PRYM VARIETIES AND MODULI OF POLARIZED NIKULIN SURFACES

GAVRIL FARKAS AND ALESSANDRO VERRA

ABSTRACT. We present a structure theorem for the moduli space $\mathcal{R}_7$ of Prym curves of genus 7 as a projective bundle over the moduli space of 7-nodal rational curves. The existence of this parametrization implies the unirationality of $\mathcal{R}_7$ and that of the moduli space of Nikulin surfaces of genus 7, as well as the rationality of the moduli space of Nikulin surfaces of genus 7 with a distinguished line. Using the results in genus 7, we then establish that $\mathcal{R}_8$ is uniruled.

A polarized Nikulin surface of genus g is a smooth polarized $K3$ surface $(S, \mathfrak{c})$, where $\mathfrak{c} \in \mathrm{Pic}(S)$ with $\mathfrak{c}^2 = 2g-2$, equipped with a double cover $f : \widetilde{S} \to S$ branched along disjoint rational curves $N_1, \ldots, N_8 \subset S$, such that $\mathfrak{c} \cdot N_i = 0$ for $i = 1, \ldots, 8$. Denoting by $e \in \mathrm{Pic}(S)$ the class defined by the equality $e^{\otimes 2} = \mathcal{O}_S(\sum_{i=1}^{8} N_i)$, one forms the *Nikulin lattice*

$$\mathfrak{N} := \Big\langle \mathcal{O}_S(N_1), \ldots, \mathcal{O}_S(N_8), e \Big\rangle$$

and obtains a primitive embedding $j : \Lambda_g := \mathbb{Z} \cdot [\mathfrak{c}] \oplus \mathfrak{N} \hookrightarrow \mathrm{Pic}(S)$. Nikulin surfaces of genus g form an irreducible 11-dimensional moduli space $\mathcal{F}_g^{\mathfrak{N}}$ which has been studied in [Do1] and [vGS]. The connection between $\mathcal{F}_g^{\mathfrak{N}}$ and the moduli space $\mathcal{R}_g$ of pairs $[C, \eta]$, where C is a curve of genus g and $\eta \in \mathrm{Pic}^0(C)[2]$ is a 2-torsion point, has been pointed out in [FV] and used to describe $\mathcal{R}_g$ in small genus. Over $\mathcal{F}_g^{\mathfrak{N}}$ one considers the open set in a tautological $\mathbf{P}^g$-bundle

$$\mathcal{P}_g^{\mathfrak{N}} := \Big\{ \big[S, j : \Lambda_g \hookrightarrow \mathrm{Pic}(S), C\big] : C \in |\mathfrak{c}| \text{ is a smooth curve of genus } g \Big\},$$

which is endowed with the two projection maps

$$\begin{array}{ccccc} & & \mathcal{P}_g^{\mathfrak{N}} & & \\ & \swarrow^{p_g} & & \searrow^{\chi_g} & \\ \mathcal{F}_g^{\mathfrak{N}} & & & & \mathcal{R}_g \end{array}$$

defined by $p_g([S, j, C]) := [S, j]$ and $\chi_g([S, j, C]) := [C, e_C := e \otimes \mathcal{O}_C]$ respectively.

Observe that $\dim(\mathcal{P}_7^{\mathfrak{N}}) = \dim(\mathcal{R}_7) = 18$. The map $\chi_7 : \mathcal{P}_7^{\mathfrak{N}} \dashrightarrow \mathcal{R}_7$ is a birational isomorphism, precisely $\mathcal{R}_7$ is birational to a Zariski locally trivial $\mathbf{P}^7$-bundle over $\mathcal{F}_7^{\mathfrak{N}}$. This is reminiscent of Mukai's result [Mu]: $\mathcal{M}_{11}$ is birational to a projective bundle over the moduli space $\mathcal{F}_{11}$ of polarized $K3$ surfaces of genus 11. Note that $\mathcal{M}_{11}$ and $\mathcal{R}_7$ are the only known examples of moduli spaces of curves admitting a non-trivial fibre bundle structure over a moduli space of polarized $K3$ surfaces. Here we describe the structure of $\mathcal{F}_7^{\mathfrak{N}}$:

Theorem 0.1. *The Nikulin moduli space $\mathcal{F}_7^{\mathfrak{N}}$ is unirational. The Prym moduli space $\mathcal{R}_7$ is birationally isomorphic to a $\mathbf{P}^7$-bundle over $\mathcal{F}_7^{\mathfrak{N}}$. It follows that $\mathcal{R}_7$ is unirational as well.*

It is well-known that $\mathcal{R}_g$ is unirational for $g \leq 6$, see [Do], [ILS], [V], and even rational for $g \leq 4$, see [Do2], [Cat]. On the other hand, the Deligne-Mumford moduli space $\overline{\mathcal{R}}_g$ of stable Prym curves of genus g is a variety of general type for $g \geq 14$, whereas $\mathrm{kod}(\overline{\mathcal{R}}_{12}) \geq 0$, see [FL]. Nothing seems to be known about the Kodaira dimension of $\overline{\mathcal{R}}_g$, for $g = 9, 10, 11$.

We now discuss the structure of $\mathcal{F}_7^{\mathfrak{N}}$. For each positive g, we denote by

$$\mathfrak{Rat}_g := \overline{\mathcal{M}}_{0,2g}/\mathbb{Z}_2^{\oplus g} \rtimes S_g$$

the moduli space of g-nodal stable rational curves of genus g. The action of the group $\mathbb{Z}_2^{\oplus g}$ is given by permuting the marked points labeled by $\{1,2\}, \ldots, \{2g-1, 2g\}$ respectively, while the symmetric group S_g acts by permuting the 2-cycles $(1,2), \ldots, (2g-1, 2g)$ respectively. The variety $\mathfrak{Rat}_g$, viewed as a subvariety of $\overline{\mathcal{M}}_g$, has been studied by Castelnuovo [Cas] at the end of the 19th century in the course of his famous attempt to prove the Brill-Noether Theorem, as well as much more recently, for instance in [GKM] [1], in the context of determining the ample cone of $\overline{\mathcal{M}}_g$. Using the identification $\mathrm{Sym}^2(\mathbf{P}^1) \cong \mathbf{P}^2$, we obtain a birational isomorphism

$$\mathfrak{Rat}_g \cong \mathrm{Hilb}^g(\mathbf{P}^2)//PGL(2),$$

where $PGL(2) \subset PGL(3)$ is regarded as the group of projective automorphisms of $\mathbf{P}^2$ preserving the image of a fixed smooth conic in $\mathbf{P}^2$.

