THE KODAIRA DIMENSION OF THE MODULI SPACE OF PRYM VARIETIES

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Prym varieties provide a correspondence between the moduli spaces of curves and abelian varieties $\mathcal{M}_g$ and $\mathcal{A}_{g-1}$, via the *Prym map* $\mathcal{P}_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$ from the moduli space $\mathcal{R}_g$ parameterizing pairs $[C, \eta]$, where $[C] \in \mathcal{M}_g$ is a smooth curve and $\eta \in \mathrm{Pic}^0(C)[2]$ is a torsion point of order 2. When $g \leq 6$ the Prym map is dominant and $\mathcal{R}_g$ can be used directly to determine the birational type of $\mathcal{A}_{g-1}$. It is known that $\mathcal{R}_g$ is rational for $g = 2, 3, 4$ (see [Dol] and references therein and [Ca] for the case of genus 4) and unirational for $g = 5$ (cf. [IGS] and [V2]). The situation in genus 6 is strikingly beautiful because $\mathcal{P}_6 : \mathcal{R}_6 \to \mathcal{A}_5$ is equidimensional (precisely $\dim(\mathcal{R}_6) = \dim(\mathcal{A}_5) = 15$). Donagi and Smith showed that $\mathcal{P}_6$ is generically finite of degree 27 (cf. [DS]) and the monodromy group equals the Weyl group WE_6 describing the incidence correspondence of the 27 lines on a cubic surface (cf. [D1]). There are three different proofs that $\mathcal{R}_6$ is unirational (cf. [D1], [MM], [V]). Verra has very recently announced a proof of the unirationality of $\mathcal{R}_7$ (see also Theorem 0.8 for a weaker result). The Prym map $\mathcal{P}_g$ is generically injective for $g \geq 7$ (cf. [FS]), although never injective. In this range, we may regard $\mathcal{R}_g$ as a partial desingularization of the moduli space $\mathcal{P}_g(\mathcal{R}_g) \subset \mathcal{A}_{g-1}$ of Prym varieties of dimension $g-1$.

The scheme $\mathcal{R}_g$ admits a suitable modular compactification $\overline{\mathcal{R}}_g$, which is isomorphic to (1) the coarse moduli space of the stack $\overline{\mathbf{R}}_g = \overline{\mathbf{M}}_g(\mathcal{B}\mathbb{Z}_2)$ of *Beauville admissible* double covers (cf. [B], [ACV]) and (2) the coarse moduli space of the stack of *Prym curves* (cf. [BCF]). The forgetful map $\pi : \mathcal{R}_g \to \mathcal{M}_g$ extends to a finite map $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$. The aim of this paper is to initiate a study of the enumerative and global geometry of $\overline{\mathcal{R}}_g$, in particular to determine its Kodaira dimension. The main result of the paper is the following:

Theorem 0.1. *The moduli space of Prym varieties $\overline{\mathcal{R}}_g$ is of general type for $g > 13$ and $g \neq 15$. The Kodaira dimension of $\overline{\mathcal{R}}_{15}$ is at least 1.*

We point out in Remark 2.9 that the existence of an effective divisor $D \in \mathrm{Eff}(\overline{\mathcal{M}}_{15})$ of slope $s(D) < 6 + 12/(g+1) = 27/4$ (that is, violating the Harris-Morrison Slope Conjecture on $\overline{\mathcal{M}}_{15}$), would imply that $\overline{\mathcal{R}}_{15}$ is of general type. There are known examples of divisors $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ satisfying $s(D) < 6 + \frac{12}{g+1}$ for every genus of the form $g = s(2s + si + i + 1)$ with $s \geq 2$ and $i \geq 0$ (cf. [F1], [F2]). No such examples have been found yet on $\overline{\mathcal{M}}_{15}$, though they are certainly expected to exist.

The normal variety $\overline{\mathcal{R}}_g$ has finite quotient singularities and an important part of the proof is concerned with showing that pluricanonical forms defined on the smooth

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part $\overline{\mathcal{R}}_g^{\mathrm{reg}} \subset \overline{\mathcal{R}}_g$ can be lifted to any resolution of singularities $\widehat{\mathcal{R}}_g \to \overline{\mathcal{R}}_g$, that is, we have isomorphisms

$$H^0\big(\overline{\mathcal{R}}_g^{\mathrm{reg}}, K_{\overline{\mathcal{R}}_g}^{\otimes l}\big) \cong H^0\big(\widehat{\mathcal{R}}_g, K_{\widehat{\mathcal{R}}_g}^{\otimes l}\big)$$

for $l \geq 0$. This is achieved in the last section of the paper. The locus of non-canonical singularities in $\overline{\mathcal{R}}_g$ is also explicitly described: A Prym curve $[X, \eta, \beta] \in \overline{\mathcal{R}}_g$ is a non-canonical singularity if and only if X has an elliptic tail C with $\mathrm{Aut}(C) = \mathbb{Z}_6$, such that the line bundle $\eta_C \in \mathrm{Pic}^0(C)[2]$ is trivial (cf. Theorem 6.7).

We outline the strategy to prove that $\overline{\mathcal{R}}_g$ is of general type for given g. If $\lambda = \pi^*(\lambda) \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$ is the pull-back of the Hodge class and $\delta_0', \delta_0^{''}, \delta_0^{\mathrm{ram}} \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$ and $\delta_i, \delta_{g-i}, \delta_{i:g-i} \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$ for $1 \leq i \leq [g/2]$ are boundary divisor classes such that

$$\pi^*(\delta_0) = \delta_0' + \delta_0^{''} + 2\delta_0^{\mathrm{ram}} \text{ and } \pi^*(\delta_i) = \delta_i + \delta_{g-i} + \delta_{i:g-i} \text{ for } 1 \leq i \leq [g/2]$$

(see Section 2 for a precise definition of these classes), then one has the formula

$$K_{\overline{\mathcal{R}}_g} \equiv 13\lambda - 2(\delta_0^{'} + \delta_0^{''}) - 3\delta_0^{\mathrm{ram}} - 2\sum_{i=1}^{[g/2]} (\delta_i + \delta_{g-i} + \delta_{i:g-i}) - (\delta_1 + \delta_{g-1} + \delta_{1:g-1}).$$

We show that this class is big by explicitly constructing effective divisors D on $\overline{\mathcal{R}}_g$ such that one can write $K_{\overline{\mathcal{R}}_g} \equiv \alpha \cdot \lambda + \beta \cdot D + \{\text{effective combination of boundary classes}\}$, for certain $\alpha, \beta \in \mathbb{Q}_{>0}$ (see (2) for the inequalities the coefficients of such D must satisfy).

We carry out an enumerative study of divisors on $\overline{\mathcal{R}}_g$ defined in terms of pairs $[C, \eta]$ such that the 2-torsion point $\eta \in \mathrm{Pic}^0(C)$ is transversal with respect to the theta divisors associated to certain stable vector bundles on C. We fix integers $k \geq 2$ and $b \geq 0$ and then define the integers

$$i := kb + k - b - 2, \; r := kb + k - 2, \; g := ik + 1 \text{ and } d := rk.$$

The Brill-Noether number $\rho(g, r, d) := g - (r+1)(g - d + r) = 0$ and a general $[C] \in \mathcal{M}_g$ carries a finite number of line bundles $L \in W_d^r(C)$. For each such line bundle L, if Q_L denotes the dual of the *Lazarsfeld bundle* defined by the exact sequence (see [L])

$$0 \longrightarrow Q_L^\vee \longrightarrow H^0(C, L) \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0,$$

we compute that $\mu(Q_L) = d/r = k$ and then $\mu(\wedge^i Q_L) = ik = g - 1$. In these circumstances we define the *Raynaud divisor* (degeneration locus of virtual codimension 1)

$$\Theta_{\wedge^i Q_L} := \{\eta \in \mathrm{Pic}^0(C) : H^0(C, \wedge^i Q_L \otimes \eta) \neq 0\}.$$

This is a virtual divisor inside $\mathrm{Pic}^0(C)$, in the sense that either $\Theta_{\wedge^i Q_L} = \mathrm{Pic}^0(C)$ or else $\Theta_{\wedge^i Q_L}$ is a divisor on $\mathrm{Pic}^0(C)$ belonging to the linear system $|\binom{r}{i}\theta|$, cf. [R]. We study the relative position of η with respect to $\Theta_{\wedge^i Q_L}$ and introduce the following locus on $\overline{\mathcal{R}}_g$:

$$\mathcal{D}_{g:k} := \{[C, \eta] \in \mathcal{R}_g : \exists L \in W_d^r(C) \text{ such that } \eta \in \Theta_{\wedge^i Q_L}\}.$$

When $k = 2, i = b$, then $g = 2i + 1, d = 2g - 2$ and $\mathcal{D}_{2i+1:2}$ has a new incarnation using the proof of the *Minimal Resolution Conjecture* [FMP]. In this case, $L = K_C$ (a genus g curve has only one $\mathfrak{g}_{2g-2}^{g-1}$!) and [FMP] gives an identification of cycles

$$\Theta_{\wedge^i Q_{K_C}} = C_i - C_i \subset \mathrm{Pic}^0(C),$$

where the right-hand-side stands for the i-th difference variety of C.

We prove in Section 2 that $\mathcal{D}_{g:k}$ is an effective divisor on $\mathcal{R}_g$. By specialization to the k-gonal locus $\mathcal{M}^1_{g,k} \subset \mathcal{M}_g$, we show that for a generic $[C,\eta] \in \mathcal{R}_g$ the vanishing $H^0(C, \wedge^i Q_L \otimes \eta) = 0$ holds for every $L \in W^r_d(C)$ (Theorem 2.3). Then we extend the determinantal structure of $\mathcal{D}_{g:k}$ to a partial compactification of $\mathcal{R}_g$ which enables us to compute the class of the compactification $\overline{\mathcal{D}}_{g:k}$. Precisely we construct two vector bundles $\mathcal{E}$ and $\mathcal{F}$ over a stack $\overline{\mathbf{R}}^0_g$ which is a partial compactification of $\mathbf{R}_g$, such that $\mathrm{rank}(\mathcal{E}) = \mathrm{rank}(\mathcal{F})$, together with a vector bundle homomorphism $\phi : \mathcal{E} \to \mathcal{F}$ such that $Z_1(\phi) \cap \mathcal{R}_g = \mathcal{D}_{g:k}$. Then we explicitly determine the class $c_1(\mathcal{F} - \mathcal{E}) \in A^1(\overline{\mathbf{R}}^0_g)$ (Theorem 2.8). The cases of interest for determining the Kodaira dimension of $\overline{\mathcal{R}}_g$ are when $k = 2, 3$ when we obtain the following results:

Theorem 0.2. *The closure of the divisor* $\mathcal{D}_{2i+1:2} = \{[C,\eta] \in \mathcal{R}_{2i+1} : h^0(C, \wedge^i Q_{K_C} \otimes \eta) \geq 1\}$ *inside* $\overline{\mathcal{R}}_{2i+1}$ *has class given by the following formula in* $\mathrm{Pic}(\overline{\mathcal{R}}_{2i+1})$*:*

$$\overline{\mathcal{D}}_{2i+1:2} \equiv \frac{1}{2i-1}\binom{2i}{i}\Big((3i+1)\lambda - \frac{i}{2}(\delta_0' + \delta_0'') - \frac{2i+1}{4}\delta_0^{\mathrm{ram}} - (3i-1)\delta_{g-1} - i(\delta_{1:g-1} + \delta_1) - \cdots\Big).$$

To illustrate Theorem 0.2 in the simplest case, $i = 1$ hence $g = 3$, we write $\mathcal{D}_{3:2} = \{[C,\eta] \in \mathcal{R}_3 : \eta = \mathcal{O}_C(x-y),\ x,y \in C\}$. The analysis carried out in Section 5 shows that the vector bundle morphism $\phi : \mathcal{E} \to \mathcal{F}$ is generically non-degenerate along the boundary divisors $\Delta_0',\ \Delta_0^{\mathrm{ram}} \subset \overline{\mathcal{R}}_3$ and degenerate (with multiplicity 1) along the divisor $\Delta_0'' \subset \overline{\mathcal{R}}_3$ of Wirtinger covers. Theorem 0.2 reads like

$$\overline{\mathcal{D}}_{3:2} \equiv c_1(\mathcal{F} - \mathcal{E}) - \delta_0'' \equiv 8\lambda - \delta_0' - 2\delta_0'' - \frac{3}{2}\delta_0^{\mathrm{ram}} - 6\delta_1 - 4\delta_2 - 2\delta_{1:2} \in \mathrm{Pic}(\overline{\mathcal{R}}_3),$$

and then $\pi_*(\overline{\mathcal{D}}_{3:2}) \equiv 56(9\lambda - \delta_0 - 3\delta_1) \equiv 56 \cdot \overline{\mathcal{M}}^1_{3,2} \in \mathrm{Pic}(\overline{\mathcal{M}}_3)$ (see Theorem 5.1). Theorem 0.2 is consistent with the following elementary fact, see e.g. [HF]: If $[\tilde{C} \to C] \in \mathcal{R}_3$ is an étale double cover, then $[\widetilde{C}] \in \mathcal{M}_5$ is hyperelliptic if and only if $[C] \in \mathcal{M}_3$ is hyperelliptic and $\eta = \mathcal{O}_C(x-y)$, with $x,y \in C$ being Weierstrass points.

Theorem 0.3. *For* $b \geq 1$ *and* $r = 3b+1$ *the class of the divisor* $\overline{\mathcal{D}}_{6b+4:3}$ *on* $\overline{\mathcal{R}}_{6b+4}$ *is given by:*

$$\overline{\mathcal{D}}_{g:3} \equiv \frac{4}{r}\binom{6b+3}{b,\,2b,\,3b+3}\Big((3b+2)(b+2)\lambda - \frac{3b^2+7b+3}{6}(\delta_0' + \delta_0'') - \frac{24b^2+47b+21}{24}\delta_0^{\mathrm{ram}} - \cdots\Big).$$

Theorems 2.8, 0.2 and 0.3 specify precisely the $\lambda, \delta_0', \delta_0^{'}$ and δ_0^{ram} coefficients in the expansion of $[\overline{\mathcal{D}}_{g:k}]$. Good lower bounds for the remaining boundary coefficients of $[\overline{\mathcal{D}}_{g:k}]$ can be obtained using Proposition 1.9. The information contained in Theorems 0.2 and 0.3 is sufficient to finish the proof of Theorem 0.1 for odd genus $g = 2i+1 \geq 15$.

When $b = 0$, hence $i = r = k-2$, Theorem 2.8 has the following interpretation:

Theorem 0.4. *We fix integers* $k \geq 3, r = k-2$ *and* $g = (k-1)^2$*. The following locus*

$$\mathcal{D}_{g:k} := \{[C,\eta] \in \mathcal{R}_g : \exists L \in W^{k-2}_{k(k-2)}(C) \ \text{ such that } H^0(C, L \otimes \eta) \neq 0\}$$

is a divisor on $\mathcal{R}_g$*. The class of its compactification inside* $\overline{\mathcal{R}}_g$ *is given by the formula*

$$\overline{\mathcal{D}}_{g:k} \equiv g!\frac{1!\,2!\,\cdots(k-2)!}{(k-1)!\,\cdots(2k-3)!\,(k^2-2k-1)}\Big(\frac{1}{2}(k^4 - 4k^3 + 11k^2 - 14k + 2)\lambda -$$

$$-\frac{k(k-2)(k^2-2k+5)}{12}(\delta_0'+\delta_0'')-\frac{(k^2-2k+3)(2k^2-4k+1)}{12}\delta_0^{\mathrm{ram}}-\cdots\Big)\in \mathrm{Pic}(\overline{\mathcal{R}}_g).$$

When $k=3$ and $g=4$, the divisor $\mathcal{D}_{4:3}$ consists of Prym curves $[C,\eta]\in\mathcal{R}_4$ for which the plane Prym-canonical model $\iota: C\stackrel{|K_C\otimes\eta|}{\longrightarrow}\mathbf{P}^2$ has a triple point. Note that for a general $[C,\eta]\in\mathcal{R}_4$, $\iota(C)$ is a 6-nodal sextic. We can then verify the formula

$$\pi_*(\overline{\mathcal{D}}_{4:3})=60(34\lambda-4\delta_0-14\delta_1-18\delta_2)=60\cdot\overline{\mathcal{GP}}^1_{4,3}\in\mathrm{Pic}(\overline{\mathcal{M}}_4),$$

where $\overline{\mathcal{GP}}^1_{4,3}\subset\overline{\mathcal{M}}_4$ is the divisor of curves with a vanishing theta-null. This is consistent with the set-theoretic equality $\pi(\mathcal{D}_{4:3})=\mathcal{GP}^1_{4,3}$ which can be easily established (see Theorem 5.4).

Another case which has a simple interpretation is when $b=1$, $i=r-1$, and then $g=(2k-1)(k-1)$, $d=2k(k-1)$. Since $\mathrm{rank}(Q_L)=r$ and $\det(Q_L)=L$, by duality we have that $\wedge^i Q_L=M_L\otimes L$, hence points $[C,\eta]\in\mathcal{D}_{(2k-1)(k-1):k}$ can be described purely in terms of multiplication maps of sections of line bundles on C:

Theorem 0.5. *We fix integers $k\geq 2$ and $g=(2k-1)(k-1)$. The following locus*

$$\mathcal{D}_{g:k}=\{[C,\eta]\in\mathcal{R}_g:\exists L\in W^{2k-2}_{2k(k-1)}(C)\text{ with } H^0(L)\otimes H^0(L\otimes\eta)\rightarrow H^0(L^{\otimes 2}\otimes\eta)\text{ not bijective}\}$$

is a divisor on $\mathcal{R}_g$. The class of its compactification inside $\overline{\mathcal{R}}_g$ equals

$$\overline{\mathcal{D}}_{g:k}\equiv g!\frac{1!\,2!\cdots(k-2)!\,(k-1)}{3(2k^2-3k-1)(2k-1)!\,(2k)!\cdots(3k-3)!\,(3k-2)}\cdot$$

$$\Big(6\big(8k^5-36k^4+78k^3-95k^2+49k-6\big)\lambda-\big(8k^5-36k^4+70k^3-71k^2+29k-2\big)\big(\delta_0'+\delta_0''\big)-$$
$$-\frac{1}{2}\big(32k^5-144k^4+262k^3-245k^2+107k-14\big)\delta_0^{\mathrm{ram}}-\cdots\Big).$$

The second class of (virtual) divisors is provided by Koszul divisors on $\overline{\mathcal{R}}_g$. For a pair (C,L) consisting of a curve $[C]\in\mathcal{M}_g$ and a line bundle $L\in\mathrm{Pic}(C)$, we denote by $K_{i,j}(C,L)$ its (i,j)-th Koszul cohomology group, cf. [L]. For a point $[C,\eta]\in\mathcal{R}_g$ we set $L:=K_C\otimes\eta$ and we stratify $\mathcal{R}_g$ using the syzygies of the Prym-canonical curve $C\stackrel{|L|}{\to}\mathbf{P}^{g-2}$. We define the stratum

$$\mathcal{U}_{g,i}:=\{[C,\eta]\in\mathcal{R}_g: K_{i,2}(C,K_C\otimes\eta)\neq\emptyset\},$$

that is, $\mathcal{U}_{g,i}$ consists of those Prym curves $[C,\eta]\in\mathcal{R}_g$ for which the Prym-canonical model $C\stackrel{|L|}{\longrightarrow}\mathbf{P}^{g-2}$ fails to satisfy the Green-Lazarsfeld property (N_i) in the sense of [GL], [L].

Theorem 0.6. *There exist two vector bundles $\mathcal{G}_{i,2}$ and $\mathcal{H}_{i,2}$ of the same rank defined over a partial compactification $\widetilde{\mathbf{R}}_{2i+6}$ of the stack $\mathbf{R}_{2i+6}$, together with a morphism $\phi:\mathcal{H}_{i,2}\rightarrow\mathcal{G}_{i,2}$ such that*

$$\mathcal{U}_{2i+6,i}:=\{[C,\eta]\in\widetilde{\mathcal{R}}_{2i+6}:K_{i,2}(C,K_C\otimes\eta)\neq 0\}$$

is the degeneracy locus of the map ϕ. The virtual class of $[\overline{\mathcal{U}}_{2i+6,i}]$ is given by the formula:

$$[\overline{\mathcal{U}}_{2i+6,i}]^{virt}=c_1(\mathcal{G}_{i,2}-\mathcal{H}_{i,2})=\binom{2i+2}{i}\Big(\frac{3(2i+7)}{i+3}\lambda-\frac{3}{2}\delta_0^{\mathrm{ram}}-(\delta_0'+\alpha\,\delta_0'')-\cdots\Big),$$

where the constant α satisfies $\alpha \geq 1$.

The compactification $\widetilde{\mathbf{R}}_g$ has the property that if $\widetilde{\mathcal{R}}_g \subset \overline{\mathcal{R}}_g$ denotes its coarse moduli space, then $\mathrm{codim}\big(\pi^{-1}(\mathcal{M}_g \cup \Delta_0) - \widetilde{\mathcal{R}}_g\big) \geq 2$. In particular Theorem 0.6 precisely determines the coefficient of $\lambda, \delta_0', \delta_0''$ and δ_0^{ram} in the expansion of $[\overline{\mathcal{U}}_{2i+6,i}]^{virt}$. We also show that when $g < 2i+6$ then $K_{i,2}(C, K_C \otimes \eta) \neq \emptyset$ for any $[C,\eta] \in \mathcal{R}_g$. By analogy with the case of canonical curves and the classical M. Green Conjecture on syzygies of canonical curves (see [Vo]), we conjecture that the morphism of vector bundles $\phi : \mathcal{G}_{i,2} \to \mathcal{H}_{i,2}$ over $\widetilde{\mathbf{R}}_{2i+6}$ is generically non-degenerate:

Conjecture 0.7. *(Prym-Green Conjecture)* For a generic point $[C,\eta] \in \mathcal{R}_g$ and $g \geq 2i+6$, we have that $K_{i,2}(C, K_C \otimes \eta) = 0$. Equivalently, the Prym-canonical curve $C \stackrel{|K_C \otimes \eta|}{\hookrightarrow} \mathbf{P}^{g-2}$ satisfies the Green-Lazarsfeld property (N_i) whenever $g \geq 2i+6$. For $g = 2i+6$, the locus $\mathcal{U}_{2i+6,i}$ is an effective divisor on $\mathcal{R}_{2i+6}$.

Proposition 3.1 shows that, if true, Conjecture 0.7 is sharp. In [F4] we verify the Prym-Green Conjecture for $g = 2i+6$ with $0 \leq i \leq 4, i \neq 1$. In particular, this together with Theorem 0.6 proves that $\overline{\mathcal{R}}_g$ is of general type for $g = 14$.

