THE UNIRULEDNESS OF THE PRYM MODULI SPACE OF GENUS 9

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Abstract. We show that the moduli space $\overline{\mathcal{R}}_9$ of Prym curves of genus 9 is uniruled. This is the largest genus for which such a result is known to hold.

The moduli space $\mathcal{R}_g$ parametrizing pairs $[C, \eta]$, where C is a smooth curve of genus g and $\eta \in \mathrm{Pic}^0(C)[2]$ is a (non-trivial) 2-torsion point in the Jacobian of C has traditionally received considerable attention in the context of finding a uniformization of the moduli space $\mathcal{A}_{g-1}$ of principally polarized abelian varieties of dimension $g-1$ via the Prym map $P_g \colon \mathcal{R}_g \to \mathcal{A}_{g-1}$, see [3, 6, 11]. In particular, in small genus when P_g is a dominant map any result on the birational geometry of $\mathcal{R}_g$ has direct consequences for $\mathcal{A}_{g-1}$. It is known that $\mathcal{R}_g$ is rational for $g = 2, 3, 4$ (see [10, 7]), whereas $\mathcal{R}_5$ is unirational [26, 30]. In genus 6 the Prym map $P_6 \colon \mathcal{R}_6 \to \mathcal{A}_5$ is finite of degree 27 and there are at least two fundamentally different ways of showing that $\mathcal{R}_6$ is unirational, see [11, 29]. Using Nikulin surfaces (that is, $K3$ surfaces endowed with a symplectic involution), we showed that $\mathcal{R}_7$ is unirational as well [18], whereas $\mathcal{R}_8$ is uniruled, see [21].

The Prym moduli space $\mathcal{R}_g$ admits a Deligne-Mumford compactification $\overline{\mathcal{R}}_g := \overline{\mathcal{M}}_g\big(\mathcal{B}\mathbb{Z}_2\big)$, which can be interpreted either as the moduli space of stable maps from curves of genus g to the classifying stack $\mathcal{B}\mathbb{Z}_2$, or in the spirit of Cornalba's work [9], as the stack of stable Prym curves of genus g, see [2, 17]. We denote by $\pi \colon \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ the morphism forgetting the Prym structure. It has been shown in [17] that $\overline{\mathcal{R}}_g$ is a variety of general type whenever $g \geqslant 14$ and $g \neq 15, 16$. Bruns [5] extended this result and showed that $\overline{\mathcal{R}}_{15}$ is of general type. By making use of non-abelian Brill-Noether theory and tropical methods, it has been recently shown in [15] that $\overline{\mathcal{R}}_{13}$ is of general type as well. By combining results from [8, 17], it also follows that the Kodaira dimension of $\overline{\mathcal{R}}_{12}$ is non-negative. On the other hand the case of $\overline{\mathcal{R}}_{16}$ remains open, due to the highly surprising expected failure of the Prym-Green Conjecture in genus 16, as discussed at length in [8, Proposition 4.4][1]. However, we expect that $\overline{\mathcal{R}}_{16}$ is of general type as well. The aim of this paper is to establish the following result:

Theorem 0.1. *The Prym moduli space $\overline{\mathcal{R}}_9$ is uniruled.*

Note that 9 is the largest genus g for which $\overline{\mathcal{R}}_g$ is known to have negative Kodaira dimension. This leaves $\overline{\mathcal{R}}_{10}$ and $\overline{\mathcal{R}}_{11}$ as the only Prym moduli spaces where there is not even a conjectural description concerning their birational nature. In comparison, the Kodaira dimension of $\overline{\mathcal{M}}_g$ remains unknown for $g = 17, \ldots, 21$, see [24, 12, 14], whereas the Kodaira dimension of both

[1] It is precisely the failure locus of the Prym-Green Conjecture on syzygies of Prym-canonical curves which is used to show in [17] that $\overline{\mathcal{R}}_g$ is of general type for a given even genus g.

components $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$ of the moduli space of spin curves is known for all g, see [20]. Note that $\overline{\mathcal{S}}_9^+$ is known to be of general type, whereas $\overline{\mathcal{S}}_9^-$ is uniruled [20] and $g = 9$ is the first case where $\overline{\mathcal{S}}_g^-$ is not known to be unirational. It remains an open question, whether one can find a unirational parametrization for $\overline{\mathcal{R}}_9$.

The proof of Theorem 0.1 relies in an essential way on the study of the following effective divisor of Brill-Noether type on the moduli space $\mathcal{R}_9$

$$\mathcal{D}_9 := \Big\{[C,\eta] \in \mathcal{R}_9 : \exists L \in W^2_8(C) \ \text{ such that } \ H^0(C, L\otimes\eta) \neq 0\Big\},$$

where $W^2_8(C)$ is the Brill-Noether variety of line bundles $L \in \mathrm{Pic}^8(C)$ with $h^0(C,L) \geqslant 3$. By standard Brill-Noether theory, a general curve C of genus 9 has a *finite* number of linear systems $L \in W^2_8(C)$ and the geometric condition defining $\mathcal{D}_9$ can be reformulated as requiring that the line bundle $L\otimes\eta \in \mathrm{Pic}^8(C)$ belongs to the theta divisor of C. As such, this condition is obviously divisorial in moduli. On the one hand, we can extend the determinantal structure defining $\mathcal{D}_9$ to the boundary of $\overline{\mathcal{R}}_9$ and thus compute the class of the closure of $\mathcal{D}_9$ in $\overline{\mathcal{R}}_9$ and we obtain the following formula, see also Theorem 2.2:

Theorem 0.2. *The class of the closure of the Brill-Noether divisor in $\overline{\mathcal{R}}_9$ is given by*

$$[\overline{\mathcal{D}}_9] = 366\lambda - 52\big(\delta_0' + \delta_0''\big) - \frac{187}{2}\delta_0^{\mathrm{ram}} - E, \tag{1}$$

where E is an effective combination of boundary divisors of $\overline{\mathcal{R}}_9$ that does not *contain the boundary classes δ_0' and δ_0^{ram}.*

Here we use the standard notation from [17] for the generators of $\mathrm{Pic}(\overline{\mathcal{R}}_g)$ and refer to Section 1 for details. In particular λ is the Hodge class on $\overline{\mathcal{R}}_g$, whereas the other boundary classes appearing in the statement of Theorem 0.2 are defined by the formula $\pi^*(\delta_0) = \delta_0' + \delta_0'' + 2\delta_0^{\mathrm{ram}}$, where $\delta_0 \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ is the class of the divisor of irreducible stable curves.

The key point in our argument is that each component of the divisor $\overline{\mathcal{D}}_9$ is uniruled and, subject to a delicate transversality assumption which is established in 3.1, we can exhibit a sweeping rational curve $R \subseteq \overline{\mathcal{D}}_9$ satisfying both

$$R \cdot K_{\overline{\mathcal{R}}_9} < 0 \quad \text{and} \quad R\cdot\overline{\mathcal{D}}_9 > 0. \tag{2}$$

If one knew that $\overline{\mathcal{D}}_9$ was irreducible, the inequalities (2) would immediately imply that $K_{\overline{\mathcal{R}}_9}$ cannot be *pseudo-effective*, that is, a limit of effective $\mathbb{Q}$-divisor classes and then using [4] (see also [19, Proposition 5.1]), it would follow that $\overline{\mathcal{R}}_9$ is uniruled. What in fact we *can* show (Theorem 3.3) is that there exists an irreducible component $\overline{\mathcal{D}'}$ of $\overline{\mathcal{D}}_9$ which is swept by rational curves $R \subseteq \overline{\mathcal{D}'}$ such that

$$R \cdot K_{\overline{\mathcal{R}}_9} < 0 \ \text{ and } \ R\cdot\overline{\mathcal{D}'} \geqslant 0,$$

which, as explained, implies via [4] (or [19] *loc.cit.*) that $\overline{\mathcal{R}}_9$ is uniruled. We are able to carry this out, while stopping short of completely determining the class of $\overline{\mathcal{D}'}$.