Let us fix once and for all a smooth rational quintic curve $R \subset \mathbf{P}^5$. For general points $x_1, y_1, \ldots, x_7, y_7 \in R$, we note that $\bigl[R, (x_1+y_1) + \cdots + (x_7+y_7)\bigr] \in \mathfrak{Rat}_7$. We denote by

$$N_1 := \langle x_1, y_1 \rangle, \ldots, N_7 := \langle x_7, y_7 \rangle \in G(2,6),$$

the corresponding bisecant lines to R and observe that $C := R \cup N_1 \cup \ldots \cup N_7$ is a nodal curve of genus 7 and degree 12 in $\mathbf{P}^5$. By writing down the Mayer-Vietoris sequence for C, we find the following identifications:

$$H^0(C, \mathcal{O}_C(1)) \cong H^0(\mathcal{O}_R(1)) \text{ and } H^0(C, \mathcal{O}_C(2)) \cong H^0(\mathcal{O}_R(2)) \oplus \Bigl(\oplus_{i=1}^7 H^0(\mathcal{O}_{N_i})\Bigr).$$

It can easily be checked that the base locus

$$S := \mathrm{Bs}\, \bigl|\mathcal{I}_{C/\mathbf{P}^5}(2)\bigr|$$

is a smooth $K3$ surface which is a complete intersection of three quadrics in $\mathbf{P}^5$. Obviously, S is equipped with the seven lines $N_1, \ldots, N_7$. In fact, S carries an eight line as well! If $H \in |\mathcal{O}_S(1)|$ is a hyperplane section, after setting

$$N_8 := 2R + N_1 + \cdots + N_7 - 2H \in \mathrm{Div}(S),$$

we compute that $N_8^2 = -2, N_8 \cdot H = 1$ and $N_8 \cdot N_i = 0$, for $i = 1, \ldots, 7$. Therefore N_8 is equivalent to an effective divisor on S, which is embedded in $\mathbf{P}^5$ as a line by the linear system $|\mathcal{O}_S(1)|$. Furthermore,

$$N_1 + \cdots + N_8 = 2(R + N_1 + \cdots + N_7 - H) \in \mathrm{Pic}(S),$$

hence by denoting $e := R + N_1 + \cdots + N_7 - H$, we obtain an embedding $\mathfrak{N} \hookrightarrow \mathrm{Pic}(S)$. Moreover $C \cdot N_i = 0$ for $i = 1, \ldots, 8$ and we may view $\Lambda_7 \hookrightarrow \mathrm{Pic}(S)$. In this way S becomes a Nikulin surface of genus 7.

We introduce the moduli space $\widehat{\mathcal{F}}_g^{\mathfrak{N}}$ of *decorated* Nikulin surfaces consisting of polarized Nikulin surfaces $\bigl[S, j : \Lambda_g \hookrightarrow \mathrm{Pic}(S)\bigr]$ of genus g, together with a distinguished line $N_8 \subset S$ viewed as a component of the branch divisor of the double covering $f : \widetilde{S} \to S$. There is an

[1] Unfortunately, in [GKM] the notation $\overline{\mathcal{R}}_g$ (reserved for the Prym moduli space) is proposed for what we denote in this paper by $\mathfrak{Rat}_g$.

obvious forgetful map $\widehat{\mathcal{F}}_g^{\mathfrak{N}} \to \mathcal{F}_g^{\mathfrak{N}}$ of degree 8. Having specified $N_8 \subset S$, we can also specify the divisor $N_1 + \cdots + N_7 \subset S$ such that $e^{\otimes 2} = \mathcal{O}_S(N_1 + \cdots + N_7 + N_8)$. We summarize what has been discussed so far:

Theorem 0.2. *The rational map $\varphi : \mathfrak{Rat}_7 \dashrightarrow \widehat{\mathcal{F}}_7^{\mathfrak{N}}$ given by*

$$\varphi\Big(\big[R, (x_1 + y_1) + \cdots + (x_7 + y_7)\big]\Big) := \Big[S, \mathcal{O}_S(R + N_1 + \cdots + N_7), N_8\Big]$$

is a birational isomorphism.

A construction of the inverse map φ^{-1} using the geometry of Prym canonical curves of genus 7 is presented in Section 2. The moduli space $\mathfrak{Rat}_g$ is related to the configuration space

$$U_g^2 := \mathrm{Hilb}^g(\mathbf{P}^2)//PGL(3)$$

of g unordered points in the plane. Using the isomorphism $PGL(3)/PGL(2) \cong \mathbf{P}^5$, we observe in Section 2 that there exists a (locally trivial) $\mathbf{P}^5$-bundle structure $\mathfrak{Rat}_g \dashrightarrow U_g^2$. In particular $\mathfrak{Rat}_g$ is rational whenever U_g^2 is. Since the rationality of U_7^2 has been established by Katsylo [Ka] (see also [Bo]), we are led to the following result:

Theorem 0.3. *The moduli space $\widehat{\mathcal{F}}_7^{\mathfrak{N}}$ of marked Nikulin surfaces of genus 7 is rational.*

Putting together Theorems 0.2 and 0.3, we conclude that there exists a dominant rational map $\mathbf{P}^{18} \dashrightarrow \mathcal{R}_7$ of degree 8. We are not aware of any dominant map from a rational variety to $\mathcal{R}_7$ of degree smaller than 8. It would be very interesting to know whether $\mathcal{R}_7$ itself is a rational variety. We recall that although $\mathcal{M}_g$ is known to be rational for $g \leq 6$ (see [Bo] and the references therein), the rationality of $\mathcal{M}_7$ is an open problem.

We sum up the construction described above in the following commutative diagram:

$$\begin{array}{ccccc}
\overline{\mathcal{M}}_{0,14} & \xrightarrow{(2^7\cdot 7!):1} & \mathfrak{Rat}_7 & & \\
\downarrow & & \downarrow \cong & & \\
\mathcal{F}_7^{\mathfrak{N}} & \xleftarrow[8:1]{} & \widehat{\mathcal{F}}_7^{\mathfrak{N}} & \xrightarrow{\mathbf{P}^5} & U_7^2
\end{array}$$

The concrete geometry of $\mathcal{R}_7$ has direct consequences concerning the Kodaira dimension of $\overline{\mathcal{R}}_8$. The projective bundle structure of $\mathcal{R}_7$ over $\mathcal{F}_7^{\mathfrak{N}}$ can be lifted to a boundary divisor of $\overline{\mathcal{R}}_8$. Denoting by $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ the map forgetting the Prym structure, one has the formula

$$\pi^*(\delta_0) = \delta_0^{'} + \delta_0^{''} + 2\delta_0^{\mathrm{ram}} \in CH^1(\overline{\mathcal{R}}_g),$$

where $\delta_0^{'} := [\Delta_0^{'}]$, $\delta_0^{''} := [\Delta_0^{''}]$, and $\delta_0^{\mathrm{ram}} := [\Delta_0^{\mathrm{ram}}]$ are boundary divisor classes on $\overline{\mathcal{R}}_g$ whose meaning will be recalled in Section 3. Note that up to a $\mathbb{Z}_2$-factor, a general point of $\Delta_0^{'}$ corresponds to a 2-pointed Prym curve of genus 7, for which we apply our Theorem 0.1. We establish the following result:

Theorem 0.4. *The moduli space $\overline{\mathcal{R}}_8$ is uniruled.*

Using the parametrization of $\mathcal{R}_7$ via Nikulin surfaces, we construct a sweeping curve Γ of the boundary divisor $\Delta_0^{'}$ of $\overline{\mathcal{R}}_8$ such that $\Gamma \cdot \delta_0^{'} > 0$ and $\Gamma \cdot K_{\overline{\mathcal{R}}_8} < 0$. This implies that the canonical class $K_{\overline{\mathcal{R}}_8}$ cannot be pseudoeffective, hence via [BDPP], the moduli space $\overline{\mathcal{R}}_8$ is uniruled. This way of showing uniruledness of a moduli space, though quite effective, does not lead to an *explicit* uniruled parametrization of $\mathcal{R}_8$. In Section 3, we sketch an alternative, more geometric way of showing that $\mathcal{R}_8$ is uniruled, by embedding a general Prym-curve of genus 8 in a certain canonical surface. A rational curve through a general point of $\overline{\mathcal{R}}_8$ is then induced by a pencil on this surface.

