REDUCED RESONANCE SCHEMES AND CHEN RANKS

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ABSTRACT. The resonance varieties are cohomological invariants that are studied in a variety of topological, combinatorial, and geometric contexts. We discuss their scheme structure in a general algebraic setting and introduce various properties that ensure the reducedness of the associated projective resonance scheme. We prove an asymptotic formula for the Hilbert series of the associated Koszul module, then discuss applications to vector bundles on algebraic curves and to Chen ranks formulas for finitely generated groups, with special emphasis on Kähler and right-angled Artin groups.

CONTENTS

1. INTRODUCTION

The concept of resonance initially appeared in topology. For a reasonably nice topological space X, its resonance variety $\mathcal{R}(X)$ is defined as the jump locus

$$\mathcal{R}(X) := \left\{ a \in H^1(X, \mathbb{C}) : H^1\big(H^*(X, \mathbb{C}), \delta_a\big) \neq 0 \right\},$$

where $\delta_a \colon H^i(X, \mathbb{C}) \to H^{i+1}(X, \mathbb{C})$ is the differential obtained by left-multiplication by a. Assuming X is 1-formal in the sense of Sullivan, $\mathcal{R}(X)$ is intimately related to the *characteristic variety* $\mathcal{V}(X)$ parametrising rank 1 local systems on X with non-vanishing homology. More precisely (see [14]), the resonance can be described as the tangent cone to the characteristic variety, that is, $\mathcal{R}(X) = \mathrm{TC}_1\big(\mathcal{V}(X)\big)$, which links the resonance to the much studied

theory of characteristic varieties. A set-theoretic description of $\mathcal{R}(X)$ for compact Kähler manifolds in terms of irrational pencils has been given by Dimca–Papadima–Suciu [14]. For complements of hyperplane arrangements a combinatorial (matroidal) description of the resonance has been found by Falk–Yuzvinsky [16], building on previous work of Libgober–Yuzvinsky [24] and Falk [15]. A purely algebraic definition of the resonance variety has been put forward by Papadima–Suciu [32] and linked to the the theory of Koszul modules. Important applications of this link, including a proof of the Generic Green's Conjecture for syzygies of canonical curves in sufficiently high characteristic have been obtained in [2, 1].

By its very definition as a jump locus, the resonance $\mathcal{R}(X)$ carries a natural scheme structure that has been however largely ignored for some time. However, even in the much studied case of hyperplane arrangements, this scheme structure comes to the fore in the formulation of Suciu's Conjecture [36] concerning the Chen ranks of the fundamental group of the hyperplane arrangement, see also [12, 35]. We aim to study systematically the scheme-theoretic properties of resonance varieties in algebraic context, introduce several natural scheme-theoretic properties that resonance varieties often satisfy and explain how they are naturally linked to other interesting concepts in the theory of Kähler groups, vector bundles on algebraic varieties, or geometric group theory.

We begin by recalling the connection between resonance varieties and Koszul modules following the set-up of [1, 2], or [32]. Let V be a finite-dimensional vector space over an algebraically closed field $\Bbbk$ of characteristic 0, and let $V^\vee$ be its $\Bbbk$-dual. We let $S := \mathrm{Sym}(V)$ denote the symmetric algebra on V. For a linear subspace $K \subseteq \bigwedge^2 V$, we define the *Koszul module* $W(V, K)$ as the middle homology of the chain complex of graded S-modules

$$K \otimes S \xrightarrow{\;\delta_2|_{K \otimes S}\;} V \otimes S(1) \xrightarrow{\;\delta_1\;} S(2).$$

Note that $W(V, K)$ is a graded S-module. Its degree q piece $W_q(V, K)$ can be canonically identified with the cohomology of the following complex of finite dimensional $\Bbbk$-vector spaces

$$K \otimes \mathrm{Sym}^q V \xrightarrow{\;\delta_2|_{K \otimes \mathrm{Sym}^q V}\;} V \otimes \mathrm{Sym}^{q+1} V \xrightarrow{\;\delta_1\;} \mathrm{Sym}^{q+2} V,$$

where δ_1 and δ_2 are the Koszul differentials. The annihilator $\mathrm{Ann}\, W(V, K)$ of the S-module $W(V, K)$ defines a *resonance scheme* $\mathcal{R}(V, K)$, whose set-theoretic support, also denoted by $\mathcal{R}(V, K)$, has been described in [32] as being

$$\mathcal{R}(V, K) := \left\{ a \in V^\vee :\ \text{there exists } b \in V^\vee \text{ such that } a \wedge b \in K^\perp \backslash \{0\} \right\} \cup \{0\}, \qquad (1.1)$$

where $K^\perp$ is the kernel of the projection $\bigwedge^2 V^\vee \twoheadrightarrow K^\vee$. We consider here the *projectivized resonance* scheme,

$$\mathbf{R}(V, K) := \mathrm{Proj}\big(S/\mathrm{Ann}\ W(V, K)\big).$$

We investigate in what follows the geometry of the resonance schemes and introduce several concepts inspired by the study of particular cases of resonance varieties in topology and Hodge theory, which turn out to be often satisfied and which ensure the reducedness of the projectivized resonance $\mathbf{R}(V, K)$.

1.1. **Separable and isotropic resonance.** We denote by $E := \bigwedge V^\vee$ the exterior algebra of $V^\vee$ and fix a linear subspace $\overline{V}^\vee \subseteq V^\vee$. We say that $\overline{V}^\vee$ is *isotropic* (with respect to $K \subseteq \bigwedge^2 V$) if $\bigwedge^2 \overline{V}^\vee \subseteq K^\perp$. The subspace $\overline{V}^\vee$ is said to be *separable* (with respect to K) if

$$K^\perp \cap \langle \overline{V}^\vee \rangle_E \subseteq \bigwedge^2 \overline{V}^\vee, \tag{1.2}$$

where $\langle \overline{V}^\vee \rangle_E$ denotes the ideal of the exterior algebra E which is generated by $\overline{V}^\vee$. Finally, $\overline{V}^\vee$ is *strongly isotropic* if it both separable and isotropic, that is, when the following equality holds

$$K^\perp \cap \langle \overline{V}^\vee \rangle_E = \bigwedge^2 \overline{V}^\vee.$$

If all the irreducible components of the resonance variety $\mathcal{R}(V, K)$ are linear subspaces of $V^\vee$, we say that $\mathcal{R}(V, K)$ is *separable* (respectively, *isotropic*, or *strongly isotropic*) if each of those components of $\mathcal{R}(V, K)$ are separable (respectively, isotropic, or strongly isotropic). We use the same terminology for the Koszul module $W(V, K)$, respectively for the resonance scheme $\mathbf{R}(V, K)$. A characterization of strongly isotropic resonance $\mathbf{R}(V, K)$ reminiscent of Petri type theorems in algebraic geometry is provided in Lemma 3.13.

With this terminology in place, we can state one of our main results:

Theorem 1.1. *Let $K \subseteq \bigwedge^2 V$ be a linear subspace, and suppose all irreducible components of $\mathcal{R}(V, K)$ are linear subspaces of $V^\vee$.*

(1) *If $W(V, K)$ is separable, then the projectivized resonance scheme $\mathbf{R}(V, K)$ is reduced and its components are disjoint.*

(2) *If $\mathbf{R}(V, K)$ is reduced and isotropic, then it is separable.*

Assume now that the resonance $\mathcal{R}(V, K)$ is linear and denote by $\overline{V}_1^\vee, \ldots, \overline{V}_k^\vee$ the (linear) components of $\mathcal{R}(V, K)$. For each $1 \leqslant t \leqslant k$, the inclusion $\overline{V}_t^\vee \subseteq V^\vee$ corresponds to a linear projection, $\pi_t \colon V \twoheadrightarrow \overline{V}_t$. Setting $\overline{K}_t := \left(\bigwedge^2 \pi_t \right)(K) \subseteq \bigwedge^2 \overline{V}_t$, we obtain in this way Koszul modules $W(\overline{V}_t, \overline{K}_t)$. As an application of Theorem 1.1, we prove the following result, which describes the dimensions of the graded pieces of a separable Koszul module.

Theorem 1.2. *Suppose $W(V, K)$ is a separable Koszul module. Then*

$$\dim W_q(V, K) = \sum_{t=1}^{k} \dim W_q(\overline{V}_t, \overline{K}_t),$$

for all $q \gg 0$.

If the resonance is separable and isotropic, then all the subspaces $\overline{K}_t \subseteq \bigwedge^2 \overline{V}_t$ appearing in the statement of Theorem 1.2 are trivial. For a vector space U we have the canonical identification $W_q(U, 0) \cong \ker\{ U \otimes \operatorname{Sym}^q U \to \operatorname{Sym}^{q+1} U \}$, and so $\dim W_q(U, 0) = \binom{q + \dim U}{q+2}$. Therefore, we obtain in this case a simple combinatorial asymptotic formula for the Hilbert series of $W(V, K)$.