The strata $\mathcal{U}_{g,i}$ have been considered before for $i = 0, 1$, in connection with the Prym-Torelli problem. Unlike the classical Torelli problem, the Prym-Torelli problem is a subtle question: Donagi's tetragonal construction shows that $\mathcal{P}_g$ fails to be injective over points $[C,\eta] \in \pi^{-1}(\mathcal{M}_{g,4}^1)$ where the curve C is tetragonal (cf. [D2]). The locus $\mathcal{U}_{g,0}$ consists of those points $[C,\eta] \in \mathcal{R}_g$ where the differential

$$(d\mathcal{P}_g)_{[C,\eta]} : H^0(C, K_C^{\otimes 2})^{\vee} \to (\mathrm{Sym}^2 H^0(C, K_C \otimes \eta))^{\vee}$$

is not injective and thus the *infinitesimal Prym-Torelli theorem* fails. It is known that $(d\mathcal{P}_g)_{[C,\eta]}$ is generically injective for $g \geq 6$ (cf. [B], or [De] Corollaire 2.3), that is, $\mathcal{U}_{g,0}$ is a proper subvariety of $\mathcal{R}_g$. In particular, for $g = 6$ the locus $\mathcal{U}_{6,0}$ is a divisor of $\mathcal{R}_6$, which gives another proof of Conjecture 0.7 for $i = 0$.

Debarre proved that $\mathcal{U}_{g,1}$ is a proper subvariety of $\mathcal{R}_g$ for $g \geq 9$ (cf. [De] Théoreme 2.2). This immediately implies that for $g \geq 9$ the Prym map $\mathcal{P}_g$ is generically injective, hence the *Prym-Torelli theorem* holds generically. Debarre's proof unfortunately does not cover the interesting case $g = 8$.

The proof of Theorem 0.1 is finished in Section 4, using in an essential way results from [F3]: We set $g' := 1 + \frac{g-1}{g}\binom{2g}{g-1}$ and then we consider the rational map which associates to a curve one of its Brill-Noether loci

$$\phi : \overline{\mathcal{M}}_{2g-1} \dashrightarrow \overline{\mathcal{M}}_{1+\frac{g-1}{g}\binom{2g}{g-1}}, \quad \phi[Y] := W_{g+1}^1(Y),$$

where $W_{g+1}^1(Y) := \{L \in \mathrm{Pic}^{g+1}(Y) : h^0(Y,L) \geq 2\}$. If $\chi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_{2g-1}$ is the map given by $\chi([C,\eta]) := [\tilde{C}]$, where $f : \tilde{C} \to C$ is the étale double cover with the property that $f_*\mathcal{O}_{\tilde{C}} = \mathcal{O}_C \oplus \eta$, then using [F3] we compute the slope of myriads of effective divisors of type $\chi^*\phi^*(A)$, where $A \in \mathrm{Ample}(\overline{\mathcal{M}}_{g'})$. This proves Theorem 0.1 for even genus $g = 2i+6 \geq 18$.

We mention in passing as an immediate application of Proposition 1.9, a different proof of the statement that $\overline{\mathcal{R}}_g$ has good rationality properties for low g (see again the Introduction for the history of this problem). Our proof is quite simple and uses only numerical properties of Lefschetz pencils of curves on $K3$ surfaces:

Theorem 0.8. *For all $g \leq 7$, the Kodaira dimension of $\overline{\mathcal{R}}_g$ is $-\infty$.*

We close by summarizing the structure of the paper. In Section 1 we introduce the stack $\overline{\mathbf{R}}_g$ of Prym curves and determine the Chern classes of certain tautological vector bundles. In Section 2 we carry out the enumerative study of the divisors $\overline{\mathcal{D}}_{g:k}$ while in Section 3 we study Koszul divisors on $\overline{\mathcal{R}}_g$ in connection with the Prym-Green Conjecture. The proof of Theorem 0.1 is completed in Section 4 while Section 5 is concerned with the enumerative geometry of $\overline{\mathcal{R}}_g$ for $g \leq 5$. In Section 6 we describe the behaviour of singularities of pluricanonical forms of $\overline{\mathcal{R}}_g$. There is a significant overlap between some of the results of this paper and those of [Be]. Among the things we use from [Be] we mention the description of the branch locus of π and the fact that $\overline{\mathcal{R}}_g$ is isomorphic to the coarse moduli space of $\overline{\mathbf{M}}_g(\mathcal{B}\mathbb{Z}_2)$ (see Section 1). However, some of the results in [Be] are not correct, in particular the statement in [Be] Chapter 3 on singularities of $\overline{\mathcal{R}}_g$. Hence we carried out a detailed study of singularities of $\overline{\mathcal{R}}_g$ in Section 6 of our paper.

1. THE STACK OF PRYM CURVES

In this section we review a few facts about compactifications of $\mathcal{R}_g$. As a matter of terminology, if $\mathbf{M}$ is a Deligne-Mumford stack, we denote by $\mathcal{M}$ its coarse moduli space (This is contrary to the convention set in [ACV] but it makes sense, at least from a historical point of view). All the Picard groups of stacks or schemes we are going to consider are with rational coefficients.

We recall that $\pi : \mathcal{R}_g \to \mathcal{M}_g$ is the $(2^{2g} - 1)$-sheeted cover which forgets the point of order 2 and we denote by $\overline{\mathcal{R}}_g$ the normalization of $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{R}_g$. By definition, $\overline{\mathcal{R}}_g$ is a normal variety and π extends to a finite ramified covering $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$. The local behaviour of this branched cover has been studied in the thesis of M. Bernstein [Be] as well as in the paper [BCF]. In particular, the scheme $\overline{\mathcal{R}}_g$ has two distinct modular incarnations which we now recall. If X is a nodal curve, a smooth rational component $E \subset X$ is said to be *exceptional* if $\#(E \cap \overline{X - E}) = 2$. The curve X is said to be *quasi-stable* if any two exceptional components of X are disjoint. Thus a quasi-stable curve is obtained from a stable curve by blowing-up each node at most once. We denote by $[st(X)] \in \overline{\mathcal{M}}_g$ the stable model of X. We have the following definition (cf. [BCF]):

Definition 1.1. A *Prym curve* of genus g consists of a triple (X, η, β), where X is a genus g quasi-stable curve, $\eta \in \mathrm{Pic}^0(X)$ is a line bundle of degree 0 such that $\eta_E = \mathcal{O}_E(1)$ for every exceptional component $E \subset X$, and $\beta : \eta^{\otimes 2} \to \mathcal{O}_X$ is a sheaf homomorphism which is generically non-zero along each non-exceptional component of X.

A *family of Prym curves* over a base scheme S consists of a triple $(\mathcal{X} \xrightarrow{f} S, \eta, \beta)$, where $f : \mathcal{X} \to S$ is a flat family of quasi-stable curves, $\eta \in \mathrm{Pic}(\mathcal{X})$ is a line bundle and

$\beta : \eta^{\otimes 2} \to \mathcal{O}_{\mathcal{X}}$ is a sheaf homomorphism, such that for every point $s \in S$ the restriction $(X_s, \eta_{X_s}, \beta_{X_s} : \eta_{X_s}^{\otimes 2} \to \mathcal{O}_{X_s})$ is a Prym curve.

We denote by $\overline{\mathbf{R}}_g$ the non-singular Deligne-Mumford stack of Prym curves of genus g. The main result of [BCF] is that the coarse moduli space of $\overline{\mathbf{R}}_g$ is isomorphic to the normalization of $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{R}_g$. On the other hand, it is proved in [Be] that $\overline{\mathcal{R}}_g$ is also isomorphic to the coarse moduli space of the Deligne-Mumford stack $\overline{\mathbf{M}}_g(\mathcal{B}\mathbb{Z}_2)$ of $\mathbb{Z}_2$-admissible double covers introduced in [B] and later in [ACV]. For intersection theory calculations the language of Prym curves is better suited than that of admissible covers. In particular, the existence of a degree 0 line bundle η over the universal Prym curve will be often used to compute the Chern classes of various tautological vector bundles defined over $\overline{\mathbf{R}}_g$. Throughout this paper we use the isomorphism between rational Picard groups $\epsilon^* : \mathrm{Pic}(\overline{\mathcal{R}}_g) \to \mathrm{Pic}(\overline{\mathbf{R}}_g)$ induced by the map $\epsilon : \overline{\mathbf{R}}_g \to \overline{\mathcal{R}}_g$ from the stack to its coarse moduli space.

Remark 1.2. If (X, η, β) is a Prym curve with exceptional components $E_1, \ldots, E_r$ and $\{p_i, q_i\} = E_i \cap \overline{X - E_i}$ for $i = 1, \ldots, r$, then obviously $\beta_{E_i} = 0$. Moreover, if $\tilde{X} := \overline{X - \bigcup_{i=1}^r E_i}$ (viewed as a subcurve of X), then we have an isomorphism of sheaves

$$(1) \qquad \eta_{\tilde{X}}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_{\tilde{X}}(-p_1 - q_1 - \cdots - p_r - q_r).$$

It is straightforward to describe all Prym curves $[X, \eta, \beta] \in \overline{\mathcal{R}}_g$ whose stable model has a prescribed topological type. We do this when $st(X)$ is a 1-nodal curve and we determine in the process the boundary components of $\overline{\mathcal{R}}_g - \mathcal{R}_g$.

Example 1.3. (*Curves of compact type*) If $st(X) = C \cup D$ is a union of two smooth curves C and D of genus i and $g - i$ respectively meeting transversally at a point, we use (1) to note that $X = C \cup D$ (that is, X has no exceptional components). The line bundle η on X is determined by the choice of two line bundles $\eta_C \in \mathrm{Pic}^0(C)$ and $\eta_D \in \mathrm{Pic}^0(D)$ satisfying $\eta_C^{\otimes 2} = \mathcal{O}_C$ and $\eta_D^{\otimes 2} = \mathcal{O}_D$ respectively. This shows that for $1 \leq i \leq [g/2]$ the pull-back under π of the boundary divisor $\Delta_i \subset \overline{\mathcal{M}}_g$ splits into three irreducible components

$$\pi^*(\Delta_i) = \Delta_i + \Delta_{g-i} + \Delta_{i:g-i},$$

where the generic point of $\Delta_i \subset \overline{\mathcal{R}}_g$ is of the form $[C \cup D, \eta_C \neq \mathcal{O}_C, \eta_D = \mathcal{O}_D]$, the generic point of Δ_{g-i} is of the form $[C \cup D, \eta_C = \mathcal{O}_C, \eta_D \neq \mathcal{O}_D])$, and finally $\Delta_{i:g-i}$ is the closure of the locus of points $[C \cup D, \eta_C \neq \mathcal{O}_C, \eta_D \neq \mathcal{O}_D]$ (see also [Be] pg. 9).

Example 1.4. (*Irreducible one-nodal curves*) If $st(X) = C_{yq} := C/y \sim q$, where $[C, y, q] \in \mathcal{M}_{g-1,2}$, then there are two possibilities, depending on whether X has an exceptional component or not. Suppose first that $X = C'$ and $\eta \in \mathrm{Pic}^0(X)$. If $\nu : C \to X$ is the normalization map, then there is an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Pic}^0(X) \xrightarrow{\nu^*} \mathrm{Pic}^0(C) \longrightarrow 0.$$

Thus η is determined by a (non-trivial) line bundle $\eta_C := \nu^*(\eta) \in \mathrm{Pic}^0(C)$ satisfying $\eta_C^{\otimes 2} = \mathcal{O}_C$ together with an identification of the fibres $\eta_C(y)$ and $\eta_C(q)$. If $\eta_C = \mathcal{O}_C$, then there is a unique way to identify the fibres $\eta_C(y)$ and $\eta_C(q)$ such that $\eta \neq \mathcal{O}_X$, and this corresponds to the classical Wirtinger cover of X. We denote by $\Delta_0^{''} = \Delta_0^{\mathrm{wir}}$ the closure in $\overline{\mathcal{R}}_g$ of the locus of Wirtinger covers. If $\eta_C \neq \mathcal{O}_C$, then for each such choice of $\eta_C \in \text{Pic}^0(C)[2]$ there are 2 ways to glue $\eta_C(y)$ and $\eta_C(q)$. This provides another $2 \times (2^{2g-2}-1)$ Prym curves having C' as their stable model. We set $\Delta_0' \subset \overline{\mathcal{R}}_g$ to be the closure of the locus of Prym curves with $\eta_C \neq \mathcal{O}_C$.

We now treat the case when $X = C \cup_{\{y,q\}} E$, with E being an exceptional component. Then $\eta_E = \mathcal{O}_E(1)$ and $\eta_C^{\otimes 2} = \mathcal{O}_C(-y-q)$. The analysis carried out in [BCF] Proposition 12, shows that π is simply ramified at each of these 2^{2g-2} Prym curves in $\pi^{-1}([C'])$. We denote by $\Delta_0^{\text{ram}} \subset \overline{\mathcal{R}}_g$ the closure of the locus of Prym curves $[C \cup_{\{y,q\}} E, \eta, \beta]$ and then Δ_0^{ram} is the ramification divisor of π. Moreover one has the relation,

$$\pi^*(\Delta_0) = \Delta_0' + \Delta_0'' + 2\Delta_0^{\text{ram}}.$$

It is easy to establish a dictionary between Prym curves and Beauville admissible covers. We explain how to do this in codimension 1 in $\overline{\mathcal{R}}_g$ (see also [D2] Example 1.9). The general point of Δ_0' corresponds to an étale double cover $[\tilde{C} \stackrel{f}{\to} C] \in \mathcal{R}_{g-1}$ induced by η_C. We denote by $y_i, q_i (i=1,2)$ the points lying in $f^{-1}(y)$ and $f^{-1}(q)$ respectively. Then

$$\overline{\mathcal{M}}_{2g-1} \ni \frac{\tilde{C}}{y_1 \sim q_1, y_2 \sim q_2} \longrightarrow \frac{C}{y \sim q} \in \overline{\mathcal{M}}_g$$

is a admissible double cover, defined up to a sign. This ambiguity is then resolved in the choice of an element in $\text{Ker}\{\nu^* : \text{Pic}^0(C_{yq})[2] \to \text{Pic}^0(C)[2]\}$.

If $[C/y \sim q, \eta, \beta]$ is a general point of Δ_0'', then we take identical copies $[C_1, y_1, q_1]$ and $[C_2, y_2, q_2]$ of $[C, y, q] \in \mathcal{M}_{g-1,2}$. The Wirtinger cover is obtained by taking

$$\overline{\mathcal{M}}_{2g-1} \ni \frac{C_1 \cup C_2}{y_1 \sim q_2, y_2 \sim q_1} \longrightarrow \frac{C}{y \sim q} \in \overline{\mathcal{M}}_g.$$

If $[C \cup_{\{y,q\}} E, \eta, \beta] \in \Delta_0^{\text{ram}}$, then $\eta_C \in \sqrt{\mathcal{O}_C(-y-q)}$ induces a $2:1$ cover $\tilde{C} \stackrel{f}{\to} C$ branched over y and q. We set $\{\tilde{y}\} := f^{-1}(y), \{\tilde{q}\} := f^{-1}(q)$. The Beauville cover is

$$\overline{\mathcal{M}}_{2g-1} \ni \frac{\tilde{C}}{\tilde{y} \sim \tilde{q}} \longrightarrow \frac{C}{y \sim q} \in \overline{\mathcal{M}}_g.$$

As usual, one denotes by $\delta_0', \delta_0'', \delta_0^{\text{ram}}, \delta_i, \delta_{g-i}, \delta_{i:g-i} \in \text{Pic}(\overline{\mathbf{R}}_g)$ the stacky divisor classes corresponding to the boundary divisors of $\overline{\mathcal{R}}_g$. We also set $\lambda := \pi^*(\lambda) \in \text{Pic}(\overline{\mathbf{R}}_g)$. Next we determine the canonical class $K_{\overline{\mathcal{R}}_g}$:

Theorem 1.5. *One has the following formula in* $\text{Pic}(\overline{\mathbf{R}}_g)$*:*

$$K_{\overline{\mathcal{R}}_g} = 13\lambda - 2(\delta_0' + \delta_0'') - 3\delta_0^{\text{ram}} - 2\sum_{i=1}^{[g/2]}(\delta_i + \delta_{g-i} + \delta_{i:g-i}) - (\delta_1 + \delta_{g-1} + \delta_{1:g-1}).$$

Proof. We use that $K_{\overline{\mathcal{M}}_g} \equiv 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{[g/2]}$ (cf. [HM]), together with the Hurwitz formula for the cover $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$. We find that $K_{\overline{\mathcal{R}}_g} = \pi^*(K_{\overline{\mathcal{M}}_g}) + \delta_0^{\text{ram}}$. □

Using this formula as well as the Appendix, we conclude that in order to prove that $\overline{\mathcal{R}}_g$ is of general type for a certain g, it suffices to exhibit a single effective divisor

$$D \equiv a\lambda - b_0'\delta_0' - b_0^{''}\delta_0^{''} - b_0^{\mathrm{ram}}\delta_0^{\mathrm{ram}} - \sum_{i=1}^{[g/2]}(b_i\delta_i + b_{g-i}\delta_{g-i} + b_{i:g-i}\delta_{i:g-i}) \in \mathrm{Eff}(\overline{\mathcal{R}}_g),$$

satisfying the following inequalities:

$$\max\Big\{\frac{a}{b_0'}, \frac{a}{b_0^{''}}\Big\} < \frac{13}{2}, \quad \max\Big\{\frac{a}{b_0^{\mathrm{ram}}}, \frac{a}{b_1}, \frac{a}{b_{g-1}}, \frac{a}{b_{1:g-1}}\Big\} < \frac{13}{3} \tag{2}$$

and

$$\max_{i\geq 1}\Big\{\frac{a}{b_i}, \frac{a}{b_{g-i}}, \frac{a}{b_{i:g-i}}\Big\} < \frac{13}{2}.$$

1.1. **The universal Prym curve.** We start by introducing the partial compactification $\widetilde{\mathcal{M}}_g := \mathcal{M}_g \cup \widetilde{\Delta}_0$ of $\mathcal{M}_g$, obtaining by adding to $\mathcal{M}_g$ the locus $\widetilde{\Delta}_0 \subset \overline{\mathcal{M}}_g$ of one-nodal irreducible curves $[C_{yq} := C/y \sim q]$, where $[C, y, q] \in \mathcal{M}_{g-1,2}$. Let $p : \widetilde{\mathbf{M}}_{g,1} \to \widetilde{\mathbf{M}}_g$ denote the universal curve. We denote $\widetilde{\mathcal{R}}_g := \pi^{-1}(\widetilde{\mathcal{M}}_g) \subset \overline{\mathcal{R}}_g$ and note that the boundary divisors $\widetilde{\Delta}_0' := \Delta_0' \cap \widetilde{\mathcal{R}}_g$, $\widetilde{\Delta}_0^{''} := \Delta_0^{''} \cap \widetilde{\mathcal{R}}_g$ and $\widetilde{\Delta}_0^{\mathrm{ram}} := \Delta_0^{\mathrm{ram}} \cap \widetilde{\mathcal{R}}_g$ become disjoint inside $\widetilde{\mathcal{R}}_g$. Finally, we set $\mathcal{Z} := \widetilde{\mathbf{R}}_g \times_{\widetilde{\mathbf{M}}_g} \widetilde{\mathbf{M}}_{g,1}$ and denote by $p_1 : \mathcal{Z} \to \widetilde{\mathbf{R}}_g$ the projection.

To obtain the universal family of Prym curves over $\widetilde{\mathbf{R}}_g$, we blow-up the codimension 2 locus $V \subset \mathcal{Z}$ corresponding to points

$$v = \big([C \cup_{\{y,q\}} E, \eta_C \in \sqrt{\mathcal{O}_C(-y-q)}],\ \eta_E = \mathcal{O}_E(1),\ \nu(y) = \nu(q)\big) \in \Delta_0^{\mathrm{ram}} \times_{\widetilde{\mathbf{M}}_g} \widetilde{\mathbf{M}}_{g,1}$$

(recall that $\nu : C \to C_{yq}$ denotes the normalization map). Suppose that $(t_1, \ldots, t_{3g-3})$ are local coordinates in an étale neighbourhood of $[C \cup_{\{y,q\}} E, \eta_C, \eta_E] \in \widetilde{\mathcal{R}}_g$ such that the local equation of Δ_0^{ram} is $(t_1 = 0)$. Then $\mathcal{Z}$ around v admits local coordinates $(x, y, t_1, \ldots, t_{3g-3})$ satisfying the equation $xy = t_1^2$. In particular, $\mathcal{Z}$ is singular along V. We denote by $\mathcal{X} := \mathrm{Bl}_V(\mathcal{Z})$ and by $f : \mathcal{X} \to \widetilde{\mathbf{R}}_g$ the induced family of Prym curves. Then for every $[X, \eta, \beta] \in \widetilde{\mathcal{R}}_g$ we have that $f^{-1}([X, \eta, \beta]) = X$.

On $\mathcal{X}$ there exists a Prym line bundle $\mathcal{P} \in \mathrm{Pic}(\mathcal{X})$ as well as a morphism of $\mathcal{O}_X$-modules $B : \mathcal{P}^{\otimes 2} \to \mathcal{O}_{\mathcal{X}}$ with the property that $\mathcal{P}_{|f^{-1}([X,\eta,\beta])} = \eta$ and $B_{|f^{-1}([X,\eta,\beta])} = \beta : \eta^{\otimes 2} \to \mathcal{O}_X$, for all points $[X, \eta, \beta] \in \widetilde{\mathcal{R}}_g$ (see e.g. [C], the same argument carries over from the spin to the Prym moduli space).

We set $\mathcal{E}_0'$, $\mathcal{E}_0^{''}$ and $\mathcal{E}_0^{\mathrm{ram}} \subset \mathcal{X}$ to be the proper transforms of the boundary divisors $p_1^{-1}(\widetilde{\Delta}_0'), p_1^{-1}(\widetilde{\Delta}_0^{''})$ and $p_1^{-1}(\widetilde{\Delta}_0^{\mathrm{ram}})$ respectively. Finally, we define $\mathcal{E}_0$ to be the exceptional divisor of the blow-up map $\mathcal{X} \to \mathcal{Z}$.

We recall that $g : \mathcal{Y} \to S$ is a family of nodal curves and L, M are line bundles on $\mathcal{Y}$, then $\langle L, M\rangle \in \mathrm{Pic}(S)$ denotes the bilinear *Deligne pairing* of L and M.

Proposition 1.6. *If $f : \mathcal{X} \to \widetilde{\mathbf{R}}_g$ is the universal Prym curve and $\mathcal{P} \in \mathrm{Pic}(\mathcal{X})$ is the corresponding Prym bundle, then one has the following relations in $\mathrm{Pic}(\widetilde{\mathbf{R}}_g)$:*

(i) $\langle \omega_f, \mathcal{P}\rangle = 0$.

(ii) $\langle \mathcal{O}_{\mathcal{X}}(\mathcal{E}_0), \mathcal{O}_{\mathcal{X}}(\mathcal{E}_0)\rangle = -2\delta_0^{\mathrm{ram}}$.
(iii) $\langle \mathcal{O}_{\mathcal{X}}(\mathcal{P}), \mathcal{O}_{\mathcal{X}}(\mathcal{P})\rangle = -\delta_0^{\mathrm{ram}}/2$.

