assuming that the inclusion $\Delta_0^{'} \subset \overline{\mathcal{Z}}_{8,2}$ holds, then $\overline{\mathcal{Z}}_{8,2} = \overline{\mathcal{R}}_8$ and the Prym-Green Conjecture fails on $\mathcal{R}_8$.*

Proof. We proceed by contradiction and assume that $\overline{\mathcal{Z}}_{8,2}$ is a divisor on $\overline{\mathcal{R}}_8$ whose class is computed in Theorem 0.6. Precisely, we have the relation

$$[\overline{\mathcal{Z}}_{8,2}]^{\mathrm{virt}} - [\overline{\mathcal{Z}}_{8,2}] = 27\lambda - 4(\delta_0^{'} + \delta_0^{''}) - 6\delta_0^{(1)} - [\overline{\mathcal{Z}}_{8,2}] \in \mathbb{Q}_{\geq 0}\langle \delta_0^{'}, \delta_0^{''}, \delta_0^{(1)} \rangle.$$

First we observe that the class $[\overline{\mathcal{Z}}_{8,2}]^{\mathrm{virt}} - 16\delta_0^{''}$ is effective, that is, the morphism of vector bundles $\varphi_{1,2} : \mathbb{H}_{1,2} \to \mathbb{G}_{1,2}$ over the stack $\widetilde{\mathsf{R}}_{8,2}$ is degenerate with multiplicity at least 16 along the boundary divisor $\Delta_0^{''}$. Indeed, assume that $[X, \eta, \phi] \in \Delta_0^{''}$ is a general point corresponding to a normalization $\mathrm{nor} : C \to X$, where $[C] \in \mathcal{M}_7$. Since $\mathrm{nor}^*(\eta) = \mathcal{O}_C$, it quickly follows that $K_{1,2}(X, \omega_X \otimes \eta) = K_{1,2}(C, K_C)$ and the latter space is 16-dimensional. This shows that the class $c_1(\mathbb{G}_{1,2} - \mathbb{H}_{1,2}) - 16\delta_0^{''}$ is effective.

Assume now that $\Delta_0^{'} \subset \overline{\mathcal{Z}}_{8,2}$, therefore $[\overline{\mathcal{Z}}_{8,2}]^{\mathrm{virt}} - \delta_0^{'} - 16\delta_0^{''} = 27\lambda - 5\delta_0^{'} - 20\delta_0^{''} - 6\delta_0^{(1)}$ is still effective. Combining this class with that of the pull-back of the Brill-Noether divisor $[\overline{\mathcal{M}}_{8,7}^2] = 22\lambda - 3\delta_0 \in \mathrm{Eff}(\widetilde{\mathsf{M}}_8)$, see [EH], after routine manipulations we can form an effective representative of the canonical class $K_{\overline{\mathcal{R}}_8}$. This is a contradiction, since $\overline{\mathcal{R}}_8$ is uniruled, see [FV3]. □

Even though we do not know whether $\Delta_0^{'}$ is unirational, we quote from [FV3]:

Theorem 5.6. [FV3] *The Prym moduli space $\mathcal{R}_7$ is unirational.*

Sketch of proof. A general Prym canonical curve $C \subset \mathbf{P}^5$ of genus 7 has the Betti table

```
       0  1  2  3 4
total: 1 11 27 24 7
    0: 1  .  .  . .
    1: .  3  .  . .
    2: .  8 27 24 7
```

The quadrics in $H^0(\mathbf{P}^5, \mathcal{I}_{C/\mathbf{P}^5}(2))$ intersect in a *Nikulin surface* $Y \subset \mathbf{P}^5$, that is, a smooth $K3$ surface containing 8 disjoint lines $L_1, \ldots, L_8$ with $L_i^2 = -2$ and $C \cdot L_i = 0$, for $i = 1, \ldots, 8$. Furthermore, $2(C - H) \equiv L_1 + \cdots + L_8$. The linear system

$$\left|\mathcal{O}_Y(C - L_1 - \ldots - L_7)\right|$$

is zero dimensional and consists of a single rational normal quintic curve $R_5 \subset \mathbf{P}^5$. The unirational parametrization of $\mathcal{R}_7$ reverses this construction. One starts with a fixed rational normal curve $R_5 \subset \mathbf{P}^5$ together with seven general secant lines $L_1, \ldots, L_8$. The union $R_5 \cup L_1 \cup \ldots \cup L_7$ is a 14-nodal stable curve which lies on three quadrics intersecting in a smooth Nikulin surface Y. A general $C \in |\mathcal{O}_Y(R_5 + L_1 + \ldots + L_7)|$ is a general curve in $\mathcal{R}_7$. This establishes a dominant rational map from a $\mathbf{P}^7$-bundle over $\overline{\mathcal{M}}_{0,14}$ onto $\mathcal{R}_7$. In the function *randomPrymCanonicalCurveOfGenus7* of our package *PrymCanonicalCurves* you find an "implementation" of the unirational parametrization. □