Corollary 1.3. *Suppose $W(V, K)$ is a separable and isotropic Koszul module and write $\mathcal{R}(V, K) = \overline{V}_1^\vee \cup \cdots \cup \overline{V}_k^\vee$. Then for all $q \gg 0$*

$$\dim W_q(V, K) = \sum_{t=1}^{k} (q + 1) \binom{q + \dim \overline{V}_t}{q + 2}.$$

1.2. **Chen ranks of groups.** One of the main applications of this theory is to the computation of the Chen ranks of large classes of finitely generated groups. Given such a group G, let G/G'' be its maximal metabelian quotient, where $G'' = \big[[G,G],[G,G]\big]$, and let $\mathrm{gr}(G/G'')$ be the associated graded Lie algebra with respect to the lower central series filtration on G/G''. The *Chen ranks* of G are defined [12, 29, 35] as the graded ranks of this (finitely generated) Lie algebra,

$$\theta_q(G) := \mathrm{rank}\,\mathrm{gr}_q(G/G'') = \mathrm{rank}\,\Gamma_q\big(G/G''\big)/\Gamma_{q+1}\big(G/G''\big). \tag{1.3}$$

From the cohomology algebra of G in low degrees, one can extract the Koszul module

$$W(G) := W(V, K)$$

of the group G, by setting $V = H_1(G, \Bbbk)$ and letting $K^\perp$ be the kernel of the cup-product map $\cup_G \colon \bigwedge^2 H^1(G, \Bbbk) \to H^2(G, \Bbbk)$. We let $\mathcal{R}(G) = \mathcal{R}(V, K)$ be the corresponding resonance variety of the group and $\mathbf{R}(G) = \mathbf{R}(V, K)$ the corresponding projectivized resonance scheme.

As shown e.g. in [29, 39], if the group G is 1-formal (that is, its pronilpotent Lie algebra admits a quadratic presentation), then

$$\theta_q(G) = \dim W_{q-2}(G) \tag{1.4}$$

for all $q \geqslant 2$. Moreover, as shown in [14], all the components of $\mathcal{R}(G)$ are (rationally defined) linear subspaces of $V^\vee$. As an immediate application of Corollary 1.3, we recover the main result of Cohen and Schenck [12, Theorem A], in a somewhat stronger form.

Corollary 1.4. *Let G be a 1-formal group, and assume $\mathbf{R}(G)$ is separable and isotropic. Denoting by $\overline{V}_1^\vee, \ldots, \overline{V}_k^\vee$ the (linear) components of $\mathcal{R}(G)$, we have*

$$\theta_q(G) = \sum_{t=1}^{k} (q-1)\binom{q + \dim \overline{V}_t - 2}{q}$$

for all $q \gg 0$.

The strengthening is two-fold: first, we drop the assumption that G admit a commutator-relators finite presentation (in view of [39], that is no longer needed); and second, we drop the assumption that the components of $\mathcal{R}(G)$ be projectively disjoint (in view of Theorem 1.1, that property follows automatically from separability). In Section 8 we will characterize the right-angled Artin group for which the resonance variety is separable respectively isotropic.

1.3. **Resonance for vector bundles on curves.** A major source of resonance varieties is provided by vector bundles in algebraic geometry. To a vector bundle E on a complex algebraic variety X, one can associate a Koszul module $W(X, E) := W(V, K)$ and a resonance variety $\mathcal{R}(X, E) := \mathcal{R}(V, K)$, by setting $V := H^0(X, E)^\vee$ and

$$K^\perp := \ker\Big\{d \colon \bigwedge^2 H^0(X, E) \longrightarrow H^0\big(X, \bigwedge^2 E\big)\Big\},$$

where d is the determinant map on global sections. It was observed in [3] that $\mathcal{R}(X, E)$ parametrizes *subpencils* of the vector bundle E, that is, line subbundles $A \hookrightarrow E$ such that

$h^0(X, A) \geqslant 2$. It is thus natural to seek to characterize geometrically those vector bundles E for which the resonance is strongly isotropic.

For simplicity, we assume X is a smooth algebraic curve and E is a rank 2 stable vector bundle of degree d on X. Following Drinfeld and Laumon [23], we say that E is *very stable* if it has no non-zero nilpotent Higgs fields, that is, the space $H^0\big(X, \omega_X \otimes \mathrm{End}(E)\big)$ contains no non-zero nilpotent elements. It has been proven in [33] that E is very stable if and only if the space $H^0\big(X, \omega_X \otimes \mathrm{End}(E)\big)$ is closed inside the moduli space $\mathrm{Higgs}_X(2, d)$ of rank 2 Higgs fields on X. For further connections between very stability and mirror symmetry, see [19]. It turns out that this concept is closely related to the strong isotropicity of $\mathcal{R}(X, E)$.

Given a line bundle L on X, let $\mathcal{SU}_X(2, L)$ be the moduli space of semistable rank 2 vector bundles E on X with $\det(E) \cong L$. The locus of stable but not very stable bundles on X is known to be a divisor in $\mathcal{SU}_X(2, L)$, see [28]. For a stable vector bundle $E \in \mathcal{SU}_X(2, L)$ and a positive integer a, we let

$$W_a^1(E) := \Big\{ A \in \mathrm{Pic}^a(X) : h^0(X, A) \geqslant 2 \text{ and } A \hookrightarrow E \Big\}$$

be the variety of degree a subpencils of E. The following result describes the structure of resonance varieties of rank 2 vector bundles on curves in terms of linear systems $|A|$:

Theorem 1.5. *Fix a general curve X of genus g and a line bundle $L \in \mathrm{Pic}^d(X)$, where $2g + 2 \leqslant d \leqslant 3g + 1$. For a general stable rank 2 vector bundle E on X with $\det(E) \cong L$, the following hold:*

(1) If $d < 3g + 1$, then $\mathcal{R}(X, E) = \{0\}$.

(2) If $d = 3g + 1$, then $\mathcal{R}(X, E)$ is strongly isotropic and moreover

$$\mathbf{R}(X, E) = \bigcup_{a = \lceil \frac{g+2}{2} \rceil}^{g+1} \mathbf{R}_a(X, E),$$

where $\mathbf{R}_a(X, E) = \bigcup \big\{ |A| : A \in W_a^1(E) \big\}$ is a disjoint union of

$$\# W_a^1(E) = \frac{2^{2a-g-2}}{g+1} \binom{g+1}{g-a+1, g-a+2, 2a-g-2}$$

disjoint projective lines.

For a bundle E of degree $3g + 1$ as in part (2), we have that $h^0(X, E) = g + 3$. The strong isotropicity of $\mathcal{R}(X, E)$ implies that the intersection $\mathbf{P} \ker(d) \cap \mathrm{Gr}_2\big(H^0(X, E)\big)$ consists of

$$\deg \mathrm{Gr}_2(g + 3) = \frac{(2g+2)!}{(g+1)! \cdot (g+2)!} = \sum_{a = \lceil \frac{g+2}{2} \rceil}^{g+1} \frac{2^{2a-g-2}}{g+1} \binom{g+1}{g-a+1, g-a+2, 2a-g-2}$$

reduced points. Furthermore, when $\deg(E) > 3g + 1$, the resonance $\mathcal{R}(X, E)$ is no longer linear, see Remark 6.9. In the particular case $a = g + 1$ we have a more precise result that holds for every very stable vector bundle, rather than for a general one.

Corollary 1.6. *Let X be a general curve of genus g and let E be a very stable vector bundle of degree $3g + 1$. Then $\mathbf{R}_{g+1}(X, E)$ consists of 2^g disjoint projective lines.*

Note that the number of maximal line subbundles of a vector bundle has been the subject of study in enumerative geometry [22,27] and recently in the context of Tevelev degrees [17].

1.4. **Resonance of Kodaira fibrations.** A Kodaira fibration is a submersion $f\colon X \to B$ from a smooth algebraic surface to a smooth projective curve B of genus $b \geqslant 2$, such that all fibres of f are smooth curves of genus g varying in moduli. Equivalently, f corresponds to a morphism $B \to M_g$ into the moduli stack of smooth genus g curves. Kodaira fibrations are objects of intense study in algebraic geometry [9,21], in the theory of surface bundles [11,20, 10,34], and in geometric group theory [26]. It is an open question posed independently by Catanese and Salter whether there are algebraic surfaces that admit more than two Kodaira fibration structures. We are confident that our Theorem 1.2 could be useful in this context. We have the following result describing the resonance of double Kodaira fibrations (for the case of surfaces admitting a unique Kodaira fibration structure, see Lemma 7.2).

Theorem 1.7. *Let X be a compact algebraic surface which admits two independent Kodaira fibrations, $\Sigma_{g_1} \hookrightarrow X \xrightarrow{f_1} B_1$ and $\Sigma_{g_2} \hookrightarrow X \xrightarrow{f_2} B_2$. Assume that the product map $f = (f_1, f_2)\colon X \to B_1 \times B_2$ induces an isomorphism on $H^1(-,\mathbb{C})$ and a monomorphism on $H^2(-,\mathbb{C})$. Then*

(1) The resonance scheme $\mathcal{R}(X) = f_1^* H^1(B_1, \mathbb{C}) \cup f_2^* H^1(B_2, \mathbb{C})$ is separable.

(2) The Chen ranks of $\pi_1(X)$ are given by the following formula, for $q \gg 0$,

$$\theta_q\bigl(\pi_1(X)\bigr) = (q-1)\left(\binom{2b_1 + q - 2}{q} + \binom{2b_2 + q - 2}{q}\right) - \binom{2b_1 + q - 3}{q - 2} - \binom{2b_2 + q - 3}{q - 2}.$$

We explain in Section 7.3 how the hypothesis of Theorem 1.7 are verified for the Atiyah–Kodaira surfaces constructed in [5,21]; consequently, the theorem provides a formula for the Chen ranks (in large degrees) of these surfaces. Finally, we also conjecture in §7.3 that the projective resonance $\mathbf{R}(G)$ of any Kähler group G is reduced and that a "Chen ranks formula" in the spirit of Theorem 1.2 and generalizing the one in Theorem 1.7 always holds.