Proof. The sheaf homomorphism $B : \mathcal{P}^{\otimes 2} \to \mathcal{O}_{\mathcal{X}}$ vanishes (with order 1) precisely along the exceptional divisor $\mathcal{E}_0$, hence $[\mathcal{E}_0] = -2c_1(\mathcal{P})$. Furthermore, we have the relations $f^*(\Delta_0^{\mathrm{ram}}) = \mathcal{E}_0^{\mathrm{ram}} + \mathcal{E}_0$ and $f_*([\mathcal{E}_0^{\mathrm{ram}}]\cdot[\mathcal{E}_0]) = 2\delta_0^{\mathrm{ram}}$ (In the fibre $f^{-1}([C\cup_{\{y,q\}} E, \eta_C])$ the divisors $\mathcal{E}_0$ and $\mathcal{E}_0^{\mathrm{ram}}$ meet over two points, corresponding to whether the marked points equals y or q. Now (ii) and (iii) follow simply from the push-pull formula. For (i), it is enough to show that $\omega_{f|\mathcal{E}_0}$ is the trivial bundle. This follows because for any point $[X,\eta,\beta]\in\widetilde{\mathcal{R}}_g$ we have that $\omega_X\otimes\mathcal{O}_E = 0$, for any exceptional component $E\subset X$. □

We now fix $i \geq 1$ and set $\mathcal{N}_i := f_*(\omega_f^{\otimes i}\otimes\mathcal{P}^{\otimes i})$. Since $R^1 f_*(\omega_f^{\otimes i}\otimes\mathcal{P}^{\otimes i}) = 0$, Grauert's theorem implies that $\mathcal{N}_i$ is a vector bundle over $\widetilde{\mathbf{R}}_g$ of rank $(g-1)(2i-1)$.

Proposition 1.7. *For each integer $i\geq 1$ the following formula in $\mathrm{Pic}(\widetilde{\mathbf{R}}_g)$ holds:*

$$c_1(\mathcal{N}_i) = \binom{i}{2}(12\lambda - \delta_0' - \delta_0'' - 2\delta_0^{\mathrm{ram}}) + \lambda - \frac{i^2}{4}\delta_0^{\mathrm{ram}}.$$

Proof. We apply Grothendieck-Riemann-Roch to the universal Prym curve $f:\mathcal{X}\to\widetilde{\mathbf{R}}_g$:

$$c_1(\mathcal{N}_i) = f_*\Big[\Big(1 + ic_1(\omega_f\otimes\mathcal{P}) + \frac{i^2c_1^2(\omega_f\otimes\mathcal{P})}{2}\Big)\Big(1 - \frac{c_1(\omega_f)}{2} + \frac{c_1^2(\omega_f)+[\mathrm{Sing}(f)]}{12}\Big)\Big]_2,$$

and then use Proposition 1.6 and Mumford's formula $(\kappa_1)_{\widetilde{\mathbf{R}}_g} = 12\lambda - \delta_0' - \delta_0'' - 2\delta_0^{\mathrm{ram}}$. □

1.2. **Inequalities between coefficients of divisors on $\overline{\mathcal{R}}_g$.** We use pencils of curves on $K3$ surfaces to establish certain inequalities between the coefficients of effective divisors on $\overline{\mathcal{R}}_g$. Using $K3$ surfaces we construct pencils that fill up the boundary divisors Δ_i, Δ_{g-i} and $\Delta_{i:g-i}$ for $1\leq i\leq [g/2]$ when $g\leq 23$. The use of such pencils in the context of $\overline{\mathcal{M}}_g$ has already been demonstrated in [FP].

We start with a Lefschetz pencil $B\subset\overline{\mathcal{M}}_i$ of curves of genus i lying on a fixed $K3$ surface S. The pencil B is induced by a family $f:\mathrm{Bl}_{i^2}(S)\to\mathbf{P}^1$ which has i^2 sections corresponding to the base points and we choose one such section σ. Using B, for each $g\geq i+1$ we create a genus g pencil $B_i\subset\overline{\mathcal{M}}_g$ of stable curves, by gluing a fixed curve $[C_2,p]\in\mathcal{M}_{g-i,1}$ along the section σ to each member of the pencil B. Then we have the following formulas on $\overline{\mathcal{M}}_g$ (cf. [FP] Lemma 2.4):

$$B_i\cdot\lambda = i+1,\ B_i\cdot\delta_0 = 6i+18,\ B_i\cdot\delta_i = -1 \text{ and } B_i\cdot\delta_j = 0 \text{ for } j\neq i.$$

We fix $1\leq i\leq[g/2]$ and lift B_i in three different ways to pencils in $\overline{\mathcal{R}}_g$. First we choose a non-trivial line bundle $\eta_2\in\mathrm{Pic}^0(C_2)[2]$. Let us denote by $A_{g-i}\subset\Delta_{g-i}\subset\overline{\mathcal{R}}_g$ the pencil of Prym curves $[C_2\cup_{\sigma(\lambda)} f^{-1}(\lambda),\ \eta_{C_2}=\eta_2,\ \eta_{f^{-1}(\lambda)}=\mathcal{O}_{f^{-1}(\lambda)}]$, with $\lambda\in\mathbf{P}^1$.

Next, we denote by $A_i\subset\Delta_i\subset\overline{\mathcal{R}}_g$ the pencil consisting of Prym curves

$$\big[C_2\cup_{\sigma(\lambda)} f^{-1}(\lambda),\ \eta_{C_2}=\mathcal{O}_{C_2},\ \eta_{f^{-1}(\lambda)}\in\overline{\mathrm{Pic}}^0(f^{-1}(\lambda))[2]\big],\ \text{ where } \lambda\in\mathbf{P}^1.$$

Clearly $\pi(A_i) = B_i$ and $\deg(A_i/B_i) = (2^{2i} - 1)$. Finally, $A_{i:g-i} \subset \Delta_{i:g-i} \subset \overline{\mathcal{R}}_g$ denotes the pencil of Prym curves $\left[C_2 \cup f^{-1}(\lambda), \eta_{C_2} = \eta_2,\ \eta_{f^{-1}(\lambda)} \in \overline{\mathrm{Pic}}^0(f^{-1}(\lambda))[2]\right]$. Again, we have that $\deg(A_{i:g-i}/B_i) = 2^{2i} - 1$.

Lemma 1.8. *If A_i, A_{g-i} and $A_{i:g-i}$ are pencils defined above, we have the following relations:*

- $A_{g-i}\cdot\lambda = i+1,\ A_{g-i}\cdot\delta_0' = 6i+18,\ A_{g-i}\cdot\delta_i = A_{g-i}\cdot\delta_0^{\mathrm{ram}} = 0,$ *and* $A_{g-i}\cdot\delta_{g-i} = -1.$
- $A_i \cdot \lambda = (i+1)(2^{2i}-1),\ A_i \cdot \delta_0' = (2^{2i-1}-2)(6i+18),\ \ A_i \cdot \delta_0'' = 6i+18,$
  $A_i \cdot \delta_0^{\mathrm{ram}} = 2^{2i-2}(6i+18)$ *and* $A_i \cdot \delta_i = -(2^{2i}-1).$
- $A_{i:g-i} \cdot \lambda = (i+1)(2^{2i}-1),\ A_{i:g-i} \cdot \delta_0' = (2^{2i-1}-1)(6i+18),$
  $A_{i:g-i} \cdot \delta_0^{\mathrm{ram}} = 2^{2i-2}(6i+18),\ A_{i:g-i} \cdot \delta_0'' = 0$ *and* $A_{i:g-i} \cdot \delta_{i:g-i} = -(2^{2i}-1).$

Note that all these intersections are computed on $\overline{\mathcal{R}}_g$. The intersection numbers of A_i, A_{g-i} and $A_{i:g-i}$ with the generators of $\mathrm{Pic}(\overline{\mathcal{R}}_g)$ not explicitly mentioned in Lemma 1.8 are all equal to 0.

Proof. We treat in detail only the case of A_i the other cases being similar. Using [FP] we find that $(A_i \cdot \lambda)_{\overline{\mathcal{R}}_g} = (\pi_*(A_i) \cdot \lambda)_{\overline{\mathcal{M}}_g} = (2^{2i}-1)(B_i \cdot \lambda)_{\overline{\mathcal{M}}_g}$. Furthermore, since $A_i \cap \Delta_{g-i} = A_i \cap \Delta_{i:g-i} = \emptyset$, we can write the formulas

$$(A_i \cdot \delta_i)_{\overline{\mathcal{R}}_g} = \big(A_i \cdot \pi^*(\delta_i)\big)_{\overline{\mathcal{R}}_g} = (2^{2i}-1)(B_i \cdot \delta_i)_{\overline{\mathcal{M}}_g}.$$

Clearly $(A_i \cdot \delta_0'')_{\overline{\mathcal{R}}_g} = (B_i \cdot \delta_0)_{\overline{\mathcal{M}}_g} = 6i+18$, whereas the intersection $A_i \cdot \delta_0'$ corresponds to choosing an element in $\mathrm{Pic}^0(f^{-1}(\lambda))[2]$, where $f^{-1}(\lambda)$ is a singular member of B. There are $2(2^{2i-2}-1)(6i+18)$ such choices. □

Proposition 1.9. *Let $D \equiv a\lambda - b_0'\delta_0' - b_0''\delta_0'' - b_0^{\mathrm{ram}}\delta_0^{\mathrm{ram}} - \sum_{i=1}^{[g/2]}(b_i\delta_i + b_{g-i}\delta_{g-i} + b_{i:g-i}\delta_{i:g-i}) \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$ be the closure in $\overline{\mathcal{R}}_g$ of an effective divisor in $\mathcal{R}_g$. Then if $1 \leq i \leq \min\{[g/2], 11\}$, we have the following inequalities:*

(1) $a(i+1) - b_0'(6i+18) + b_{g-i} \geq 0.$

(2) $a(i+1) - b_0^{\mathrm{ram}}(6i+18)\frac{2^{2i-2}}{2^{2i}-1} - b_0'(6i+18)\frac{2^{2i-1}-1}{2^{2i}-1} + b_{i:g-i} \geq 0.$

(3) $a(i+1) - b_0^{\mathrm{ram}}(6i+18)\frac{2^{2i-2}}{2^{2i}-1} - b_0'\,(6i+18)\frac{2^{2i-1}-2}{2^{2i}-1} - b_0''(6i+18)\frac{1}{2^{2i}-1} + b_i \geq 0.$

Proof. We use that that in this range the pencils A_i, A_{g-i} and $A_{i:g-i}$ fill-up the boundary divisors Δ_i, Δ_{g-i} and $\Delta_{i:g-i}$ respectively, hence $A_i \cdot D,\ A_{g-i} \cdot D,\ A_{i:g-i} \cdot D \geq 0$. □

Proof of Theorem 0.8. We lift the Lefschetz pencil $B \subset \overline{\mathcal{M}}_g$ corresponding to a fixed $K3$ surface, to a pencil $\tilde{B} \subset \overline{\mathcal{R}}_g$ of Prym curves by taking Prym curves $\tilde{B} := \{[C_\lambda, \eta_{C_\lambda}] \in \overline{\mathcal{R}}_g : [C_\lambda] \in B, \eta_{C_\lambda} \in \overline{\mathrm{Pic}}^0(C_\lambda)[2]\}$. We have the following formulas

$\tilde{B}\cdot\lambda = (2^{2g}-1)(g+1), \tilde{B}\cdot\delta_0' = (2^{2g-1}-2)(6g+18),\ \tilde{B}\cdot\delta_0'' = 6g+18,\ \tilde{B}\cdot\delta_0^{\mathrm{ram}} = 2^{2g-2}(6g+18).$

Furthermore, $\tilde{B}$ is disjoint from all the remaining boundary classes of $\overline{\mathcal{R}}_g$. One now verifies that $\tilde{B} \cdot K_{\overline{\mathcal{R}}_g} < 0$ precisely when $g \leq 7$. Since $\tilde{B}$ is a covering curve for $\overline{\mathcal{R}}_g$ in the range $g \leq 11, g \neq 10$, we find that $\kappa(\overline{\mathcal{R}}_g) = -\infty$. □

2. Theta divisors for vector bundles and geometric loci in $\overline{\mathcal{R}}_g$

We present a general method of constructing geometric divisors on $\overline{\mathcal{R}}_g$. For a fixed point $[C,\eta]\in\mathcal{R}_g$ we shall study the relative position of $\eta\in \mathrm{Pic}^0(C)[2]$ with respect to certain pluri-theta divisors on $\mathrm{Pic}^0(C)$.

We start by fixing a smooth curve C. If $E\in U_C(r,d)$ is a semistable vector bundle on C of integer slope $\mu(E):=d/r\in\mathbb{Z}$, then following Raynaud [R], we introduce the determinantal cycle

$$\Theta_E:=\{\eta\in \mathrm{Pic}^{g-\mu-1}(C): H^0(C,E\otimes\eta)\neq 0\}.$$

Either $\Theta_E=\mathrm{Pic}^{g-\mu-1}(C)$, or else, Θ_E is a divisor on $\mathrm{Pic}^{g-\mu-1}(C)$ and then $\Theta_E\equiv r\cdot\theta$. In the latter case we say that Θ_E is the *theta divisor* of the vector bundle E. Clearly, Θ_E is a divisor if and only if $H^0(C,E\otimes\eta)=0$, for a general bundle $\eta\in\mathrm{Pic}^{g-\mu-1}(C)$.

Let us now fix a globally generated line bundle $L\in\mathrm{Pic}^d(C)$ such that $h^0(C,L)=r+1$. The *Lazarsfeld vector bundle* M_L of L is defined using the exact sequence on C

$$0\longrightarrow M_L\longrightarrow H^0(C,L)\otimes\mathcal{O}_C\longrightarrow L\longrightarrow 0$$

(see also [GL], [L], [Vo], [F1], [FMP] for many applications of these bundles). It is customary to denote $Q_L:=M_L^{\vee}$, hence $\mu(Q_L)=d/r$. When $L=K_C$, one writes $Q_C:=Q_{K_C}$. The vector bundles Q_L (and all its exterior powers) are semistable under mild genericity assumptions on C (see [L] or [F1] Proposition 2.1). In the case $\mu(\wedge^i Q_L)=g-1$, when we expect $\Theta_{\wedge^i Q_L}$ to be a divisor on $\mathrm{Pic}^0(C)$, we may ask whether for a given point $[C,\eta]\in\mathcal{R}_g$ the condition $\eta\in\Theta_{\wedge^i Q_L}$ is satisfied or not. Throughout this section we denote by $\mathfrak{G}^r_d\to\mathcal{M}_g$ the Deligne-Mumford stack parameterizing pairs $[C,l]$, where $[C]\in\mathcal{M}_g$ and $l=(L,V)\in G^r_d(C)$ is a linear series of type $\mathfrak{g}^r_d$.

We fix integers $k\geq 2$ and $b\geq 0$. We set integers $i:=kb+k-b-2$,

$$r:=kb+k-2,\ g:=k(kb+k-b-2)+1=ik+1\ \text{ and } d:=k(kb+k-2).$$

Since $\rho(g,r,d)=0$, a general curve $[C]\in\mathcal{M}_g$ carries a finite number of (obviously complete) linear series $l\in G^r_d(C)$. We denote this number by

$$N:=g!\frac{1!\,2!\,\cdots r!}{(k-1)!\,\cdots\,(k-1+r)!}=\deg(\mathfrak{G}^r_d/\mathcal{M}_g).$$

We also note that we can write $g=(r+1)(k-1)$ and $d=rk$, and moreover, each line bundle $L\in W^r_d(C)$ satisfies $h^1(C,L)=k-1$. Furthermore, we compute $\mu(\wedge^i Q_L)=ik=g-1$ and then we introduce the following virtual divisor on $\mathcal{R}_g$:

$$\mathcal{D}_{g:k}:=\{[C,\eta]\in\mathcal{R}_g:\exists L\in W^r_d(C)\ \text{ such that }\ h^0(C,\wedge^i Q_L\otimes\eta)\geq 1\}.$$

¿From the definition it follows that $\mathcal{D}_{g:k}$ is either pure of codimension 1 in $\mathcal{R}_g$, or else $\mathcal{D}_{g:k}=\mathcal{R}_g$. We shall prove that the second possibility does not occur.

For $[C,\eta]\in\mathcal{R}_g$ and $L\in W^r_d(C)$ one has the following exact sequence on C

$$0\longrightarrow\wedge^i M_L\otimes K_C\otimes\eta\longrightarrow\wedge^i H^0(C,L)\otimes K_C\otimes\eta\longrightarrow\wedge^{i-1}M_L\otimes L\otimes K_C\otimes\eta\longrightarrow 0,$$

from which, using Serre duality, one derives the following equivalences:

$$[C,\eta]\in\mathcal{D}_{g:k}\Leftrightarrow h^1(C,\wedge^i M_L\otimes K_C\otimes\eta)\geq 1\Leftrightarrow$$

(3) $\wedge^i H^0(C,L)\otimes H^0(C,K_C\otimes\eta)\to H^0(C,\wedge^{i-1}M_L\otimes L\otimes K_C\otimes\eta)$ is not an isomorphism.

Note that obviously $\operatorname{rank}\big(\wedge^i H^0(C,L)\otimes H^0(C,K_C\otimes\eta)\big)=\binom{r+1}{i}(g-1)$, while

$$h^0(C,\wedge^{i-1}M_L\otimes L\otimes K_C\otimes\eta)=\chi(C,\wedge^{i-1}M_L\otimes L\otimes K_C\otimes\eta)=$$
$$=\binom{r}{i-1}\big(-k(i-1)+d+g-1\big)=\binom{r+1}{i}(g-1)$$

(use that M_L is a semistable vector bundle and that $\mu(\wedge^{i-1}M_L\otimes L\otimes K_C\otimes\eta)>2g-1$).

Remark 2.1. As pointed out in the Introduction, an important particular case is $k=2$, when $i=b, g=2i+1, r=2i, d=4i=2g-2$. Since $W^{g-1}_{2g-2}(C)=\{K_C\}$, it follows that $[C,\eta]\in\mathcal{D}_{2i+1,2}\Leftrightarrow\eta\in\Theta_{\wedge^i Q_C}$. The main result from [FMP] states that for any $[C]\in\mathcal{M}_g$ the Raynaud locus $\Theta_{\wedge^i Q_C}$ is a divisor in $\mathrm{Pic}^0(C)$ (that is, $\wedge^i Q_C$ has a theta divisor) and we have an equality of cycles

$$\Theta_{\wedge^i Q_C}=C_i-C_i\subset\mathrm{Pic}^0(C),\tag{4}$$

where the right-hand-side denotes the *i-th difference variety* of C, that is, the image of the difference map

$$\phi: C_i\times C_i\to\mathrm{Pic}^0(C),\ \ \phi(D,E):=\mathcal{O}_C(D-E).$$

Using Lazarsfeld's filtration argument [L] Lemma 1.4.1, one finds that for a generic choice of distinct points $x_1,\ldots,x_{g-2}\in C$, there is an exact sequence

$$0\longrightarrow\oplus_{l=1}^{g-2}\mathcal{O}_C(x_l)\longrightarrow Q_C\longrightarrow K_C\otimes\mathcal{O}_C(-x_1-\cdots-x_{g-2})\longrightarrow 0,$$

which implies the inclusion $C_i-C_i\subset\Theta_{\wedge^i Q_C}$. The importance of (4) is that it shows that $\Theta_{\wedge^i Q_C}$ is a divisor on $\mathrm{Pic}^0(C)$, that is, $H^0(C,\wedge^i Q_C\otimes\eta)=0$ for a generic $\eta\in\mathrm{Pic}^0(C)$.

Theorem 2.2. *For every genus $g=2i+1$ we have the following identification of cycles on $\mathcal{R}_g$:*

$$\mathcal{D}_{2i+1:2}:=\{[C,\eta]\in\mathcal{R}_g:\eta\in C_i-C_i\}.$$

Next we prove that $\mathcal{D}_{g:k}$ is an actual divisor on $\mathcal{R}_g$ for any $k\geq 2$ and we achieve this by specialization to the k-gonal locus $\mathcal{M}^1_{g,k}$ in $\mathcal{M}_g$.

Theorem 2.3. *Fix $k\geq 2, b\geq 1$ and g,r,d,i defined as above. Then $\mathcal{D}_{g:k}$ is a divisor on $\mathcal{R}_g$. Precisely, for a generic $[C,\eta]\in\mathcal{R}_g$ we have that $H^0(C,\wedge^i Q_L\otimes\eta)=0$, for every $L\in W^r_d(C)$.*

Proof. Since there is a unique irreducible component of $\mathfrak{G}^r_d$ mapping dominantly onto $\mathcal{M}_g$, to prove that $\mathcal{D}_{g:k}$ is a divisor it suffices to exhibit a single element $[C,L,\eta]\in\mathfrak{G}^r_d$ such that (1) the Petri map

$$\mu_0(C,L): H^0(C,L)\otimes H^0(C,K_C\otimes L^\vee)\to H^0(C,K_C)$$

is an isomorphism, and (2) for each point $\eta\in\mathrm{Pic}^0(C)[2]$, we have that $\eta\notin\Theta_{\wedge^i Q_L}$.

Proposition 2.1.1 from [CM] ensures that for a generic k-gonal curve $[C,A]\in\mathfrak{G}^1_k$ of genus $g=(r+1)(k-1)$ one has that $h^0(C,A^{\otimes j})=j+1$ for $1\leq j\leq r+1$. In particular there is an isomorphism $\mathrm{Sym}^j H^0(C,A)\cong H^0(C,A^{\otimes j})$. Using this and Riemann-Roch, we obtain that $h^0(C,K_C\otimes A^{\otimes(-j)})=(k-1)(r+1-j)$ for $0\leq j\leq r+1$. Thus there is a generically injective rational map $\mathfrak{G}^1_k\dashrightarrow\mathfrak{G}^r_d$ given by $[C,A]\mapsto[C,A^{\otimes r}]$ (The use of such a map has been first pointed out to me in a different context by S. Keel). We claim that $\mathfrak{G}^1_k$ maps into the "main component" of $\mathfrak{G}^r_d$ which maps dominantly onto $\overline{\mathcal{M}}_g$. To prove this it suffices to check that the Petri map

$$\mu_0(C, A^{\otimes r}) : H^0(C, A^{\otimes r}) \otimes H^0(C, K_C \otimes A^{\otimes(-r)}) \to H^0(C, K_C)$$

is an isomorphism (Remember that $H^0(C, A^{\otimes r}) \cong \mathrm{Sym}^r H^0(C, A)$). We use the base point free pencil trick to write down the exact sequence

$$0 \longrightarrow H^0(K_C \otimes A^{\otimes -(j+1)}) \longrightarrow H^0(A) \otimes H^0(K_C \otimes A^{\otimes(-j)}) \stackrel{\mu_j(A)}{\longrightarrow} H^0(K_C \otimes A^{\otimes -(j-1)}).$$

One can now easily check that $\mu_j(A)$ is surjective for $1 \leq j \leq r$ by using the formulas $h^0(C, K_C \otimes A^{\otimes(-j)}) = (k-1)(r+1-j)$ valid for $0 \leq j \leq r+1$. This in turns implies that $\mu_0(C, A^{\otimes r})$ is surjective, hence an isomorphism.

We now check condition (2) and note that for $[C, L = A^{\otimes r}] \in \mathfrak{G}^r_d$, the Lazarsfeld bundle splits as $Q_L \cong A^{\oplus r}$. In particular, $\wedge^i Q_L \cong \oplus_{\binom{r}{i}} A^{\otimes i}$, hence the condition $H^0(C, \wedge^i Q_L \otimes \eta) \neq 0$ is equivalent to $H^0(C, A^{\otimes i} \otimes \eta) \neq 0$, that is, the translate of the theta divisor $W_{g-1}(C) - A^{\otimes i} \subset \mathrm{Pic}^0(C)$ cannot contain any point of order 2 on $\mathrm{Pic}^0(C)$. Using that the moduli space of triples $[C, A, \eta]$, where $[C, A] \in \mathfrak{G}^1_k$ and $\eta \in \mathrm{Pic}^0(C)[2]$ is irreducible for each $k \geq 3$, it suffices to prove the statement for a single such triple.