1. Polarized Nikulin surfaces

We briefly recall some basics on Nikulin surfaces, while referring to [vGS], [GS] for details. A *symplectic involution* ι on a smooth $K3$ surface Y has 8 fixed points and we denote by $\bar{Y} := Y/\langle \iota \rangle$ the quotient. The surface $\bar{Y}$ has 8 nodes. Letting $\sigma : \widetilde{S} \to Y$ be the blow-up of the fixed points, the involution ι lifts to an involution $\tilde{\iota} : \widetilde{S} \to \widetilde{S}$ fixing the eight (-1)-curves $E_1, \ldots, E_8 \subset \widetilde{S}$. Denoting by $f : \widetilde{S} \to S$ the quotient map by the involution $\widetilde{\iota}$, we obtain a smooth $K3$ surface S, together with a primitive embedding of the Nikulin lattice $\mathfrak{N} \cong E_8(-2) \hookrightarrow \mathrm{Pic}(S)$, where $N_i = f(E_i)$ for $i = 1, \ldots, 8$. In particular, the sum of rational curves $N := N_1 + \cdots + N_8$ is an even divisor on S, that is, there exists a class $e \in \mathrm{Pic}(S)$ such that $e^{\otimes 2} = \mathcal{O}_S(N_1 + \cdots + N_8)$. The cover $f : \widetilde{S} \to S$ is branched precisely along the curves $N_1, \ldots, N_8$. The following diagram summarizes the notation introduced so far and will be used throughout the paper:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\sigma} & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ S & \longrightarrow & \bar{Y} \end{array} \tag{1}$$

Nikulin [Ni] showed that the possible configurations of even sets of disjoint (-2)-curves on a $K3$ surface S are only those consisting of either 8 curves (in which case S is a Nikulin surface as defined in this paper), or of 16 curves, in which case S is a Kummer surface. From this point of view, Nikulin surfaces appear naturally as the *Prym analogues* of $K3$ surfaces.

Definition 1.1. A *polarized Nikulin surface* of genus g consists of a smooth $K3$ surface and a primitive embedding j of the lattice $\Lambda_g = \mathbb{Z} \cdot \mathfrak{c} \oplus \mathfrak{N} \hookrightarrow \mathrm{Pic}(S)$, such that $\mathfrak{c}^2 = 2g - 2$ and the class $j(\mathfrak{c})$ is nef.

Polarized Nikulin surfaces of genus g form an irreducible moduli 11-dimensional moduli space $\mathcal{F}_g^{\mathfrak{N}}$, see for instance [Do1]. Structure theorems for $\mathcal{F}_g^{\mathfrak{N}}$ for genus $g \leq 6$ have been established in [FV]. For instance the following result is proven in *loc.cit.* for Nikulin surfaces of genus $g = 6$. Let $V = \mathbb{C}^5$ and fix a smooth quadric $Q \subset \mathbf{P}(V)$. Then one has a birational isomorphism, which, in particular, shows that $\mathcal{F}_6^{\mathfrak{N}}$ is unirational:

$$\mathcal{F}_6^{\mathfrak{N}} \overset{\cong}{\dashrightarrow} G\Big(7, \bigwedge^2 V\Big)^{\mathrm{ss}} /\!/ \mathrm{Aut}(Q).$$

On the other hand, fundamental facts about $\mathcal{F}_g^{\mathfrak{N}}$ are still not known. For instance, it is not clear whether $\mathcal{F}_g^{\mathfrak{N}}$ is a variety of general type for large g. Nikulin surfaces have been recently used decisively in [FK] to prove the Prym-Green Conjecture on syzygies of general Prym-canonical curves of even genus.

For a polarized Nikulin surface (S, j) of genus g as above, we set $C := j(\mathfrak{c})$ and then $H \equiv C - e \in \mathrm{Pic}(S)$. It is shown in [GS], that for any Nikulin surface S having minimal Picard lattice $\mathrm{Pic}(S) = \Lambda_g$, the linear system $\mathcal{O}_S(H)$ is very ample for $g \geq 6$. We compute that $H^2 = 2g - 6$ and denote by $\phi_H : S \to \mathbf{P}^{g-2}$ the corresponding embedding. Since $N_i \cdot H = 1$ for $i = 1, \ldots, 8$, it follows that the images $\phi_H(N_i) \subset \mathbf{P}^{g-2}$ are lines. The existence of two closely linked distinguished polarizations $\mathcal{O}_S(C)$ and $\mathcal{O}_S(H)$ of genus g and $g-2$ respectively on any Nikulin surface is one of the main sources for the rich geometry of the moduli space $\mathcal{F}_g^{\mathfrak{N}}$ for $g \leq 6$, see [FV] and [vGS].

Suppose that $\big[S, j : \Lambda_7 \hookrightarrow \mathrm{Pic}(S)\big]$ is a polarized Nikulin surface of genus 7. In this case

$$\phi_H : S \hookrightarrow \mathbf{P}^5$$

is a surface of degree 8 which is a complete intersection of three quadrics. For each smooth curve $C \in |\mathcal{O}_S(j(\mathfrak{c}))|$, we have that $[C, \eta := e_C] \in \mathcal{R}_7$. Since $\mathcal{O}_C(1) = K_C \otimes \eta$, it follows that the restriction $\phi_{H|C} : C \hookrightarrow \mathbf{P}^5$ is a Prym-canonically embedded curve of genus 7. This assignment gives rise to the map $\chi_7 : \mathcal{P}_7^{\mathfrak{N}} \to \mathcal{R}_7$.

Conversely, to a general Prym curve $[C, \eta] \in \mathcal{R}_7$ we associate a unique Nikulin surface of genus 7 as follows. We consider the Prym-canonical embedding $\phi_{K_C \otimes \eta} : C \hookrightarrow \mathbf{P}^5$ and observe that $S := \mathrm{bs}(|\mathcal{I}_{C/\mathbf{P}^5}(2)|)$ is a complete intersection of three quadrics, that is, if smooth, a $K3$ surfaces of degree 8. In fact, S is smooth for a general choice of $[C, \eta] \in \mathcal{R}_7$, see [FV] Proposition 2.3. We then set $N \equiv 2(C - H) \in \mathrm{Pic}(S)$ and note that $N^2 = -16$ and $N \cdot H = 8$. Using the cohomology exact sequence

$$0 \longrightarrow H^0(S, \mathcal{O}_S(N - C)) \longrightarrow H^0(S, \mathcal{O}_S(N)) \longrightarrow H^0(C, \mathcal{O}_C(N)) \longrightarrow 0,$$

since $\mathcal{O}_C(N)$ is trivial, we conclude that the divisor N is effective on S. It is shown in *loc.cit.* that for a general $[C, \eta] \in \mathcal{R}_7$, we have a splitting $N = N_1 + \ldots + N_8$ into a sum of 8 disjoint lines with $C \cdot N_i = 0$ for $i = 1, \ldots, 8$. This turns S into a Nikulin surface and explains the birational isomorphisms

$$\chi_7^{-1} : \mathcal{P}_7^{\mathfrak{N}} \overset{\cong}{\dashrightarrow} \mathcal{R}_7$$

referred to in the Introduction.