In this paper we refrain from studying the Koszul modules and the Chen ranks of hyperplane arrangements. They will form the subject of the forthcoming paper [4].

2. Koszul modules and resonance schemes

2.1. **Koszul modules.** We now recast some of the notions introduced in the previous section in a more convenient setting, following the approach adopted in [32] and developed in [2,1,3].

Once again, let V be a finite-dimensional vector space over an algebraically closed field $\Bbbk$ of characteristic 0, and let $K \subseteq \bigwedge^2 V$ be a subspace. We also let $K^\perp \subseteq \bigwedge^2 V^\vee$ be the subspace of all linear functionals vanishing on K. Unless otherwise specified, all tensor products will be over $\Bbbk$.

Set $S := \operatorname{Sym}(V)$. Upon picking a basis for V, this algebra may be identified with the polynomial ring $\Bbbk[x_1, \ldots, x_n]$, where $n = \dim V$. Let $\left(\bigwedge^\bullet V \otimes S, \delta\right)$ be the corresponding Koszul complex. As already mentioned, the *Koszul module* $W(V, K)$ is the middle homology of the chain complex

$$ K \otimes S \xrightarrow{\ \delta_2|_{K \otimes S}\ } V \otimes S(1) \xrightarrow{\ \delta_1\ } S(2). \tag{2.1} $$

It is readily seen that $W(V, K)$ is the cokernel of the S-linear map

$$ \left(\textstyle\bigwedge^3 V \oplus K\right) \otimes S \longrightarrow \textstyle\bigwedge^2 V \otimes S \tag{2.2} $$

with matrix $\delta_3 + \iota \otimes \operatorname{id}_S$, where $\iota \colon K \hookrightarrow \bigwedge^2 V$ is the inclusion map. One has an alternate presentation of $W(V, K)$ as a cokernel of a morphism of graded S-modules:

$$ \textstyle\bigwedge^3 V \otimes S(-1) \longrightarrow \left(\textstyle\bigwedge^2 V / K\right) \otimes S \longrightarrow W(V, K) \longrightarrow 0 \tag{2.3} $$

A $\Bbbk$-linear map $\varphi \colon V \to \overline{V}$ induces a linear map $\bigwedge^2 \varphi \colon \bigwedge^2 V \to \bigwedge^2 \overline{V}$ and we let $\overline{K}$ be the image of K under this map. Writing $\overline{S} := \operatorname{Sym}(\overline{V})$, the map φ induces a ring morphism $\pi \colon S \to \overline{S}$. Denoting by $\widetilde{W}(\overline{V}, \overline{K})$ the cokernel of the composed map

$$ (\textstyle\bigwedge^3 V \oplus K) \otimes S \longrightarrow \textstyle\bigwedge^2 V \otimes S \longrightarrow \textstyle\bigwedge^2 \overline{V} \otimes S $$

we obtain a morphism of graded S-modules (which by abuse of notation we also call φ),

$$ \varphi \colon W(V, K) \longrightarrow\!\!\!\!\rightarrow \widetilde{W}(\overline{V}, \overline{K}). \tag{2.4} $$

Clearly, $W(\overline{V}, \overline{K}) = \widetilde{W}(\overline{V}, \overline{K}) \otimes_S \overline{S}$. Moreover, if $\varphi \colon V \to \overline{V}$ is surjective, then the morphism (2.4) is also surjective.

2.2. Koszul modules and differentials.
Next, we explain the relationship between Koszul modules and the sheaf $\Omega = \Omega^1_{\mathbf{P}}$ of differential forms on the projective space $\mathbf{P} := \mathbf{P}(V^\vee)$. Consider the Euler sequence

$$ 0 \longrightarrow \Omega \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow 0. \tag{2.5} $$

If $p = [f] \in \mathbf{P}$, where $0 \neq f \in V^\vee$, then the restriction of (2.5) to p identifies with

$$ 0 \longrightarrow \ker(f) \longrightarrow V \xrightarrow{\ f\ } \Bbbk \longrightarrow 0. $$

Using the left exactness of the global sections functor, and the fact that $W(V, 0) = \ker(\delta_1)$ (as follows from (2.1)), we obtain from (2.5) that

$$ W_q(V, 0) = H^0\bigl(\mathbf{P}, \Omega(q + 2)\bigr). \tag{2.6} $$

for all $q \in \mathbb{Z}$. Using (2.1), we can also write

$$W_q(V, K) = \operatorname{coker}\Big\{ K \otimes \operatorname{Sym}^q V \longrightarrow W_q(V, 0)\Big\}. \tag{2.7}$$

The graded S-module $W(V, K)$ induces a coherent sheaf $\mathcal{W}(V, K)$ on the projective space $\mathbf{P} = \mathbf{P}\big(V^\vee\big)$. We refer to $\mathcal{W}(V, K)$, as the *Koszul sheaf* associated with the pair (V, K).

From the natural surjection $\bigwedge^2 V \otimes \mathcal{O}_{\mathbf{P}}(-2) \twoheadrightarrow \Omega$, coupled with the inclusion $K \subseteq \bigwedge^2 V$, we obtain a map $K \otimes \mathcal{O}_{\mathbf{P}}(-2) \to \Omega$, which, upon twisting, taking global sections, and using (2.7) yields the identification

$$W_q(V, K) = \operatorname{coker}\Big\{ H^0\big(\mathbf{P}, K \otimes \mathcal{O}_{\mathbf{P}}(q)\big) \longrightarrow H^0\big(\mathbf{P}, \Omega(q + 2)\big)\Big\},$$

for all $q \in \mathbb{Z}$. Accordingly, the associated Koszul sheaf can be realized as

$$\mathcal{W}(V, K) = \operatorname{coker}\Big\{ K \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \Omega(2)\Big\}, \tag{2.8}$$

and in particular we have the identification $\mathcal{W}(V, 0) = \Omega(2)$ of sheaves on $\mathbf{P}$.

If $i\colon \overline{V}^\vee \subseteq V^\vee$ is a linear subspace, let $\pi\colon V \twoheadrightarrow \overline{V}$ be the dual map. Set $\overline{K} := \bigwedge^2 \pi(K)$. By the discussion above, we have a surjective morphism of graded S-modules,

$$\pi\colon W(V, K) \longrightarrow \widetilde{W}(\overline{V}, \overline{K}). \tag{2.9}$$

The morphism π from (2.9) corresponds to a surjective morphism of coherent $\mathcal{O}_{\mathbf{P}}$-sheaves,

$$\pi\colon \mathcal{W}(V, K) \longrightarrow i_* \mathcal{W}(\overline{V}, \overline{K}). \tag{2.10}$$

2.3. **The presentation of the Koszul sheaf.** Let $\overline{\mathbf{P}} = \mathbf{P}\big(\overline{V}^\vee\big) \hookrightarrow \mathbf{P} = \mathbf{P}\big(V^\vee\big)$ be the inclusion map and let p be a point in $\overline{\mathbf{P}}$. Write $p = [e_1]$ for $0 \neq e_1 \in \overline{V}^\vee$ and complete $\{e_1\}$ to bases of $\overline{V}^\vee$ and $V^\vee$. We let $X_1, \ldots, X_n$ denote the homogeneous coordinates on $\mathbf{P}$ corresponding to the choice of basis for $V^\vee$, and we let $x_i := X_i/X_1$ denote the local coordinates at the point p. Letting $A = \mathcal{O}_{\mathbf{P}, p}$ denote the local ring of $\mathbf{P}$ at p, we have

$$A = \Bbbk[x_2, \ldots, x_n]_{\mathfrak{m}}, \tag{2.11}$$

where $\mathfrak{m} = (x_2, \ldots, x_n)$. Using (2.5), we have that the stalk of Ω at p can be described as

$$\Omega_p = \bigoplus_{j=2}^n A \cdot dv_j, \tag{2.12}$$

where $dv_j := v_j - x_j \cdot v_1$. Here we identify v_j and $v_j \otimes \frac{1}{X_1} \in V \otimes \mathcal{O}_{\mathbf{P}, p}(-1)$.

Proposition 2.1. *When viewed as an A-module, the stalk of the Koszul sheaf $\mathcal{W} = \mathcal{W}(V, K)$ at the point $p \in \overline{\mathbf{P}}$ has presentation*

$$K \otimes A \xrightarrow{\ \partial\ } \Omega_p \xrightarrow{\ \nu\ } \mathcal{W}_p, \tag{2.13}$$

where ∂ is the restriction to $K \otimes A$ of the A-linear map $\delta_2 \colon \bigwedge^2 V \otimes A \to \Omega_p$ by

$$\begin{aligned}
\delta_2(v_s \wedge v_t) &= x_s \cdot dv_t - x_t \cdot dv_s \quad &\textit{for } 2 \leqslant s, t \leqslant n, \\
\delta_2(v_1 \wedge v_t) &= dv_t \quad &\textit{for } 2 \leqslant t \leqslant n.
\end{aligned} \tag{2.14}$$

Proof. The Koszul differential $\delta_2 \colon \bigwedge^2 V \otimes S \to V \otimes S$ yields a map $\delta_2 \colon \bigwedge^2 V \otimes A \to V \otimes A$, which in turn restricts to a map $\partial \colon K \otimes A \to \Omega_p$. We then obtain the commuting diagram

$$\begin{array}{ccc}
K \otimes A & \xrightarrow{\ \partial\ } \Omega_p \xrightarrow{\ \nu\ } \mathcal{W}_p \\
\Big\downarrow & \Big\| \quad \nearrow \\
\bigwedge^2 V \otimes A & \xrightarrow{\ \delta_2\ } \Omega_p
\end{array} \tag{2.15}$$

and it is now readily verified that the map ∂ has the desired form. $\qquad\square$

By abuse of notation, we will write dv_t for the element of $\mathcal{W}_p$ which is the image under ν of the corresponding element $dv_t \in \Omega_p$.