We assume by contradiction that for *any* $[C, A] \in \mathfrak{G}^1_k$ and *any* $\eta \in \mathrm{Pic}^0(C)[2]$, we have that $H^0(C, A^{\otimes i} \otimes \eta) \geq 1$. We specialize C to a hyperelliptic curve and choose $A = \mathfrak{g}^1_2 \otimes \mathcal{O}_C(x_1 + \cdots + x_{k-2})$, with $x_1, \ldots, x_{k-2} \in C$ being general points. Finally we take $\eta := \mathcal{O}_C(p_1 + \cdots + p_{i+1} - q_1 - \cdots - q_{i+1}) \in \mathrm{Pic}^0(C)[2]$, with $p_1, \ldots, p_{i+1}, q_1, \ldots, q_{i+1}$ being distinct ramification points of the hyperelliptic $\mathfrak{g}^1_2$. It is now straightforward to check that $H^0(C, A^{\otimes i} \otimes \eta) = 0$. $\square$

In order to compute the class $[\overline{\mathcal{D}}_{g:k}] \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$ we extend the determinantal description of $\mathcal{D}_{g:k}$ to the boundary of $\overline{\mathcal{R}}_g$. We start by setting some notation. We denote by $\mathbf{M}^0_g \subset \mathbf{M}_g$ the open substack classifying curves $[C] \in \mathcal{M}_g$ such that $W^r_{d-1}(C) = \emptyset$ and $W^{r+1}_d(C) = \emptyset$. We know that $\mathrm{codim}(\mathcal{M}_g - \mathcal{M}^0_g, \mathcal{M}_g) \geq 2$. We further denote by $\Delta^0_0 \subset \Delta_0 \subset \overline{\mathcal{M}}_g$ the locus of curves $[C/y \sim q]$ where $[C] \in \mathcal{M}_{g-1}$ is a curve that satisfies the Brill-Noether theorem and where $y, q \in C$ are arbitrary points. Note that every Brill-Noether general curve $[C] \in \mathcal{M}_{g-1}$ satisfies

$$W^r_{d-1}(C) = \emptyset, \;\; W^{r+1}_d(C) = \emptyset \text{ and } \dim W^r_d(C) = \rho(g-1, r, d) = r.$$

We set $\overline{\mathbf{M}}^0_g := \mathbf{M}^0_g \cup \Delta^0_0 \subset \overline{\mathbf{M}}_g$. Then we consider the Deligne-Mumford stack

$$\sigma_0 : \mathfrak{G}^r_d \to \overline{\mathbf{M}}^0_g$$

classifying pairs $[C, L]$ with $[C] \in \overline{\mathcal{M}}^0_g$ and $L \in G^r_d(C)$ (cf. [EH], [F2], [Kh] -note that it is essential that $\rho(g, r, d) = 0$. At the moment there is no known extension of this stack over the entire $\overline{\mathbf{M}}_g$). We remark that for any curve $[C] \in \overline{\mathcal{M}}^0_g$ and $L \in W^r_d(C)$ we have that $h^0(C, L) = r+1$, that is, $\mathfrak{G}^r_d$ parameterizes only complete linear series. Indeed, for a smooth curve $[C] \in \mathcal{M}^0_g$ we have that $W^{r+1}_d(C) = \emptyset$, so necessarily $W^r_d(C) = G^r_d(C)$. For a point $[C_{yq} := C/y \sim q] \in \Delta^0_0$ we have the identification

$$\sigma_0^{-1}[C_{yq}] = \{L \in W^r_d(C) : h^0(C, L \otimes \mathcal{O}_C(-y-q)) = r\},$$

where we note that since the normalization $[C] \in \mathcal{M}_{g-1}$ is assumed to be Brill-Noether general, any sheaf $L \in \sigma_0^{-1}[C_{yq}]$ satisfies $h^0(C, L \otimes \mathcal{O}_C(-y)) = h^0(C, L \otimes \mathcal{O}_C(-q)) = r$ and $h^0(C, L) = r+1$. Furthermore, $\sigma_0 : \mathfrak{G}^r_d \to \overline{\mathbf{M}}_g^0$ is proper, which is to say that $\overline{W}^r_d(C_{yq}) = W^r_d(C_{yq})$, where the left-hand-side denotes the closure of $W^r_d(C_{yq})$ in the variety $\overline{\mathrm{Pic}}^d(C_{yq})$ of torsion-free sheaves on C_{yq}. This follows because a non-locally free torsion-free sheaf in $\overline{W}^r_d(C_{yq}) - W^r_d(C_{yq})$ is of the form $\nu_*(A)$, where $A \in W^r_{d-1}(C)$ and $\nu : C \to C_{yq}$ is the normalization map. But we know that $W^r_{d-1}(C) = \emptyset$, because $[C] \in \mathcal{M}_{g-1}$ satisfies the Brill-Noether theorem. Since $\rho(g, r, d) = 0$, by general Brill-Noether theory, there exists a unique irreducible component of $\mathfrak{G}^r_d$ which maps onto $\overline{\mathbf{M}}_g^0$. It is certainly not the case that $\mathfrak{G}^r_d$ is irreducible, unless $k \leq 3$, when either $\mathfrak{G}^r_d = \mathbf{M}_g$ ($k = 2$), or $\mathfrak{G}^r_d$ is isomorphic to a Hurwitz stack ($k = 3$). We denote by $f^r_d : \mathfrak{C}^r_{g,d} := \overline{\mathbf{M}}_{g,1}^0 \times_{\overline{\mathbf{M}}_g^0} \mathfrak{G}^r_d \to \mathfrak{G}^r_d$ the pull-back of the universal curve $\overline{\mathbf{M}}_{g,1}^0 \to \overline{\mathbf{M}}_g^0$ to $\mathfrak{G}^r_d$. Once we have chosen a Poincaré bundle $\mathcal{L}$ on $\mathfrak{C}^r_{g,d}$ we can form the three codimension 1 tautological classes in $A^1(\mathfrak{G}^r_d)$:

(5)

$$\mathfrak{a} := (f^r_d)_*\big(c_1(\mathcal{L})^2\big),\ \mathfrak{b} := (f^r_d)_*\big(c_1(\mathcal{L}) \cdot c_1(\omega_{f^r_d})\big),\ \mathfrak{c} := (f^r_d)_*\big(c_1(\omega_{f^r_d})^2\big) = (\sigma_0)^*\big((\kappa_1)_{\overline{\mathbf{M}}_g^0}\big).$$

These classes depend on the choice of $\mathcal{L}$ and behave functorially with respect to base change, see also Remark 2.7 on the precise statement regarding the choice of $\mathcal{L}$. We set $\overline{\mathbf{R}}_g^0 := \pi^{-1}(\widetilde{\mathbf{M}}_g^0) \subset \widetilde{\mathbf{R}}_g$ and introduce the stack of $\mathfrak{g}^r_d$'s on Prym curves

$$\sigma : \mathfrak{G}^r_d(\widetilde{\mathbf{R}}_g^0/\widetilde{\mathbf{M}}_g^0) := \overline{\mathbf{R}}_g^0 \times_{\overline{\mathbf{M}}_g^0} \mathfrak{G}^r_d \to \overline{\mathbf{R}}_g^0.$$

By a slight abuse of notation we denote the boundary divisors by the same symbols, that is, $\Delta_0' := \sigma^*(\Delta_0'), \Delta_0'' := \sigma^*(\Delta_0'')$ and $\Delta_0^{\mathrm{ram}} := \sigma^*(\Delta_0^{\mathrm{ram}})$. Finally, we introduce the universal curve over the stack of $\mathfrak{g}^r_d$'s on Prym curves:

$$f' : \mathcal{X}^r_d := \mathcal{X} \times_{\overline{\mathbf{R}}_g^0} \mathfrak{G}^r_d(\overline{\mathbf{R}}_g^0/\overline{\mathbf{M}}_g^0) \to \mathfrak{G}^r_d(\overline{\mathbf{R}}_g^0/\overline{\mathbf{M}}_g^0).$$

On $\mathcal{X}^r_d$ there are two tautological line bundles, the universal Prym bundle $\mathcal{P}^r_d$ which is the pull-back of $\mathcal{P} \in \mathrm{Pic}(\mathcal{X})$ under the projection $\mathcal{X}^r_d \to \mathcal{X}$, and a Poincaré bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}^r_d)$ characterized by the property $\mathcal{L}_{|f'^{-1}[X,\eta,\beta,L]} = L \in W^r_d(C)$, for each point $[X, \eta, \beta, L] \in \mathfrak{G}^r_d(\overline{\mathcal{R}}_g^0/\overline{\mathcal{M}}_g^0)$. Note that we also have the codimension 1 classes $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A^1(\mathfrak{G}^r_d(\overline{\mathbf{R}}_g^0/\overline{\mathbf{M}}_g^0))$ defined by the formulas (5).

Proposition 2.4. *Let C be a curve of genus g and let $L \in W^r_d(C)$ be a globally generated complete linear series. Then for any integer $0 \leq j \leq r$ and for any line bundle $A \in \mathrm{Pic}^a(C)$ such that $a \geq 2g + d - r + j - 1$, we have that $H^1(C, \wedge^j M_L \otimes A) = 0$.*

Proof. We use a filtration argument due to Lazarsfeld [L]. Having fixed L and A, we choose general points $x_1, \ldots, x_{r-1} \in C$ such that $h^0\big(C, L \otimes \mathcal{O}_C(-x_1 - \cdots - x_{r-1})\big) = 2$ and then there is an exact sequence on C

$$0 \longrightarrow L^\vee(x_1 + \cdots + x_{r-1}) \longrightarrow M_L \longrightarrow \oplus_{l=1}^{r-1} \mathcal{O}_C(-x_l) \longrightarrow 0.$$

Taking the j-th exterior powers and tensoring the resulting exact sequence with A, we find that in order to conclude that $H^1(C, \wedge^i M_L \otimes A) = 0$ for $i \leq r$, it suffices to show that for $1 \leq i \leq r$ the following hold:

(1) $H^1\big(C, A\otimes \mathcal{O}_C(-D_j)\big)=0$ for each effective divisor $D_j\in C_j$ with support in the set $\{x_1,\ldots,x_{r-1}\}$, and

(2) $H^1\big(C, A\otimes L^\vee\otimes \mathcal{O}_C(D_{r-j})\big)=0$, for any effective divisor $D_{r-j}\in C_{r-j}$ with support contained in $\{x_1,\ldots,x_{r-1}\}$.

Both (1) and (2) hold for degree reasons since $\deg(C, A\otimes\mathcal{O}_C(-D_j))\geq 2g-1$ and $\deg(C, A\otimes L^\vee\otimes\mathcal{O}_C(D_{r-j})\geq 2g-1$ and the points $x_1,\ldots,x_{r-1}\in C$ are general. $\square$

Next we use Proposition 2.4 to prove a vanishing result for Prym curves.

Proposition 2.5. *For each point $[X,\eta,\beta,L]\in\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$ and $0\leq a\leq i-1$, we have that*

$$H^1(X,\wedge^a M_L\otimes L^{\otimes(i-a)}\otimes\omega_X\otimes\eta)=0.$$

Proof. If X is smooth, then the vanishing follows directly from Proposition 2.4. Assume now that $[X,\eta,\beta]\in\Delta_0'\cup\Delta_0''$, that is, $st(X)=X$ and $\eta\in\mathrm{Pic}^0(X)[2]$. As usual, we denote by $\nu: C\to X$ the normalization map, and $L_C:=\nu^*(L)\in W^r_d(C)$ satisfies $h^0(C, L_C\otimes\mathcal{O}_C(-y-q))=r$, hence $H^0(X,L)\cong H^0(C,L_C)$, which implies that $\nu^*(M_L)=M_{L_C}$. Tensoring the usual exact sequence on X

$$0\longrightarrow\mathcal{O}_X\longrightarrow\nu_*\mathcal{O}_C\longrightarrow\nu_*\mathcal{O}_C/\mathcal{O}_X\longrightarrow 0,$$

by the line bundle $\wedge^a M_L\otimes L^{(i-a)}\otimes\omega_X\otimes\eta$, we find that a sufficient condition for the vanishing $H^1(X,\wedge^a M_L\otimes L^{\otimes(i-a)}\otimes\omega_X\otimes\eta)=0$ to hold, is to show that

$$H^1(C,\wedge^a M_{L_C}\otimes L_C^{\otimes(i-a)}\otimes K_C\otimes\eta_C)=H^1(C,\wedge^a M_{L_C}\otimes L_C^{\otimes(i-a)}\otimes K_C(y+q)\otimes\eta_C)=0.$$

Since $i<r$, this follows directly from Proposition 2.4.

We are left with the case when $[X,\eta,\beta]\in\Delta_0^{\mathrm{ram}}$, when $X:=C\cup_{\{q,y\}}E$, with E being a smooth rational curve, $L_C\in W^r_d(C), L_E=\mathcal{O}_E$ and $\eta_C^{\otimes 2}=\mathcal{O}_C(-y-q)$. We also have that $M_{L|C}=M_{L_C}$ and $M_{L|E}=H^0(C,L_C\otimes\mathcal{O}_C(-y-q))\otimes\mathcal{O}_E$. A standard argument involving the Mayer-Vietoris sequence on X shows that the vanishing of the group $H^1(X,\wedge^a M_L\otimes L^{\otimes(i-a)}\otimes\omega_X\otimes\eta)$ is implied by the following vanishing conditions

$$H^1(C,\wedge^a M_{L_C}\otimes L_C^{\otimes(i-a)}\otimes K_C(y+q)\otimes\eta_C)=H^1(C,\wedge^a M_{L_C}\otimes L_C^{\otimes(i-a)}\otimes K_C\otimes\eta_C)=0.$$

The conditions of Proposition 2.4 being satisfied $(i\leq r-1)$, we finish the proof. $\square$

Proposition 2.5 enables us to define a sequence of tautological vector bundles on $\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$: First, we set $\mathcal{H}:=f'_*(\mathcal{L})$. By Grauert's theorem, it follows that $\mathcal{H}$ is a vector bundle of rank $r+1$ with fibre $\mathcal{H}[X,\eta,\beta,L]=H^0(X,L)$. For $j\geq 0$ we set

$$\mathcal{A}_{0,j}:=f'_*(\mathcal{L}^{\otimes j}\otimes\omega_{f'}\otimes\mathcal{P}^r_d).$$

Since $R^1f'_*(\mathcal{L}^{\otimes j}\otimes\omega_{f'}\otimes\mathcal{P}^r_d)=0$ we find that $\mathcal{A}_{0,j}$ is a vector bundle over $\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$ of rank equal to $h^0(X,L^{\otimes j}\otimes\omega_X\otimes\eta)=jd+g-1$. Next we introduce the global Lazarsfeld vector bundle $\mathcal{M}$ over $\mathcal{X}^r_d$ by the exact sequence

$$0\longrightarrow\mathcal{M}\longrightarrow f'^*(\mathcal{H})\longrightarrow\mathcal{L}\longrightarrow 0,$$

hence $\mathcal{M}_{f'^{-1}[X,\eta,\beta,L]} = M_L$ for each $[X,\eta,\beta,L] \in \mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$. Then for integers $a, j \geq 1$ we define the sheaf

$$\mathcal{A}_{a,j} := f'_*(\wedge^a \mathcal{M} \otimes \mathcal{L}^{\otimes j} \otimes \omega_{f'} \otimes \mathcal{P}^r_d).$$

For each $1 \leq a \leq i-1$, we have proved that $R^1 f'_*(\wedge^a \mathcal{M} \otimes \mathcal{L}^{\otimes(i-a)} \otimes \omega_{f'} \otimes \mathcal{P}^r_d) = 0$ (cf. Proposition 2.5), therefore $\mathcal{A}_{a,i-a}$ is a vector bundle over $\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$ having rank

$$\mathrm{rk}(\mathcal{A}_{a,i-a}) = \chi\big(X, \wedge^a M_L \otimes L^{\otimes(i-a)} \otimes \omega_X \otimes \eta\big) = \binom{r}{a} k(i-a)(r+1).$$

Proposition 2.5 also shows that for all integers $1 \leq a \leq i-1$, the vector bundles $\mathcal{A}_{a,i-a}$ sit in exact sequences

$$(6) \qquad 0 \longrightarrow \mathcal{A}_{a,i-a} \longrightarrow \wedge^a \mathcal{H} \otimes \mathcal{A}_{0,i-a} \longrightarrow \mathcal{A}_{a-1,i-a+1} \longrightarrow 0.$$

We shall need the expression for the Chern numbers of $\mathcal{A}_{a,i-a}$. Using (6) it will be sufficient to compute $c_1(\mathcal{A}_{0,j})$ for all $j \geq 0$.

Proposition 2.6. *For all $j \geq 0$ one has the following formula in $A^1(\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g))$:*

$$c_1(\mathcal{A}_{0,j}) = \lambda + \frac{j}{2}B + \frac{j^2}{2}A - \frac{1}{4}\delta_0^{\mathrm{ram}}.$$

Proof. We apply Grothendieck-Riemann-Roch to the morphism $f' : \mathcal{X}^r_d \to \mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$:

$$c_1(\mathcal{A}_{0,j}) = c_1\big(f'_!(\omega_{f'} \otimes \mathcal{L}^{\otimes j} \otimes \mathcal{P}^r_d)\big) =$$

$$= f'_*\Big[\Big(1+c_1(\omega_{f'}\otimes\mathcal{L}^{\otimes j}\otimes\mathcal{P}^r_d)+\frac{c_1^2(\omega_{f'} \otimes \mathcal{L}^{\otimes j} \otimes \mathcal{P}^r_d)}{2}\Big)\Big(1-\frac{c_1(\omega_{f'})}{2}+\frac{c_1^2(\omega_{f'}) + [\mathrm{Sing}(f')]}{12}\Big)\Big]_2,$$

where $\mathrm{Sing}(f') \subset \mathcal{X}^r_d$ denotes the codimension 2 singular locus of the morphism f', therefore $f'_*[\mathrm{Sing}(f')] = \Delta_0' + \Delta_0'' + 2\Delta_0^{\mathrm{ram}}$. We finish the proof using Mumford's formula $\kappa_1 = f'_*(c_1^2(\omega_{f'})) = 12\lambda - (\delta_0' + \delta_0'' + 2\delta_0^{\mathrm{ram}})$ and noting that $f'_*(c_1(\mathcal{L}) \cdot c_1(\mathcal{P}^r_d)) = 0$ (the restriction of $\mathcal{L}$ to the exceptional divisor of $f' : \mathcal{X}^r_d \to \mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$ is trivial) and $f'_*(c_1(\omega_{f'}) \cdot c_1(\mathcal{P}^r_d)) = 0$,. Finally, according to Proposition 1.6 we have that $f'_*(c_1^2(\mathcal{P}^r_d)) = -\delta_0^{\mathrm{ram}}/2$. □

Remark 2.7. While the construction of the vector bundles $\mathcal{A}_{a,j}$ depends on the choice of the Poincaré bundle $\mathcal{L}$ and that of the Prym bundle $\mathcal{P}^r_d$, it is easy to check that if we set the vector bundles $\mathcal{A} := \wedge^i \mathcal{H} \otimes \mathcal{A}_{0,0}$ and $\mathcal{B} := \mathcal{A}_{i-1,i}$, then the vector bundle $Hom(\mathcal{A}, \mathcal{B})$ on $\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$, as well as the morphism

$$\phi \in H^0\big(\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g), Hom(\mathcal{A}, \mathcal{B})\big)$$

whose degeneracy locus is the virtual divisor $\overline{\mathcal{D}}_{g:k}$, are independent of such choices. More precisely, let us denote by Ξ the collection of triples $\alpha := \big(\pi_\alpha, \mathcal{L}_\alpha, (\mathcal{P}^r_d)_\alpha\big)$, where $\pi_\alpha : \Sigma_\alpha \to \mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$ is an étale surjective morphism from a scheme Σ_α, $(\mathcal{P}^r_d)_\alpha$ is a Prym bundle and $\mathcal{L}_\alpha$ is a Poincaré bundle on $p_{2,\alpha} : \mathcal{X}^r_d \times_{\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)} \Sigma_\alpha \to \Sigma_\alpha$. Recall that if $\Sigma \to \mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$ is an étale surjection from a scheme and $\mathcal{L}$ and $\mathcal{L}'$ are two Poincaré bundles on $p_2 : \mathcal{X}^r_d \times_{\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)} \Sigma \to \Sigma$, then the sheaf $\mathcal{N} := p_{2*}Hom(\mathcal{L}, \mathcal{L}')$ is invertible and there is a canonical isomorphism $\mathcal{L} \otimes p_2^* \mathcal{N} \cong \mathcal{L}'$. For every $\alpha \in \Xi$ we construct the morphism between vector bundles of the same rank $\phi_\alpha : \mathcal{A}_\alpha \to \mathcal{B}_\alpha$ as above. Then since a straightforward cocycle condition is met, we find that there exists a vector bundle $Hom(\mathcal{A}, \mathcal{B})$ on $\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$ together with a section $\phi \in H^0(\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g), Hom(\mathcal{A}, \mathcal{B}))$ such that for every $\alpha = (\pi_\alpha, \mathcal{L}_\alpha, (\mathcal{P}^r_d)_\alpha) \in \Xi$ we have that

$$\pi^*_\alpha(Hom(\mathcal{A}, \mathcal{B})) = Hom(\mathcal{A}_\alpha, \mathcal{B}_\alpha) \text{ and } \pi^*_\alpha(\phi) = \phi_\alpha.$$

We are finally in a position to compute the class of the divisor $\overline{\mathcal{D}}_{g:k}$.