Suppose now that $\big[S, \mathcal{O}_S(C), N_8\big] \in \widehat{\mathcal{F}}_7^{\mathfrak{N}}$, that is, we single out a (-2)-curve in the Nikulin lattice. Writing $e^{\otimes 2} = \mathcal{O}_C(N_1 + \cdots + N_8)$, the choice of N_8 also determines the sum of the seven remaining lines $N_1 + \cdots + N_7$, where $H \cdot N_i = 1$, for $i = 1, \ldots, 8$. We compute

$$(C - N_1 - \ldots - N_7)^2 = -2 \quad \text{and} \quad (C - N_1 - \ldots - N_7) \cdot H = 5,$$

in particular, there exists an effective divisor R on S, with $R \equiv C - N_1 - \ldots - N_7$. Note also that $R \cdot N_i = 2$, for $i = 1, \ldots, 7$, that is, $R \subset \mathbf{P}^5$ comes endowed with seven bisecant lines.

Proposition 1.2. *For a decorated Nikulin surface* $\big[S, \mathcal{O}_S(C), N_8\big] \in \widehat{\mathcal{F}}_7^{\mathfrak{N}}$ *satisfying* $\mathrm{Pic}(S) = \Lambda_7$, *we have that* $H^1(S, \mathcal{O}_S(C - N_1 - \cdots - N_7)) = 0$. *In particular,*

$$R \in |\mathcal{O}_S(C - N_1 - \cdots - N_7)|$$

is a smooth rational quintic curve on S.

Proof. Assume by contradiction that the curve $R \subset S$ is reducible. In that case, there exists a smooth irreducible (-2)-curve $Y \subset S$, such that $Y \cdot R < 0$ and $H^0(S, \mathcal{O}_S(R - Y)) \neq 0$. Assuming $\mathrm{Pic}(S)$ is generated by $C, N_1, \ldots, N_8$ and the class $e = (N_1 + \cdots + N_8)/2$, there exist integers $a, b, c_1, \ldots, c_8 \in \mathbb{Z}$, such that

$$Y \equiv a \cdot C + \left(c_1 + \frac{b}{2}\right) \cdot N_1 + \cdots + \left(c_8 + \frac{b}{2}\right) \cdot N_8.$$

Setting $b_i := c_i + \frac{b}{2}$, the numerical hypotheses on Y can be rewritten in the following form:

(2) $$b_1^2 + \cdots + b_8^2 = 6a^2 + 1 \ \text{ and } 6a + b_1 + \cdots + b_8 \leq -1.$$

Since Y is effective, we find that $a \geq 0$ (use that $C \subset S$ is nef). Applying the same considerations to the effective divisor $R - Y$, we obtain that $a \in \{0, 1\}$.

If $a = 0$, then $Y \equiv b_1 N_1 + \cdots + b_8 N_8 \geq 0$, hence $b_i \geq 0$ for $i = 1, \ldots, 8$, which contradicts the inequality $b_1 + \cdots + b_8 \leq -1$, so this case does not appear.

If $a = 1$, then $R - Y \equiv -(1 + b_1)N_1 - \cdots - (1 + b_7)N_7 - b_8 N_8 \geq 0$, therefore $b_8 \leq 0$ and $b_i \leq -1$ for $i = 1, \ldots, 7$. From (2), we obtain that $b_8 = 0$ and $b_1 = \cdots = b_7 = -1$. Thus $Y \equiv R$, which is a contradiction, for Y was assumed to be a proper irreducible component of R. □

Retaining the notation above, we obtain a map $\psi : \widehat{\mathcal{F}}_7^{\mathfrak{N}} \dashrightarrow \mathfrak{Rat}_7$, defined by

$$\psi\Big([S, \mathcal{O}_S(C), N_8]\Big) := [R,\ N_1 \cdot R + \ldots + N_7 \cdot R],$$

where the cycle $N_i \cdot R \in \mathrm{Sym}^2(R)$ is regarded as an effective divisor of degree 2 on R. The map ψ is regular over the dense open subset of $\widehat{\mathcal{F}}_7^{\mathfrak{N}}$ consisting of Nikulin surfaces having the minimal Picard lattice Λ_7. We are going to show that ψ is a birational isomorphism by explicitly constructing its inverse.

We fix a smooth rational quintic curve $R \subset \mathbf{P}^5$ and recall the canonical identification

$$\big|\mathcal{I}_{R/\mathbf{P}^5}(2)\big| = \big|\mathcal{O}_{\mathrm{Sym}^2(R)}(3)\big| \tag{3}$$

between the linear system of quadrics containing $R \subset \mathbf{P}^5$ and that of plane cubics. Here we use the isomorphism $\mathrm{Sym}^2(R) \xrightarrow{\cong} \mathbf{P}^2$, under which to a quadric $Q \in H^0(\mathbf{P}^5, \mathcal{I}_{R/\mathbf{P}^5}(2))$ one assigns the symmetric correspondence

$$\Sigma_Q := \{x + y \in \mathrm{Sym}^2(R) : \langle x, y\rangle \subset Q\},$$

which is a cubic curve in $\mathrm{Sym}^2(R)$.

Let $N_1, \ldots, N_7$ general bisecant lines to R and consider the semi-stable curve of genus 7

$$C := R \cup N_1 \cup \ldots \cup N_7 \subset \mathbf{P}^5.$$

Proposition 1.3. *For a general choice of the bisecants $N_1, \ldots, N_7$ of the curve $R \subset \mathbf{P}^5$, the base locus*

$$S := \mathrm{Bs}\ \big|\mathcal{I}_{C/\mathbf{P}^5}(2)\big|$$

is a smooth $K3$ surface of degree 8.

Proof. The bisecant line N_i is determined by the degree 2 divisor $N_i \cdot R \in \mathrm{Sym}^2(R)$. Under the identification (3), the quadrics containing the line N_i are identified with the cubics in $\big|\mathcal{O}_{\mathrm{Sym}^2(R)}(3)\big|$ that pass through the point $N_i \cdot R$. It follows that the linear system $\big|\mathcal{I}_{C/\mathbf{P}^5}(2)\big|$ corresponds to the linear system of cubics in $\mathrm{Sym}^2(R)$ passing through 7 general points. Since the secants N_i (and hence the points $N_i \cdot R \in \mathrm{Sym}^2(R)$) have been chosen to be general, we obtain that $\dim |\mathcal{I}_{C/\mathbf{P}^5}(2)| = 2$.