Based on Theorem 5.6 we have strong evidence that the inclusion $\Delta_0' \subset \mathcal{R}_8 \cap \mathcal{Z}_1$ holds, therefore also for the equality $\overline{\mathcal{Z}}_{8,2} = \overline{\mathcal{R}}_8$. We start with a randomly chosen curve in $C \in \mathcal{R}_7(\mathbb{F}_p)$ for a prime p with $10^4 < p < 3 \cdot 10^4$. By the Hasse-Weil theorem, one has the following estimate for the number of rational points of a curve C over $\mathbb{F}_p$:

$$p + 1 - 2g\sqrt{p} \leq |C(\mathbb{F}_p)| \leq p + 1 + 2g\sqrt{p}.$$

Rational points are easy to find: We consider a plane model $C_{\rm pl} \subset \mathbf{P}^2$ of C defined over $\mathbb{F}_p$. At least about 50% of the lines $L \in (\mathbf{P}^2)^\vee(\mathbb{F}_p)$ intersect C in at least one $\mathbb{F}_p$-rational point. Indeed there are $p^2 + O(p)$ lines. The set

$$\{(L, P) \in (\mathbf{P}^2)^\vee(\mathbb{F}_p) \times C(\mathbb{F}_p) \mid P \in L\}$$

has about $p^2 + O(p^{3/2})$ points, since there are $p + 1$ lines through a given point. The secant line spanned by two points of $C(\mathbb{F}_p)$ are counted twice in this set. Thus at least half the lines, i.e. $p^2 - \frac{1}{2}p^2 + O(p^{3/2})$ intersect C in at least one $\mathbb{F}_p$-rational point. (A better estimate should be $1 - \frac{1}{2} + \frac{1}{3!} - \frac{1}{4!} + \ldots = 1 - \frac{1}{e} \approx 63\%$ for the fraction of $\mathbb{F}_p$-rational lines which intersect C in at least one $\mathbb{F}_p$-rational point.)

We find the $\mathbb{F}_p$-rational intersection points by computing the primary decomposition of $I_{C_{\rm pl}} + I_L \subset \mathbb{F}_p[x, y, z]$. A probabilistic algorithm gives us two points $P, Q \in C(\mathbb{F}_p)$. Consider now the curve $[C' = \frac{C}{P \sim Q}] \in \Delta_0 \subset \overline{\mathcal{M}}_8$ obtained by identifying P and Q, and let $\mathrm{nor} : C \to C'$ be the normalization map. We now apply the following:

Proposition 5.7. *Let* $\mathrm{nor} : C \to C'$ *be the partial normalization of a single node of a irreducible nodal curve* C' *of genus* g *defined over a field* $\mathbb{F}$ *and let* $P, Q \in C(\mathbb{F})$ *be the preimages of the node of* C'. *Let* $\eta = \mathcal{O}_C(D_1 - D_2) \in \mathrm{Pic}^0 C[2]$ *be a 2-torsion line bundle on* C *curve with* D_1, D_2 *being effective divisors defined over* $\mathbb{F}$ *with support disjoint from* $\{P, Q\}$. *Then there exists a two torsion line bundle* $\eta' \in \mathrm{Pic}^0 C[2]$ *defined over* $\mathbb{F}$ *with* $\mathrm{nor}^*\eta' \cong \eta$ *if and only if* $\frac{f(P)}{f(Q)} \in (\mathbb{F}^*)^2$ *where* $f \in \mathbb{F}(C)$ *is the rational function with* $(f) = 2D_1 - 2D_2$.

Proof. Consider an embedding $C \subset \mathbf{P}^r$ defined over $\mathbb{F}$, e.g. a (pluri)canonical model. Suppose $\frac{f(P)}{f(Q)} = \alpha^2$. Consider an arbitrary rational function $g = \frac{g_0}{g_1} \in \mathbb{F}(C)$, where g_0, g_1

are linear forms on $\mathbf{P}^r$. We choose a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{F})$$

such that the rational function $h = \frac{ag_0+bg_1}{cg_0+dg_1}$ takes values $h(P) = 1$ and $h(Q) = \alpha$. Let $(h) = (h)_0 - (h)_\infty$ be the principal divisor of h on C. The divisor $E_1 - E_2$ with $E_1 = D_1+(h)_0$ and $E_2 = D_2+(h)_\infty$ is 2-torsion on C' since $\frac{(fh)(P)}{(fh)(Q)} = 1$ and $\mathrm{nor}^*\mathcal{O}_{C'}(E_1-E_2) = \mathcal{O}_C(D_1 - D_2)$, because $(E_1 - E_2) - (D_1 - D_2)$ is a principal divisor on C. This proves that the condition is sufficient. To establish necessity, we note that $1 \in (\mathbb{F}^*)^2$ $\square$

Thus for a finite field of $\mathrm{char}(\mathbb{F}_p) \neq 2$, in about 50% of the cases we can descend the 2-torsion bundle from C to C'. In this way we find many $\mathbb{F}_p$-rational points on Δ_0'. Computing the syzygies of the resulting Prym canonical curves we always land in the component $\mathfrak{Kosz}_6$, see *randomOneNodalPrymCanonicalCurveOfGenus8* in our package *PrymCanonicalCurves*. This does not establish the inclusion $\Delta_0' \subset \mathfrak{Kosz}_6$ (hence the equality $\mathcal{Z}_{g,2} = \mathcal{R}_8$), since for any fixed prime p, we only have finitely many points in $\Delta_0'(\mathbb{F}_p)$. However, the possibility that $\Delta_0' \not\subset \mathfrak{Kosz}_6$ appears to be exceedingly unlikely.

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