Theorem 2.8. *We fix integers $k \geq 2, b \geq 0$ and set*

$$i := kb - b + k - 2,\ r := kb + k - 2,\ g := ik + 1, d := rk$$

as above. Then there exists a morphism $\phi : \wedge^i \mathcal{H} \otimes \mathcal{A}_{0,0} \to \mathcal{A}_{i-1,1}$ between vector bundles of the same rank over $\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)$, such that the push-forward under σ of the restriction to $\mathfrak{G}^r_d(\boldsymbol{R}^0_g/\boldsymbol{M}^0_g)$ of the degeneration locus of ϕ is precisely the effective divisor $\mathcal{D}_{g:k}$. Moreover we have the following expression for its class in $A^1(\overline{\boldsymbol{R}}^0_g)$:

$$\sigma_*\big(c_1(\mathcal{A}_{i-1,1} - \wedge^i\mathcal{H}\otimes\mathcal{A}_{0,0})\big) \equiv \binom{r}{b}\frac{N}{(r+k)(kr+k-r-3)}\Big(\mathfrak{A}\lambda - \frac{\mathfrak{B}_0}{6}(\delta_0' + \delta_0'') - \frac{\mathfrak{B}_0^{\mathrm{ram}}}{12}\delta_0^{\mathrm{ram}}\Big),$$

where

$$\mathfrak{A} = (k^5 - 4k^4 + 5k^3 - 2k^2)b^3 + (3k^5 - 13k^4 + 24k^3 - 23k^2 + 9k)b^2 +$$
$$+ (3k^5 - 14k^4 + 34k^3 - 45k^2 + 24k - 4)b + k^5 - 5k^4 + 15k^3 - 25k^2 + 16k - 2,$$
$$\mathfrak{B}_0 = (k^5 - 4k^4 + 5k^3 - 2k^2)b^3 + (3k^5 - 13k^4 + 22k^3 - 17k^2 + 5k)b^2 +$$
$$+ (3k^5 - 14k^4 + 30k^3 - 33k^2 + 14k - 2)b + k^5 - 5k^4 + 13k^3 - 19k^2 + 10k$$

and

$$\mathfrak{B}_0^{\mathrm{ram}} = (4k^5 - 16k^4 + 20k^3 - 8k^2)b^3 + (12k^5 - 52k^4 + 85k^3 - 65k^2 + 20k)b^2 +$$
$$+ (12k^5 - 56k^4 + 111k^3 - 114k^2 + 53k - 8)b + 4k^5 - 20k^4 + 46k^3 - 58k^2 + 34k - 6.$$

Proof. To compute the class of the degeneracy locus of ϕ we use the exact sequence (6) and Proposition 2.6. We write the following identities in $A^1(\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g))$:

$$c_1\big(\mathcal{A}_{i-1,1} - \wedge^i\mathcal{H}\otimes\mathcal{A}_{0,0}\big) = \sum_{l=0}^{i}(-1)^{l-1}c_1(\wedge^{i-l}\mathcal{H}\otimes\mathcal{A}_{0,l}) =$$

$$= \sum_{l=0}^{i}(-1)^{l-1}\Big((ld+g-1)\binom{r}{i-l-1}c_1(\mathcal{H}) + \binom{r+1}{i-l}c_1(\mathcal{A}_{0,l})\Big) =$$

$$= -k\binom{kb+k-4}{b-1}c_1(\mathcal{H}) + \frac{1}{2}\binom{kb+k-3}{b}\,\mathfrak{b} -$$

$$-\binom{kb+k-2}{b}\lambda - \frac{kb+k-2b-3}{2(kb+k-3)}\binom{kb+k-3}{b}\,\mathfrak{a} + \frac{1}{4}\binom{kb+k-2}{b}\delta_0^{\mathrm{ram}} =$$

$$= \binom{r-1}{b}\Big(-\frac{kb}{r-1}\,c_1(\mathcal{H}) + \frac{1}{2}\,\mathfrak{b} - \frac{r-2b-1}{2(r-1)}\,\mathfrak{a} - \frac{r}{r-b}\,\lambda + \frac{r}{4(r-b)}\delta_0^{\mathrm{ram}}\Big),$$

where $\delta_0^{\mathrm{ram}} = \sigma^*(\delta_0^{\mathrm{ram}}) \in A^1(\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g))$. The classes $\mathfrak{a}, \mathfrak{b} \in A^1(\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g))$ and the line bundle $\mathcal{H} \in \mathrm{Pic}\big(\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)\big)$ are defined in terms of a Poincaré bundle $\mathcal{L}$: If

$\mathcal{L}' := \mathcal{L} \otimes f'^*(\mathcal{M})$ is another Poincaré bundle with $\mathcal{M} \in \mathrm{Pic}\big(\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)\big)$ and if $\mathfrak{a}', \mathfrak{b}', \mathcal{H}'$ denote the classes defined in terms of $\mathcal{L}'$ using (5), then we have formulas:

$$\mathfrak{a}' = \mathfrak{a} + 2dc_1(\mathcal{M}),\ \mathfrak{b}' = \mathfrak{b} + (2g-2)c_1(\mathcal{M}) \text{ and } c_1(\mathcal{H}') = c_1(\mathcal{H}) + (r+1)c_1(\mathcal{M}).$$

A straightforward calculation shows that the class

$$\Xi := -\frac{kb}{r-1}\, c_1(\mathcal{H}) + \frac{1}{2}\, \mathfrak{b} - \frac{r-2b-1}{2(r-1)}\, \mathfrak{a} \in A^1(\mathfrak{G}^r_d(\overline{\mathbf{R}}^0_g/\overline{\mathbf{M}}^0_g)) \tag{7}$$

is independent of the choice of $\mathcal{L}$ and $\sigma_*(\Xi) = \pi^*\big((\sigma_0)_*(\Xi_0)\big)$, where the $\Xi_0 \in A^1(\mathfrak{G}^r_d)$ is defined by the same formula (7) but inside $\mathrm{Pic}(\mathfrak{G}^r_d)$. We outline below the computation of $\pi^*\big((\sigma_0)_*(\Xi_0)\big)$ which uses [F2] in an essential way.

We follow closely [F2] and denote by $\overline{\mathbf{M}}^1_g := \mathbf{M}^0_g \cup \Delta^0_0 \cup \Delta^0_1$ the partial compactification of $\mathbf{M}^0_g$ obtained from $\overline{\mathbf{M}}^0_g$ by adding the stack $\Delta^0_1 \subset \Delta_1$ consisting of curves $[C \cup_y E]$, where $[C, y] \in \mathcal{M}_{g-1,1}$ is a Brill-Noether general pointed curve and $[E, y] \in \overline{\mathcal{M}}_{1,1}$. We extend $\sigma_0 : \mathfrak{G}^r_d \to \overline{\mathbf{M}}^0_g$ to a proper map $\sigma_1 : \widetilde{\mathfrak{G}}^r_d \to \overline{\mathbf{M}}^1_g$ from the Deligne-Mumford stack of limit linear series $\mathfrak{g}^r_d$ (cf. [EH], [F2], [Kh]). Then for each $n \geq 1$ we consider the vector bundles $\mathcal{G}_{0,n}$ over $\widetilde{\mathfrak{G}}^r_d$ defined in [F2] Proposition 2.8 and which has the following description of its fibres:

- $\mathcal{G}_{0,n}(C, L) = H^0(C, L^{\otimes n})$, for each $[C] \in \mathcal{M}^0_g$ and $L \in W^r_d(C)$.
- $\mathcal{G}_{0,n}(t) = H^0\big(C, L^{\otimes n}(-y-q)\big) \oplus \mathbb{C} \cdot u^n \subset H^0(C, L^{\otimes n})$, where the point $t = \big(C_{yq},\ L \in W^r_d(C)\big) \in \sigma_0^{-1}([C_{yq}])$, with $u \in H^0(C, L)$ being a section such that
$$H^0(C, L) = H^0(C, L(-y-q)) \oplus \mathbb{C} \cdot u.$$
- $\mathcal{G}_{0,n}(t) = H^0(C, L^{\otimes n}(-2y)) \oplus \mathbb{C} \cdot u^n \subset H^0(C, L^{\otimes n})$, where $t = \big(C \cup_y E, l_C, l_E\big) \in \sigma_0^{-1}([C \cup_y E])$ and $(l_C, l_E) \in G^r_d(C) \times G^r_d(E)$ being a limit linear series $\mathfrak{g}^r_d$ with $l_C = (L, H^0(C, L))$ and $u \in H^0(C, L)$ a section such that
$$H^0(C, L) = H^0(C, L(-2y)) \oplus \mathbb{C} \cdot u.$$

We extend the classes $\mathfrak{a}, \mathfrak{b} \in A^1(\mathfrak{G}^r_d)$ over the stack $\widetilde{\mathfrak{G}}^r_d$ by choosing a Poincaré bundle over $\overline{\mathbf{M}}^1_{g,1} \times_{\overline{\mathbf{M}}^1_g} \widetilde{\mathfrak{G}}^r_d$ which restricts to line bundles of bidegree $(d, 0)$ on curves $[C \cup_y E] \in \Delta^0_1$. Grothendieck-Riemann-Roch applied to the universal curve over $\widetilde{\mathfrak{G}}^r_d$ gives that

$$c_1(\mathcal{G}_{0,n}) = \lambda - \frac{n}{2}\mathfrak{b} + \frac{n^2}{2}\mathfrak{a} \in A^1(\widetilde{\mathfrak{G}}^r_d), \ \text{ for all } n \geq 2 \tag{8}$$

while obviously $\sigma^*(\mathcal{G}_{0,1}) = \mathcal{H}$. We now fix a general pointed curve $[C, q] \in \mathcal{M}_{g-1}$ and an elliptic curve $[E, y] \in \mathcal{M}_{1,1}$ and consider the test curves (see also [F2] p. 7)

$$C^0 := \{C/y \sim q\}_{y \in C} \subset \Delta^0_0 \subset \overline{\mathcal{M}}^1_g \ \text{ and } \ C^1 := \{C \cup_y E\}_{y \in C} \subset \Delta^0_1 \subset \overline{\mathcal{M}}^1_g.$$

For $n \geq 1$, the intersection numbers $C^0 \cdot (\sigma_0)_*(c_1(\mathcal{G}_{0,n}))$ and $C^1 \cdot (\sigma_0)_*(c_1(\mathcal{G}_{0,n}))$ can be computed using [F2] Lemmas 2.6 and 2.13 and Proposition 2.12. Together with the relation (cf. [F2] p. 15 for details)

$$(\sigma_0)_*\big(c_1(\mathcal{G}_{0,n})\big)_\lambda - 12(\sigma_0)_*\big(c_1(\mathcal{G}_{0,n})\big)_{\delta_0} + (\sigma_0)_*\big(c_1(\mathcal{G}_{0,n})\big)_{\delta_1} = 0,$$

this completely determine the classes $(\sigma_0)_*\big(c_1(\mathcal{G}_{0,n})\big) \in A^1(\widetilde{\mathfrak{G}}^r_d)$. Then using (8) we find

$$(\sigma_0)_*(\mathfrak{a}) \equiv N\Big(-\frac{rk(r^2k^2-3r^2k+3rk^2+2r^2+2k^2+4k-7rk-4r-10)}{(rk-r+k-3)(rk-r+k-2)}\lambda +$$

$$+\frac{rk(r^2k^2-3r^2k+3rk^2-8rk+2r^2+2k^2+r-k-3)}{6(rk-r+k-3)(rk-r+k-2)}\delta_0+\cdots\Big),$$

$$(\sigma_0)_*(\mathfrak{b}) \equiv N\Big(\frac{6rk}{rk-r+k-2}\lambda - \frac{rk}{2(rk-r+k-2)}\delta_0+\cdots\Big),$$

and this completes the computation of the class $(\sigma_0)_*(\Xi)$ and finishes the proof. $\square$

The rather unwieldy expressions from Theorem 2.8 simplify nicely when $k=2,3$ when we obtain Theorems 0.2 and 0.3.

Proof of Theorem 0.1 when $g=2i+1$. We construct an effective divisor on $\overline{\mathcal{R}}_g$ satisfying the inequalities (2) as follows: The pull-back to $\overline{\mathcal{R}}_g$ of the Harris-Mumford divisor $\overline{\mathcal{M}}^1_{g,i+1}$ of curves of genus $2i+1$ with a $\mathfrak{g}^1_{i+1}$ is given by the formula: $\pi^*(\overline{\mathcal{M}}^1_{g,i+1}) \equiv$

$$\equiv \frac{(2i-2)!}{(i+1)!(i-1)!}\Big(6(i+2)\lambda-(i+1)(\delta_0'+\delta_0''+2\delta_0^{\mathrm{ram}})-\sum_{j=1}^{i}3j(g-j)(\delta_j+\delta_{g-j}+\delta_{j:g-j})\Big).$$

We split $\overline{\mathcal{D}}_{2i+1:2}$ into boundary components of compact type and their complement

$$\overline{\mathcal{D}}_{2i+1:2} \equiv E+\sum_{j=1}^{i}\big(a_j\delta_j+a_{g-j}\delta_{g-j}+a_{j:g-j}\delta_{j:g-j}\big),$$

where $a_j, a_{g-j}, a_{j:g-j} \geq 0$ and $\Delta_j, \Delta_{g-j}, \Delta_{j:g-j} \subsetneq \mathrm{supp}(E)$ for $1 \leq j \leq i$, we consider the following positive linear combination on $\overline{\mathcal{R}}_g$:

$$A := \frac{i!\,(i-1)!}{(2i-1)\,(2i-3)!}\cdot\pi^*(\overline{\mathcal{M}}^1_{2i+1,i+1})+4\frac{(i!)^2}{(2i)!}\cdot E \equiv \frac{4(3i+5)}{i+1}\,\lambda-2(\delta_0'+\delta_0'')-3\delta_0^{\mathrm{ram}}-\cdots,$$

where each of the coefficients of δ_j, δ_{g-j} and $\delta_{j:g-j}$ in the expansion of A are at least

$$\frac{6(i-1)j(2i+1-j)}{(2i-1)(i+1)} \geq 2.$$

Since $\frac{4(3i+5)}{i+1} < 13$ for $i \geq 8$, the conclusion now follows using (2). For $i=7$ we find that $A \equiv 13\lambda-2(\delta_0'+\delta_0'')-3\delta_0^{\mathrm{ram}}-\cdots$, hence $\kappa(\overline{\mathcal{R}}_{15}) \geq 0$. To obtain that $\kappa(\overline{\mathcal{R}}_{15}) \geq 1$, we use the fact that on $\overline{\mathcal{M}}_{15}$ there exists a Brill-Noether divisor other than $\overline{\mathcal{M}}^1_{15,8}$, namely the divisor $\overline{\mathcal{M}}^3_{15,14}$ of curves $[C] \in \mathcal{M}_{15}$ with a $\mathfrak{g}^3_{14}$. This divisor has the same slope $s(\overline{\mathcal{M}}^3_{15,14}) = s(\overline{\mathcal{M}}^1_{15,8}) = 27/4$, but $\mathrm{supp}(\overline{\mathcal{M}}^3_{15,14}) \neq \mathrm{supp}(\overline{\mathcal{M}}^1_{15,8})$. It follows that there exist constants $\alpha, \beta, \gamma, m \in \mathbb{Q}_{>0}$ such that

$$\alpha\cdot E+\beta\cdot\pi^*(\overline{\mathcal{M}}^1_{15,8}) \equiv \alpha\cdot E+\gamma\cdot\pi^*(\overline{\mathcal{M}}^3_{15,14}) \in |mK_{\overline{\mathcal{R}}_{15}}|.$$

Thus we have found distinct multicanonical divisors on $\overline{\mathcal{M}}_{15}$, that is, $\kappa(\overline{\mathcal{M}}_{15}) \geq 1$. $\square$

Remark 2.9. The same numerical argument shows that if one replaces $\overline{\mathcal{M}}^1_{15,8}$ with any divisor $D \in \mathrm{Eff}(\overline{\mathcal{M}}_{15})$ with $s(D) < s(\overline{\mathcal{M}}^1_{15,8}) = 27/4$, then $\overline{\mathcal{R}}_{15}$ is of general type. Any counterexample to the Slope Conjecture on $\overline{\mathcal{M}}_{15}$ makes $\overline{\mathcal{R}}_{15}$ of general type.

3. Koszul cohomology of Prym canonical curves

We recall that for a curve C, a line bundle $L \in \mathrm{Pic}^d(C)$ and integers $i, j \geq 0$, the Koszul cohomology group $K_{i,j}(C, L)$ is obtained from the complex

$$\wedge^{i+1} H^0(L) \otimes H^0(L^{\otimes(j-1)}) \stackrel{d_{i+1,j-1}}{\longrightarrow} \wedge^i H^0(L) \otimes H^0(L^{\otimes j}) \stackrel{d_{i,j}}{\longrightarrow} \wedge^{i-1} H^0(L) \otimes H^0(L^{\otimes(j+1)}),$$

where the maps are the Koszul differentials (cf. [GL]). There is a well-known connection between Koszul cohomology groups and Lazarsfeld bundles. Assuming that L is globally generated, a diagram chasing argument involving exact sequences of the type

$$0 \longrightarrow \wedge^a M_L \otimes L^{\otimes b} \to \wedge^a H^0(L) \otimes L^{\otimes b} \longrightarrow \wedge^{a-1} M_L \otimes L^{\otimes(b+1)} \longrightarrow 0$$

for various $a, b \geq 0$, yields the following identification (see also [GL] Lemma 1.10)

$$K_{i,j}(C, L) = \frac{H^0(C, \wedge^i M_L \otimes L^{\otimes j})}{\mathrm{Image}\{\wedge^{i+1} H^0(C, L) \otimes H^0(C, L^{\otimes(j-1)})\}}. \tag{9}$$

We fix $[C, \eta] \in \mathcal{R}_g$, set $L := K_C \otimes \eta \in W^{g-2}_{2g-2}(C)$ and consider the *Prym-canonical* map $C \stackrel{|L|}{\to} \mathbf{P}^{g-2}$. We denote by $\mathcal{I}_C \subset \mathcal{O}_{\mathbf{P}^{g-2}}$ the ideal sheaf of the Prym-canonical curve.

By analogy with [F2] we study the Koszul stratification of $\mathcal{R}_g$ and define the strata

$$\mathcal{U}_{g,i} := \{[C, \eta] \in \mathcal{R}_g : K_{i,2}(C, K_C \otimes \eta) \neq \emptyset\}.$$

Using (9) we write the series of equivalences

$$[C, \eta] \in \mathcal{U}_{g,i} \Leftrightarrow H^1(C, \wedge^{i+1} M_L \otimes L) \neq \emptyset \Leftrightarrow h^0(C, \wedge^{i+1} M_L \otimes L) >$$
$$> \binom{g-2}{i+1}\Big(-\frac{(i+1)(2g-2)}{g-2} + (g-1)\Big).$$

Next we write down the exact sequence

$$0 \longrightarrow H^0(\wedge^{i+1} M_{\mathbf{P}^{g-2}}(1)) \stackrel{a}{\longrightarrow} H^0(C, \wedge^{i+1} M_L \otimes L) \longrightarrow H^1(\wedge^{i+1} M_{\mathbf{P}^{g-2}} \otimes \mathcal{I}_C(1)) \longrightarrow 0,$$

and then also

$$\mathrm{Coker}(a) = H^1\big(\mathbf{P}^{g-2}, \wedge^{i+1} M_{\mathbf{P}^{g-2}} \otimes \mathcal{I}_C(1)\big) = H^0\big(\mathbf{P}^{g-2}, \wedge^i M_{\mathbf{P}^{g-2}} \otimes \mathcal{I}_C(2)\big).$$

Using the well-known fact that $h^0(\mathbf{P}^{g-2}, \wedge^{i+1} M_{\mathbf{P}^{g-2}}(1)) = \binom{g-1}{i+2}$ (use for instance the Bott vanishing theorem), we end-up with the following equivalence:

$$[C, \eta] \in \mathcal{U}_{g,i} \Leftrightarrow h^0(\mathbf{P}^{g-2}, \wedge^i M_{\mathbf{P}^{g-2}} \otimes \mathcal{I}_C(2)) > \binom{g-3}{i}\frac{(g-1)(g-2i-6)}{i+2}. \tag{10}$$

Proposition 3.1. *(1) For $g < 2i + 6$, we have that $K_{i,2}(C, K_C \otimes \eta) \neq \emptyset$ for any $[C, \eta] \in \mathcal{R}_g$, that is, the Prym-canonical curve $C \stackrel{|K_C+\eta|}{\longrightarrow} \mathbf{P}^{g-2}$ does not satisfy property (N_i).*

(2) For $g = 2i + 6$, the locus $\mathcal{U}_{g,i}$ is a virtual divisor on $\mathcal{R}_g$, that is, there exist vector bundles $\mathcal{G}_{i,2}$ and $\mathcal{H}_{i,2}$ over $\mathbf{R}_g$ such that $\mathrm{rank}(\mathcal{G}_{i,2}) = \mathrm{rank}(\mathcal{H}_{i,2})$, together with a bundle morphism $\phi : \mathcal{H}_{i,2} \to \mathcal{G}_{i,2}$ such that $\mathcal{U}_{g,i}$ is the degeneracy locus of ϕ.

Proof. Part (1) is an immediate consequence of (10), since we have the equivalence

$$K_{i,2}(C, K_C \otimes \eta) = 0 \Leftrightarrow h^0(\mathbf{P}^{g-2}, \wedge^i M_{\mathbf{P}^{g-2}} \otimes \mathcal{I}_C(2)) = \binom{g-3}{i} \frac{(g-1)(g-2i-6)}{i+2}.$$

For part (2) one constructs two vector bundles $\mathcal{G}_{i,2}$ and $\mathcal{H}_{i,2}$ over $\mathbf{R}_g$ having fibres

$$\mathcal{G}_{i,2}[C,\eta] = H^0(C, \wedge^i M_{K_C\otimes\eta}(2)) \text{ and } \mathcal{H}_{i,2}[C,\eta] = H^0(\mathbf{P}^{g-2}, \wedge^i M_{\mathbf{P}^{g-2}}(2)).$$

There is a natural morphism $\phi : \mathcal{H}_{i,2} \to \mathcal{G}_{i,2}$ given by restriction. We have that

$$\text{rank}(\mathcal{G}_{i,2}) = \binom{g-2}{i}\Big(-\frac{i(2g-2)}{g-2} + 3(g-1)\Big) \text{ and } \text{rank}(\mathcal{H}_{i,2}) = (i+1)\binom{g}{i+2}$$

and the condition that $\text{rank}(\mathcal{G}_{i,2}) = \text{rank}(\mathcal{H}_{i,2})$ is equivalent to $g = 2i+6$. □

We describe a set-up that will be used to define certain tautological sheaves over $\widetilde{\mathbf{R}}_g$ and compute the class $[\overline{\mathcal{U}}_{g,i}]^{virt}$. We use the notation from Subsection 1.1, in particular from Proposition 1.7 and recall that $f : \mathcal{X} \to \widetilde{\mathbf{R}}_g$ is the universal Prym curve, $\mathcal{P} \in \text{Pic}(\mathcal{X})$ denotes the universal Prym line bundle and $\mathcal{N}_i = f_*(\omega_f^{\otimes i} \otimes \mathcal{P}^{\otimes i})$. We denote by $T := \mathcal{E}_0'' \cap \text{Sing}(f)$ the codimension 2 subvariety corresponding to Wirtinger covers $[C_{yq}, \eta \in \text{Pic}^0(C_{yq})[2], \nu(y) = \nu(q)] \in \mathcal{X}$ (where $\nu^*(\eta) = \mathcal{O}_C$), with the marked point being the node of the underlying curve C_{yq}. Let us fix a point $[X := C_{yq}, \eta, \beta] \in \widetilde{\Delta}_0' \cup \widetilde{\Delta}_0''$ where as usual $\nu : C \to X$ is the normalization map. Then we have an identification

(11) $$\mathcal{N}_1[X,\eta,\beta] = \text{Ker}\big\{H^0\big(C, \omega_C(y+q) \otimes \eta_C\big) \to (\nu_*\mathcal{O}_C/\mathcal{O}_X) \otimes \omega_X \otimes \eta \cong \mathbb{C}_{y\sim q}\big\},$$

where the map is given by taking the difference of residues at y and q. Note that when $\eta_C = \mathcal{O}_C$, that is when$[X,\eta,\beta] \in \widetilde{\Delta}_0''$, we have that $\mathcal{N}_1[X,\eta,\beta] = H^0(C,\omega_C)$. For a point

$$[X = C \cup_{\{y,q\}} E,\ \eta_C \in \sqrt{\mathcal{O}_C(-y-q)}, \eta_E] \in \widetilde{\Delta}_0^{\text{ram}}$$

we have an identification

(12) $$\mathcal{N}_1[X,\eta,\beta] = \text{Ker}\big\{H^0(C, \omega_C(y+q) \otimes \eta_C) \oplus H^0(E, \mathcal{O}_E(1)) \to (\omega_X \otimes \eta)_{y,q} \cong \mathbb{C}^2_{y,q}\big\}.$$

We set

$$\mathcal{M} := \text{Ker}\{f^*(\mathcal{N}_1) \to \omega_f \otimes \mathcal{P}\}.$$

¿From the discussion above it is clear that the image of $f^*(\mathcal{N}_1) \to \omega_f \otimes \mathcal{P}$ is $\omega_f \otimes \mathcal{P} \otimes \mathcal{I}_T$. Since $T \subset \mathcal{X}$ is smooth of codimension 2 it follows that $\mathcal{M}$ is locally free. For $a, b \geq 0$, we define the sheaf $\mathcal{E}_{a,b} := f_*(\wedge^a \mathcal{M} \otimes \omega_f^{\otimes b} \otimes \mathcal{P}^{\otimes b})$ over $\widetilde{\mathbf{R}}_g$. Clearly $\mathcal{E}_{a,b}$ is locally free. We have that $\mathcal{E}_{0,b} = \mathcal{N}_b$ for $b \geq 0$, and we always have left-exact sequences

(13) $$0 \longrightarrow \mathcal{E}_{a,b} \longrightarrow \wedge^a \mathcal{E}_{0,1} \otimes \mathcal{E}_{0,b} \longrightarrow \mathcal{E}_{a-1,b+1},$$

which are right-exact off the divisor $\widetilde{\Delta}_0''$ (to be proved later). We then define inductively a sequence of vector bundles $\{\mathcal{H}_{a,b}\}_{a,b\geq 0}$ over $\widetilde{\mathbf{R}}_g$ in the following way: We set $\mathcal{H}_{0,b} := \text{Sym}^b \mathcal{N}_1$ for each $b \geq 0$. Then having defined $\mathcal{H}_{a-1,b}$ for all $b \geq 0$, we define the vector bundle $\mathcal{H}_{a,b}$ by the exact sequence

(14) $$0 \longrightarrow \mathcal{H}_{a,b} \longrightarrow \wedge^a \mathcal{H}_{0,1} \otimes \text{Sym}^b \mathcal{H}_{0,1} \longrightarrow \mathcal{H}_{a-1,b+1} \longrightarrow 0.$$

For a point $[X,\eta,\beta] \in \widetilde{\mathcal{R}}_g$, if we use the identification $H^0(X, \omega_X \otimes \eta) = H^0(\mathbf{P}^{g-2}, \mathcal{O}_{\mathbf{P}^{g-2}}(1))$, we have a natural identification of the fibre

$$\mathcal{H}_{a,b}[X,\eta,\beta] = H^0(\mathbf{P}^{g-2}, \wedge^a M_{\mathbf{P}^{g-2}}(b)).$$

By induction on $a \geq 0$, there exist vector bundle morphisms $\phi_{a,b} : \mathcal{H}_{a,b} \to \mathcal{E}_{a,b}$.