We have showed in Proposition 1.2, that for a general Nikulin surface S,

$$H^1(S, \mathcal{O}_S(H - N_1 - \cdots - \hat{N}_i - \cdots - N_8)) = 0,$$

for all $i = 1, \ldots, 8$. In particular, the morphism ψ is defined on all components of $\widehat{\mathcal{F}}_7^{\mathfrak{N}}$ and the image of each component is an element of $\mathfrak{Rat}_7$. For such a point in $\mathrm{Im}(\psi)$, it follows that bs $\big|\mathcal{I}_{C/\mathbf{P}^5}(2)\big|$ is a smooth surface, in fact a general Nikulin surface of genus 7. □

Proof of Theorem 0.2. As explained in the Introduction, the map $\varphi : \mathfrak{Rat}_7 \dashrightarrow \widehat{\mathcal{F}}_7^{\mathfrak{N}}$ is well-defined and clearly the inverse of ψ. In particular, it follows that $\widehat{\mathcal{F}}_7^{\mathfrak{N}}$ is also irreducible (and in fact unirational). □

2. CONFIGURATION SPACES OF POINTS IN THE PLANE

Throughout this section we use the identification $\mathrm{Sym}^2(\mathbf{P}^1) \cong \mathbf{P}^2$ induced by the map $\rho : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ obtained by taking the projection of the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ to the space of symmetric tensors, that is, $\rho\big([a_0, a_1], [b_0, b_1]\big) = [a_0b_0, a_1b_1, a_0b_1 + a_1b_0]$. We identify the diagonal $\Delta \subset \mathbf{P}^1 \times \mathbf{P}^1$ with its image $\rho(\Delta)$ in $\mathbf{P}^2$. We view $PGL(2)$ as the subgroup of automorphisms of $\mathbf{P}^2$ that preserve the conic Δ. Furthermore, the choice of Δ induces a canonical identification

$$PGL(3)/PGL(2) = |\mathcal{O}_{\mathbf{P}^2}(2)| = \mathbf{P}^5.$$

For $g \geq 5$, we consider the projection

$$\beta : \mathfrak{Rat}_g := \mathrm{Hilb}^g(\mathbf{P}^2)//SL(2) \to \mathrm{Hilb}^g(\mathbf{P}^2)//SL(3) =: U_g^2.$$

Definition 2.1. If X is a del Pezzo surface of degree 2, a *contraction* of X is the blow-up $f : X \to \mathbf{P}^2$ of 7 points in general position in $\mathbf{P}^2$.

Specifying a pair (X, f) as above, amounts to giving a *plane model* of the del Pezzo surface, that is, a pair (X, L), where X is a del Pezzo surface with $K_X^2 = 2$ and $L \in \mathrm{Pic}(S)$ is such that $L^2 = 1$ and $K_X \cdot L = -2$. Therefore U_7^2 is the GIT moduli space of pairs (X, f) (or equivalently of pairs (X, L)) as above.

Proposition 2.2. *The morphism $\beta : \mathrm{Hilb}^g(\mathbf{P}^2)//SL(2) \to U_g^2$ is a locally trivial $\mathbf{P}^5$-fibration.*

Proof. Having fixed the conic $\Delta \subset \mathbf{P}^2$, we have an identification $\mathbf{P}^2 \cong \mathrm{Sym}^2(\Delta) \cong (\mathbf{P}^2)^\vee$, that is, we view points in $\mathrm{Sym}^2(\Delta)$ as lines in $\mathbf{P}^2$. A general point $D \in \mathrm{Hilb}^g(\mathbf{P}^2)$ corresponds to a union $D = \ell_1 + \cdots + \ell_g$ of g lines in $\mathbf{P}^2$, such that $\mathrm{Aut}(\{\ell_1, \ldots, \ell_g\}) = 1$. We consider the rank 6 vector bundle $\mathcal{E}$ over $\mathrm{Hilb}^g(\mathbf{P}^2)$ with fibre

$$\mathcal{E}(\ell_1 + \cdots + \ell_g) := H^0\big(\mathcal{O}_{\ell_1 + \cdots + \ell_g}(2)\big).$$

Clearly $\mathcal{E}$ descends to a vector bundle E over the quotient U_g^2. We then observe that one has a canonical identification $\mathbf{P}(E) \cong \mathrm{Hilb}^g(\mathbf{P}^2)//SL(2)$, or more geometrically, $\mathfrak{Rat}_g$ is the moduli space of pairs consisting of an unordered configuration of g lines and a conic in $\mathbf{P}^2$. The birational isomorphism $\mathbf{P}(E) \to \mathrm{Hilb}^g(\mathbf{P}^2)//SL(2)$ is given by the assignment

$$\Big(\ell_1 + \cdots + \ell_g, Q\Big) \bmod SL(3) \mapsto \sigma(\ell_1) + \cdots + \sigma(\ell_g) \bmod SL(2),$$

where $\sigma \in SL(3)$ is an automorphism such that $\sigma(Q) = \Delta$. □

Proof of Theorem 0.3. We have established that the moduli space $\widehat{\mathcal{F}}_7^{\mathfrak{N}}$ is birationally isomorphic to the projectivization of a $\mathbf{P}^5$-bundle over U_7^2. Since U_7^2 is rational, cf. [Bo] Theorem 2.2.4.2, we conclude. □

Remark 2.3. In view of Theorem 0.3, it is natural to ask whether there exists a rational *modular* degree 8 cover $\widehat{\mathcal{R}}_7 \to \mathcal{R}_7$ which is a locally trivial $\mathbf{P}^7$-bundle over the rational variety $\widehat{\mathcal{F}}_7^{\mathfrak{N}}$, such that the following diagram is commutative:

$$\begin{array}{ccccc} \widehat{\mathcal{R}}_7 & \xrightarrow{?} & \widehat{\mathcal{F}}_7^{\mathfrak{N}} & \xrightarrow{\cong} & \mathfrak{Rat}_7 \\ {\scriptstyle 8:1}\downarrow & & {\scriptstyle 8:1}\downarrow & & \\ \mathcal{R}_7 & \xrightarrow{\mathbf{P}^7} & \mathcal{F}_7^{\mathfrak{N}} & & \end{array}$$

One candidate for the cover $\hat{\mathcal{R}}_7$ is the universal singular locus of the Prym-theta divisor,

$$\widehat{\mathcal{R}}_7 := \Big\{[C,\eta,L] \in \mathcal{R}_7 : [C,\eta] \in \mathcal{R}_7 \text{ and } L \in \mathrm{Sing}(\Xi)/\pm\Big\},$$

where $\mathrm{Sing}(\Xi) = \{L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) : \mathrm{Nm}_f(L) = K_C, h^0(C,L) \geq 4, h^0(C,L) \equiv 0 \bmod 2\}$. It is shown in [De] that for a general point $[C,\eta] \in \mathcal{R}_7$, the locus $\mathrm{Sing}(\Xi)$ is reduced and consists of 16 points, so indeed $\deg(\widehat{\mathcal{R}}_7/\mathcal{R}_7) = 8$.