2.4. Resonance varieties and schemes.
It has been proven in [32] that the set-theoretic support of the S-module $W(V, K)$ is given by the *resonance*,

$$\mathcal{R}(V, K) := \left\{ a \in V^{\vee} : \text{ there exists } b \in V^{\vee} \text{ such that } a \wedge b \in K^{\perp} \backslash \{0\} \right\} \cup \{0\}.$$

where $K^{\perp} \subseteq \bigwedge^2 V^{\vee}$. In other words, for a given $K \subseteq \bigwedge^2 V$ we have that $W_q(V, K) = 0$ for $q \gg 0$ if and only if $\mathcal{R}(V, K) = 0$. The annihilator of the Koszul module $W(V, K)$, which we denote by $I(V, K)$, is a homogeneous ideal in S. We let

$$\mathcal{R}(V, K) := \operatorname{Spec}\big(S/I(V, K)\big) \tag{2.16}$$

denote the *affine resonance scheme*, which is the scheme-theoretic support of $W(V, K)$ inside $V^{\vee}$. Since this is the only scheme structure we will use, there is no ambiguity in using the same notation for the scheme $\mathcal{R}(V, K)$ and its underlying variety.

Remark 2.2. There is another possible scheme structure on $\mathcal{R}(V, K)$, given by the Fitting ideal $\operatorname{Fitt}_0 W(V, K)$. However, the scheme structure given by the annihilator $I(V, K)$ is the minimal one, and is invariant under closed embeddings of ambient affine spaces. This property of the annihilator support will be used in the proof of Theorem 4.5. For a comparison of these two scheme structures in the case of right-angled Artin groups, see Example 8.6.

The scheme-theoretic support of the Koszul sheaf $\mathcal{W}(V, K)$ defined in (2.8) is called the *projective resonance scheme*, and is denoted by

$$\mathbf{R}(V, K) := \operatorname{Proj}\big(S/I(V, K)\big). \tag{2.17}$$

Using the same convention as above, we will denote the underlying projective resonance variety by $\mathbf{R}(V, K)$, although our main interest is in the scheme structure.

We isolate here and in the later sections certain desirable properties of the resonance schemes and their underlying varieties that arose from the study of resonance varieties of complements of hyperplane arrangements.

Definition 2.3. We say that the resonance of (V, K) is *linear* if $\mathcal{R}(V, K)$ is a union of linear subspaces of $V^\vee$. Furthermore, we say that $\mathcal{R}(V, K)$ is *projectively disjoint* if the irreducible components of $\mathbf{R}(V, K)$ are pairwise disjoint. Finally, we say that the resonance is *projectively reduced* if $\mathbf{R}(V, K)$ is a reduced scheme.

Remark 2.4. If $\mathbf{R}(V, K)$ is a finite union of lines, then these lines are necessarily projectively disjoint.

2.5. **Projective geometry interpretation of resonance.** The projectivized resonance has a simple description in terms of projective geometry, using the incidence variety of the Grassmannian. Let $\mathbf{G} := \mathrm{Gr}_2(V^\vee)$ be the Grassmannian of 2-planes in $V^\vee$, viewed as a subset of $\mathbf{P}\big(\bigwedge^2 V^\vee\big)$ via the Plücker embedding. Consider the diagram

$$\mathbf{P} \times \mathbf{G} \longleftarrow \Xi \xrightarrow{\ \mathrm{pr}_2\ } \mathbf{G} \lhook\joinrel\longrightarrow \mathbf{P}\big(\textstyle\bigwedge^2 V^\vee\big), \tag{2.18}$$
$$\Big\downarrow{\scriptstyle \mathrm{pr}_1}$$
$$\mathbf{P}$$

where $\Xi = \big\{(p, L) \in \mathbf{P} \times \mathbf{G} : p \in L\big\}$ is the incidence variety and pr_1 and pr_2 are the two projections as in (2.18). The natural bijection between the points of $\mathbf{G}$ and the set of lines in $\mathbf{P}$ is given by the correspondence $\mathrm{pr}_1 \circ \mathrm{pr}_2^{-1}$, which maps a point $[a \wedge b] \in \mathbf{G}$ to the line L_{ab} in $\mathbf{P}$ passing through $[a]$ and $[b]$. The inverse of this map is given by $L_{ab} \mapsto \mathrm{pr}_2(\mathrm{pr}_1^{-1}([a]) \cap \mathrm{pr}_1^{-1}([b]))$. The next lemma readily follows.

Lemma 2.5. *Set-theoretically, the projective resonance variety is given by*

$$\mathbf{R}(V, K) = \mathrm{pr}_1\big(\mathrm{pr}_2^{-1}(\mathbf{G} \cap \mathbf{P}K^\perp)\big). \tag{2.19}$$

Moreover, the following inclusion holds

$$\mathbf{G} \cap \mathbf{P}K^\perp \subseteq \mathrm{pr}_2\big(\mathrm{pr}_1^{-1}(\mathbf{R}(V, K))\big). \tag{2.20}$$

In the particular case of hyperplane arrangements, the equality (2.19) was previously established by Lima-Filho and Schenck in [25, Proposition 2.1]. As illustrated in Example 3.3 below, the inclusion (2.20) is not an equality in general.

Since the projection pr_2 realizes Ξ as the projectivization of the universal rank-two bundle on the Grassmannian, the projectivized resonance is covered by lines. More precisely, if $[a] \in \mathbf{R}(V, K)$ and $0 \neq a \wedge b \in K^\perp$, then the line L_{ab} joining $[a]$ and $[b]$ is included in $\mathbf{R}(V, K)$. Note that a point in the resonance may be contained in a higher-dimensional linear subspace of $\mathbf{R}(V, K)$. Specifically, if $[a] \in \mathbf{R}(V, K)$, then

$$H_a := \big\{[b] \in \mathbf{P} : a \wedge b \in K^\perp\big\} \tag{2.21}$$

is a linear subspace of $\mathbf{P}$ of dimension at least one and completely contained in $\mathbf{R}(V, K)$.

The projective geometry description of resonance yields plenty of examples of projective varieties that are not resonance varieties.

Example 2.6. Even among varieties covered by lines, there are simple examples that are not resonance varieties. Let $\Gamma = \{p_1, \ldots, p_m\} \subseteq \mathbf{G}$ be a set of points that are not contained

in any hyperplane of $\mathbf{P}(\bigwedge^2 V^\vee)$. Then the set $X := \operatorname{pr}_1 \circ \operatorname{pr}_2^{-1}(\Gamma)$ is the union of m disjoint lines, $X = L_1 \sqcup \cdots \sqcup L_m$. Suppose $X = \mathbf{R}(V, K)$, for some subspace $K \subseteq \bigwedge^2 V$. By the above discussion, for any $[a] \in \mathbf{R}(V, K)$, the linear subspace H_a is contained in $\mathbf{R}(V, K)$. It follows that for any j and any $[a], [b] \in L_j$ we have $a \wedge b \in K^\perp$, which implies that $\Gamma = \mathbf{G} \cap \mathbf{P}K^\perp$, which is a contradiction. Hence, X is not a resonance variety.

Remark 2.7. The above situation is in stark contrast with what happens for the closely related characteristic varieties. For instance, given any integrally defined hypersurface $Y \subseteq (\mathbb{C}^*)^n$, there exists a finitely presented group G with $H_1(G, \mathbb{Z}) = \mathbb{Z}^n$ for which the characteristic variety $\mathcal{V}(G) := \{\rho \in \operatorname{Hom}(G, \mathbb{C}^*) : H_1(G, \mathbb{C}_\rho) \neq 0\}$ is isomorphic to $Y \cup \{1\}$; see [38, Lemma 10.3].

3. Isotropic and separable subspaces of resonance

Recall that V is a finite-dimensional vector space and $K \subseteq \bigwedge^2 V$ is a linear subspace.

Definition 3.1. A linear subspace $\overline{V}^\vee \subseteq V^\vee$ is said to be *isotropic* (with respect to K) if $\bigwedge^2 \overline{V}^\vee \subseteq K^\perp$.

The isotropicity property can be described by passing to the quotient. If $\pi \colon V \twoheadrightarrow \overline{V}$ is the corresponding projection, recalling that $K^\perp$ is the kernel of the projection $\bigwedge^2 V^\vee \twoheadrightarrow K^\vee$, setting $\overline{K} := \bigwedge^2(\pi)(K)$ we observe that $\overline{V}^\vee$ is isotropic if and only if $\overline{K} = 0$.

Definition 3.2. We say that the resonance variety $\mathcal{R}(V, K)$ is *isotropic* if it is linear and each of its irreducible components is isotropic.

By definition, any isotropic subspace $\overline{V}^\vee \subseteq V^\vee$ is automatically contained in the resonance variety $\mathcal{R}(V, K)$. Moreover, $\mathbf{R}(V, K)$ is a union of isotropic lines; more precisely,

$$\mathbf{R}(V, K) = \bigcup_{a,b \in V^\vee : a \wedge b \in K^\perp} L_{ab}. \tag{3.1}$$

On the other hand, a linear component of the resonance variety is not necessarily isotropic, as shown by the following example.