Proposition 3.2. *For $b \geq 2$ and $a \geq 0$ we have the vanishing of the higher direct images*

$$R^1 f_*\big(\wedge^a \mathcal{M} \otimes \omega_f^{\otimes b} \otimes \mathcal{P}^{\otimes b}\big)_{| \ \boldsymbol{R}_g \cup \widetilde{\Delta}_0' \cup \widetilde{\Delta}_0^{\mathrm{ram}}} = 0.$$

It follows that the sequences (13) are right-exact off the divisor $\widetilde{\Delta}_0''$ of $\widetilde{\mathbf{R}}_g$.

Proof. Over the locus $\mathbf{R}_g$ the vanishing is a consequence of Proposition 2.4. For simplicity we prove that $R^1 f_*\big(\wedge^a \mathcal{M} \otimes \omega_f^{\otimes b} \otimes \mathcal{P}^{\otimes b}\big) \otimes \mathcal{O}_{\widetilde{\Delta}_0^{\mathrm{ram}}} = 0$, the vanishing over $\widetilde{\Delta}_0'$ being similar. We fix a point $[X = C \cup_{\{y,q\}} E, \eta_C, \eta_E] \in \widetilde{\Delta}_0^{\mathrm{ram}}$, with $\eta_C^{\otimes 2} = \mathcal{O}_C(-y-q)$, $\eta_E = \mathcal{O}_E(1)$ and set $L := \omega_X \otimes \eta \in \mathrm{Pic}^{2g-2}(X)$. We show that $H^1(X, \wedge^a M_L \otimes L^{\otimes b}) = 0$ for all $a \geq 0$ and $b \geq 2$. A Mayer-Vietoris argument shows that it suffices to prove that

$$H^1(C, \wedge^a M_L \otimes L^{\otimes b} \otimes \mathcal{O}_C) = 0, \ \ H^1(E, \wedge^a M_L \otimes L^{\otimes b} \otimes \mathcal{O}_E) = 0, \text{ and} \tag{15}$$

$$H^1(C, \wedge^a M_L \otimes L^{\otimes b} \otimes \mathcal{O}_C(-y-q)) = 0. \tag{16}$$

For $L_C := L \otimes \mathcal{O}_C = K_C(y+q) \otimes \eta_C$ and $L_E := L_E \otimes \mathcal{O}_E$, we write the exact sequences

$$0 \longrightarrow H^0(C, L_C(-y-q)) \otimes \mathcal{O}_E \longrightarrow M_L \otimes \mathcal{O}_E \longrightarrow M_{L_E} \longrightarrow 0, \ \text{ and}$$

$$0 \longrightarrow H^0(E, L_E(-y-q)) \otimes \mathcal{O}_C \longrightarrow M_L \otimes \mathcal{O}_C \longrightarrow M_{L_C} \longrightarrow 0,$$

and we find that $M_L \otimes \mathcal{O}_C = M_{L_C}$ while obviously $M_{L_E} = \mathcal{O}_E(-1)$. We conclude that the statements (15) and (16) for all $a \geq 0$ and $b \geq 2$ can be reduced to showing that

$$H^1(C, \wedge^a M_{L_C} \otimes L_C^{\otimes b}) = H^1(C, \wedge^a M_{L_C} \otimes L_C^{\otimes b} \otimes \mathcal{O}_C(-y-q)) = 0, \text{ for all } a \geq 0, b \geq 2.$$

This is now a direct application of Proposition 2.4. □

Proof of Theorem 0.6. We have constructed the vector bundle morphism $\phi_{i,2} : \mathcal{H}_{i,2} \to \mathcal{E}_{i,2}$ over $\widetilde{\mathbf{R}}_g$. For $g = 2i+6$ we have that $\mathrm{rank}(\mathcal{H}_{i,2}) = \mathrm{rank}(\mathcal{E}_{i,2})$ and the virtual Koszul class $[\overline{\mathcal{U}}_{g,i}]^{virt}$ is given by $c_1(\mathcal{E}_{i,2} - \mathcal{H}_{i,2})$. We recall that for a rank e vector bundle $\mathcal{E}$ over a variety X and for $i \geq 1$, we have the formulas $c_1(\wedge^i \mathcal{E}) = \binom{e-1}{i-1} c_1(\mathcal{E})$ and $c_1(\mathrm{Sym}^i(\mathcal{E})) = \binom{e+i-1}{e} c_1(\mathcal{E})$. Using (13) we find that there exists a constant $\alpha \geq 0$ such that

$$c_1(\mathcal{E}_{i,2}) - \alpha \cdot \delta_0'' = \sum_{l=0}^{i} (-1)^l c_1(\wedge^{i-l} \mathcal{E}_{0,1} \otimes \mathcal{E}_{0,l+2}) = \sum_{l=0}^{i} (-1)^l \binom{g-1}{i-l} c_1(\mathcal{E}_{0,l+2}) +$$

$$+ \sum_{l=0}^{i} (-1)^l \big((g-1)(2l+3)\big) \binom{g-2}{i-l-1} c_1(\mathcal{E}_{0,1}),$$

while a repeated application of the exact sequence (14) gives that

$$c_1(\mathcal{H}_{i,2}) = \sum_{l=0}^{i} (-1)^l c_1(\wedge^{i-l} \mathcal{H}_{0,1} \otimes \mathrm{Sym}^{l+2} \mathcal{H}_{0,1}) =$$

$$= \sum_{l=0}^{i} (-1)^l \Big(\binom{g-1}{i-l} c_1(\mathrm{Sym}^{l+2}(\mathcal{H}_{0,1})) + \binom{g+l}{l+2} c_1(\wedge^{i-l} \mathcal{H}_{0,1}) \Big)$$

$$= \sum_{l=0}^{i} (-1)^l \Bigl(\binom{g-1}{i-l}\binom{g+l}{g-1} + \binom{g+l}{l+2}\binom{g-2}{i-l-1} \Bigr) c_1(\mathcal{H}_{0,1}),$$

with $\mathcal{E}_{0,1} = \mathcal{H}_{0,1} = \mathcal{N}_1$ and $\mathcal{E}_{0,l+2} = \mathcal{N}_{l+2}$ for $l \geq 0$. Proposition 1.7 finishes the proof. $\Box$

Comparing these formulas to the canonical class of $\overline{\mathcal{R}}_g$, one obtains that $\overline{\mathcal{R}}_g$ is of general type for $g > 12$.

4. EFFECTIVE DIVISORS ON $\overline{\mathcal{R}}_g$

We now use in an essential way results from [F3] to produce myriads of effective divisors on $\overline{\mathcal{R}}_g$. This construction, though less explicit than that of $\overline{\mathcal{U}}_{2i+6}$ and $\overline{\mathcal{D}}_{g:k}$, is still very effective and we use it to show $\overline{\mathcal{R}}_{18}, \overline{\mathcal{R}}_{20}$ and $\overline{\mathcal{R}}_{22}$ are of general type.

We consider the morphism $\chi : \overline{\mathcal{R}}_g \rightarrow \overline{\mathcal{M}}_{2g-1}$ given by $\chi([C,\eta]) := [\tilde{C}]$, where $f : \tilde{C} \rightarrow C$ is the étale double cover determined by η. Thus one has

$$f_*\mathcal{O}_{\tilde{C}} = \mathcal{O}_C \oplus \eta \ \text{ and } \ H^i(\tilde{C}, f^*L) = H^i(C,L) \oplus H^i(C, L\otimes \eta) \text{ for any } L \in \mathrm{Pic}(C),\ i=0,1.$$

The pullback map χ^* at the level of Picard groups has been determined by M. Bernstein in [Be] Lemma 3.1.3. We record her results:

Proposition 4.1. *The pullback map* $\chi^* : \mathrm{Pic}(\overline{\mathcal{R}}_g) \rightarrow \mathrm{Pic}(\overline{\mathcal{M}}_{2g-1})$ *is given as follows:*

$$\chi^*(\lambda) = 2\lambda - \frac{1}{4}\delta_0^{\mathrm{ram}},\ \chi^*(\delta_0) = \delta_0^{\mathrm{ram}} + 2\bigl(\delta_0' + \delta_0'' + \sum_{i=1}^{[g/2]} \delta_{i:g-i}\bigr),\ \chi^*(\delta_i) = 2\delta_{g-i} \textit{ for } 1 \leq i \leq g-1.$$

Proof. The formula for $\chi^*(\delta_i)$ when $1 \leq i \leq g-1$ is immediate. To determine $\chi^*(\lambda)$ one notices that $\chi^*((\kappa_1)_{\overline{\mathcal{M}}_{2g-1}}) = 2(\kappa_1)_{\overline{\mathcal{R}}_g}$ and the rest follows from Mumford's formulas $(\kappa_1)_{\overline{\mathcal{M}}_{2g-1}} = 12\lambda - \delta \in \mathrm{Pic}(\overline{\mathcal{M}}_{2g-1})$ and $(\kappa_1)_{\overline{\mathcal{R}}_g} = 12\lambda - \pi^*(\delta) \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$. $\Box$

We set the integer $g' := 1 + \frac{g-1}{g}\binom{2g}{g-1}$. In [F3] we have studied the rational map

$$\phi : \overline{\mathcal{M}}_{2g-1} - - > \overline{\mathcal{M}}_{1+\frac{g-1}{g}\binom{2g}{g-1}}, \quad \phi[Y] := W^1_{g+1}(Y),$$

and determined the pullback map at the level of divisors $\phi^* : \mathrm{Pic}(\overline{\mathcal{M}}_{g'}) \rightarrow \mathrm{Pic}(\overline{\mathcal{M}}_{2g-1})$. In particular, we proved that if $A \in \mathrm{Pic}(\overline{\mathcal{M}}_{g'})$ is a divisor of slope $s(A) = s$, then the slope of the pullback $\phi^*(A)$ is equal to (cf. [F3] Theorem 0.2)

$$(17) \qquad s\bigl(\phi^*(A)\bigr) = 6 + \frac{8g^3s - 32g^3 - 19g^2s + 66g^2 + 6gs - 16g + 3s + 6}{(g-1)(g+1)(g^2s - 2gs - 4g^2 + 7g + 3)}.$$

To obtain effective divisors of small slope on $\overline{\mathcal{R}}_g$ we shall consider pullbacks $(\phi\chi)^*(A)$, where $A \in \mathrm{Ample}(\overline{\mathcal{M}}_{g'})$. (Of course, one can consider the cone $\chi^*(\mathrm{Ample}(\overline{\mathcal{M}}_{2g-1}))$, but a quick look at Proposition (4.1) shows that it is impossible to obtain in this way divisors on $\overline{\mathcal{R}}_g$ satisfying the inequalities (2). Pulling back merely *effective* divisors $\overline{\mathcal{M}}_{2g-1}$ rather than ample ones, is problematic since $\chi(\overline{\mathcal{R}}_g)$ tends to be contained in many geometric divisors on $\overline{\mathcal{M}}_{2g-1}$.) In order for the pullbacks $\chi^*\phi^*(A)$ to be well-defined as effective divisors on $\overline{\mathcal{R}}_g$ we prove the following result:

Proposition 4.2. *If* $\mathrm{dom}(\phi) \subset \overline{\mathcal{M}}_{2g-1}$ *is the domain of definition of the rational morphism* $\phi : \overline{\mathcal{M}}_{2g-1} \to \overline{\mathcal{M}}_{g'}$*, then* $\chi(\overline{\mathcal{R}}_g) \cap \mathrm{dom}(\phi) \neq \emptyset$*. It follows that for any ample divisor* $A \in \mathrm{Ample}(\overline{\mathcal{M}}_{g'})$*, the pullback* $\chi^*\phi^*(A) \in \mathrm{Eff}(\overline{\mathcal{R}}_g)$ *is well-defined.*

Proof. We take a general point $[C \cup_y E, \eta_C = \mathcal{O}_C, \eta_E] \in \Delta_1 \subset \overline{\mathcal{R}}_g$. The corresponding admissible double cover is then $f : C_1 \cup_{y_1} \widetilde{E} \cup_{y_2} C_2 \to C \cup_y E$, where $[C_1, y_1]$ and $[C_2, y_2]$ are copies of $[C, y]$ mapping isomorphically to $[C, y]$, and $f : \widetilde{E} \to E$ is the étale double cover induced by the torsion point $\eta_E \in \mathrm{Pic}^0(E)[2]$. We have that $C_i \cap \widetilde{E} = \{y_i\}$, where $f_{\widetilde{E}}(y_1) = f_{\widetilde{E}}(y_2) = y$. Thus $\chi[C \cup E, \mathcal{O}_C, \eta_E] := [C_1 \cup_{y_1} \widetilde{E} \cup_{y_2} C_2]$, where $y_1, y_2 \in \widetilde{E}$ are such that $\mathcal{O}_{\widetilde{E}}(y_1 - y_2)$ is a 2-torsion point in $\mathrm{Pic}^0(\widetilde{E})$.

Suppose now that $X := C_1 \cup_{y_1} E \cup_{y_2} C_2$ is a curve of compact type such that $[C_i, y_i] \in \mathcal{M}_{g-1,1}$ $(i = 1, 2)$ and $[E, y_1, y_2] \in \mathcal{M}_{1,2}$ are all Brill-Noether general. In particular, the class $y_1 - y_2 \in \mathrm{Pic}^0(E)$ is not torsion. Then $\phi([X]) := [\overline{W}^1_{g+1}(X)]$ is the variety of limit linear series $\mathfrak{g}^1_{g+1}$ on X. The general point of each irreducible component of $\overline{W}^1_{g+1}(X)$ corresponds to a refined linear series l on X satisfying the following compatibility conditions in terms of Brill-Noether numbers (see also [EH], [F3]):
(18)
$$1 = \rho(l_{C_1}, y_1) + \rho(l_{C_2}, y_2) + \rho(l_E, y_1, y_2) = 1 \ \text{ and } \rho(l_{C_1}, y_1), \rho(l_{C_2}, y_2), \rho(l_E, y_1, y_2) \geq 0.$$

If $\rho(l_{C_2}, y_2) = 1$, we find two types of components of $\overline{W}^1_{g+1}(X)$ which we describe: Since $\rho(l_{C_1}, y_1) = 0$, there exists an integer $0 \leq a \leq g/2$ such that $a^{l_{C_1}}(y_1) = (a, g + 2 - a)$. On E there are two choices for $l_E \in G^1_{g+1}(E)$ such that $a^{l_E}(y_1) = (a - 1, g + 1 - a)$. Either $a^{l_E}(y_2) = (a, g+1-a)$ (there is a unique such l_E), and then l_{C_2} belongs to the connected curve $T_a := \{l_{C_2} \in G^1_{g+1}(C_2) : a^{l_{C_2}}(y_2) \geq (a, g+1-a)\}$, or else, $a^{l_E}(y_2) = (a-1, g+2-a)$ (again, there is a unique such l_E), and then the C_2-aspect of l belongs to the curve $T'_a := \{l_{C_2} \in G^1_{g+1}(C_2) : a^{l_{C_2}}(y_2) \geq (a-1, g+2-a)\}$. Thus $\{l_{C_1}\} \times T_a$ and $\{l_{C_2}\} \times T'_a$ are irreducible components of $\overline{W}^1_{g+1}(X)$. When $\rho(l_E, y_1, y_2) = 1$, then there are three types of irreducible components of $\overline{W}^1_{g+1}(X)$ corresponding to the cases

$$a^{l_E}(y_1) = (a - 1, g + 1 - a), \ a^{l_E}(y_2) = (a - 1, g + 1 - a) \ \text{ for } 0 \leq a \leq g/2,$$

$$a^{l_E}(y_1) = (a - 1, g + 1 - a), \ a^{l_E}(y_2) = (a, g - a) \text{ for } 1 \leq a \leq (g - 1)/2, \ \text{ and}$$

$$a^{l_E}(y_1) = (a - 1, g + 1 - a), \ a^{l_E}(y_2) = (a - 2, g + 2 - a) \ \text{ for } 2 \leq a \leq (g - 1)/2.$$

Finally, the case $\rho(l_{C_1}, y_1) = 1$ is identical to the case $\rho(l_{C_2}, y_2) = 1$ by reversing the role of the curves C_1 and C_2. The singular points of $\overline{W}^1_{g+1}(X)$ correspond to (necessarily) crude limit $\mathfrak{g}^1_{g+1}$'s satisfying $\rho(l_{C_1}, y_1) = \rho(l_{C_2}, y_2) = \rho(l_E, y_1, y_2) = 0$. For such l, there must exist two irreducible components of X, say Y and Z, for which $Y \cap Z = \{x\}$ and such that $a_0^{l_Y}(x) + a_1^{l_Z}(x) = g + 2$ and $a_1^{l_Y}(x) + a_0^{l_Z}(x) = g + 1$. The point l lies precisely on the two irreducible components of $\overline{W}^1_{g+1}(X)$: The one corresponding to refined limit $\mathfrak{g}^1_{g+1}$ with vanishing sequence on Y equal to $(a_0^{l_Y}(x) - 1, a_1^{l_Y}(x))$, and the one with vanishing $(a_0^{l_Z}(x), a_1^{l_Z}(x) - 1)$ on Z. Thus $\overline{W}^1_{g+1}(X)$ is a stable curve of compact type, so $[X] \in \mathrm{dom}(\phi)$. Using [F3], this set-theoretic description applies to the image under ϕ of any point $[C_1 \cup_{y_1} E \cup_{y_2} C_2]$, in particular to $[C_1 \cup_{y_1} \widetilde{E} \cup_{y_2} C_2] = \chi([C \cup_y E])$. We have showed that $\chi(\Delta_1) \cap \mathrm{dom}(\phi) \neq \emptyset$. □

Proof of Theorem 0.1 for genus $g = 18, 20, 22$. We construct an effective divisor on $\overline{\mathcal{R}}_g$ which satisfies the inequalities (2) and which is of the form

$$\mu\pi^*(D) + \epsilon\chi^*\phi^*(A) = \alpha\lambda - 2(\delta_0' + \delta_0'') - 3\delta_0^{\mathrm{ram}} - \sum_{i=1}^{[g/2]}(b_i\delta_i + b_{g-i}\delta_{g-i} + b_{i:g-i}\delta_{i:g-i}),$$

where $A \equiv s\lambda - \delta \in \mathrm{Pic}(\overline{\mathcal{M}}_{g'})$ is an ample class (which happens precisely when $s > 11$, cf. [CH]), $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ and $\mu, \epsilon > 0$ and $\alpha < 13$. We solve this linear system using Proposition 4.1 and find that we must have

$$\epsilon = \frac{8}{12 - s(\phi^*(A))} \text{ and } \mu = \frac{16 - 2s(\phi^*(A))}{12 - s(\phi^*(A))}.$$

To conclude that $\overline{\mathcal{R}}_g$ is of general type, it suffices to check that the inequality

$$\alpha = \frac{8s\big(\phi^*(A)\big)}{12 - s\big(\phi^*(A)\big)} + \Big(6 + \frac{12}{g+1}\Big)\frac{16 - 2s\big(\phi^*(A)\big)}{12 - s\big(\phi^*(A)\big)} < 13$$

has a solution $s = s(A) \geq 11$. Using (17), we find that this is the case for $g \geq 18$. □

5. The enumerative geometry of $\overline{\mathcal{R}}_g$ in small genus

In this Section we describe the divisors $\mathcal{D}_{g:k}$ and $\mathcal{U}_{g,i}$ for small g. We start with the case $g = 3$. This result has been first obtained by M. Bernstein [Be] Theorem 3.2.3 using test curves inside $\overline{\mathcal{R}}_3$. Our method is more direct and uses the identification of cycles $C - C = \Theta_{Q_C} \subset \mathrm{Pic}^0(C)$, valid for all curves $[C] \in \mathcal{M}_3$.

Theorem 5.1. *The divisor $\mathcal{D}_{3:2} = \{[C, \eta] \in \mathcal{R}_3 : \eta \in C - C\}$ is equal to the locus of étale double covers $[\tilde{C} \stackrel{f}{\to} C] \in \mathcal{R}_3$ such that $[\tilde{C}] \in \mathcal{M}_5$ is hyperelliptic. We have the equality of cycles $\overline{\mathcal{D}}_{3:2} \equiv 8\lambda - \delta_0' - 2\delta_0'' - \frac{3}{2}\delta_0^{\mathrm{ram}} - 6\delta_1 - 4\delta_2 - 2\delta_{1:2} \in \mathrm{Pic}(\overline{\mathcal{R}}_3)$. Moreover,*

$$\pi_*(\overline{\mathcal{D}}_{3:2}) \equiv 56 \cdot \overline{\mathcal{M}}_{3,2}^1 = 56 \cdot (9\lambda - \delta_0 - 3\delta_1) \in \mathrm{Pic}(\overline{\mathcal{M}}_3).$$

This equality corresponds to the fact that for an étale double cover $f : \tilde{C} \to C$, the source $\tilde{C}$ is hyperelliptic if and only if C is hyperelliptic and $\eta \in C - C \subset \mathrm{Pic}^0(C)$.