We now discuss the construction of the pencil R of Prym curves sweeping every component of the divisor $\overline{\mathcal{D}}_9$. We start with a sufficiently general point $[C,\eta] \in \mathcal{D}_9$. Therefore there exists a linear system $L \in W^2_8(C)$ such that $H^0(C, L \otimes \eta) \neq 0$, which by Riemann-Roch also implies that $H^0(C, \overline{L}) \neq 0$, where $\overline{L} := \omega_C \otimes L^\vee \in W^2_8(C)$ is the residual linear system. We write

$$\overline{L} \otimes \eta \cong \mathcal{O}_C(y_1 + \cdots + y_8), \tag{3}$$

where $y_1, \ldots, y_8 \in C$. We may assume that the points $y_1, \ldots, y_8$ are mutually distinct and that L is globally generated and induces a birational map $\varphi_L \colon C \to \Gamma \subseteq \mathbf{P}^2$, where Γ is a nodal plane octic (this last claim follows from e.g. [EH2, Theorem 2]). We denote by $o_1, \ldots, o_{12} \in \mathbf{P}^2$ the nodes of Γ and set $\{x_i, x_i'\} = \varphi_L^{-1}(o_i)$ for $i = 1, \ldots, 12$. Because of the generality of $[C, \eta]$ we may assume that the sets $\{x_1, x_1', \ldots, x_{12}, x_{12}'\}$ and $\{y_1, \ldots, y_8\}$ are disjoint. By adjunction, we have

$$\overline{L} \cong \mathcal{O}_C(4)\Big(-\sum_{i=1}^{12}(x_i + x_i')\Big),$$

where $L = \mathcal{O}_C(1)$, that is, the linear system $|\overline{L}|$ is cut out on Γ by a quartic plane curve passing through the nodes $o_1, \ldots, o_{12}$. From (3), we obtain that

$$\mathcal{O}_C(2y_1 + \cdots + 2y_8) \cong \overline{L}^{\otimes 2} \cong \mathcal{O}_C(8)\Big(-2\sum_{i=1}^{12}(x_i + x_i')\Big), \tag{4}$$

which amounts to saying that there exists an octic curve $\Gamma' \subseteq \mathbf{P}^2$ nodal at the points $o_1, \ldots, o_{12}$ and tangent to Γ at the points $y_1, \ldots, y_8$. Quite remarkably, the curve Γ' has the same numerical characteristics as Γ and we can consider the pencil $\{\Gamma_t\}_{t \in \mathbf{P}^1}$ of octics spanned by Γ and Γ'. Each curve in this pencil has singularities at the points $o_1, \ldots, o_{12}$ and passes through the points $y_1, \ldots, y_8$, therefore its normalization is a curve of genus 9. Because of condition (4), we can lift this pencil to a pencil R of Prym curves in $\overline{\mathcal{R}}_9$, by taking

$$R := \Big\{\big[C_t,\ \eta_t = \omega_{C_t}(-1)(-y_1 - \cdots - y_8)\big]\Big\}_{t \in \mathbf{P}^1} \subseteq \overline{\mathcal{R}}_9, \tag{5}$$

where $\varphi_t \colon C_t \to \Gamma_t \subseteq \mathbf{P}^2$ is the normalization map.

We will establish in Section 3 that we can choose a Prym curve $[C, \eta] \in \mathcal{D}_9$ filling up a codimension one subvariety of $\overline{\mathcal{R}}_9$, that is, an irreducible component $\overline{\mathcal{D}'}$ of $\overline{\mathcal{D}}_9$, such that every curve C_t in the pencil R passing through the point $[C, \eta]$ is *irreducible* and *nodal*. We then compute the intersection of R with the generators of $\mathrm{Pic}(\overline{\mathcal{R}}_9)$. First observe that because Γ and Γ' share a common tangent line at each of the points $y_1, \ldots, y_8$, there will be precisely one curve in the pencil R that has a nodal singularity at y_i. The corresponding Prym curve lies in the boundary divisor Δ_0^{ram} (which can be viewed as the ramification divisor of the finite map $\pi \colon \overline{\mathcal{R}}_9 \to \overline{\mathcal{M}}_9$). Moreover, these are the only points of intersection of R and Δ_0^{ram} and at each of these points the intersection is transverse, which shows that $R \cdot \delta_0^{\mathrm{ram}} = 8$. Calculation to be performed in Theorem 3.1 then imply that

$$R \cdot \lambda = 9,\ R \cdot \delta_0' = 47,\ R \cdot \delta_0'' = 0 \ \text{ and } \ R \cdot \delta_i = R \cdot \delta_{9-i} = R \cdot \delta_{i:9-i} = 0, \text{ for } i = 1, \ldots, 4, \tag{6}$$

which leads to the following intersection number with the canonical class

$$R \cdot K_{\overline{\mathcal{R}}_9} = R \cdot \Big(13\lambda - 2(\delta_0' + \delta_0'') - 3\delta_0^{\mathrm{ram}} - \cdots\Big) = 13 \cdot 9 - 2 \cdot 47 - 3 \cdot 8 = -1. \tag{7}$$

The rational curve R is a moving curve for the divisor $\overline{\mathcal{D}'}$. Since in Section 3 (Theorem 3.3) we also show that any curve $R \subseteq \overline{\mathcal{R}}_9$ having the intersection numbers given by (6) has to intersect non-negatively any divisor disjoint from a generic pencil of Prym curves on a general Nikulin surface, we conclude that $R \cdot \overline{\mathcal{D}'} \geqslant 0$. This implies that $K_{\overline{\mathcal{R}}_9}$ is not pseudo-effective, thus proving Theorem 0.1.

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1. The Brill-Noether divisor on $\overline{\mathcal{R}}_9$

In this Section, after recalling basic facts about the moduli space $\overline{\mathcal{R}}_g$, we compute the class of the divisor $\overline{\mathcal{D}}_9$, then establish various facts about the class of every irreducible component of $\overline{\mathcal{D}}_9$, which will prove to be essential in the proof of Theorem 0.1. It turns out that the formula for the class $[\overline{\mathcal{D}}_9]$ is stated without proof as part of [17, Theorem 0.4]. Due to the importance this class plays in the course of our argument, and because the formula in *loc.cit.* contains several missprints, we shall present all details of the calculation of $[\overline{\mathcal{D}}_9]$.

Recall [3, 2, 17] that $\overline{\mathcal{R}}_g$ denotes the moduli stack of stable Prym curves of genus g, that is, consisting of triples $[X, \eta, \beta]$, where X is a quasi-stable genus g curve, η is a locally free sheaf of total degree 0 on X such that $\eta_{|E} \cong \mathcal{O}_E(1)$ for every smooth rational component $E \subseteq X$ with $|E \cap \overline{X \backslash E}| = 2$ (such a component being called *exceptional*) and $\beta \colon \eta^{\otimes 2} \to \mathcal{O}_X$ is a sheaf morphism that is non-zero along each non-exceptional component of X. There exists a finite branch map $\pi \colon \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ assigning to a triple $[X, \eta, \beta]$ as above the stabilization of X, obtained from X by contracting all of its exceptional components.

The Picard group of $\overline{\mathcal{R}}_g$ is freely generated by the Hodge class λ and the boundary classes $\delta_0', \delta_0'', \delta_0^{\mathrm{ram}}$ and $\delta_i, \delta_{g-i}, \delta_{i:g-i}$, where $i = 1, \ldots, \lfloor \frac{g}{2} \rfloor$. Denoting by $\delta_0 \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ the class of the locus of irreducible stable curves and by $\delta_i \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ for $i = 1, \ldots, \lfloor \frac{g}{2} \rfloor$ the class of the closure of the locus of the union of two smooth curves of genus i and $g - i$ meeting at one point, one has the following relations:

$$\pi^*(\delta_0) = \delta_0^{'} + \delta_0^{''} + 2\delta_0^{\mathrm{ram}} \quad \text{and} \quad \pi^*(\delta_i) = \delta_i + \delta_{g-i} + \delta_{i:g-i}, \text{ for } i \geqslant 1. \tag{8}$$

We now recall the meaning of the classes $\delta_0^{'} := [\Delta_0^{'}]$, $\delta_0^{''} := [\Delta_0^{''}]$ respectively $\delta_0^{\mathrm{ram}} := [\Delta_0^{\mathrm{ram}}]$, while referring to [17] for details. If we fix a general point $[C_{xy}] \in \Delta_0$ induced by a 2-pointed curve $[C, x, y] \in \mathcal{M}_{g-1,2}$ and the normalization map $\nu \colon C \to C_{xy}$, where $\nu(x) = \nu(y)$, a general point of the irreducible divisor Δ_0' (respectively of Δ_0'') corresponds to a stable Prym curve $[C_{xy}, \eta]$, where $\eta \in \mathrm{Pic}^0(C_{xy})[2]$ and $\nu^*(\eta) \in \mathrm{Pic}^0(C)$ is non-trivial (respectively trivial). A general point of Δ_0^{ram} is of the form $[X, \eta]$, where $X := C \cup_{\{x,y\}} \mathbf{P}^1$ is a quasi-stable curve and $\eta \in \mathrm{Pic}^0(X)$ satisfies $\eta_{\mathbf{P}^1} \cong \mathcal{O}_{\mathbf{P}^1}(1)$ and $\eta_C^{\otimes 2} \cong \mathcal{O}_C(-x - y)$. Note that Δ_0^{ram} corresponds generically to 1-nodal irreducible curves where the Prym structure is not free at the node of the underlying curve.