Example 3.3. Consider a subspace $K \subseteq \bigwedge^2 V$ of dimension $1 \leqslant m \leqslant n - 2$, where $n = \dim V$. Since the sheaf $\Omega^1_{\mathbf{P}}(2)$ has rank $n - 1$, formula (2.8) implies that $\mathcal{R}(V, K)$ coincides with $V^\vee$. However, $\mathcal{R}(V, K)$ is not isotropic, as $K^\perp$ is a proper subspace of $\bigwedge^2 V^\vee$.

Remark 3.4. Another instance when the resonance coincides with the ambient space is when V decomposes as a non-trivial direct sum of $\Bbbk$-vector subspaces, $V = U_1 \oplus U_2$, so that $U_1^\vee \wedge U_2^\vee \subseteq K^\perp$. Then it follows that $\mathcal{R}(V, K) = V^\vee$, see also [30].

For the next definition, let $E := \bigwedge V^\vee$ be the exterior algebra on the dual vector space $V^\vee$, and write $\langle U \rangle_E$ for the ideal in E generated by a subset $U \subseteq E$.

Definition 3.5. A linear space $\overline{V}^\vee \subseteq V^\vee$ is *separable* (with respect to $K \subseteq \bigwedge^2 V$) if

$$K^\perp \cap \langle \overline{V}^\vee \rangle_E \subseteq \bigwedge^2 \overline{V}^\vee. \tag{3.2}$$

We say that $\overline{V}^{\vee}$ is *strongly isotropic* if it is separable and isotropic, that is,

$$K^{\perp} \cap \langle \overline{V}^{\vee} \rangle_E = \bigwedge{}^2 \overline{V}^{\vee}.$$

We say that the resonance variety $\mathcal{R}(V, K)$ is *separable* if it is linear and each of its irreducible components is separable. Likewise, the resonance is *strongly isotropic* if it is isotropic and separable.

Example 3.6. If $K^{\perp} = 0$, then any subspace is separable with respect to K. At the other end of the spectrum, if $K^{\perp} = \bigwedge^2 V^{\vee}$, then the only separable subspace is the trivial one.

The next example will be used in the proof of Theorem 4.5.

Example 3.7. Suppose $\mathcal{R}(V, K) = V^{\vee}$. Then the resonance variety is separable. Moreover, the scheme structure is clearly reduced. If $\mathcal{R}(V, K) = \{0\}$, then the resonance variety is automatically separable. The scheme structure is not necessarily reduced; however, the projectivized resonance is empty, and thus reduced.

Example 3.8. An example of non-separable resonance is obtained if $V^{\vee} = \langle e_1, e_2, e_3, e_4 \rangle$ and $K^{\perp} = \langle e_1 \wedge e_2, e_1 \wedge e_3 + e_2 \wedge e_4 \rangle$. Then $\mathcal{R}(V, K) = \langle e_1, e_2 \rangle$, which is not separable.

As shown in the next example, separable subspaces are not necessarily contained in the resonance.

Example 3.9. Take again $V^{\vee} = \langle e_1, e_2, e_3, e_4 \rangle$ and $K^{\perp} = \langle e_1 \wedge e_2 + e_3 \wedge e_4 \rangle$. Then the subspace $\overline{V}^{\vee} = \langle e_1, e_2 \rangle$ is separable, yet $\mathcal{R}(V, K) = \{0\}$.

3.1. **A local view of separability.** If $n = \dim V$ and $\overline{n} = \dim \overline{V}$, fix a basis $(e_1, \dots, e_n)$ of $V^{\vee}$ such that $(e_1, \dots, e_{\overline{n}})$ is a basis for $\overline{V}^{\vee}$. Set $U := \ker \{\pi \colon V \to \overline{V}\}$. Letting $(v_1, \dots, v_n)$ denote the dual basis of V, we obtain a direct-sum decomposition,

$$\bigwedge{}^2 V = L \oplus M \oplus H, \tag{3.3}$$

where

$$\begin{aligned}
L &= \langle v_s \wedge v_t : s, t \leqslant \overline{n} \rangle \cong \bigwedge{}^2 \overline{V}, \\
M &= \langle v_s \wedge v_t : s \leqslant \overline{n} \text{ and } t > \overline{n} \rangle \cong \overline{V} \otimes U, \\
H &= \langle v_s \wedge v_t : s, t > \overline{n} \rangle \cong \bigwedge{}^2 U.
\end{aligned} \tag{3.4}$$

Observe that

$$M^{\vee} = \left(\bigwedge{}^2 V^{\vee} \cap \langle \overline{V}^{\vee} \rangle_E \right) / \bigwedge{}^2 \overline{V}^{\vee}. \tag{3.5}$$

Consider now the map $\bigwedge^2 \pi \colon \bigwedge^2 V \to \bigwedge^2 \overline{V}$. Then $\ker \left(\bigwedge^2 \pi \right)^{\vee} = \bigwedge^2 V^{\vee} / \bigwedge^2 \overline{V}^{\vee}$, and hence $M^{\vee} \subseteq \ker \left(\bigwedge^2 \pi \right)^{\vee}$, inducing a surjection $\overline{\pi} \colon \ker \left(\bigwedge^2 \pi \right) \twoheadrightarrow M$. Let

$$p_M \colon K \cap \ker \left(\bigwedge{}^2 \pi \right) \longrightarrow M \tag{3.6}$$

be the restriction of $\overline{\pi}$ to the subspace $K \cap \ker \left(\bigwedge^2 \pi \right)$. The next lemma provides a convenient local criterion for verifying the separability of $\overline{V}^{\vee}$, that we will often use for concrete applications.

Lemma 3.10. *With notation as above,*

(1) $\operatorname{coker}(p_M) \cong \left((K^\perp \cap \langle \overline{V}^\vee \rangle_E)/(K^\perp \cap \bigwedge^2 \overline{V}^\vee) \right)^\vee.$

(2) The subspace $\overline{V}^\vee \subset V^\vee$ is separable if and only if the map p_M is surjective.

Proof. The exact sequence

$$0 \longrightarrow K \cap \ker\left(\textstyle\bigwedge^2 \pi\right) \longrightarrow K \longrightarrow \textstyle\bigwedge^2 \overline{V}$$

gives rise by dualizing to the exact sequence

$$K^\perp \cap \textstyle\bigwedge^2 \overline{V}^\vee \longrightarrow \textstyle\bigwedge^2 \overline{V}^\vee \longrightarrow K^\vee \longrightarrow (K \cap \ker(\textstyle\bigwedge^2 \pi))^\vee \longrightarrow 0, \qquad (3.7)$$

from which we infer that the kernel of the map $K^\vee \to (K \cap \ker(\bigwedge^2 \pi))^\vee$ is equal to $\bigwedge^2 \overline{V}^\vee/(K^\perp \cap \bigwedge^2 \overline{V}^\vee)$. Consider now the diagram

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow K^\perp \cap \bigwedge^2 \overline{V}^\vee & \longrightarrow & \bigwedge^2 \overline{V}^\vee & \longrightarrow & \bigwedge^2 \overline{V}^\vee/(K^\perp \cap \bigwedge^2 \overline{V}^\vee) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow K^\perp \cap \langle \overline{V}^\vee \rangle_E & \longrightarrow & \bigwedge^2 V^\vee \cap \langle \overline{V}^\vee \rangle_E & \longrightarrow & K^\vee \\
\downarrow & & \downarrow & & \downarrow \\
0 \to (K^\perp \cap \langle \overline{V}^\vee \rangle_E)/(K^\perp \cap \bigwedge^2 \overline{V}^\vee) & \longrightarrow & M^\vee & \xrightarrow{p_M^\vee} & \left(K \cap \ker(\bigwedge^2 \pi) \right)^\vee \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$
$$(3.8)$$

By (3.5), the middle column is exact; the left columns and the top row are clearly exact; finally, the middle row is also exact. Thus, the bottom row is also exact and the diagram commutes, by the Snake Lemma. Dualizing the bottom row in diagram (3.8) yields the first claim. The second claim now follows from claim (1) and Definition 3.5. $\qquad \square$

When $\overline{V}^\vee$ is isotropic with respect to K, the above separability criterion simplifies. First observe that $\overline{V}^\vee$ is isotropic if and only if $K \subseteq \ker(\bigwedge^2 \pi)$. We then have the following:

Corollary 3.11. *An isotropic subspace $\overline{V}^\vee \subseteq V^\vee$ is separable if and only if the map $p_M \colon K \to M$ given by (3.6) is surjective.*

We refer to Lemma 3.13 below for a dual version of Corollary 3.11. With the help of diagram (2.18), we obtain now the following projective-geometric interpretation of separability.