Proof. We use the set-up from Theorem 2.8 and recall that there exists a vector bundle morphism $\phi : \mathcal{H} \otimes \mathcal{A}_{0,0} \to \mathcal{A}_{0,1}$ over $\overline{\mathbf{R}}_3^0$ such that $Z_1(\phi) \cap \mathcal{R}_3 = \mathcal{D}_{3:2}$. Here $\mathcal{H} = \pi^*(\mathbb{E})$, $\mathcal{A}_{0,0}[X, \eta, \beta] = H^0(X, \omega_X \otimes \beta)$ and $\mathcal{A}_{0,1}[X, \eta, \beta] = H^0(X, \omega_X^{\otimes 2} \otimes \beta)$, for each point $[X, \eta, \beta] \in \widetilde{\mathcal{R}}_g$. Using (11) and (12) we check that both $\phi_{|\Delta_0'}$ and $\phi_{|\Delta_0^{\mathrm{ram}}}$ are generically non-degenerate. Over a point $t = [C_{yq},\ \eta, \beta] \in \Delta_0''$ corresponding to a Wirtinger covering (i.e. $\nu : C \to C_{yq}$, with $[C] \in \mathcal{M}_2$ and $\nu^*(\eta) = \mathcal{O}_C$), we have that

$$\phi(t) : H^0(C, K_C) \otimes H^0\big(C, K_C \otimes \mathcal{O}_C(y+q)\big) \to \mathcal{A}_{0,1}(t) \subset H^0\big(C, \omega_C^{\otimes 2} \otimes \mathcal{O}_C(2y+2q)\big).$$

From the base point free pencil trick we find that $\mathrm{Ker}(\phi(t)) = H^0(C, \mathcal{O}_C(y+q))$, that is, $\phi_{|\Delta_0''}$ is everywhere degenerate and the class $c_1(\mathcal{A}_{0,1} - \mathcal{H} \otimes \mathcal{A}_{0,0}) - \delta_0'' \in \mathrm{Pic}(\overline{\mathbf{R}}_3^0)$ is effective. From the formulas $\pi_*(\lambda) = 63\lambda,\ \pi_*(\delta_0') = 30\delta_0,\ \pi_*(\delta_0^{''}) = \delta_0$ and $\pi_*(\delta_0^{\mathrm{ram}}) = 16\delta_0$, we obtain that

$$s\big(\pi_*(c_1(\mathcal{A}_{0,1} - \mathcal{H}\otimes\mathcal{A}_{0,0}) - \delta_0^{''})\big) = 9.$$

The hyperelliptic locus $\overline{\mathcal{M}}_{3,2}^1$ is the only divisor on $D \in \mathrm{Eff}(\overline{\mathcal{M}}_3)$ with $\Delta_i \subsetneq \mathrm{supp}(D)$ for $i = 0, 1$ and $s(D) \leq 9$, which leads to the formula $\pi_*(\overline{\mathcal{D}}_{3:2}) = 56 \cdot \overline{\mathcal{M}}_{3,2}^1$. □

Theorem 5.2. *The divisor $\overline{\mathcal{D}}_{5:2} := \{[C, \eta] \in \mathcal{R}_5 : \eta \in C_2 - C_2\}$ equals the locus of étale double covers $[\tilde{C} \stackrel{f}{\to} C] \in \mathcal{R}_5$ such that the genus 9 curve $\tilde{C}$ is tetragonal. We have the formula $\overline{\mathcal{D}}_{5:2} = 14\lambda - 2(\delta_0' + \delta_0^{''}) - \frac{5}{2}\delta_0^{\mathrm{ram}} - 10\delta_4 - 4\delta_{1:4} - \cdots \in \mathrm{Pic}(\overline{\mathcal{R}}_5)$.*

Proof. We start with an étale cover $f : \tilde{C} \stackrel{2:1}{\to} C$ corresponding to the torsion point $\eta = \mathcal{O}_C(D - E)$, with $D, E \in C_2$. Then

$$H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(f^*D)) = H^0(C, \mathcal{O}_C(D)) \oplus H^0(C, \mathcal{O}_C(E)),$$

that is, $|f^*D| \in G_4^1(\tilde{C})$ and $[\tilde{C}] \in \overline{\mathcal{M}}_{9,4}^1$. Conversely, if $l \in G_4^1(\tilde{C})$, then l must be invariant under the involution of $\tilde{C}$ and then $f_*(l) \in G_4^1(C)$ contains two divisors of the type $2x + 2y \equiv 2p + 2q$. Then we take $\eta = \mathcal{O}_C(x + y - p - q)$, that is, $[C, \eta] \in \mathcal{D}_{5:2}$. □

Remark 5.3. Since $\mathrm{codim}(\overline{\mathcal{M}}_{9,4}^1, \overline{\mathcal{M}}_9) = 3$ while $\mathcal{D}_{5:2}$ is a divisor in $\mathcal{R}_3$, there seems to be a dimensional discrepancy in Theorem 5.2. This is explained by noting that for an étale double covering $f : \tilde{C} \to C$ over a general curve $[C] \in \mathcal{M}_5$, the codimension 1 condition $\mathrm{gon}(\tilde{C}) \leq 5$ is equivalent to the seemingly stronger condition $\mathrm{gon}(\tilde{C}) \leq 4$. Indeed, if $l \in G_5^1(\tilde{C})$ is base point free, then l is not invariant under the involution of $\tilde{C}$ and $\dim |f_* l| \geq 2$ so $G_5^2(C) \neq \emptyset$, a contradiction with the genericity assumption on C.

Theorem 5.4. *The divisor $\mathcal{D}_{4:3} = \{[C, \eta] \in \mathcal{R}_4 : \exists A \in W_3^1(C) \text{ with } H^0(C, A \otimes \eta) \neq 0\}$ can be identified with the locus of Prym curves $[C, \eta] \in \mathcal{R}_4$ such that the Prym-canonical model $C \stackrel{|K_C \otimes \eta|}{\longrightarrow} \mathbf{P}^2$ is a plane sextic curve with a triple point. We also have the class formula*

$$\overline{\mathcal{D}}_{4:3} \equiv 8\lambda - \delta_0' - 2\delta_0^{''} - \frac{7}{4}\delta_0^{\mathrm{ram}} - 4\delta_3 - 7\delta_1 - 3\delta_{1:3} - \cdots \in \mathrm{Pic}(\overline{\mathcal{R}}_4),$$

hence $\pi_*(\overline{\mathcal{D}}_{4:3}) = 60 \cdot \overline{\mathcal{GP}}_{4,3}^1 = 60(34\lambda - 4\delta_0 - 14\delta_1 - 18\delta_2) \in \mathrm{Pic}(\overline{\mathcal{M}}_4)$, where

$$\mathcal{GP}_{4,3}^1 \subset \mathcal{M}_4 := \{[C] \in \mathcal{M}_4 : \exists A \in W_3^1(C), A^{\otimes 2} = K_C\}$$

is the Gieseker-Petri divisor of curves $[C] \in \mathcal{M}_4$ with a vanishing theta-null.

Proof. We start with a Prym curve $[C, \eta] \in \mathcal{R}_4$ such that there exists $A \in W_3^1(C)$ with $H^0(C, A \otimes \eta) \neq 0$. We claim that $A^{\otimes 2} = K_C$, that is, $[C] \in \mathcal{GP}_{4,3}^1$. Indeed, assuming the opposite, we find *disjoint* divisors $D_1, D_2 \in C_3$ such that $D_1 \in |A \otimes \eta|$ and $D_2 \in |K_C \otimes A^\vee \otimes \eta|$. In particular, the subspaces $H^0(C, K_C \otimes \eta(-D_i)) \subset H^0(C, K_C)$ are both of dimension 2, hence they intersect non-trivially, that is $H^0\big(C, K_C \otimes \eta(-D_1 - D_2)\big) \neq 0$. Since $D_1 + D_2 \equiv K_C$, this implies $\eta = 0$, a contradiction.

The proof that the vector bundle morphism $\phi : \mathcal{H} \otimes \mathcal{A}_{0,0} \to \mathcal{A}_{0,1}$ constructed in the proof of Theorem 2.8 is degenerate with order 1 along the divisor $\Delta_0^{''} \subset \overline{\mathcal{R}}_4$ follows from (11). Thus $c_1\big(\mathcal{A}_{0,1} - \mathcal{H}\otimes \mathcal{A}_{0,0}\big) - \delta_0^{''} \in \mathrm{Pic}(\overline{\mathcal{R}}_4)$ is an effective class and its push-forward to $\overline{\mathcal{M}}_4$ has slope $17/2$. The only divisor $D\in \mathrm{Eff}(\overline{\mathcal{M}}_4)$ with $\Delta_i \subsetneq \mathrm{supp}(D)$ for $i=0,1,2$ and $s(D)\leq 17/2$, is the theta-null divisor $\overline{\mathcal{GP}}^1_{4,3}$ (cf. [F3] Theorem 5.1). □

Remark 5.5. For a general point $[C,\eta]\in \mathcal{R}_4$, the Prym-canonical curve $\iota: C \stackrel{|K_C\otimes\eta|}{\longrightarrow} \mathbf{P}^2$ is a plane sextic with 6 nodes which correspond to the preimages in $\phi^{-1}(\eta)$ under the second difference map

$$C_2\times C_2 \to \mathrm{Pic}^0(C),\ \ (D_1,D_2)\mapsto \mathcal{O}_C(D_1-D_2).$$

Note that $W_2(C)\cdot (W_2(C)+\eta)=6$. For a general $[C,\eta]\in \mathcal{D}_{4:3}$, the model $\iota(C)\subset \mathbf{P}^2$ has a triple point. For a hyperelliptic curve $[C]\in \mathcal{M}^1_{4,2}$, out of the $255=2^{2g}-1$ étale double covers of C, there exist 210 for which $C\stackrel{|K_C\otimes\eta|}{\longrightarrow} \mathbf{P}^2$ has an ordinary 4-fold point and no other singularity. The remaining $45=\binom{2g+2}{2}$ coverings correspond to the case $\eta=\mathcal{O}_C(x-y)$, with $x,y\in C$ being Weierstrass points, when $|K_C\otimes \eta|$ has 2 base points and ι is a degree 2 map onto a conic.

6. THE SINGULARITIES OF THE MODULI SPACE OF PRYM CURVES

The moduli space $\overline{\mathcal{R}}_g$ is a normal variety with finite quotient singularities. To determine its Kodaira dimension we consider a smooth model $\widehat{\mathcal{R}}_g$ of $\overline{\mathcal{R}}_g$ and then analyze the growth of the dimension of the spaces $H^0\big(\widehat{\mathcal{R}}_g, K_{\widehat{\mathcal{R}}_g}^{\otimes l}\big)$ of pluricanonical forms for all $l\geq 0$. In this section we show that in doing so one only needs to consider forms defined on $\overline{\mathcal{R}}_g$ itself.

Theorem 6.1. *We fix $g\geq 4$ and let $\widehat{\mathcal{R}}_g\to\overline{\mathcal{R}}_g$ be any desingularisation. Then every pluricanonical form defined on the smooth locus $\overline{\mathcal{R}}_g^{\mathrm{reg}}$ of $\overline{\mathcal{R}}_g$ extends holomorphically to $\widehat{\mathcal{R}}_g$, that is, for all integers $l\geq 0$ we have isomorphisms*

$$H^0\big(\overline{\mathcal{R}}_g^{\mathrm{reg}}, K_{\overline{\mathcal{R}}_g}^{\otimes l}\big)\cong H^0\big(\widehat{\mathcal{R}}_g, K_{\widehat{\mathcal{R}}_g}^{\otimes l}\big).$$

A similar statement has been proved for the moduli space $\overline{\mathcal{M}}_g$ of curves cf. [HM] Theorem 1, and for the moduli space $\overline{\mathcal{S}}_g$ of spin curves, cf. [Lud] Theorem 4.1. We start by explicitly describing the locus of non-canonical singularities in $\overline{\mathcal{R}}_g$, which has codimension 2. At a non-canonical singularity there exist *local* pluricanonical forms that do acquire poles on a desingularisation. We show that this situation does not occur for forms defined on the smooth locus $\overline{\mathcal{R}}_g^{\mathrm{reg}}$, and they extend holomorphically to $\widehat{\mathcal{R}}_g$.

Definition 6.2. An *automorphism* of a Prym curve (X,η,β) is an automorphism $\sigma\in\mathrm{Aut}(X)$ such that there exists an isomorphism of sheaves $\gamma:\sigma^*\eta\to\eta$ making the following diagram commutative.

$$\begin{array}{ccc} (\sigma^*\eta)^{\otimes 2} & \xrightarrow{\gamma^{\otimes 2}} & \eta^{\otimes 2} \\ \downarrow{\scriptstyle \sigma^*\beta} & & \downarrow{\scriptstyle \beta} \\ \sigma^*\mathcal{O}_X & \xrightarrow{\simeq} & \mathcal{O}_X \end{array}$$

If $C := st(X)$ denotes the stable model of X then there is a group homomorphism $\mathrm{Aut}(X,\eta,\beta) \to \mathrm{Aut}(C)$ given by $\sigma \mapsto \sigma_C$. The kernel $\mathrm{Aut}_0(X,\eta,\beta)$ of this homomorphism is called the subgroup of *inessential automorphisms* of (X,η,β).

Remark 6.3. The subgroup $\mathrm{Aut}_0(X,\eta,\beta)$ is isomorphic to $\{\pm 1\}^{CC(\widetilde{X})}/\pm 1$, where $CC(\widetilde{X})$ is the set of connected components of the non-exceptional subcurve $\widetilde{X}$ (compare [CCC] Lemma 2.3.2 and [Lud] Proposition 2.7). Given $\gamma_j \in \{\pm 1\}$ for every connected component $\widetilde{X}_j$ of $\widetilde{X}$ consider the automorphism $\widetilde{\gamma}$ of $\widetilde{\eta} = \eta_{|\widetilde{X}}$ which is multiplication by γ_j in every fibre over $\widetilde{X}_j$. Then there exists a unique inessential automorphism σ such that $\widetilde{\gamma}$ extends to an isomorphism $\gamma : \sigma^*\eta \to \eta$ compatible with the morphisms $\sigma^*\beta$ and β. Considering $(-\gamma_j)_j$ instead of $(\gamma_j)_j$ gives the same automorphism σ.

Definition 6.4. For a quasi-stable curve X, an irreducible component C_j is called an *elliptic tail* if $p_a(C_j) = 1$ and $C_j \cap \overline{(X - C_j)} = \{p\}$. The node p is then an *elliptic tail node*. A non-trivial automorphism σ of X is called an *elliptic tail automorphism* (with respect to C_j) if $\sigma_{|X-C_j}$ is the identity.

Theorem 6.5. *Let (X,η,β) be a Prym curve of genus $g \geq 4$. The point $[X,\eta,\beta] \in \overline{\mathcal{R}}_g$ is smooth if and only if $\mathrm{Aut}(X,\eta,\beta)$ is generated by elliptic tail involutions.*

Throughout this Appendix, X denotes a quasi-stable curve of genus $g \geq 2$ and $C := st(X)$ is its stable model. We denote by $N \subset \mathrm{Sing}(C)$ the set of exceptional nodes and $\Delta := \mathrm{Sing}(C) - N$. Then X is the support of a Prym curve if and only if N considered as a subgraph of the dual graph $\Gamma(C)$ is *eulerian*, that is, every vertex of $\Gamma(C)$ is incident to an even number of edges in N (cf. [BCF] Proposition 0.4).

Locally at a point $[X,\eta,\beta]$, the moduli space $\overline{\mathcal{R}}_g$ is isomorphic to the quotient of the versal deformation space $\mathbb{C}_\tau^{3g-3}$ of (X,η,β) modulo the action of the automorphism group $\mathrm{Aut}(X,\eta,\beta)$. If $\mathbb{C}_t^{3g-3} = \mathrm{Ext}^1(\Omega_C^1, \mathcal{O}_C)$ denotes the versal deformation space of C, then the map $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_t^{3g-3}$ is given by $t_i = \tau_i^2$ if $(t_i = 0) \subset \mathbb{C}_t^{3g-3}$ is the locus where the exceptional node $p_i \in N$ persists and $t_i = \tau_i$ otherwise. The morphism $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ is given locally by the map $\mathbb{C}_\tau^{3g-3}/\mathrm{Aut}(X,\eta,\beta) \to \mathbb{C}_t^{3g-3}/\mathrm{Aut}(C)$. One has the following decomposition of the versal deformation space of (X,η,β)

$$\mathbb{C}_\tau^{3g-3} = \bigoplus_{p_i \in N} \mathbb{C}_{\tau_i} \oplus \bigoplus_{p_i \in \Delta} \mathbb{C}_{\tau_i} \oplus \bigoplus_{C_j \subset C} H^1\big(C_j^\nu, T_{C_j^\nu}(-D_j)\big),$$

where for a node $p_i \in N$ we denote by $(\tau_i = 0) \subset \mathbb{C}_\tau^{3g-3}$ the locus where the corresponding exceptional component E_i persists, while for a node $p_i \in \Delta$ we denote by $(\tau_i = 0) \subset \mathbb{C}_\tau^{3g-3}$ the locus of those deformations in which p_i persists. Finally, for a component $C_j \subset C$ with normalization C_j^ν, if D_j consists of the inverse images of the nodes of C under the normalization map $C_j^\nu \to C_j$, the group $H^1(C_j^\nu, T_{C_j^\nu}(-D_j))$ parameterizes deformations of the pair (C_j^ν, D_j). This decomposition is compatible with the decomposition

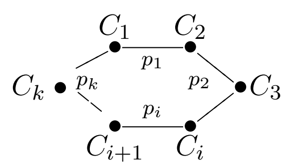
$$\mathbb{C}_t^{3g-3} = \big(\bigoplus_{p_i \in \mathrm{Sing}(C)} \mathbb{C}_{t_i}\big) \oplus \big(\bigoplus_{C_j} H^1(C_j^\nu, T_{C_j^\nu}(-D_j))\big)$$

as well as with the actions of the automorphism groups on $\mathbb{C}_\tau^{3g-3}$ and $\mathbb{C}_t^{3g-3}$, see also [Lud] pg. 5. The point $[X,\eta,\beta] \in \overline{\mathcal{R}}_g$ is smooth if and only if the action of $\mathrm{Aut}(X,\eta,\beta)$ on $\mathbb{C}_\tau^{3g-3}$ is generated by quasi-reflections, that is, elements $\sigma \in \mathrm{Aut}(X,\eta,\beta)$ having 1 as an eigenvalue of multiplicity precisely $3g-4$. Theorem 6.5 follows from the following proposition.

Proposition 6.6. *Let $\sigma \in \mathrm{Aut}(X,\eta,\beta)$ be an automorphism of a Prym curve (X,η,β) of genus $g \geq 4$. Then σ acts on $\mathbb{C}_\tau^{3g-3}$ as a quasi-reflection if and only if X has an elliptic tail C_j such that σ is the elliptic tail involution with respect to C_j.*

Proof. Let σ be an elliptic tail involution with respect to C_j. The induced automorphism σ_C is an elliptic tail involution of C and acts on the versal deformation space $\mathbb{C}_t^{3g-3}$ of C as $t_1 \mapsto -t_1$ and $t_i \mapsto t_i$, $i \neq 1$. Here t_1 is the coordinate corresponding to the node $p_1 \in C_j \cap \overline{(C - C_j)}$. The node p_1 being non-exceptional, we have that $t_1 = \tau_1$ hence $\sigma \cdot \tau_1 = -\tau_1$. If $\tau_i = t_i (i \neq 1)$, then $\sigma \cdot \tau_i = \tau_i$. For coordinates $t_i = \tau_i^2$, σ is the identity in a neighbourhood of the corresponding exceptional component E_i, thus $\sigma \cdot \tau_i = \tau_i$.

Now let $\sigma \in \mathrm{Aut}(X,\eta,\beta)$ act as a quasi-reflection with eigenvalues ζ and 1. As in the proof of [Lud] Proposition 2.15, there exists a node $p_1 \in C$ such that the action of σ is given by $\sigma \cdot \tau_1 = \zeta\tau_1$ and $\sigma \cdot \tau_j = \tau_j$ for $j \neq 1$. When $p_1 \in N$, the induced automorphism σ_C acts via $t_1 \mapsto \zeta^2 t_1$ and $\sigma_C \cdot t_j = t_j$ for $j \neq 1$. If $\zeta^2 \neq 1$, then σ_C acts as a quasi-reflection and p_1 is an elliptic tail node, which contradicts the assumption $p_1 \in N$. Therefore $\sigma_C = \mathrm{Id}_C$ and the exceptional component E_1 over p_1 is the only component on which σ acts non-trivially. The graph $N \subset \Gamma(C)$ is eulerian and there exists a circuit of edges in N containing p_1.



By Remark 6.3, σ corresponds to an element $\pm(\gamma_j)_j \in \{\pm 1\}^{CC(\widetilde{X})}/\pm 1$. Since σ acts non-trivially on E_1 we find that $\gamma_1 = -\gamma_2$. In particular, there exists $i \neq 1$ such that σ acts non-trivially on E_i. This is a contradiction which shows that the node p_1 is non-exceptional, $\tau_1 = t_1$ and $\sigma_C \cdot t_1 = \zeta t_1$ and $\sigma_C \cdot t_i = t_i$ for $i \neq 1$. Thus σ_C is an elliptic tail involution of C with respect to an elliptic tail through the node p_1 and $\zeta = -1$. Since σ fixes every coordinate corresponding to an exceptional component of X, it follows that σ is an elliptic tail involution of X. □

Theorem 6.7. *We fix $g \geq 4$. A point $[X,\eta,\beta] \in \overline{\mathcal{R}}_g$ is a non-canonical singularity if and only if X has an elliptic tail C_j with j-invariant 0 and η is trivial on C_j.*

The proof is similar to that of the analogous statement for $\overline{\mathcal{S}}_g$ and we refer to [Lud] Theorem 3.1 for a detailed outline of the proof and background on quotient singularities. Locally at $[X,\eta,\beta]$, the space $\overline{\mathcal{R}}_g$ is isomorphic to a neighbourhood of the origin in $\mathbb{C}_\tau^{3g-3}/\mathrm{Aut}(X,\eta,\beta)$. We consider the normal subgroup H of $\mathrm{Aut}(X,\eta,\beta)$ generated by automorphisms acting as quasi-reflections on $\mathbb{C}_\tau^{3g-3}$. The map $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_\tau^{3g-3}/H =$

$\mathbb{C}_{\upsilon}^{3g-3}$ is given by $\upsilon_i = \tau_i^2$ if p_i is an elliptic tail node and $\upsilon_i = \tau_i$ otherwise. The automorphism group $\mathrm{Aut}(X, \eta, \beta)$ acts on $\mathbb{C}_{\upsilon}^{3g-3}$ and the quotient $\mathbb{C}_{\upsilon}^{3g-3}/\,\mathrm{Aut}(X, \eta, \beta)$ is isomorphic to $\mathbb{C}_{\tau}^{3g-3}/\,\mathrm{Aut}(X, \eta, \beta)$. Since $\mathrm{Aut}(X, \eta, \beta)$ acts on $\mathbb{C}_{\upsilon}^{3g-3}$ without quasi-reflections the Reid–Shepherd-Barron–Tai criterion applies to this new action.