Applying the Hurwitz formula to the branch cover $\pi\colon \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$, we also have

$$K_{\overline{\mathcal{R}}_g} = 13\lambda - 2\big(\delta_0' + \delta_0''\big) - 3\delta_0^{\mathrm{ram}} - 3\big(\delta_1 + \delta_{g-1} + \delta_{1:g-1}\big) - 2\sum_{i=2}^{\lfloor \frac{g}{2} \rfloor}\big(\delta_i + \delta_{g-i} + \delta_{i:g-i}\big). \tag{9}$$

1.1. **The divisor $\mathcal{D}_9$.** We now specialize to the case of $\overline{\mathcal{R}}_9$. From the Brill-Noether Theorem [1] it follows that for a general curve $[C] \in \mathcal{M}_9$ the Brill-Noether variety $G^2_8(C)$ is finite and consists of 42 distinct points. We denote by $\mathcal{G}^2_8 \to \mathcal{M}_9$ the Deligne-Mumford stack of linear systems classifying pairs $[C, \ell]$, where $[C] \in \mathcal{M}_9$ and $\ell = (L, V) \in G^2_8(C)$.

We introduce the *Brill-Noether divisor* on $\mathcal{R}_9$

$$\mathcal{D}_9 := \Big\{[C,\eta] \in \mathcal{R}_9 : \exists L \in W^2_8(C) \;\text{ such that }\; h^0(C, L\otimes \eta) \geqslant 1\Big\}.$$

It follows from [17, Theorem 2.3] that $\mathcal{D}_9$ is an effective divisor on $\mathcal{R}_9$.

In order to realize $\mathcal{D}_9$ as the degeneracy locus of two vector bundles of the same rank over $\mathcal{R}_9$, for a pair $[C, L] \in \mathcal{G}^2_8$ such that L is globally generated and $h^0(C, L) = 3$, let M_L be the rank 2 syzygy bundle defined by the exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(C, L) \otimes \mathcal{O}_C \longrightarrow L \longrightarrow 0. \tag{10}$$

Tensoring the resulting exact sequence with η, for a triple $[C, L, \eta] \in \mathcal{RG}^2_8 := \mathcal{G}^2_8 \times_{\mathcal{M}_9} \mathcal{R}_9$ the condition $H^0(C, L \otimes \eta) \neq 0$ is then equivalent to the non-injectivity of the following map

$$\chi_{C,L,\eta}\colon H^1\big(C, M_L \otimes \eta\big) \longrightarrow H^0(C, L\otimes \eta) \otimes H^1(C, \eta). \tag{11}$$

Since $H^0\big(C, M_L\otimes\eta\big) = 0$, by Riemann-Roch $h^1\big(C, M_L \otimes \eta\big) = -\deg(M_L) + 2\cdot(g-1) = \deg(L) + 2\cdot 8 = 24$, whereas clearly $\dim H^0(C, L) \otimes H^1(C, \eta) = h^0(C, L)\cdot h^0(C, \omega_C\otimes\eta) = 3\cdot 8 = 24$, that is, $\chi_{C,L,\eta}$ is a map between vector spaces of the same dimension. We globalize the description (11) to a morphism of vector bundles of the same rank over the moduli stack of Prym curves.

Let $\mathcal{M}_9^\circ \subseteq \mathcal{M}_9$ be the open substack of smooth curves C of genus 9 such that $G^2_7(C) = \varnothing$. It is easy to see that $\mathrm{codim}(\mathcal{M}_9 - \mathcal{M}_g^\circ, \mathcal{M}_9) \geqslant 2$. In particular, $h^0(C, L) = 3$, for every $[C] \in \mathcal{M}_9^\circ$ and $L \in W^2_8(C)$. We denote by $\Delta_0^\circ \subseteq \Delta_0 \subset \overline{\mathcal{M}}_9$ the locus of nodal curves $[C_{xy} = C/x \sim y]$, where C is a smooth curve of genus 8 such that $G^2_7(C) = \varnothing$ and $x, y \in C$ (note that we allow for the possibility $x = y$, in which case we attach an elliptic tail to C at the point x). Set

$$\widetilde{\mathcal{M}}_9 := \mathcal{M}_9^\circ \cup \Delta_0^\circ \subseteq \overline{\mathcal{M}}_9 \quad\text{and}\quad \widetilde{\mathcal{R}}_9 := \pi^{-1}(\widetilde{\mathcal{M}}_9) \subseteq \overline{\mathcal{R}}_9.$$

Observe that $\mathrm{Pic}(\widetilde{\mathcal{R}}_9)_{\mathbb{Q}}$ is freely generated by the classes $\lambda, \delta_0', \delta_0''$ and δ_0^{ram}. Let

$$\widetilde{\mathcal{G}}^2_8 \to \widetilde{\mathcal{M}}_9$$

be the stack classifying pairs $[C, L]$, where $[C] \in \widetilde{\mathcal{M}}_9$ and L is a torsion free sheaf on C having degree 8 and with $h^0(C, L) = 3$. Note that in the case of an element $[C_{xy}, L]$, where C_{xy} is a

1-nodal curve, the assumption $G^2_7(C) = \emptyset$ guarantees that L is necessarily locally free, else $L = \nu_*(A)$, where $A \in G^2_7(C)$, where we recall that $\nu\colon C \to C_{xy}$ denotes the normalization map. For further details, we refer to [17, page 770].

We finally introduce the stack of linear series over Prym curves

$$\sigma\colon \widetilde{\mathcal{RG}}^2_8 := \widetilde{\mathcal{R}}_9 \times_{\widetilde{\mathcal{M}}_9} \widetilde{\mathcal{G}}^2_8 \to \widetilde{\mathcal{R}}_9$$

and consider the universal Prym curve of genus 9 over it

$$f\colon \mathcal{C} \to \widetilde{\mathcal{RG}}^2_8.$$

Note that, if $t = \bigl[C \cup_{\{x,y\}} \mathbf{P}^1, \eta, L\bigr] \in \sigma^{-1}(\Delta_0^{\mathrm{ram}})$, where C is a smooth curve of genus 8, then $f^{-1}(t) = C \cup_{\{x,y\}} \mathbf{P}^1$, cf. [17, 1.1].

At the level of $\mathcal{C}$ we have a universal *Prym* line bundle $\mathcal{P}$ and a *Poincaré* line bundle which are characterized by the property $\mathcal{P}_{|f^{-1}[X,\eta,\beta,L]} = \eta \in \mathrm{Pic}^0(X)$ and $\mathcal{L}_{|f^{-1}[X,\eta,\beta,L]} = L \in \mathrm{Pic}^8(X)$, for each point $[X, \eta, \beta, L] \in \widetilde{\mathcal{RG}}^2_8$.

Following [17, 13], we introduce the codimension 1 following tautological classes in $\mathrm{Pic}\bigl(\widetilde{\mathcal{RG}}^2_8\bigr)$:

$$\mathfrak{a} := f_*\bigl(c_1(\mathcal{L})^2\bigr) \quad \text{and} \quad \mathfrak{b} := f_*\bigl(c_1(\mathcal{L}) \cdot c_1(\omega_f)\bigr). \tag{12}$$

We shall also need the tautological rank 3 vector bundle on $\widetilde{\mathcal{RG}}^2_8$

$$\mathcal{V} := f_*(\mathcal{L}).$$

The fact that $\mathcal{V}$ is locally free follows from Grauert's theorem, since as we already explained, $h^0(C, L) = 3$, for every $[C, L] \in \widetilde{\mathcal{G}}^2_8$.

We record the following formulas describing the push-forward of these tautological classes under the generically finite morphism σ, see [13]:

$$\sigma_*(\mathfrak{a}) = -564\lambda + 83\bigl(\delta_0' + \delta_0'' + 2\delta_{\mathrm{ram}}\bigr), \quad \text{and} \quad \sigma_*(\mathfrak{b}) = 252\lambda - 21\bigl(\delta_0' + \delta_0'' + 2\delta_0^{\mathrm{ram}}\bigr). \tag{13}$$

The class $\sigma_*\bigl(c_1(\mathcal{V})\bigr)$ can also be determined, see [17], but it will not be used in what follows.