3. The uniruledness of $\overline{\mathcal{R}}_8$

We now explain how our structure results on $\mathcal{F}_7^{\mathfrak{N}}$ and $\mathcal{R}_7$ lead to an easy proof of the uniruledness of $\overline{\mathcal{R}}_8$. We begin by reviewing a few facts about the compactification $\overline{\mathcal{R}}_g$ of $\mathcal{R}_g$ by means of stable Prym curves, see [FL] for details. The geometric points of the coarse moduli space $\overline{\mathcal{R}}_g$ are triples (X,η,β), where X is a quasi-stable curve of genus g, $\eta \in \mathrm{Pic}(X)$ is a line bundle of total degree is 0 such that $\eta_E = \mathcal{O}_E(1)$ for each smooth rational component $E \subset X$ with $|E \cap \overline{X-E}| = 2$ (such a component is said to be *exceptional*), and $\beta : \eta^{\otimes 2} \to \mathcal{O}_X$ is a sheaf homomorphism whose restriction to any non-exceptional component is an isomorphism. If $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ is the map dropping the Prym structure, one has the formula [FL]

$$\pi^*(\delta_0) = \delta_0^{'} + \delta_0^{''} + 2\delta_0^{\mathrm{ram}} \in CH^1(\overline{\mathcal{R}}_g), \tag{4}$$

where $\delta_0^{'} := [\Delta_0^{'}]$, $\delta_0^{''} := [\Delta_0^{''}]$, and $\delta_0^{\mathrm{ram}} := [\Delta_0^{\mathrm{ram}}]$ are irreducible boundary divisor classes on $\overline{\mathcal{R}}_g$, which we describe by specifying their respective general points.

We choose a general point $[C_{xy}] \in \Delta_0 \subset \overline{\mathcal{M}}_g$ corresponding to a smooth 2-pointed curve (C,x,y) of genus $g-1$ and consider the normalization map $\nu : C \to C_{xy}$, where $\nu(x) = \nu(y)$. A general point of $\Delta_0^{'}$ (respectively of $\Delta_0^{''}$) corresponds to a pair $[C_{xy},\eta]$, where $\eta \in \mathrm{Pic}^0(C_{xy})[2]$ and $\nu^*(\eta) \in \mathrm{Pic}^0(C)$ is non-trivial (respectively, $\nu^*(\eta) = \mathcal{O}_C$). A general point of Δ_0^{ram} is a Prym curve of the form (X,η), where $X := C \cup_{\{x,y\}} \mathbf{P}^1$ is a quasi-stable curve with $p_a(X) = g$ and $\eta \in \mathrm{Pic}^0(X)$ is a line bundle such that $\eta_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}(1)$ and $\eta_C^{\otimes 2} = \mathcal{O}_C(-x-y)$. In this case, the choice of the homomorphism β is uniquely determined by X and η. Therefore, we drop β from the notation of such a Prym curve. There are similar decompositions of the pull-back $\pi^*([\Delta_i])$ of the other boundary divisors $\Delta_i \subset \overline{\mathcal{M}}_g$ for $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$, see again [FL] for details.

Via Nikulin surfaces we construct a sweeping curve for the divisor $\Delta_0^{'} \subset \overline{\mathcal{R}}_8$. Let us start with a general element of $\Delta_0^{'}$ corresponding to a smooth 2-pointed curve $[C,x,y] \in \mathcal{M}_{7,2}$ and a 2-torsion point $\eta \in \mathrm{Pic}^0(C_{xy})[2]$ and set $\eta_C := \nu^*(\eta) \in \mathrm{Pic}^0(C)[2]$. Using [FV] Theorem 0.2, there exists a Nikulin surface $f : \widetilde{S} \to S$ branched along 8 rational curves $N_1,\ldots,N_8 \subset S$ and an embedding $C \subset S$, such that $C \cdot N_i = 0$ for $i = 1,\ldots,8$ and $\eta_C = e_C$, where $e \in \mathrm{Pic}(S)$ is the even class with $e^{\otimes 2} = \mathcal{O}_S(N_1 + \cdots + N_8)$. We can also assume that $\mathrm{Pic}(S) = \Lambda_7$. By moving C in its linear system on S, we may assume that $x, y \notin N_1 \cup \ldots \cup N_8$, and we set $\{x_1,x_2\} = f^{-1}(x)$ and $\{y_1,y_2\} = f^{-1}(y)$.

We pick a Lefschetz pencil $\Lambda := \{C_t\}_{t\in \mathbf{P}^1}$ consisting of curves on S passing through the points x and y. Since the locus $\big\{D \in |\mathcal{O}_S(C)| : D \supset N_i\big\}$ is a hyperplane in $|\mathcal{O}_S(C)|$, it follows that there are precisely eight distinct values $t_1, \ldots, t_8 \in \mathbf{P}^1$ such that

$$C_{t_i} =: C_i = N_i + D_i,$$

where D_i is a smooth curve of genus 6 which contains x and y and intersects N_i transversally at two points. For each $t \in \mathbf{P}^1 - \{t_1, \ldots, t_8\}$, we may assume that C_t is a smooth curve and denoting $[\bar{C}_t := C_t/x \sim y] \in \overline{\mathcal{M}}_8$, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pic}^0(\bar{C}_t)[2] \longrightarrow \mathrm{Pic}^0(C_t)[2] \longrightarrow 0.$$

In particular, there exist two distinct line bundles $\eta_t^{'}, \eta_t^{''} \in \mathrm{Pic}^0(\bar{C}_t)$ such that

$$\nu_t^*(\eta_t^{'}) = \nu_t^*(\eta_t^{''}) = e_{C_t}.$$

Using the Nikulin surfaces, we can consistently distinguish $\eta_t^{'}$ from $\eta_t^{''}$. Precisely, $\eta_t^{'}$ corresponds to the admissible cover

$$f^{-1}(C_t)/x_1 \sim y_1, x_2 \sim y_2 \stackrel{2:1}{\longrightarrow} \bar{C}_t$$

whereas $\eta_t^{''}$ corresponds to the admissible cover

$$f^{-1}(C_t)/x_1 \sim y_2, x_2 \sim y_1 \stackrel{2:1}{\longrightarrow} \bar{C}_t.$$

First we construct the pencil $R := \{\bar{C}_t\}_{t\in \mathbf{P}^1} \hookrightarrow \overline{\mathcal{M}}_8$. Formally, we have a fibration $u : \mathrm{Bl}_{2g-2}(S) \to \mathbf{P}^1$ induced by the pencil Λ by blowing-up S at its $2g-2$ base points (two of which being x and y respectively), which comes endowed with sections E_x and E_y given by the corresponding exceptional divisors. The pencil R is obtained from u, by identifying inside the surface $\mathrm{Bl}_{2g-2}(S)$ the sections E_x and E_y respectively.