We fix an automorphism $\sigma \in \mathrm{Aut}(X, \eta, \beta)$ of order n and a primitive n-th root of unity ζ_n. If the action of σ on $\mathbb{C}_{\upsilon}^{3g-3}$ has eigenvalues $\zeta_n^{a_1}, \ldots, \zeta_n^{a_{3g-3}}$ with $0 \leq a_i < n$ for $i = 1, \ldots, 3g-3$, then following [Re2] we define the *age* of σ by

$$\mathrm{age}(\sigma, \zeta_n) := \frac{1}{n} \sum_{i=1}^{n} a_i.$$

We say that σ satisfies the Reid–Shepherd-Barron–Tai inequality if $\mathrm{age}(\sigma, \zeta_n) \geq 1$. The Reid–Shepherd-Barron–Tai criterion states that $\mathbb{C}_{\upsilon}^{3g-3}/\,\mathrm{Aut}(X, \eta, \beta)$ has canonical singularities if and only if every $\sigma \in \mathrm{Aut}(X, \eta, \beta)$ satisfies the Reid–Shepherd-Barron–Tai inequality (cf. [Re], [T],[Re2]).

Proof of the if-part of Theorem 6.7. Let (X, η, β) be a Prym curve, $C = st(X)$ and $C_j \subset X$ an elliptic tail with $\mathrm{Aut}(C_j) = \mathbb{Z}_6$ and we assume $\eta_{C_j} = \mathcal{O}_{C_j}$. We fix an elliptic tail automorphism σ_C with respect to $C_j \subset C$ such that $\mathrm{ord}(\sigma_C) = 6$. Then σ_C acts on $\mathbb{C}_t^{3g-3}$ by $t_1 \mapsto \zeta_6 t_1$, $t_2 \mapsto \zeta_6^2 t_2$ for an appropriate sixth root of unity ζ_6 and $\sigma \cdot t_i = t_i$ for $i \neq 1, 2$. Here $t_1, t_2 \in \mathrm{Ext}^1(\Omega_C^1, \mathcal{O}_C)$ correspond to smoothing the node $p_1 \in C_j \cap \overline{(C - C_j)}$ and deforming the curve $[C_j, p_1] \in \overline{\mathcal{M}}_{1,1}$ respectively. Since $\eta_{C_j} = \mathcal{O}_{C_j}$, the automorphism σ_C lifts to an automorphism $\sigma \in \mathrm{Aut}(X, \eta, \beta)$ such that $\sigma_{\overline{X - C_j}}$ is the identity. Then σ acts on $\mathbb{C}_{\tau}^{3g-3}$ as $\sigma \cdot \tau_1 = \zeta_6 \tau_1$, $\sigma \cdot \tau_2 = \zeta_6^2 \tau_2$ and $\sigma \cdot \tau_i = \tau_i$ for $i \neq 1, 2$. Since $\upsilon_1 = \tau_1^2$ and $\upsilon_2 = \tau_2$, the action of σ on $\mathbb{C}_{\upsilon}^{3g-3}$ is $\upsilon_1 \mapsto \zeta_6^2 \upsilon_1$, $\upsilon_2 \mapsto \zeta_6^2 \upsilon_2$ and $\upsilon_i \mapsto \upsilon_i$, $i \neq 1, 2$. We compute $\mathrm{age}(\sigma, \zeta_6^2) = \frac{1}{3} + \frac{1}{3} + 0 + \cdots + 0 = \frac{2}{3} < 1$, that is, $[X, \eta, \beta] \in \overline{\mathcal{R}}_g$ is a non-canonical singularity. Similarly, an elliptic tail automorphism of order 3 with respect to C_j acts via $\tau_1 \mapsto \zeta_3^2 \tau_1$, $\tau_2 \mapsto \zeta_3 \tau_2$ and $\tau_i \mapsto \tau_i$, $i \neq 1, 2$, and then for the action on $\mathbb{C}_{\upsilon}^{3g-3}$ as $\upsilon_1 \mapsto \zeta_3 \upsilon_1$, $\upsilon_2 \mapsto \zeta_3 \upsilon_2$ and $\upsilon_i \mapsto \upsilon_i$ for $i \neq 1, 2$. This gives a value of $\frac{2}{3}$ for the age. ☐

Suppose that $[X, \eta, \beta] \in \overline{\mathcal{R}}_g$ is a non-canonical singularity. Then there exists an automorphism $\sigma \in \mathrm{Aut}(X, \eta, \beta)$ of order n which acts on $\mathbb{C}_{\upsilon}^{3g-3}$ such that $\mathrm{age}(\sigma, \zeta_n) < 1$. Let $p_{i_0}, p_{i_1} = \sigma_C(p_{i_0}), \ldots, p_{i_{m-1}} = \sigma_C^{m-1}(p_{i_0})$ be distinct nodes of C which are cyclicly permuted by the induced automorphism σ_C and p_{i_j} is *not* an elliptic tail node. The action of σ on the subspace $\bigoplus_j \mathbb{C}_{\tau_{i_j}} \subset \mathbb{C}_{\tau}^{3g-3}$ is given by the matrix

$$B = \begin{pmatrix} 0 & c_1 & & \\ \vdots & & \ddots & \\ 0 & & & c_{m-1} \\ c_m & 0 & \cdots & 0 \end{pmatrix}$$

for appropriate scalars $c_j \neq 0$. The pair $\big((X, \eta, \beta), \sigma\big)$ is said to be *singularity reduced* if for every such cycle we have that $\prod_{j=1}^{m} c_j \neq 1$.

Proposition 6.8. *(*[HM], [Lud] Proposition 3.6*) There exists a deformation (X', η', β') of (X, η, β) such that σ deforms to an automorphism $\sigma' \in \mathrm{Aut}(X', \eta', \beta')$ and the nodes of every* *cycle of nodes as above with $\prod_{j=1}^m c_j = 1$ are smoothed. The pair $\big((X', \eta', \beta'), \sigma'\big)$ is then singularity reduced and the action of σ on $\mathbb{C}_v^{3g-3}$ and that of σ' on $\mathbb{C}_{v'}^{3g-3}$ have the same eigenvalues and hence the same age.*

We fix a singularity reduced pair $\big((X, \eta, \beta), \sigma\big)$ with $n := \operatorname{ord}(\sigma) \geq 2$ and assume that $\operatorname{age}(\sigma, \zeta_n) < 1$. We denote this assumption by $(\star)$. Using [Lud] Proposition 3.7 we obtain that if $(\star)$ holds, the induced automorphism σ_C of $C = st(X)$ fixes every node with the possible exception of two nodes which are interchanged.

Proposition 6.9. *If $(\star)$ holds, then σ_C fixes all components of the stable model C of X.*

Proof. Let $C_{i_0}, C_{i_1} = \sigma_C(C_{i_0}), \ldots, C_{i_{m-1}} = \sigma_C^{m-1}(C_{i_0})$ be distinct components of C, $\sigma_C^m(C_{i_0}) = C_{i_0}$ and assume that $m \geq 2$. Most of the proof of Proposition 3.8. in [Lud] applies to the case of Prym curves and implies that the normalization $C_{i_0}^\nu$ is rational and there are exactly three preimages of nodes $p_1^+, p_2^+, p_3^+ \in C_{i_0}^\nu$ mapping to different nodes of C. By [Lud] Proposition 3.7 at least one of p_1, p_2, p_3 is fixed by σ_C. If either one or all three nodes are fixed, then $g(C) = 2$, impossible. Thus two nodes, say p_1 and p_2, are fixed by σ_C while p_3 is interchanged with a fourth node p_4. Interchanging p_3 and p_4 gives a contribution of $\frac{1}{2}$ to age(σ, ζ_n). Now consider the action of σ_C near p_1 and let $xy = 0$ be a local equation of C at p_1. We have that $t_1 = xy \mapsto yx = t_1$ and $\tau_1 \mapsto \pm\tau_1$, where the minus sign is only possible if $p_1 \in N$. Since p_1 is not an elliptic tail node and $\big((X, \eta, \beta), \sigma\big)$ is singularity reduced, we have $\tau_1 \mapsto -\tau_1$, which gives an additional contribution of $\frac{1}{2}$ to the age, that is, $\operatorname{age}(\sigma, \zeta_n) \geq \frac{1}{2} + \frac{1}{2} = 1$, contradicting $(\star)$. ☐

Proposition 6.10 ([HM] p. 28, 36, [Lud] Proposition 3.9)**.** *We assume that $(\star)$ holds and denote by $\varphi_j = \sigma^\nu_{|C_j^\nu}$ the induced automorphism of the normalization C_j^ν of the irreducible component C_j of C. Then the pair $\big(C_j^\nu, \varphi_j\big)$ is one of the following types:*

(i) *$\varphi_j = \mathrm{Id}_{C_j^\nu}$ and C_j^ν arbitrary.*
(ii) *C_j^ν is rational and $\operatorname{ord}(\varphi_j) = 2, 4$.*
(iii) *C_j^ν is elliptic and $\operatorname{ord}(\varphi_j) = 2, 4, 3, 6$.*
(iv) *C_j^ν is hyperelliptic of genus 2 and φ_j is the hyperelliptic involution.*
(v) *C_j^ν is hyperelliptic of genus 3 and φ_j is the hyperelliptic involution.*
(vi) *C_j^ν is bielliptic of genus 2 and φ_j is the associated involution.*

The possibility of σ_C interchanging two nodes does not appear, cf. [Lud] Prop. 3.10:

Proposition 6.11. *Under the assumption $(\star)$, the automorphism σ_C fixes all the nodes of C.*

Proposition 6.12. *Assume $(\star)$ holds. Let C_j be a component of C with normalization C_j^ν, D_j the divisor of the marked points on C_j^ν and $\varphi_j = \sigma^\nu_{|C_j^\nu}$. Then $(C_j^\nu, D_j, \varphi_j)$ is of one of the following types and the contribution to $\operatorname{age}(\sigma, \zeta_n)$ coming from $H^1\big(C_j^\nu, T_{C_j^\nu}(-D_j)\big) \subset \mathbb{C}_v^{3g-3}$ is at least the following quantity w_j:*

(i) *Identity component: $\varphi_j = \mathrm{Id}_{C_j^\nu}$, arbitrary pair (C_j^ν, D_j) and $w_j = 0$*
(ii) *Elliptic tail: C_j^ν is elliptic, $D_j = p_1^+$ and p_1^+ is fixed by φ_j.*
*order 2***:** $\operatorname{ord}(\varphi_j) = 2$ *and* $w_j = 0$

order 4: C_j^ν *has* j*-invariant* 1728, $\mathrm{ord}(\varphi_j) = 4$ *and* $w_j = \frac{1}{2}$
order 3, 6: C_j^ν *has* j*-invariant* 0, $\mathrm{ord}(\varphi_j) = 3$ *or* 6 *and* $w_j = \frac{1}{3}$

(iii) *Elliptic ladder:* C_j^ν *is elliptic,* $D_j = p_1^+ + p_2^+$*, with* p_1^+ *and* p_2^+ *both fixed by* φ_j*.*
order 2: $\mathrm{ord}(\varphi_j) = 2$ *and* $w_j = \frac{1}{2}$
order 4: C_j^ν *has* j*-invariant* 1728, $\mathrm{ord}(\varphi_j) = 4$ *and* $w_j = \frac{3}{4}$
order 3: C_j^ν *has* j*-invariant* 0, $\mathrm{ord}(\varphi_j) = 3$ *and* $w_j = \frac{2}{3}$

(iv) *Hyperelliptic tail:* C_j^ν *has genus* 2*,* φ_j *is the hyperelliptic involution,* D_j *is of the form* $D_j = p_1^+$ *with* p_1^+ *fixed by* φ_j *and* $w_j = \frac{1}{2}$*.*

Proof. The proof is along the lines of the proof of Proposition 3.11 in [Lud]. The only difference occurs in the case of a singular elliptic tail on which σ acts with order 2. Assume that C_j^ν is rational, $D_j = p_1^+ + p_1^- + p_2$, with $\mathrm{ord}(\varphi_j) = 2$ which fixes p_2^+ and interchanges p_1^+ and p_1^-. If $xy = 0$ is an equation for C at p_1, then σ_C acts via $t_1 = xy \mapsto yx = t_1$. Since p_1 is not an elliptic tail node and $\big((X, \eta, \beta), \sigma\big)$ is singularity reduced, the node p_1 must be exceptional and $\sigma \cdot \tau_1 = -\tau_1$.

A deformation of (X, η, β) over the locus $(\tau_i = 0)_{i \neq 1} \subset \mathbb{C}_\tau^{3g-3}$ smooths p_1. Furthermore, σ deforms to an automorphism σ' of a general Prym curve (X', η', β') over this locus, φ_j deforms to the involution φ_j' on the smooth elliptic tail C_j' such that it fixes the line bundle $\eta'_{C_j'}$, and the restrictions of σ and σ' to the complement of C_j resp. C_j' coincide. Over the non-exceptional subcurve $\widetilde{X} \subset X$ we have $(\widetilde{\sigma}')^* \widetilde{\eta}' \cong \widetilde{\eta}'$. Thus $\sigma \cdot \tau_1 = \tau_1$ which is a contradiction. The case of a singular elliptic tail is thus excluded. □

Proposition 6.13. *Under the hypothesis* $(\star)$*, the hyperelliptic tail case does not occur.*

Proof. Let C_j be a genus 2 tail of C and $C_{j'}$ the second component through p_1. The action of σ on $H^1(C_j^\nu, T_{C_j^\nu}(-D_j))$ contributes $\frac{1}{2}$ to the age of σ and $C_{j'}$ has to be one of the cases of Proposition 6.12. If $C_{j'}$ is elliptic, then $g(C) = 3$. If $C_{j'}$ is a hyperelliptic tail or an elliptic ladder, the action on $H^1(C_{j'}^\nu, T_{C_{j'}^\nu}(-D_{j'}))$ contributes at least $\frac{1}{2}$. Therefore $C_{j'}$ is an identity component. If $xy = 0$ is an equation for C at p_1, then σ_C acts via $t_1 = xy \mapsto -xy = -t_1$. The node p_1 is disconnecting, hence non-exceptional, and it is not an elliptic tail node. Therefore, $\upsilon_1 = \tau_1 = t_1$ and σ acts as $\sigma \cdot \upsilon_1 = -\upsilon_1$. This gives an additional contribution of $\frac{1}{2}$ to the age of σ finishing the proof. □

Proposition 6.14. *In situation* $(\star)$ *the elliptic ladder cases do not occur.*

Proof. Let C_j be an elliptic ladder of C of order $n_j = \mathrm{ord}(\varphi_j)$ and denote by $C_{j'}$ resp. $C_{j''}$ the second component through the node p_1 resp. p_2. Since every elliptic ladder contributes at least $\frac{1}{2}$ to the age, $C_{j'}$ and $C_{j''}$ can only be elliptic tails or identity components. If both are elliptic tails, then $g(C) = 3$, hence we may assume that $C_{j'}$ is an identity component. If $xy = 0$ is an equation for C at p_1, then σ_C acts as $x \mapsto x$, $y \mapsto \alpha y$ and $t_1 \mapsto \alpha t_1$, where α is a primitive n_j-th root of 1. If p_1 is non-exceptional then $v_1 = \tau_1 = t_1$ and the space $H^1(C_j^\nu, T_{C_j^\nu}(-D_j)) \oplus \mathbb{C} \cdot v_1$ contributes to the age at least

$$1 = \begin{cases} \frac{1}{2} + \frac{1}{2} & \text{if } n_j = 2 \\ \frac{3}{4} + \frac{1}{4} & \text{if } n_j = 4 \\ \frac{2}{3} + \frac{1}{3} & \text{if } n_j = 3 \end{cases}$$

Therefore $p_1 \in N$. Since $N \subset \Gamma(C)$ is an eulerian subgraph, the node p_2 is also exceptional, both p_1 and p_2 are non-disconnecting and $C_{j''}$ is an identity component as well. Moreover $\sigma_C \cdot t_i = \alpha t_i$, $i = 1, 2$. Since $v_i = \tau_i$ and $\tau_i^2 = t_i$ for $i = 1, 2$, we find that $\sigma \cdot v_i = \alpha_i v_i$, $i = 1, 2$, where α_i is a square root of α. Therefore, the contribution to the age of σ coming from $H^1(C_j^\nu, T_{C_j^\nu}(-D_j)) \oplus \mathbb{C} \cdot v_1 \oplus \mathbb{C} \cdot v_2$ is at least

$$1 = \begin{cases} \frac{1}{2} + \frac{1}{4} + \frac{1}{4} & \text{if } n_j = 2 \\ \frac{3}{4} + \frac{1}{8} + \frac{1}{8} & \text{if } n_j = 4 \\ \frac{2}{3} + \frac{1}{6} + \frac{1}{6} & \text{if } n_j = 3 \end{cases}$$

and the case of elliptic ladders is excluded. □

Proposition 6.15. *Under hypothesis* $(\star)$*, the case of an elliptic tail of order* 4 *does not occur.*

Proof. Let C_j be an elliptic tail of order 4 and $C_{j'}$ another component of C through p_1. Then $\sigma_{C|C'_j} = \mathrm{Id}_{C'_j}$ and σ_C acts as $t_1 = xy \mapsto \zeta_4 xy = \zeta_4 t_1$ for a suitable fourth root ζ_4 of 1. Since p_1 is an elliptic tail node, we have $v_1 = t_1^2$ and $\sigma \cdot v_1 = -v_1$. The action of σ on $H^1(C_j^\nu, T_{C_j^\nu}(-D_j)) \oplus \mathbb{C} \cdot v_1$ contributes $\geq \frac{1}{2} + \frac{1}{2} = 1$ to age(σ, ζ_4) excluding this case. □

Proposition 6.16. *In situation* $(\star)$ *there has to be at least one elliptic tail of order* 3 *or* 6*.*

Proof. Assume to the contrary that every component of C is either an identity component or an elliptic tail of order 2. The action of σ on every space $H^1(C_j^\nu, T_{C_j^\nu}(-D_j))$ is trivial. If p_1 is the node of an elliptic tail of order 2, then $\sigma_C \cdot t_1 = -t_1$ and we have $v_1 = \tau_1^2 = t_1^2$ and $\sigma \cdot v_1 = v_1$. In case p_1 is non-exceptional but not an elliptic tail node, $\sigma_C \cdot t_1 = t_1$. Since $v_1 = \tau_1 = t_1$, we find that σ fixes v_1. If $p_1 \in N$, then $\sigma_C \cdot t_1 = t_1$ and $v_1^2 = \tau_1^2 = t_1$ and σ acts as $v_1 \mapsto \pm v_1$. Since age$(\sigma, \zeta_n) < 1$, there is exactly one node p_1 such that $\sigma \cdot v_1 = -v_1$, that is, σ acts as quasi-reflection on $\mathbb{C}_v^{3g-3}$, a contradiction. □

Proof of the only-if-part of Theorem 6.7. We proved, that if $\big((X, \eta, \beta), \sigma\big)$ is a singularity reduced pair and $\mathrm{age}(\sigma, \zeta_n) < 1$, where $n = \mathrm{ord}(\sigma)$, there exists an elliptic tail $C_j \subset C$ with Aut$(C_j) = \mathbb{Z}_6$ such that ord$(\sigma_{C_j}) \in \{3, 6\}$. Since $\sigma_{C_j}^*(\eta_{C_j}) \cong \eta_{C_j}$, we find that $\eta_{C_j} = \mathcal{O}_{C_j}$. Let $\big((X, \eta, \beta), \sigma\big)$ be a pair consisting of a Prym curve and an automorphism such that the age$(\sigma, \zeta_n) < 1$. By Proposition 6.8 we may deform $((X, \eta, \beta), \sigma)$ to a singularity reduced pair $((X', \eta', \beta'), \sigma')$ such that the actions of σ on $\mathbb{C}_v^{3g-3}$ and σ' on $\mathbb{C}_{v'}^{3g-3}$ have the same ages. Therefore X' has an elliptic tail C'_j with Aut$(C'_j) = \mathbb{Z}_6$ such that $\eta'_{C'_j}$ is trivial and σ' acts on C'_j of order 3 or 6. In the deformation of (X, η, β) to (X', η', β') elliptic tails are preserved hence $\big((X, \eta, \beta), \sigma\big)$ enjoys the same properties. □

Remark 6.17. If $\sigma \in$ Aut(X, η, β) satisfies the inequality $\mathrm{age}(\sigma, \zeta_n) < 1$ (with respect to the action on $\mathbb{C}_v^{3g-3}$), then σ is an elliptic tail automorphism and ord$(\sigma) \in \{3, 6\}$. Indeed, we already know that $\sigma_C \in$ Aut(C) acts with order 3 or 6 on an elliptic tail C_j.

The action of σ on $H^1(C_j^\nu, T_{C_j^\nu}(-D_j))$ and the υ-coordinate corresponding to the elliptic tail node on C_j contributes at least $\frac{2}{3}$ to $\mathrm{age}(\sigma, \zeta_n)$. Thus there is exactly one elliptic tail of order 3 or 6 and σ_C is an elliptic tail automorphism of the same order. If σ is not an elliptic tail automorphism of X, then there exists an exceptional component $E_1 \subset X$ on which σ acts non-trivially. Since E_1 connects two non-exceptional components of X on which σ acts trivially, $\sigma \cdot \upsilon_1 = -\upsilon_1$, giving a contribution of $\frac{1}{2}$ and an age $\geq \frac{2}{3} + \frac{1}{2} \geq 1$.

Proof of Theorem 6.1. We start with a pluricanonical form ω on $\overline{\mathcal{R}}_g^{\mathrm{reg}}$ and show that ω lifts to a desingularization of a neighbourhood of every point $[X, \eta, \beta] \in \overline{\mathcal{R}}_g$. We may assume that $[X, \eta, \beta]$ is a general non-canonical singularity of $\overline{\mathcal{R}}_g$, hence $X = C_1 \cup_p C_2$, where $[C_1, p] \in \mathcal{M}_{g-1,1}$ is general and $[C_2, p] \in \mathcal{M}_{1,1}$ has j-invariant 0. Furthermore $\eta_{C_2} = \mathcal{O}_{C_2}$ and $\eta_1 := \eta_{C_1} \in \mathrm{Pic}^0(C_1)[2]$. We consider the pencil $\phi : \overline{\mathcal{M}}_{1,1} \longrightarrow \overline{\mathcal{R}}_g$ given by $\phi[C', p] = [C' \cup_p C_1, \eta_{C'} = \mathcal{O}_{C'}, \eta_{C_1} = \eta_1]$. Since $\phi(\overline{\mathcal{M}}_{1,1}) \cap \Delta_0^{\mathrm{ram}} = \emptyset$, we imitate [HM] pg. 41-44 and construct an *explicit* open neighbourhood $\overline{\mathcal{R}}_g \supset S \supset \phi(\overline{\mathcal{M}}_{1,1})$ such that the restriction to S of $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ is an isomorphism and every form $\omega \in H^0(\overline{\mathcal{R}}_g^{\mathrm{reg}}, K_{\overline{\mathcal{R}}_g^{\mathrm{reg}}}^{\otimes l})$ extends to a resolution $\widehat{S}$ of S. For an arbitrary non-canonical singularity we show that ω extends locally to a desingularizaton along the lines of [Lud] Theorem 4.1. $\square$

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