We consider the global syzygy bundle on $\mathcal{C}$ defined by the exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow f^*(\mathcal{V}) \longrightarrow \mathcal{L} \longrightarrow 0.$$

From this sequence, we obtain the following formulas which we record:

$$c_1(\mathcal{M}) = f^*c_1(\mathcal{V}) - c_1(\mathcal{L}) \quad \text{and} \quad c_2(\mathcal{M}) = f^*c_2(\mathcal{M}) + c_1^2(\mathcal{L}) - f^*c_1(\mathcal{V}) \cdot c_1(\mathcal{L}). \tag{14}$$

We then introduce the following vector bundles over $\widetilde{\mathcal{RG}}^2_8$:

$$\mathcal{A} := R^1 f_*\bigl(\mathcal{M} \otimes \mathcal{P}\bigr) \quad \text{and} \quad \mathcal{B} := f^*\bigl(\mathcal{V} \otimes R^1 f_*(\mathcal{P})\bigr),$$

having fibres $\mathcal{A}\bigl([X, \eta, \beta, L]\bigr) = H^1(X, M_L \otimes \eta)$ and $\mathcal{B}\bigl([X, \eta, \beta, L]\bigr) = H^0(X, L) \otimes H^1(X, \eta)$. Note that both $\mathcal{A}$ and $\mathcal{B}$ are locally free sheaves of rank 24 and there exists a natural morphism

$$\chi\colon \mathcal{A}\to\mathcal{B}, \tag{15}$$

which over a point $[C,\eta,L]$ corresponding to a smooth curve C globalizes the maps (11). Let $\widetilde{Z}$ be the degeneracy locus of the morphism χ and $Z:=\tilde{Z}\cap(\pi\circ\sigma)^{-1}\big(\mathcal{M}_9^{\circ}\big)$. Therefore, $\sigma(Z)$ coincides over $\mathcal{R}_9^{\circ}=\pi^{-1}(\mathcal{M}_9^{\circ})$ with the divisor $\mathcal{D}_9$.

Theorem 1.1. *The class of the degeneracy locus $\widetilde{Z}$ of the morphism $\chi\colon\mathcal{A}\to\mathcal{B}$ equals*

$$[\widetilde{Z}]=c_1(\mathcal{B}-\mathcal{A})=-\lambda-\frac{\mathfrak{a}}{2}+\frac{\mathfrak{b}}{2}+\frac{1}{4}\sigma^*(\delta_0^{\mathrm{ram}}).$$

Proof. Calculating the class of $\mathcal{B}$ is straightforward. Using [17, Proposition 1.7] we have that $c_1\big(R^1f_*(\mathcal{P})\big)=-c_1\big(f_*(\omega_f\otimes\mathcal{P}^{\vee})\big)=-\lambda+\frac14\sigma^*(\delta_0^{\mathrm{ram}})$, therefore

$$c_1(\mathcal{B})=8f^*c_1(\mathcal{V})+3c_1\big(R^1f_*(\mathcal{P})\big)=8f^*\big(c_1(\mathcal{V})\big)-3\lambda+\frac{3}{4}\sigma^*(\delta_0^{\mathrm{ram}}). \tag{16}$$

In order to calculate $c_1(\mathcal{A})$ we apply Grothendieck-Riemann-Roch to the universal curve $f\colon\mathcal{C}\to\widetilde{\mathcal{RG}}_8^2$ and to the vector bundle $\mathcal{M}\otimes\mathcal{P}$ and we write:

$$-c_1(\mathcal{A})=-c_1\big(R^1f_*(\mathcal{M}\otimes\mathcal{P})\big)=f_*\Big[\Big(2+c_1(\mathcal{M}\otimes\mathcal{P})+\frac{c_1^2(\mathcal{M}\otimes\mathcal{P})-2c_2(\mathcal{M}\otimes\mathcal{P})}{2}\Big)\cdot$$
$$\Big(1-\frac{c_1(\omega_f)}{2}+\frac{c_1^2(\omega_f)+[\mathrm{Sing}(f)]}{12}\Big)\Big]_2,$$

where $\mathrm{Sing}(f)\subseteq\mathcal{C}$ denotes the codimension 2 singular locus (that is, the locus of nodes) of the universal curve f. Clearly $f_*[\mathrm{Sing}(f)]=\sigma^*\big(\Delta_0'+\Delta_0''+2\Delta_0^{\mathrm{ram}}\big)$. To estimate the degree 2 terms appearing in the right hand side of this formula, we use Mumford's formula [24] $f_*\big(c_1^2(\omega_f)\big)=12\lambda-\sigma^*(\delta_0'+\delta_0''+2\delta_0^{\mathrm{ram}})$, coupled with the formulas (14), as well as with the following formulas, see [17, Proposition 1.6]:

$$f_*\big(c_1^2(\mathcal{P})\big)=-\frac{\delta_0^{\mathrm{ram}}}{2},\quad f_*(c_1(\omega_f)\cdot c_1(\mathcal{P}))=0,\quad f_*\big(c_1(\omega_f)\cdot f^*c_1(\mathcal{V})\big)=16c_1(\mathcal{V}),$$
$$f_*\big(c_1(\mathcal{L})\cdot c_1(\mathcal{P})\big)=f_*\big(f^*(c_1(\mathcal{V})\cdot c_1(\mathcal{P})\big)=0,\quad f_*\big(c_1(\mathcal{L})\cdot f^*c_1(\mathcal{V})\big)=8c_1(\mathcal{V}).$$

Making also use of the formula $c_2(\mathcal{M}\otimes\mathcal{P})=c_2(\mathcal{M})+c_1^2(\mathcal{P})+c_1(\mathcal{M})\cdot c_1(\mathcal{P})$, we conclude that

$$-c_1(\mathcal{A})=2\lambda+\frac{\mathfrak{b}}{2}-\frac{\mathfrak{a}}{2}-8c_1(\mathcal{V})-\frac{1}{2}\sigma^*(\delta_0^{\mathrm{ram}}). \tag{17}$$

Combining (17) and (16), we find $[\widetilde{Z}]=c_1(\mathcal{B})-c_1(\mathcal{A})=-\lambda+\frac{\mathfrak{b}}{2}-\frac{\mathfrak{a}}{2}+\frac14\sigma^*(\delta_0^{\mathrm{ram}})$, which finishes the proof. $\square$

2. Nikulin surfaces and the divisor $\overline{\mathcal{D}}_9$

In this section we show that a general pencil of Prym curves lying on a Nikulin surface is disjoint from the divisor $\overline{\mathcal{D}}_9$. We begin by recalling basic facts about the connection between Nikulin surfaces and Prym curves, our main references being [18, 22].

A *Nikulin surface* is a $K3$ surface S endowed with a symplectic automorphism, or equivalently with a non-trivial double cover

$$f\colon \tilde{S} \to S$$

having a branch divisor $N := N_1 + \cdots + N_8$ consisting of 8 disjoint smooth rational curves $N_i \subseteq S$. The class $[N]$ is divisible by 2 and we set $\mathfrak{e} := \frac{1}{2}\big([N_1] + \cdots + [N_8]\big) \in \mathrm{Pic}(S)$ and define the *Nikulin lattice* to be the rank 8 lattice $\mathfrak{N} \subseteq \mathrm{Pic}(S)$ generated by $[N_1], \ldots, [N_8]$ and by $\mathfrak{e}$. A *polarized* Nikulin surface of genus g is a Nikulin surface $f\colon \tilde{S} \to S$ as above, together with a smooth curve $C \subset S$ of genus g such that $C \cdot N_i = 0$, for $i = 1, \ldots, 8$. There is an irreducible 11-dimensional moduli space $\mathcal{F}_g^{\mathfrak{N}}$ of polarized Nikulin surfaces of genus g and for a general such surface one has $\mathrm{Pic}(S) \cong \mathbb{Z} \cdot [C] \oplus \mathfrak{N}$. If $\tilde{C} := f^{-1}(C)$, then $f_C := f_{|\tilde{C}}\colon \tilde{C} \to C$ is an étale double covering and $\mathfrak{e}_C := \mathcal{O}_C(\mathfrak{e}) \in \mathrm{Pic}^0(C)$ is the non trivial 2-torsion element defining the covering f_C.