Lemma 3.12. *Let $\overline{V}^\vee \subseteq V^\vee$ be a linear subspace and denote by $\overline{\mathbf{P}} \subseteq \mathbf{P}$ the associated projective subspace. Then $\overline{V}^\vee$ is separable if and only if*

$$\operatorname{Span}(\operatorname{pr}_2 \operatorname{pr}_1^{-1}(\overline{\mathbf{P}})) \cap \mathbf{P}(K^\perp) \subseteq \mathbf{P}\left(\textstyle\bigwedge^2 \overline{V}^\vee\right). \qquad (3.9)$$

Proof. Note that the projective subspace $\operatorname{Span}(\operatorname{pr}_2 \operatorname{pr}_1^{-1}(\overline{\mathbf{P}})) \subseteq \mathbf{P}\left(\bigwedge^2 V^\vee\right)$ spanned by $\operatorname{pr}_2 \operatorname{pr}_1^{-1}(\overline{\mathbf{P}})$ is the projectivization of the space of quadrics in the exterior ideal $\langle \overline{V}^\vee \rangle_E$. The claim follows. $\qquad \square$

Lemma 3.10 provides useful information regarding the restriction of the Koszul sheaf over the projectivization $\overline{\mathbf{P}} := \mathbf{P}(\overline{V})$ of a separable subspace $\overline{V}$. Recall that $\overline{K}$ denotes the image of K under the projection $\bigwedge^2 V \twoheadrightarrow \bigwedge^2 \overline{V}$ and that the conormal bundle $N^\vee_{\overline{\mathbf{P}}/\mathbf{P}}$ is isomorphic to $\ker(\pi) \otimes \mathcal{O}_{\mathbf{P}}(-1)$. We infer that the morphism $\mathcal{W}(V,K)|_{\overline{\mathbf{P}}} \to \mathcal{W}(\overline{V}, \overline{K})$ is an isomorphism provided $(K \cap \ker(\bigwedge^2 \pi)) \otimes \mathcal{O}_{\overline{\mathbf{P}}} \to \ker(\pi) \otimes \mathcal{O}_{\overline{\mathbf{P}}}(1)$ is surjective. Given that M is naturally identified with $\ker(\pi) \otimes \overline{V}$, Lemma 3.10 ensures that if $\overline{V}^\vee$ is separable, this morphism is already surjective on global sections, and hence the isomorphism $\mathcal{W}(V,K)|_{\overline{\mathbf{P}}} \xrightarrow{\simeq} \mathcal{W}(\overline{V}, \overline{K})$ is guaranteed. In the next section we shall prove a stronger statement, namely, that the map $\mathcal{W}(V,K) \to \mathcal{W}(\overline{V}, \overline{K})$ is a local isomorphism at any point of $\overline{\mathbf{P}}$.

3.2. **Strong isotropy via the multiplication map.** We now recast Corollary 3.11 in a setting that will prove to be useful when studying Koszul modules associated to vector bundles on varieties. Assume $\overline{V}^\vee \subseteq V^\vee$ is an isotropic subspace, put $U^\vee = V^\vee / \overline{V}^\vee$ with projection map $\rho \colon V^\vee \to U^\vee$, and denote by $\phi \colon \bigwedge^2 V^\vee \to K^\vee$ the dual to the inclusion map $K \hookrightarrow \bigwedge^2 V$. By the isotropy hypothesis, the map ϕ induces a *multiplication map*,

$$\mu \colon \overline{V}^\vee \otimes U^\vee \longrightarrow K^\vee,$$

by setting $\mu\big(\alpha \otimes \rho(\beta)\big) := \phi(\alpha \wedge \beta)$. Note that absent the isotropy condition, μ is not well-defined. With the notation from the previous sections, under the natural identification $M \cong \overline{V} \otimes U$, the map μ is dual to p_M. The following result which can be deduced from Lemma 3.10. For the convenience of the reader, we also give here a self-contained proof.

Lemma 3.13. *The subspace $\overline{V}^\vee$ is strongly isotropic if and only if μ is injective.*

Proof. Assume first that μ is injective. Let $\omega = \sum \alpha_i \wedge \beta_i \in K^\perp \cap \langle \overline{V} \rangle_E$, with $\alpha_i \in \overline{V}^\vee$ linearly independent. Set $b_i := \rho(\beta_i)$. Since $\phi(\omega) = 0$, it follows that $\mu(\sum \alpha_i \otimes b_i) = 0$, which implies $\sum \alpha_i \otimes b_i = 0$. Since α_i are linearly independent, the elements b_i all vanish, hence $\beta_i \in \overline{V}^\vee$ for all i. In particular, $\omega \in \bigwedge^2 \overline{V}^\vee$.

Conversely, if $\overline{V}^\vee$ is strongly isotropic, choose $w = \sum \alpha_i \otimes b_i \in \ker(\mu)$ with $\alpha_i \in \overline{V}^\vee$ linearly independent. Lifting each b_i to $\beta_i \in V^\vee$, we obtain an element $\omega = \sum \alpha_i \wedge \beta_i \in \ker(\phi) = K^\perp$. Since ω belongs also to $\langle \overline{V}^\vee \rangle$ and since $\overline{V}^\vee$ is isotropic, it follows that $\omega \in \bigwedge^2 \overline{V}^\vee$, and hence $\omega = \sum \alpha'_j \wedge \beta'_j$ with $\alpha'_j, \beta'_j \in \overline{V}^\vee$. Since $\rho(\beta'_j) = 0$, we have that $w = 0$. $\qquad\square$

This form of Corollary 3.11 is particularly useful in the case of vector bundles. We refer to Section 6 for a more detailed discussion.

4. Koszul modules with separable resonance

Our goal in this section is to study those Koszul modules whose resonance varieties are separable, and to prove Theorem 1.1, part (1) and Theorem 1.2 from the Introduction.

4.1. **Bases in K with respect to separable subspaces.** As usual, let V be a finite-dimensional vector space and let $K \subseteq \bigwedge^2 V$ be a subspace. The next lemma provides explicit bases in K, related to a given separable linear subspace $\overline{V}^\vee \subseteq V^\vee$. Let $\bigwedge^2 V = L \oplus M \oplus H$ be the direct-sum decomposition given by (3.4).

Lemma 4.1. *Let $\overline{V}^\vee$ be a separable subspace of $V^\vee$ with respect to K. There exists then a basis of K of the form $\{\alpha_{s,t} : s \leqslant \overline{n}, t > \overline{n}\} \cup \{\beta_1, \dots, \beta_N\}$, where N is a non-negative integer, such that, for each $s \leqslant \overline{n}$, $t > \overline{n}$, and $1 \leqslant j \leqslant N$, we have*

$$\alpha_{s,t} = v_s \wedge v_t + h_{s,t}, \quad \beta_j = \ell_j + h_j, \tag{4.1}$$

for some collection of elements $\ell_j \in L$ and $h_{s,t}, h_j \in H$.

Proof. By Lemma 3.10, the map p_M is surjective. Hence, we can lift each $v_s \wedge v_t \in M$ to an element $\alpha_{s,t} \in K \cap (M \oplus H)$. We write $\alpha_{s,t} = v_s \wedge v_t + h_{s,t}$, where $h_{s,t} \in H$. If we take $(\beta_1, \dots, \beta_N)$ to be a basis of an algebraic complement of $K \cap (M \oplus H)$ in K, the elements β_j have the desired form, namely, $\beta_j = \ell_j + h_j$, for some $\ell_j \in L$ and $h_j \in H$. $\qquad\square$

Note that for some of the elements β_j in the above lemma (e.g., those contained in the kernel of p_M from (3.6)), the corresponding element ℓ_j is zero.

4.2. **Koszul sheaves and separable components.** Consider the map of Koszul modules $\pi \colon W(V, K) \to W(\overline{V}, \overline{K})$ from (2.9) and the corresponding map of Koszul sheaves,

$$\pi \colon \mathcal{W}(V, K) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{W}(\overline{V}, \overline{K}). \tag{4.2}$$

We write $\mathcal{W} = \mathcal{W}(V, K)$ and $\overline{\mathcal{W}} = \mathcal{W}(\overline{V}, \overline{K})$, and we let $\overline{\mathbf{P}} = \mathbf{P}(\overline{V}^\vee) \hookrightarrow \mathbf{P} = \mathbf{P}(V^\vee)$ be the inclusion map. Fix a point $p \in \mathbf{P}$. Write $p = [e_1]$ for $0 \neq e_1 \in \overline{V}^\vee$ and complete $\{e_1\}$ to bases of $\overline{V}^\vee$ and $V^\vee$ as before. By Proposition 2.1, the stalk of $\mathcal{W}$ at p has presentation $K \otimes A \xrightarrow{\partial} \Omega_p \xrightarrow{\nu} \mathcal{W}_p \to 0$, where ∂ is the restriction to $K \otimes A$ of the map defined on $\bigwedge^2 V \otimes A$ by (2.14).

Assume now that $\overline{V}^\vee$ is a separable subspace of $V^\vee$ with respect to K, and consider the basis of K obtained in Lemma 4.1.

Lemma 4.2. *For $t = \overline{n} + 1, \dots, n$, we have that $dv_t = 0$ in $\mathcal{W}_p$.*

Proof. We let $\mathcal{M} \subseteq \mathcal{W}_p$ denote the A-submodule generated by $dv_{\overline{n}+1}, \dots, dv_n$. To prove that $\mathcal{M} = 0$, it suffices by Nakayama's lemma to show that $\mathcal{M} \subseteq \mathfrak{m} \cdot \mathcal{M}$. We note that by (3.3) and (2.14) we have that

$$\nu(\partial(h)) \in \mathfrak{m} \cdot \mathcal{M} \ \text{ for all } h \in H. \tag{4.3}$$

Since

$$0 = \nu(\partial(\alpha_{1,t})) = \nu(\delta_2(\alpha_{1,t})) = \nu(dv_t) + \nu(\delta_2(h_{1,t})),$$

it follows that $dv_t \in \mathfrak{m} \cdot \mathcal{M}$ for $t = \overline{n} + 1, \dots, n$, showing that $\mathcal{M} \subseteq \mathfrak{m} \cdot \mathcal{M}$ and concluding the proof. $\qquad\square$

Combining Lemma 4.2 with the following result will allow us to conclude that locally at p, the Koszul sheaf $\mathcal{W}$ is scheme-theoretically supported on $\overline{\mathbf{P}} = \mathbf{P}(\overline{V}^\vee)$.