Lemma 3.1. *The pencil $R \subset \overline{\mathcal{M}}_8$ has the following numerical characters:*

$$R\cdot \lambda = g+1 = 8, \ \ R\cdot \delta_0 = 6g+16 = 58, \ \ \textit{and} \ \ R\cdot \delta_j = 0 \ \ \textit{for } j = 1, \ldots, 4.$$

Proof. We observe that $(R\cdot\lambda)_{\overline{\mathcal{M}}_8} = (\Lambda\cdot\lambda)_{\overline{\mathcal{M}}_7} = g+1 = 8$ and $(R\cdot\delta_j)_{\overline{\mathcal{M}}_8} = (\Lambda\cdot\delta_j)_{\overline{\mathcal{M}}_7} = 0$ for $j \geq 1$. Finally, in order to determine the degree of the normal bundle of Δ_0 along R, we write:

$$(R\cdot\delta_0)_{\overline{\mathcal{M}}_8} = (\Lambda\cdot\delta_0)_{\overline{\mathcal{M}}_7} + E_x^2 + E_y^2 = 6g+18-2 = 58,$$

where we have used the well-known fact that a Lefschetz pencil of curves of genus g on a $K3$ surface possesses $6g+18$ singular fibres (counted with their multiplicities) and that $E_x^2 = E_y^2 = -1$. □

Next, note that the family $\Big\{[\bar{C}_t, \eta_t] : \nu_t^*(\eta_t) = e_{C_t}\Big\}_{t\in\mathbf{P}^1} \hookrightarrow \overline{\mathcal{R}}_8$ splits into two irreducible components meeting in eight points. We consider one of the irreducible components, say

$$\Gamma := \Big\{[\bar{C}_t, \eta_t^{'}]\Big\}_{t\in\mathbf{P}^1} \hookrightarrow \overline{\mathcal{R}}_8.$$

Lemma 3.2. *The curve $\Gamma \subset \overline{\mathcal{R}}_8$ constructed above has the following numerical features:*

$$\Gamma\cdot\lambda = 8, \ \ \Gamma\cdot\delta_0^{'} = 42, \ \ \Gamma\cdot\delta_0^{''} = 0 \ \textit{and} \ \Gamma\cdot\delta_0^{\mathrm{ram}} = 8.$$

Furthermore, Γ is disjoint from all boundary components contained in $\pi^*(\Delta_j)$ for $j = 1, \ldots, 4$.

Proof. First we observe that Γ intersects the divisor Δ_0^{ram} transversally at the points corresponding to the values $t_1, \ldots, t_8 \in \mathbf{P}^1$, when the curve C_i acquires the (-2)-curve N_i as a component. Indeed, for each of these points $e_{D_i}^{\otimes(-2)} = \mathcal{O}_{D_i}(-N_i)$ and $e_{N_i}^{\vee} = \mathcal{O}_{N_i}(1)$, therefore $[C_i, e_{C_i}] \in \Delta_0^{\mathrm{ram}}$. Furthermore, using Lemma 3.1 we write $(\Gamma \cdot \lambda)_{\overline{\mathcal{R}}_8} = \pi_*(\Gamma) \cdot \lambda = 8$ and

$$\Gamma \cdot (\delta_0^{'} + \delta_0^{''} + 2\delta_0^{\mathrm{ram}}) = \Gamma \cdot \pi^*(\delta_0) = R \cdot \delta_0 = 58.$$

Furthermore, for $t \in \mathbf{P}^1 - \{t_1, \ldots, t_8\}$, the curve $f^{-1}(C_t)$ cannot split into two components, else $\mathrm{Pic}(S) \supsetneq \Lambda_7$. Therefore $\gamma \cdot \delta_0^{''} = 0$ and hence $\Gamma \cdot \delta_0^{'} = 42$. $\square$

Proof of Theorem 0.4. The curve $\Gamma \subset \overline{\mathcal{R}}_8$ constructed above is a sweeping curve for the irreducible boundary divisor $\Delta_0^{'}$, in particular it intersects non-negatively every irreducible effective divisor D on $\overline{\mathcal{R}}_8$ which is different from $\Delta_0^{'}$. Since $\Gamma \cdot \delta_0^{'} > 0$, it follows that D intersects non-negatively *every* pseudoeffective divisor on $\overline{\mathcal{R}}_8$. Using the formula for the canonical divisor [FL]

$$K_{\overline{\mathcal{R}}_8} = 13\lambda - 2(\delta_0^{'} + \delta_0^{''}) - 3\delta_0^{\mathrm{ram}} - \cdots \in CH^1(\overline{\mathcal{R}}_8),$$

applying Lemma 3.2 we obtain that $\Gamma \cdot K_{\overline{\mathcal{R}}_8} = -4 < 0$, thus $K_{\overline{\mathcal{R}}_8} \notin \mathrm{Eff}(\overline{\mathcal{R}}_8)$. Using [BDPP], we conclude that $\overline{\mathcal{R}}_8$ is uniruled, in particular its Kodaira dimension is negative. $\square$

3.1. **The uniruledness of the universal singular locus of the theta divisor over $\overline{\mathcal{R}}_8$.** In what follows, we sketch a second, more geometric proof of Theorem 0.4, skipping some details. This proof provides a *concrete* way of constructing a rational curve through a general point of $\overline{\mathcal{R}}_8$. We fix a general element $[C, \eta] \in \mathcal{R}_8$ and denote by $f : \widetilde{C} \to C$ the corresponding unramified double cover and by $\iota : \widetilde{C} \to \widetilde{C}$ the involution exchanging the sheets of f. Following [W], we consider the singular locus of the Prym theta divisor, that is, the locus

$$V^3(C,\eta) = \mathrm{Sing}(\Xi) := \big\{L \in \mathrm{Pic}^{14}(\tilde{C}) : \mathrm{Nm}_f(L) = K_C,\ h^0(C,L) \geq 4 \text{ and } h^0(C,L) \equiv 0 \bmod 2\big\}.$$

It follows from [W], that $V^3(C,\eta)$ is a smooth curve. We pick a line bundle $L \in V^3(C,\eta)$ with $h^0(\widetilde{C}, L) = 4$, a general point $\tilde{x} \in \widetilde{C}$ and consider the ι-invariant part of the Petri map, that is,

$$\mu_0^+\big(L(-\tilde{x})\big) : \mathrm{Sym}^2 H^0(\widetilde{C}, L(-\tilde{x})) \to H^0(C, K_C(-x)), \quad s \otimes t + t \otimes s \mapsto s \cdot \iota^*(t) + t \cdot \iota^*(s),$$

where $x := f(\tilde{x}) \in C$. We set $\mathbf{P}^2 := \mathbf{P}\big(H^0(L(-\tilde{x}))^{\vee}\big)$, and similarly to [FV] Section 2.2, we consider the map $q : \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^5$ obtained from the Segre embedding $\mathbf{P}^2 \times \mathbf{P}^2 \hookrightarrow \mathbf{P}^8$ and then projecting onto the space of symmetric tensors. We have the following commutative diagram:

$$\begin{array}{ccccc}
\widetilde{C} & \xrightarrow{\big(L(-\tilde{x}),\, \iota^*(L(-\tilde{x}))\big)} & \mathbf{P}^2 \times \mathbf{P}^2 & \longrightarrow & \\
\downarrow f & & \downarrow q & & \mathbf{P}^8 = \mathbf{P}\big(H^0(L(-\tilde{x}))^{\vee} \otimes H^0(L(-\tilde{x}))^{\vee}\big) \\
C & \xrightarrow{\ |\mu_0^+(L(-\tilde{x}))|\ } & \mathbf{P}^5 = \mathbf{P}\big(\mathrm{Sym}^2 H^0\big(L(-\tilde{x})\big)^{\vee}\big) & \dashleftarrow &
\end{array}$$