We define the *Nikulin pencil* in $\overline{\mathcal{R}}_g$ to be the pencil $\Xi_g \subseteq \overline{\mathcal{R}}_g$ consisting of Prym curves $\big\{[C_t, \mathfrak{e}_{C_t}]\big\}_{t\in\mathbf{P}^1}$ induced by a Lefschetz pencil $\big\{C_t\big\}_{t\in\mathbf{P}^1}$ on a general polarized Nikulin surface S. The following formulas hold, see [18, Proposition 1.4].

$$\Xi_g \cdot \lambda = g + 1, \quad \Xi_g \cdot \delta_0' = 6g + 2, \quad \Xi_g \cdot \delta_0'' = 0 \quad \text{and} \quad \Xi_g \cdot \delta_0^{\mathrm{ram}} = 8. \tag{18}$$

All elements of Ξ_g are irreducible curves, therefore the intersection of Ξ_g with the other boundary divisors in $\overline{\mathcal{R}}_g$ are equal to zero.

2.1. **Moduli of vector bundles on Nikulin surfaces.** Given a smooth $K3$ surface S, the *Mukai pairing* [25, 28] on $H^\bullet(S)$ is defined by

$$(v_0, v_1, v_2) \cdot (w_0.w_1, w_2) := v_1 \cdot w_1 - v_2 \cdot w_0 - v_0 \cdot w_2 \in H^4(S, \mathbb{Z}) \cong \mathbb{Z}.$$

For a sheaf F on S, let $v(F) := \Big(\mathrm{rk}(F), c_1(F), \chi(F) - \mathrm{rk}(F)\Big)$ be its Mukai vector. For a polarization $H \in \mathrm{Pic}(S)$, let $M_H(v)$ be the moduli space of S-equivalence classes of (Gieseker) H-semistable sheaves F on S with Mukai vector $v(F) = v$. Let $M_H^s(v)$ the open subset of $M_H(v)$ corresponding to H-stable sheaves. Then it is known that $M_H^s(v)$ is pure dimensional and $\dim M_H^s(v) = v^2 + 2$. In particular, if $v(F)^2 < -2$, then F is not stable.

Specializing now to the case when S is a Nikulin surface, we now show that the pencil Ξ_9 is disjoint from the divisor $\overline{\mathcal{D}}_9$. The following result uses essential input from M. Lelli-Chiesa:

Theorem 2.1. *Let $f\colon \tilde{S} \to S$ be a polarized Nikulin surface of genus 9 with $\mathrm{Pic}(S) \cong \mathbb{Z}\cdot C \oplus \mathfrak{N}$ and let $\Xi_9 \subseteq \overline{\mathcal{R}}_9$ be an induced Lefschetz pencil of Prym curves on S. Then $\Xi_9 \cap \overline{\mathcal{D}}_9 = \emptyset$.*

Proof. Let $C \subseteq S$ be a curve in the polarization class $|C|$ of S, thus $C \cdot N_i = 0$ for $i = 1, \ldots, 8$. We fix $L \in W_8^2(C)$ such that $H^0(C, L \otimes \mathfrak{e}_C) \neq 0$ and set again $\overline{L} := \omega_C \otimes L$. Each curve in the linear system $|C|$ is Brill-Noether general, cf. [16, Lemma 5.1], in particular L is globally generated and we consider the associated *Lazarsfeld-Mukai bundle* [27]

$$0 \longrightarrow E_{C,L}^\vee \longrightarrow H^0(C, L) \otimes \mathcal{O}_S \longrightarrow C \longrightarrow 0.$$

By dualizing one has the following exact sequence

$$0 \longrightarrow H^0(C, L)^\vee \otimes \mathcal{O}_C \longrightarrow E_{C,L} \longrightarrow \overline{L} \longrightarrow 0. \tag{19}$$

Observe that $E_{C,L}$ is globally generated. Tensoring the exact sequence (19) by $\mathfrak{e}^\vee$, taking global section and using that $H^i\big(S, \mathfrak{e}^\vee\big) = 0$ for all i (cf. [18, Lemma 1.3]), we conclude that $H^1\big(S, E_{C,L} \otimes \mathfrak{e}^\vee\big) \xrightarrow{\cong} H^1\big(C, \overline{L} \otimes \mathfrak{e}_C^\vee\big)$, therefore $H^1\big(S, E_{C,L} \otimes \mathfrak{e}^\vee\big) \neq 0$. It follows that $\mathrm{Ext}^1(E_{C,L}, \mathfrak{e}) \neq 0$, that is, there exist a non-trivial extension

$$0 \longrightarrow \mathfrak{e} \longrightarrow E \longrightarrow E_{C,L} \longrightarrow 0. \tag{20}$$

Since $c_1(E_{C,L}) = [C]$, whereas $h^0\big(S, E_{C,L}\big) = h^0(C, L) + h^0(C, \overline{L}) = 6$ and $h^i\big(S, E_{C,L}\big) = 0$ for $i = 1, 2$, we compute the Mukai vector

$$v(E) = \big(4, [C] + \mathfrak{e}, 2\big),$$

therefore $v(E)^2 = -4 < -2$, in particular the vector bundle E is not stable and hence also not $\mu = \mu_C$-stable, that is, slope stable with respect to the polarization defined as $\mu_C(F) := \frac{c_1(F)\cdot C}{\mathrm{rk}(F)}$, for any coherent sheaf F on S[2]

Let E_1 be a maximally destabilizing subsheaf of E of maximal rank $r := \mathrm{rk}(E_1) \leqslant 3$. We may assume that the quotient $G = E/E_1$ is torsion free, hence E_1 is locally free and we have the following diagram:

$$\begin{tikzcd}
 & & E_1 \arrow[d, hook] \arrow[dr, "\phi"] & & \\
0 \arrow[r] & \mathfrak{e} \arrow[r] & E \arrow[r] \arrow[d, two heads] & E_{C,L} \arrow[r] & 0 \\
 & & G & &
\end{tikzcd} \tag{21}$$

We write $c_1(E_1) = a[C] + N'$, where $N' \in \mathfrak{N}$, in particular $C \cdot N' = 0$. Then we write $\mu(E_1) = \frac{16a}{r} \geqslant \mu(E) = 4$, which yields $a \geqslant 1$.

We claim that $\mathrm{Hom}(E_1, \mathfrak{e}) = 0$, therefore the image $\phi \in \mathrm{Hom}(E_1, E_{C,L})$ of the injection $E_1 \hookrightarrow E$ is a non-zero morphism. Indeed, assuming $0 \neq h \in \mathrm{Hom}(E_1, \mathfrak{e})$, set $E_1' := \mathrm{Ker}(h)$ and write down an exact sequence

$$0 \longrightarrow E_1' \longrightarrow E_1 \xrightarrow{h} \mathfrak{e}(-D) \otimes \mathcal{I}_{\xi/S} \longrightarrow 0,$$

where ξ is a 0-dimensional subscheme and D is an effective divisor on S respectively. In particular, $c_1(E_1')\cdot C = c_1(E_1)\cdot C + D\cdot C \geqslant c_1(E_1)\cdot C$ and therefore $\mu(E_1') = \frac{c_1(E_1')\cdot C}{\mathrm{rk}\, E_1'} > \frac{c_1(E_1)\cdot C}{\mathrm{rk}(E_1)}$, thus contradicting the maximality of E_1 among all destabilizing subsheaves of E.

[2] Usually the Mumford-Takemoto $\mu := \mu_C$-stability is defined with respect to an ample line bundle on S, but as pointed out in both [23] and [25, Remark 4.C.4] this assumption is too strong and can be replaced with the one that C be big and nef, which is precisely the case at hand when C is the polarization of a Nikulin surface.

Therefore $\phi\colon E_1 \to E_{C,L}$ is a non-zero morphism. Since the Lazarsfeld-Mukai bundle $E_{C,L}$ is easily shown to be μ-semistable (the same proof as in the case of $K3$ surfaces of Picard number one treated in [27] applies here as well), it follows that $c_1\big(E_{C,L} \otimes E_1^\vee\big) \cdot C \geqslant 0$, which yields $r = 3$ and $a = 1$, that is, $c_1(E_1') = C + N'$.

Assume first ϕ is injective. Accordingly we have an exact sequence

$$0 \longrightarrow E_1 \longrightarrow E_{C,L} \longrightarrow Q \longrightarrow 0,$$

where $Q = \mathcal{I}_{\xi/S}(-N')$. Since $E_{C,L}$ is globally generated, Q must be globally generated as well, which is impossible, for no sheaf of type $\mathcal{I}_\xi\big(b_1 N_1 + \cdots + b_8 N_8\big)$ can be globally generated. In the general case, set $\mathcal{K} := \mathrm{Ker}(\phi)$, in particular $\mathcal{K} \hookrightarrow \mathcal{M}$. Setting $Q := \mathrm{Coker}(\phi)$, we again conclude that $c_1(Q) \in \mathfrak{N}$, whereas Q must be globally generated, which is a contradiction.