Lemma 4.3. *We have that $x_t \cdot dv_s = 0$ in $\mathcal{W}_p$, for all $2 \leqslant s \leqslant \overline{n}$ and $t > \overline{n}$.*

Proof. As in the proof of Lemma 4.2, we have that

$$0 = \nu(\partial(\alpha_{s,t})) = \nu(x_s \cdot dv_t - x_t \cdot dv_s) + \nu(\delta_2(h_{s,t})). \tag{4.4}$$

Using (4.3) together with Lemma 4.2 we conclude that $\nu(x_t \cdot dv_s) = 0$, as desired. □

Proposition 4.4. *Assume $\overline{V}^{\vee}$ is a separable linear subspace contained in $\mathcal{R}(V, K)$. Then the map $\pi \colon \mathcal{W}(V, K) \twoheadrightarrow \mathcal{W}(\overline{V}, \overline{K})$ induces an isomorphism on stalks at each point $p \in \overline{\mathbf{P}}$.*

Proof. If $B := A/(x_{\overline{n}+1}, \ldots, x_n)$ denotes the local ring $\mathcal{O}_{\overline{\mathbf{P}},p}$ of $\overline{\mathbf{P}}$ at p, then Lemmas 4.2 and 4.3 show that the map $\nu \colon \Omega_p \twoheadrightarrow \mathcal{W}_p$ factors through $\overline{\Omega}_p$, where $\overline{\Omega} = \Omega_{\overline{\mathbf{P}}}$. We thus obtain an alternative presentation of $\mathcal{W}_p$, this time over B, given by

$$K \otimes B \xrightarrow{\ \partial\ } \overline{\Omega}_p \longrightarrow \mathcal{W}_p. \tag{4.5}$$

Observe that $\partial(v_s \wedge v_t) = 0$ if $t > \overline{n}$, and therefore the map ∂ factors through $\overline{K} \otimes B$. Since $\overline{\mathcal{W}}_p$ has presentation

$$\overline{K} \otimes B \longrightarrow \overline{\Omega}_p \longrightarrow \overline{\mathcal{W}}_p, \tag{4.6}$$

we conclude that the natural map $\mathcal{W}_p \to \overline{\mathcal{W}}_p$ is an isomorphism, thereby completing the proof of Proposition 4.4. □

4.3. Separability and reduced scheme structure.

Next, we apply Proposition 4.4 to obtain a characterization of the separable irreducible components of the resonance scheme.

Theorem 4.5. *Each separable irreducible component of $\mathbf{R}(V, K)$ is a reduced, isolated component of projectivised resonance.*

Proof. Let $P = \mathbf{P}(\overline{V}^{\vee})$ be a separable component of $\mathbf{R}(V, K)$ (possibly non-reduced). Suppose that P intersects some other component Q (not necessarily linear), and let $p \in P \cap Q$ be any point. Using Proposition 4.4, we infer that locally at p the resonance scheme $\mathbf{R}(V, K)$ is a closed subscheme of $\overline{\mathbf{P}} = P_{\mathrm{red}}$, which is a contradiction, since P_{red} does not contain Q. Therefore, P is an isolated irreducible component, i.e., a connected component of $\mathbf{R}(V, K)$.

To prove that the scheme P is reduced, consider any point $p \in P$. Applying Proposition 4.4 as in the previous paragraph, we deduce that locally at p, the component P is contained in $\overline{\mathbf{P}} = P_{\mathrm{red}}$, and so $P = P_{\mathrm{red}}$. □

A consequence of this theorem proves (in a stronger form) Theorem 1.1, part (1).

Corollary 4.6. *If the whole resonance is separable, then it is also projectively disjoint and reduced.*

As we shall show in Example 8.6, the converse of Theorem 4.5 is not necessarily true.

4.4. **Decomposition of separable Koszul modules.** Suppose $\mathcal{R}(V,K)$ is linear, with components $\overline{V}_1^{\vee}, \ldots, \overline{V}_k^{\vee}$. For each $1 \leqslant t \leqslant k$, the inclusion $\overline{V}_t^{\vee} \subseteq V^{\vee}$ corresponds to a linear projection, $\pi_t \colon V \twoheadrightarrow \overline{V}_t$. We set $\overline{K}_t := \big(\bigwedge^2 \pi_t\big)(K)$, and obtain in this way Koszul modules $W(\overline{V}_t, \overline{K}_t)$, together with natural surjective maps $\pi_t \colon W(V,K) \twoheadrightarrow \widetilde{W}(\overline{V}_t, \overline{K}_t)$ as in (2.9), as well as corresponding maps of Koszul sheaves, $\pi_t \colon \mathcal{W} \twoheadrightarrow \mathcal{W}_t =: \mathcal{W}(\overline{V}_t, \overline{K}_t)$.

The next theorem proves Theorem 1.2 from the Introduction.

Theorem 4.7. *Suppose $W(V,K)$ is a separable Koszul module. Then the morphism*

$$\Pi := (\pi_1, \ldots, \pi_k) \colon W(V,K) \longrightarrow \bigoplus_{t=1}^{k} \widetilde{W}(\overline{V}_t, \overline{K}_t), \tag{4.7}$$

is an isomorphism in sufficiently large degrees.

Proof. Consider the map of sheaves

$$\mathbf{\Pi} := (\pi_1, \ldots, \pi_k) \colon \mathcal{W} \longrightarrow \bigoplus_{t=1}^{k} \mathcal{W}_t. \tag{4.8}$$

Let $p \in \mathbf{R}(V,K)$ a point; then p belongs to, say, $\overline{V}_t^{\vee}$. Since $\overline{V}_t^{\vee}$ is a separable subspace, Proposition 4.4 implies that π_t is an isomorphism at p. Moreover, outside $\mathbf{R}(V,K)$, the map π_t is also an isomorphism, since the stalks on each side are zero. Corollary 4.6 now shows that the map $\mathbf{\Pi}$ is an isomorphism. Since $\mathbf{\Pi}$ is the sheafification of Π, it follows that Π is an isomorphism in degrees $q \gg 0$. □

5. Isotropic components of the resonance

As mentioned in Section 4.3, the projective resonance can be reduced in the absence of separability. We will show in this section that if the resonance is isotropic, then the two conditions—projectively reduced and, respectively, separable—are actually equivalent.

Suppose $\overline{V}^{\vee} \subseteq V^{\vee}$ is a linear, irreducible component of $\mathcal{R}(V,K)$ which is isotropic, and let P denote the corresponding linear component of the resonance scheme $\mathbf{R}(V,K)$. The reduced subscheme structure on P is given by $P_{\mathrm{red}} = \overline{\mathbf{P}}$, where $\overline{\mathbf{P}} = \mathbf{P}\big(\overline{V}^{\vee}\big)$. The goal of this section is to analyze the condition that P is reduced. We prove the following theorem.

Theorem 5.1. *Suppose that $\overline{V}$ is an isotropic component of $\mathcal{R}(V,K)$, and let P denote the corresponding component of $\mathbf{R}(V,K)$. The following are equivalent:*

(1) P is reduced.
(2) P is generically reduced.
(3) $\overline{V}$ is strongly isotropic.

Proof. It is clear that (1) $\Rightarrow$ (2). Since strongly isotropic components are separable, it follows from Theorem 4.5 that the implication (3) $\Rightarrow$ (1) also holds. The only implication left to prove is therefore (2) $\Rightarrow$ (3). Let $\pi \colon V \to \overline{V}$ be the associated surjective homomorphism. Recall that $\overline{V}^{\vee}$ is isotropic if and only if $(\bigwedge^2 \pi)(K) = 0$. We then obtain as in (2.9) a surjective homomorphism of Koszul modules, $\pi \colon W(V,K) \twoheadrightarrow W(\overline{V}, 0)$, and a corresponding surjection at the level of sheaves, $\pi \colon \mathcal{W}(V,K) \twoheadrightarrow \mathcal{W}(\overline{V}, 0) = \overline{\Omega}(2)$, where $\overline{\Omega} = \Omega_{\overline{\mathbf{P}}}$ is viewed as a sheaf on $\mathbf{P}$ via the closed immersion $\overline{\mathbf{P}} \hookrightarrow \mathbf{P}$.

Suppose now that $p = [e_1]$ is a reduced point of P, where $0 \neq e_1 \in \overline{V}^{\vee}$. Upon choosing bases for $\overline{V}^{\vee}$ and $V^{\vee}$ as in Section 4, consider the decomposition $\bigwedge^2 V = L \oplus M \oplus H$ given by (3.3) and (3.4), and let $p_M \colon K \to M$ denote the restriction to K of the second-component projection $\bigwedge^2 V \to M$. As shown in Corollary 3.11, condition (3) is equivalent to p_M being surjective. Since the reduced locus of P is non-empty, it is dense in P, and so it is not contained in any other irreducible component of $\mathbf{R}(V, K)$. We will therefore assume that the point p is chosen to lie only on the component P of $\mathbf{R}(V, K)$. It follows that $\mathcal{W}(V, K)$ is supported on $\overline{\mathbf{P}}$ locally at p. We prove:

Lemma 5.2. *The map* $\pi \colon \mathcal{W}(V, K) \twoheadrightarrow \overline{\Omega}(2)$ *is a local isomorphism at the point* p.