Let $\Sigma := \mathrm{Im}(q) \subset \mathbf{P}^5$ be the determinantal cubic surface; its singular locus is the Veronese surface V_4. For a general choice of $[C,\eta] \in \mathcal{R}_8$, $L \in V^3(C,\eta)$ and of $\tilde{x} \in \tilde{C}$, the map

$\mu_0^+(L(-\tilde{x}))$ is injective and let $W \subset H^0(C, K_C(-x))$ be its 6-dimensional image. Comparing dimensions, we observe that the kernel of the multiplication map

$$\mathrm{Sym}^2(W) \longrightarrow H^0(C, K_C^{\otimes 2}(-2x))$$

is at least 2-dimensional. In particular, there exist distinct quadrics $Q_1, Q_2 \subset \mathbf{P}^5$ such that

$$C \subset S := Q_1 \cap Q_2 \cap \Sigma \subset \mathbf{P}^5.$$

Since $\mathrm{Sing}(\Sigma) = V_4$, the surface S is singular at the 16 points of intersection $Q_1 \cap Q_2 \cap V_4$. Assuming we can find $(C, \eta, L, \tilde{x})$ such that $\mathrm{Sing}(S) = Q_1 \cap Q_2 \cap V_4$, we obtain that S is a canonical surface, that is, $K_S = \mathcal{O}_S(1)$.

Using the exact sequence $0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(C)) \to H^0(\mathcal{O}_C(C)) \to 0$, since $\mathcal{O}_C(C) = \mathcal{O}_C(x)$, we obtain that $\dim |\mathcal{O}_S(C)| = 1$, that is, C moves on S. Moreover the pencil $|\mathcal{O}_S(C)|$ has $x \in S$ as a base point.

We denote by $\widetilde{S} := q^{-1}(S) \subset \mathbf{P}^2 \times \mathbf{P}^2$. For each curve $C_t \in |\mathcal{O}_S(C)|$, we denote by $\widetilde{C}_t := q^{-1}(C_t) \subset \widetilde{S}$ the corresponding double cover. Furthermore, we define a line bundle $L_t \in \mathrm{Pic}^{14}(\widetilde{C}_t)$, by setting $\mathcal{O}_{\widetilde{C}_t}(1,0) = L_t(-\tilde{x})$ (in which case, $\mathcal{O}_{\widetilde{C}_t}(0,1) = \iota^*(L_t(-\tilde{x}))$).

The construction we just explained induces a uniruled parametrization of the universal singular locus of the Prym theta divisor in genus 8 (which dominates $\mathcal{R}_8$). Our result is conditional to a (very plausible) transversality assumption:

Theorem 3.3. *Assume there exists $[C, \eta, L, x]$ as above, such that $S = Q_1 \cap Q_2 \cap \Sigma \subset \mathbf{P}^5$ is a 16-nodal canonical surface. Then the moduli space*

$$\mathcal{R}_8^3 := \Big\{[C, \eta, L] : [C, \eta] \in \mathcal{R}_8,\ L \in V^3(C, \eta)\Big\}$$

is uniruled.

Proof. The assignment $\mathbf{P}^1 \ni t \mapsto [\widetilde{C}_t/C_t, L_t] \in \mathcal{R}_8^3$ described above provides a rational curve passing through a general point of $\mathcal{R}_8^3$. □

References

[BDPP] S. Boucksom, J.P. Demailly, M. Păun and T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, Journal of Algebraic Geometry **22** (2013), 201-248.

[Bo] C. Böhning, *The rationality problem in algebraic geometry*, Habilitationsschrift, Göttingen 2009.

[Cas] G. Castelnuovo, *Numero delle involuzioni razionali giacenti sopra una curva di dato genere*, Rendiconti R. Accad. Lincei **5** (1889) 130-133.

[Cat] F. Catanese, *On the rationality of certain moduli spaces related to curves of genus* 4, Springer Lecture Notes in Mathematics **1008** (1983), 30-50.

[De] O. Debarre, *Sur le probleme de Torelli pour les varietes de Prym*, American J. Math. **111** (1989), 111-134.

[Do1] I. Dolgachev, *Mirror symmetry for lattice polarized $K3$ surfaces*, J. Math. Sciences **81** (1996), 2599-2630.

[Do2] I. Dolgachev, *Rationality of $\mathcal{R}_2$ and $\mathcal{R}_3$*, Pure and Applied Math. Quarterly, **4** (2008), 501-508.

[Do3] I. Dolgachev, *Classical algebraic geometry: A modern view*, Cambridge University Press 2012.

[Do] R. Donagi, *The unirationality of $\mathcal{A}_5$*, Annals of Mathematics **119** (1984), 269-307.

[FK] G. Farkas and M. Kemeny, *The generic Green-Lazarsfeld conjecture*, arXiv:1408.4164.

[FL] G. Farkas and K. Ludwig, *The Kodaira dimension of the moduli space of Prym varieties*, Journal of the European Mathematical Society **12** (2010), 755-795.

[FV] G. Farkas and A. Verra, *Moduli of theta-characteristics via Nikulin surfaces*, Mathematische Annalen **354** (2012), 465-496.

[GS] A. Garbagnati and A. Sarti, *Projective models of $K3$ surfaces with an even set*, Advances in Geometry **8** (2008), 413-440.

[GKM] A. Gibney, S. Keel and I. Morrison, *Towards the ample cone of* $\overline{\mathcal{M}}_{g,n}$, Journal of the American Mathematical Society **15** (2002), 273-294.

[vGS] B. van Geemen and A. Sarti, *Nikulin involutions on* $K3$ *surfaces*, Mathematische Zeitschrift **255** (2007), 731-753.

[ILS] E. Izadi, M. Lo Giudice and G. Sankaran, *The moduli space of étale double covers of genus* 5 *is unirational*, Pacific Journal of Mathematics **239** (2009), 39-52.

[Ka] P. Katsylo, *On the birational geometry of* $(\boldsymbol{P}^n)^{(m)}/GL_{n+1}$, Max-Planck Institut Preprint, MPI/94-144 (1994).

[Mu] S. Mukai, *Curves and* $K3$ *surfaces of genus eleven*, in: Moduli of vector bundles, Lecture Notes in Pure and Applied Mathematics Vol. 179, Dekker (1996), 189-197.

[Ni] V.V. Nikulin, *Kummer surfaces*, Izvestia Akad. Nauk SSSR **39** (1975), 278-293.

[V] A. Verra, *A short proof of the unirationality of* $\mathcal{A}_5$, Indagationes Math. **46** (1984), 339-355.

[W] G. Welters, *A theorem of Gieseker-Petri type for Prym varieties*, Annales Sci. École Normale Supérieure **18** (1985), 671–683.

HUMBOLDT-UNIVERSITÄT ZU BERLIN, INSTITUT FÜR MATHEMATIK, UNTER DEN LINDEN 6
10099 BERLIN, GERMANY
E-mail address: farkas@math.hu-berlin.de

UNIVERSITÁ ROMA TRE, DIPARTIMENTO DI MATEMATICA, LARGO SAN LEONARDO MURIALDO
1-00146 ROMA, ITALY
E-mail address: verra@mat.uniroma3.it