We finish the argument by noticing that the same reasoning works for a 1-nodal curve $C \in |C|$ and for a globally generated line bundle $L \in W^2_8(C)$, therefore $\Xi_9 \cap \overline{\mathcal{D}}_9 = \varnothing$. □

We are now in a position to finish the calculation of the class of the closure of the divisor $\mathcal{D}_9$.

Theorem 2.2. *One has the following formula for the closure $\widetilde{\mathcal{D}}_9$ in $\widetilde{\mathcal{R}}_9$ of the divisor $\mathcal{D}_9$:*

$$[\widetilde{\mathcal{D}}_9] = 366\lambda - 52\big(\delta_0' + \delta_0''\big) - \frac{187}{2}\delta_0^{\mathrm{ram}} - \alpha \cdot \delta_0'' \in \mathrm{Pic}(\widetilde{\mathcal{R}}_9),$$

where $\alpha \geqslant 0$.

Proof. We determined in Theorem 1.1 the class of the degeneracy locus Z of the morphism $\chi\colon \mathcal{A} \to \mathcal{B}$ of vector bundles over $\widetilde{\mathcal{RG}}^2_8$ defined in (15). Using the formulas (13), we conclude that $\sigma_*\big([\widetilde{Z}]\big) = 366\lambda - 52\big(\delta_0' + \delta_0''\big) - \frac{187}{2}\delta_0^{\mathrm{ram}}$. Furthermore, Theorem 2.1 shows that the morphism χ is generically non-degenerate along (each component of) the boundary divisors $\sigma^*(\delta_0')$ and $\sigma^*(\delta_0^{\mathrm{ram}})$, that is, the class of $\sigma_*(\widetilde{Z})$ and that of $\widetilde{\mathcal{D}}_9$ coincide up to a multiple of δ_0''. The precise value of this multiple plays no further role in our argument. □

3. The uniruled parametrization of $\overline{\mathcal{D}}_9$

We now spell out the uniruled parametrization of the divisor $\overline{\mathcal{D}}_9$ that was sketched in the Introduction. Note that we do not establish that $\overline{\mathcal{D}}_9$ is irreducible, instead we show that each irreducible component of $\overline{\mathcal{D}}_9$ is uniruled.

Let $[C, L] \in \mathcal{G}^2_8$ be a general element corresponding to a smooth curve C of genus 9 and a complete, globally generated linear system $L \in G^2_8(C)$. Let $\varphi_L\colon C \to \mathbf{P}^2$ be the map induced by $|L|$ and set $\Gamma := \varphi_L(C)$ and $\overline{L} := \omega_C \otimes L^\vee$. Since C may be assumed to be Petri general, $H^1(C, \overline{L}^2) = 0$, therefore $h^0\big(C, \overline{L}^2\big) = 8$. Due to our generality assumptions, Γ is a nodal octic and let $o_1, \ldots, o_{12}$ be its nodes and denote $\xi := o_1 + \cdots + o_{12} \in \mathrm{Hilb}^{12}(\mathbf{P}^2)$. Set

$$\epsilon\colon X := \mathrm{Bl}_\xi(\mathbf{P}^2) \longrightarrow \mathbf{P}^2$$

and denote by $E_1, \dots, E_{12}$ the exceptional divisors corresponding to the 12 points and by $h := \epsilon^*(\mathcal{O}_{\mathbf{P}^2}(1)) \in \mathrm{Pic}(X)$ the hyperplane class. Note that X is also endowed with a regular degree 4 cover

$$q := \varphi_{|4h-E_1-\cdots-E_{12}|}\colon X \longrightarrow \mathbf{P}^2. \tag{22}$$

Conversely, we introduce the universal parameter space of 12-nodal plane octics, that is,

$$\mathcal{P} := \Big\{\big(o_1+\cdots+o_{12}, C\big) : o_1+\cdots+o_{12} \in \mathrm{Hilb}^{12}(\mathbf{P}^2),\ C \in \big|8h-2E_1-\cdots-2E_{12}\big|\Big\},$$

then observe that $\mathcal{P} \to \mathrm{Hilb}^{12}(\mathbf{P}^2)$ is generically a $\mathbf{P}^8$-bundle and the map

$$\mathcal{P}/\!/SL(3) \dashrightarrow \mathcal{G}^2_8, \quad \big(o_1+\cdots+o_{12},\ C\big) \mapsto [C, \mathcal{O}_C(h)]$$

is a birational isomorphism.

By adjunction $\overline{L} \cong \mathcal{O}_C\big(4h-E_1-\cdots-E_{12}\big)$ and the restriction map $\big|\mathcal{O}_X(C)\big| \to \big|\mathcal{O}_C(C)\big| = \big|\overline{L}^2\big|$ is dominant, in particular every element of the linear system $|\overline{L}^2|$ is cut out by an plane octic singular at $o_1, \dots, o_{12}$.

Note that the resolution of the scheme $\xi = o_1+\cdots+o_{12}$ of 12 general points in $\mathbf{P}^2$ has the following form described by the Hilbert-Burch theorem:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-6)^2 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-4)^3 \longrightarrow \mathcal{I}_\xi \longrightarrow 0.$$

Precisely, one picks general quadratic forms $a, b, c, a', b', c' \in H^0\big(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2)\big)$ and then ξ can be described by the following condition in $\mathbf{P}^2 = \mathbf{P}^2_{[x_1,x_2,x_3]}$:

$$\mathrm{rk}\begin{pmatrix} a(x_1,x_2,x_3) & b(x_1,x_2,x_3) & c(x_1,x_2,x_3) \\ a'(x_1,x_2,x_3) & b'(x_1,x_2,x_3) & c'(x_1,x_2,x_3) \end{pmatrix} \leqslant 1 \tag{23}$$

The quadruple cover $q\colon X \to \mathbf{P}^2$ defined by (22) is then the resolution of the rational map $\mathbf{P}^2 \overset{4:1}{\dashrightarrow} \mathbf{P}^2$ given by $(ab'-a'b,\ ac'-a'c,\ bc'-b'c)$.

Assume $\big[C, L, \eta\big] \in \mathcal{RG}^2_8$ is an element such that $H^0(C, L\otimes\eta) \neq 0$, that is, $[C,\eta] \in \mathcal{D}_9$. Write $\overline{L}\otimes\eta = \mathcal{O}_C(y_1+\cdots+y_8)$, therefore $\overline{L}^{\otimes 2} \cong \mathcal{O}_C(2y_1+\cdots+2y_8)$. As long as C is a Petri general curve, it follows that there exists a curve on X

$$C' \in \big|8h-2E_1-\cdots-2E_{12}\big| \quad \text{such that} \quad C'\cdot C = 2y_1+\cdots+2y_8.$$

As explained in the Introduction, we consider the pencil $R \subseteq \overline{\mathcal{R}}_9$ generated by the curves C and C'. Its general element is of the form $\big[C_t, \mathcal{O}_{C_t}\big(4h-E_1-\cdots-E_{12}\big)(-y_1-\cdots-y_8)\big]$.

Working globally, we can consider the closure $\mathcal{Y}$ inside $\mathcal{P}\times_{\mathrm{Hilb}^{12}(\mathbf{P}^2)}\mathcal{P}$ of the following locus

$$\Big\{(\xi, C, C') \in \mathcal{P}\times_{\mathrm{Hilb}^{12}(\mathbf{P}^2)}\mathcal{P} : C, C' \in \big|8h-2E_1-\cdots-2E_{12}\big|,\ C' \neq C \ \text{ with } \ C\cdot C' = 2y_1+\cdots+2y_8\Big\}.$$

The condition defining $\mathcal{Y}$ can be reformulated as saying that there exist points $y_1, \ldots, y_8 \in X$, such that the curves C and C' intersect non-transversally at $y_1, \ldots, y_8$. Observe that the quotient $\mathcal{Y}/\!/SL(3)$ has two $\mathbf{P}^1$-bundle structures over the Brill-Noether divisor

$$\begin{tikzcd}[column sep=small] & \mathcal{Y}/\!/SL(3) \arrow[dl, "\upsilon_1"'] \arrow[dr, "\upsilon_2"] & \\ \overline{\mathcal{D}}_9 & & \overline{\mathcal{D}}_9 \end{tikzcd} \tag{24}$$

defined by forgetting either C or C', that is, $\upsilon_1\big(\xi, C, C'\big) := \big[C, \omega_C(-h)(-y_1 - \cdots - y_8)\big]$ respectively $\upsilon_2\big(\xi, C, C'\big) := \big[C', \omega_{C'}(-h)(-y_1 - \cdots - y_8)\big]$. The rational curve R passing through a general point $[C, \eta]$ in a component of $\overline{\mathcal{D}}_9$ corresponds precisely to $\upsilon_2\big(\upsilon_1^{-1}\big([C,\eta]\big)\big)$.

We make now the assumption (†) that for a given element $[C, L, \eta]$ as above, every curve in the pencil R spanned by C and C' is irreducible and at most 1-nodal. Subject to this assumption, which we establish later along at least one irreducible component of the divisor $\overline{\mathcal{D}}_9$, we can compute the intersection numbers of R and establish (6) from the Introduction.

Theorem 3.1. *Assuming (†) holds, one has the following intersection numbers:*

$$R \cdot \lambda = 9, \quad R \cdot \delta_0' = 47, \quad R \cdot \delta_0^{\mathrm{ram}} = 8, \quad R \cdot \delta_0'' = 0.$$

Proof. The assumption (†) implies that every curve in the pencil $R = \{C_t\}_{t \in \mathbf{P}^1}$ spanned by C and C' is irreducible and nodal. Furthermore, all curves in R have a common tangent line ℓ_i at each of the points y_i for $i = 1, \ldots, 8$. We consider the blow-up of the rational surface X at the points $y_1, \ldots, y_8$, as well as the points $\ell_1, \ldots, \ell_8$ regarded as points on the exceptional divisors $E_{y_1}, \ldots, E_{y_8}$ and set $X' := \mathrm{Bl}_{\{y_1, \ldots, y_8, \ell_1, \ldots, \ell_8\}}(X)$. We have an induced fibration

$$u \colon X' \longrightarrow \mathbf{P}^1,$$

where $u^{-1}(t)$ is the proper transform of C_t. Note that in the pencil R there exists for each $i = 1, \ldots, 8$ precisely one curve $C_i = C_{t_i}$ that is singular at y_i (and smooth at the remaining points y_j, with $j \neq i$). In this case $u^{-1}(t_i) = C'_{t_i} + E_{y_i}$, where C'_{t_i} is a smooth curve of genus 8 meeting E_{y_j} transversally at two distinct points. The Prym curve structure on $u^{-1}(t_i)$ is provided by a line bundle $\eta_{t_i} \in \mathrm{Pic}^0\big(u^{-1}(t_i)\big)$ such that $\eta_{E_{y_i}} \cong \mathcal{O}_{E_{y_i}}(1)$, that is, each point $u^{-1}(t_i)$ gives rise to a point in the boundary divisor Δ_0^{ram}. It is also clear that these are the only points in the pencil R, where the sheaf inducing the Prym structure on the plane curve $\epsilon\big(u^{-1}(t)\big)$ is not locally free and that the intersection of R with Δ_0^{ram} at each of the points $\big[u^{-1}(t_i), \eta_{u^{-1}(t_i)}\big]$ is transversal, therefore

$$R \cdot \delta_0^{\mathrm{ram}} = 8. \tag{25}$$

To evaluate the remaining intersection numbers is now relatively easy. Observe first that $\chi\big(X', \mathcal{O}_{X'}\big) = \chi(X, \mathcal{O}_X) = 1$, therefore

$$R \cdot \lambda = \chi\big(X', \mathcal{O}_{X'}\big) + g - 1 = 9. \tag{26}$$

Furthermore, $K_{X'}^2 = K_X^2 - 16 = K_{\mathbf{P}^2} - (12+16) = -19$. Then applying Noether's formula we write $c_2(X') = 12\chi\big(X', \mathcal{O}_{X'}\big) - K_{X'}^2 = 12 + 19 = 31$, implying that

$$R \cdot \big(\delta_0' + \delta_0'' + 2\delta_0^{\mathrm{ram}}\big) = c_2(X') + 4(g-1) = 31 + 4 \cdot 8 = 63. \tag{27}$$

Clearly $R \cdot \delta_0'' = 0$, for a point in the intersection would imply the existence of a plane quartic passing through all the points $o_1, \ldots, o_{12}$ and $y_1, \ldots, y_8$, which yields $\eta \cong \mathcal{O}_C$, which is a contradiction. Putting this, (25) and (27) together, we obtain $R \cdot \delta_0' = 63 - 2 \cdot 8 = 47$, which finishes the proof. □

Corollary 3.2. *One has $R \cdot K_{\overline{\mathcal{R}}_9} = -1$ and $R \cdot [\overline{\mathcal{D}}_9] = 102$.*

Proof. This is a direct consequence of Theorems 3.1, 1.1 and of (9). □

Doe to the (unlikely) possibility that the Brill-Noether divisor $\overline{\mathcal{D}}_9$ may be reducible, we need a somewhat stronger statement than Corollary 3.2 to conclude that $K_{\overline{\mathcal{R}}_9}$ is not pseudo-effective.

Theorem 3.3. *Let $\overline{\mathcal{D}'}$ be any irreducible component of the divisor $\overline{\mathcal{D}}_9$. Then $R \cdot \overline{\mathcal{D}'} \geqslant 0$.*

Proof. We consider an irreducible component $\overline{\mathcal{D}'}$ of $\overline{\mathcal{D}}_9$ and write

$$[\overline{\mathcal{D}'}] = a\lambda - b_0'\delta_0' - b_0''\delta_0'' - b_0^{\mathrm{ram}}\delta_0^{\mathrm{ram}} - \sum_{i=1}^{4}\big(b_i\delta_i + b_{9-i}\delta_{9-i} + b_{i:9-i}\delta_{i:9-i}\big) \in \mathrm{Pic}(\overline{\mathcal{R}}_9).$$

We may clearly assume that $\overline{\mathcal{D}'} \neq \Delta_0''$. We apply Theorem 2.1 and observe that $\overline{\mathcal{D}'}$ is disjoint from a general pencil Ξ_9 of Prym curves on a general Nikulin surface. Using (18) we obtain that

$$10a - 56b_0' - 8b_0^{\mathrm{ram}} = 0. \tag{28}$$

Note that $b_0'' \geqslant 0$. Indeed, one considers the following sweeping curve A_0'' of the divisor Δ_0''. Fix a general curve $[C, p] \in \mathcal{M}_{8,1}$ and consider the family of Prym curves

$$A_0'' := \Big\{\big[C_{xp} = C/x \sim p,\ \eta_{xp}\big] : x \in C,\ \eta_{xp} \in \mathrm{Pic}^0(C_{xp})[2],\ \ \nu^*(\eta_{xp}) \cong \mathcal{O}_C\Big\}.$$

Here $\nu\colon C \to C_{xp}$ denotes the normalization. Then $A_0'' \cdot \overline{\mathcal{D}'} = (2g-2)b_0'' - b_1 \geqslant 0$. Since clearly $b_1 \geqslant 0$ (use that the intersection of $\overline{\mathcal{D}'}$ with the test curve given by a ruling of the boundary divisor $\Delta_1 \subseteq \overline{\mathcal{R}}_g$ is non-negative), one concludes that $b_0'' \geqslant 0$, as claimed.

We next use the lift to $\overline{\mathcal{R}}_9$ of a general pencil of curves of genus 9 on a fixed $K3$ surface under the map $\pi\colon \overline{\mathcal{R}}_9 \to \overline{\mathcal{M}}_9$. Given a general $K3$ surface S with $\mathrm{Pic}(S) = \mathbb{Z} \cdot [C]$ where $C^2 = 16$, we take a Lefschetz pencil $\big\{C_t\big\}_{t \in \mathbf{P}^1}$ in the linear system $\big|\mathcal{O}_S(C)\big|$, then consider the curve

$$A := \Big\{[C_t, \eta_t] : \eta_t \in \overline{\mathrm{Pic}}^0(C_t)[2],\ \ t \in \mathbf{P}^1\Big\} \subseteq \overline{\mathcal{R}}_9.$$

We record the intersection numbers of A with the generators of $\mathrm{Pic}(\overline{\mathcal{R}}_9)$, cf. [17, Lemma 1.8]:

$$A \cdot \lambda = (g+1)(2^{2g}-1),\ \ A \cdot \delta_0' = (6g+18)(2^{2g-1}-2),\ A \cdot \delta_0'' = 6g+18,\ A \cdot \delta_0^{\mathrm{ram}} = (6g+18)2^{2g-2},$$

where $g = 9$. The intersection numbers of A with the remaining generators of $\mathrm{Pic}(\overline{\mathcal{R}}_9)$ are all zero. Clearly A is a sweeping curve in $\overline{\mathcal{R}}_9$, therefore it intersects every effective divisor in $\overline{\mathcal{R}}_9$ non-negatively. In particular, combining the relation (28) with the inequality $A \cdot \overline{\mathcal{D}'} \geqslant 0$, we obtain that $5a \leqslant 36b_0'$. But then using (3.1) coupled again with (28), we write

$$R \cdot \overline{\mathcal{D}'} = 9a - 47b_0' - 8b_0^{\mathrm{ram}} = 9a - 47b_0' + 56b_0' - 10a = 9b_0' - a \geqslant \Big(\frac{45}{36} - 1\Big)a \geqslant 0,$$

since clearly $a \geqslant 0$ (the class λ is big and nef), which brings the proof to an end. □

3.1. **The existence of a good sweeping rational curve inside $\overline{\mathcal{D}}_9$.** We are left with establishing the assumption (†), that plays a crucial role in the proof of both Theorems 3.1 and 3.3. Recall that Z was defined as the degeneracy locus inside $\mathcal{RG}^2_8$ of the morphism χ considered in (15). Although not strictly needed in the proof, note that it follows from [5, Proposition 3.4] and [17, Theorem 2.3] that $\mathcal{RG}^2_8$ is irreducible.

Keeping the notation from the beginning of Section 3, we are going to exhibit a point $[C, L, \eta] \in Z$, inducing a plane octic $\varphi_L \colon C \to \Gamma$ such that:

(1) Γ is nodal at 12 points $o_1, \ldots, o_{12}$ and has no further singularities.
(2) If $\overline{L} \otimes \eta \cong \mathcal{O}_C(y_1 + \cdots + y_8)$, the points $y_1, \ldots, y_8$ are pairwise distinct and disjoint from the set $\big\{\varphi_L^{-1}(o_1), \ldots, \varphi_L^{-1}(o_{12})\big\}$.
(3) Each curve in the pencil spanned by Γ and Γ' is irreducible and nodal.

Since each of the conditions (1)-(3) is open in $Z \subseteq \mathcal{RG}^2_8$ and each component of Z maps generically finite under the map σ onto a component of $\overline{\mathcal{D}}_9$, exhibiting *one* such point $[C, L, \eta]$, implies the existence of a component $\overline{\mathcal{D}'}$ of $\overline{\mathcal{D}}_9$, for which Theorem 3.3 can be applied.

We begin with the following observation:

Lemma 3.4. *Fix a general point $\xi \in \mathrm{Hilb}^{12}(\boldsymbol{P}^2)$. Then the locus of reducible curves in the linear system $\big|8h - 2E_1 - \cdots - 2E_{12}\big|$ on X has codimension at least 4.*

Proof. Assume $C = C_1 + C_2$ is such a reducible curve. Since the points $o_1, \ldots, o_{12} \in \mathbf{P}^2$ are general, there is no cubic passing through all of them, nor a plane septic curve nodal at each of these points. Therefore the possibility yielding the largest number of moduli is when both C_1 and C_2 are elements of the linear system $\big|4h - E_1 - \ldots - E_{12}\big|$. But such curves depend on at most $2 \cdot \dim \big|4h - E_1 - \cdots - E_{12}\big| = 4$ parameters. □

Lemma 3.5. *There exists a point $[C, L, \eta] \in Z$, such that no curve in the pencil spanned by C and C' has either cusps or singularities of order at least 3.*

Proof. We consider a general point $\xi = o_1 + \cdots + o_{12} \in \mathrm{Hilb}^{12}(\mathbf{P}^2)$ such that the quadruple cover $q \colon X \to \mathbf{P}^1$ considered in (22) has a branch curve with only nodes and cusps as singularities. We choose a general element

$$D_0 \in \big|4h - E_1 - \cdots - E_{12}\big|,$$

that is, D_0 is a smooth quartic curve passing through $o_1, \ldots, o_{12}$. Further, we take a general pencil of quartics $\Lambda := \big\{D_t\big\}_{t \in \mathbf{P}^1}$ in the linear system $\big|4h - E_1 - \cdots - E_{12}\big|$, then consider the pencil of *reducible* octics in X

$$\left\{C_t = D_0 + D_t \subseteq X : t \in \mathbf{P}^1\right\}$$

having D_0 as a base component. We may assume that each curve D_t is irreducible and at most 1-nodal and that the intersection of D_0 with two generators of the pencil Λ is everywhere transverse. It thus follows that no curve $D_0 + D_t$ in X has cusps or singularities of order at least 3. Observe however, that in this pencil there are 12 curves with a tacnode, corresponding to the situation when the plane quartics $\epsilon(D_0)$ and $\epsilon(D_{t_i})$ have a common tangent at the points o_i, for $i = 1, \ldots, 12$. Note furthermore that we can pick *general* points $y_1, \ldots, y_8 \in D_0$ and then with the notation of (24), observe that the curves $D_0 + D_{t_1}$ and $D_0 + D_{t_2}$ obviously have non-transverse intersection at each of the points $y_1, \ldots, y_8$ (in fact they even have a common component), that is, the point $\left(\xi, D_0 + D_{t_1}, D_0 + D_{t_2}\right)$ can be regarded as an element of $\mathcal{Y}$. □

Lemma 3.6. *There exists a point $[C, L, \eta] \in Z$, such that no curve in the pencil spanned by C and C' has tacnodal singularities.*

Proof. We keep the notation of Lemma 3.5 and assume that we chosen $o_1 + \cdots + o_{12}$ such that the quadruple cover $q\colon X \to \mathbf{P}^2$ has a branch curve $\Delta \subseteq \mathbf{P}^2$ with only nodes and cusps as singularities. We consider two points $\xi_a, \xi_b \in \mathrm{Hilb}^2(\mathbf{P}^2)$ corresponding to distinct points a and b and tangent directions $\xi_a \in \mathbf{P}\big(T_a(\mathbf{P}^2)\big)$ at a and $\xi_b \in \mathbf{P}\big(T_b(\mathbf{P}^2)\big)$ at b respectively. We denote by $\Lambda := \left|\mathcal{I}_{\xi_a+\xi_b}(2)\right|$ the pencil of conics passing through the cluster $\xi_a + \xi_b$ in $\mathbf{P}^2$ and the family of octics

$$\left\{q^*(D) : D \in \Lambda\right\}. \tag{29}$$

Observe that for distinct elements $D, D' \in \Lambda$, the curves $q^*(D)$ and $q^*(D')$ are mutually tangent at the 8 points in $q^*(\xi_a)$ and $q^*(\xi_b)$, therefore $\big(o_1 + \cdots + o_{12}, q^*(D), q^*(D')\big) \in \mathcal{Y}$. Furthermore, no conic D in the pencil Λ is tacnodal, implying also that no curve in (29) has a tacnode. This last claim can also be easily verified by picking $\xi \in \mathrm{Hilb}^{12}(\mathbf{P}^2)$ generically via (23). This finishes the proof. □

Proof of Theorem 0.1. Applying Lemmas 3.4, 3.5 and 3.6, we conclude that there exists an irreducible component $\overline{\mathcal{D}'}$ of the divisor $\overline{\mathcal{D}}_9$ such that the assumption (†) is satisfied for the sweeping curve $R \subseteq \overline{\mathcal{D}'}$, in particular Theorems 3.1 and 3.3 can be applied. The conclusion follows as described in the Introduction. The canonical class $K_{\overline{\mathcal{R}}_9}$ is not pseudo-effective. Indeed, otherwise we write $K_{\overline{\mathcal{R}}_9} \equiv a \cdot \overline{\mathcal{D}'} + M$, where $a \geqslant 0$ and M is a pseudo-effective $\mathbb{R}$-divisor class on $\overline{\mathcal{R}}_9$ not containing $\overline{\mathcal{D}'}$ in its support, therefore $R \cdot M \geqslant 0$. It follows that $0 > R \cdot K_{\overline{\mathcal{R}}_9} = aR \cdot \overline{\mathcal{D}'} + R \cdot M \geqslant 0$, a contradiction. Applying [4], it follows that $\overline{\mathcal{R}}_9$ is uniruled.

□

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