Proof. Since $\mathcal{W}(V, K)$ is scheme-theoretically supported on $\overline{\mathbf{P}}$ at p, and since $\overline{\Omega}$ is a locally free sheaf on $\overline{\mathbf{P}}$, we can think of π locally at p as a split surjection of sheaves on $\overline{\mathbf{P}}$. To conclude, it is then enough to check that it is an isomorphism on the fiber at p.

Recall from (2.8) that $\mathcal{W}(V, K) = \operatorname{coker}\{K \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \Omega(2)\}$. We need to prove the exactness on the fiber at p of the exact sequence

$$K \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \Omega(2) \longrightarrow \overline{\Omega}(2) \longrightarrow 0 \,. \tag{5.1}$$

Exactness on the right follows because $\overline{\mathbf{P}} \hookrightarrow \mathbf{P}$ is a closed immersion. For the next argument, which is independent on the earlier choice of bases, we write $f = e_1 \in V^{\vee}$, so that $p = [f]$. Recalling that $\overline{V}^{\vee} \subseteq V^{\vee}$, there exists a linear form $\overline{f} \colon \overline{V} \to \Bbbk$ such that $f = \overline{f} \circ \pi$. The restriction of (5.1) yields a complex,

$$K \longrightarrow \ker(f) \longrightarrow \ker(\overline{f}),$$

where the second map is induced by π, and the first one is induced by the Koszul differential $a \wedge b \mapsto f(a) \cdot b - f(b) \cdot a$. By duality, its exactness reduces to that of

$$\ker(\overline{f})^{\vee} \longrightarrow \ker(f)^{\vee} \longrightarrow K^{\vee}, \tag{5.2}$$

where the second is the map ψ in the commutative diagram:

$$\begin{array}{ccccc} V^{\vee} & \xrightarrow{\;\wedge f\;} & \bigwedge^2 V^{\vee} & \twoheadrightarrow & K^{\vee}. \\ & \searrow & \big\uparrow{\scriptstyle \wedge f} & \nearrow{\scriptstyle \psi} & \\ & & \ker(f)^{\vee} & & \end{array} \tag{5.3}$$

The exactness of (5.2) amounts to the following statement: if $g \in V^{\vee}$ has the property that $g \wedge f \in K^{\perp}$, then the restriction of g to $\ker(f)$ is obtained via composition with π from a linear form on $\ker(\overline{f})$. This follows if we can prove that $g \in \overline{V}^{\vee}$. In view of (2.1), the condition that $g \wedge f \in K^{\perp}$ implies that $\langle g, f \rangle \subseteq \mathcal{R}(V, K)$, which in turn implies that the line $L_{f,g}$ through $p = [f]$ and $[g]$ lies in $\mathbf{R}(V, K)$. Since the only component of $\mathbf{R}(V, K)$ containing p is P, it follows that $L_{f,g} \subseteq P$. In particular, $g \in \overline{V}^{\vee}$, completing the proof. $\quad\square$

We let $A = \mathcal{O}_{\mathbf{P},p}$ as in (2.11). Using the presentation of $\mathcal{W}_p$ from (2.13), the description (2.12) of the stalk at p of Ω, and the analogous description of $\overline{\Omega}_p$, we derive from Lemma 5.2 an exact sequence

$$K \otimes A \xrightarrow{\ \partial\ } \bigoplus_{j=2}^{n} A \cdot dv_j \longrightarrow \bigoplus_{j=2}^{\overline{n}} B \cdot d\overline{v}_j \longrightarrow 0 \,, \tag{5.4}$$

where $B = A/(x_{\overline{n}+1}, \ldots, x_n) \cong \mathcal{O}_{\overline{\mathbf{P}},p}$, $\overline{v}_j = \pi(v_j) \in \overline{V}$, and $d\overline{v}_j = \overline{v}_j - x_j \cdot \overline{v}_1$. Since $\overline{V}^{\vee}$ is isotropic, it follows that $K^{\perp} \supseteq L^{\vee}$, and therefore $K \subseteq M \oplus H$, which yields

$$\ker(p_M) = K \cap H. \tag{5.5}$$

If we let $\mathfrak{m} = (x_2, \ldots, x_n)$ denote the maximal ideal of A, we infer using (2.14) that ∂ sends $(K \cap H) \otimes A$ into $\bigoplus_{j=\overline{n}+1}^{n} \mathfrak{m} \cdot dv_j$. We then obtain from (5.4) a commutative diagram,

$$\begin{array}{ccccc}
(K \cap H) \otimes A & \longrightarrow & \displaystyle\bigoplus_{j=\overline{n}+1}^{n} \mathfrak{m} \cdot dv_j & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
K \otimes A & \longrightarrow & \displaystyle\bigoplus_{j=2}^{n} A \cdot dv_j & \longrightarrow & \displaystyle\bigoplus_{j=2}^{\overline{n}} B \cdot d\overline{v}_j \longrightarrow 0 \,,
\end{array} \tag{5.6}$$

where the vertical maps are inclusions. Using (5.5), we infer that $p_M(K) \cong K/(K \cap H)$. Taking cokernels of the vertical maps in diagram (5.6) gives rise to an exact sequence,

$$p_M(K) \otimes A \xrightarrow{\ \partial\ } \left(\bigoplus_{j=2}^{\overline{n}} A \cdot dv_j \right) \oplus \left(\bigoplus_{j=\overline{n}+1}^{n} \Bbbk \cdot dv_j \right) \longrightarrow \bigoplus_{j=2}^{\overline{n}} B \cdot d\overline{v}_j \longrightarrow 0 \,, \tag{5.7}$$

where $A/\mathfrak{m} \cong \Bbbk$. Tensoring over A with $A/\mathfrak{m}^2$ preserves right exactness; therefore, since $\Bbbk \otimes_A A/\mathfrak{m}^2 = \Bbbk$ and $B \otimes_A A/\mathfrak{m}^2 = B/\mathfrak{m}^2$, we obtain a further exact sequence,

$$p_M(K) \otimes A/\mathfrak{m}^2 \xrightarrow{\ \partial\ } \left(\bigoplus_{j=2}^{\overline{n}} A/\mathfrak{m}^2 \cdot dv_j \right) \oplus \left(\bigoplus_{j=\overline{n}+1}^{n} \Bbbk \cdot dv_j \right) \longrightarrow \bigoplus_{j=2}^{\overline{n}} B/\mathfrak{m}^2 \cdot d\overline{v}_j \longrightarrow 0 \,. \tag{5.8}$$

It follows from (2.14) that $p_M(K) \otimes (\mathfrak{m}/\mathfrak{m}^2) \subseteq \ker(\partial)$, so (5.8) yields the exact sequence,

$$p_M(K) \otimes A/\mathfrak{m} \xrightarrow{\ \partial\ } \left(\bigoplus_{j=2}^{\overline{n}} A/\mathfrak{m}^2 \cdot dv_j \right) \oplus \left(\bigoplus_{j=\overline{n}+1}^{n} \Bbbk \cdot dv_j \right) \longrightarrow \bigoplus_{j=2}^{\overline{n}} B/\mathfrak{m}^2 \cdot d\overline{v}_j \longrightarrow 0 \,. \tag{5.9}$$

Viewing this as an exact sequence of $\Bbbk$-vector spaces, and noting that $\dim_{\Bbbk}(A/\mathfrak{m}^2) = n$ and $\dim_{\Bbbk}(B/\mathfrak{m}^2) = \overline{n}$, we conclude that

$$\dim(p_M(K)) \geqslant (\overline{n} - 1) \cdot n + (n - \overline{n}) - (\overline{n} - 1) \cdot \overline{n} = \overline{n} \cdot (n - \overline{n}) = \dim(M). \tag{5.10}$$

Since $p_M(K) \subseteq M$, this shows that p_M is surjective, thus concluding the proof of Theorem 5.1. $\qquad\square$

As a consequence of this theorem, we infer that separability is equivalent to reducedness in the isotropic case, thereby completing the proof of Theorem 1.1, part (2).

Corollary 5.3. *Suppose that $\mathcal{R}(V, K)$ is isotropic. Then $\mathcal{R}(V, K)$ is strongly isotropic if and only if $\mathbf{R}(V, K)$ is reduced.*

Proof. If the resonance variety is strongly isotropic, then it is separable, so by Theorem 1.1 part (1), we conclude that $\mathbf{R}(V, K)$ is reduced. The converse follows from Theorem 5.1. $\square$

6. Resonance for vector bundles on curves

6.1. **Vector bundles on projective varieties.** In this section we study the scheme-theoretic properties of resonance varieties associated to vector bundles. We fix a smooth projective variety X, a vector bundle E on X and consider the determinant map

$$d\colon \textstyle\bigwedge^2 H^0(X, E) \longrightarrow H^0\big(X, \textstyle\bigwedge^2 E\big). \tag{6.1}$$

Set $V := H^0(X, E)^\vee$ and $K^\perp := \ker(d) \subseteq \bigwedge^2 V^\vee$. Following [3], we consider the associated Koszul module $W(X, E) := W(V, K)$ and resonance variety $\mathcal{R}(X, E) := \mathcal{R}(V, K)$. As pointed out in [3], a section $0 \neq s \in H^0(X, E)$ lies in $\mathcal{R}(X, E)$ if an only it spans a *subpencil* of E, that is, $s \in H^0(X, A) \subseteq H^0(X, E)$, where $A \hookrightarrow E$ is a line subbundle with $h^0(X, A) \geqslant 2$. Thus one is led to describe the resonance variety $\mathcal{R}(X, E)$ in terms of the variety of subpencils of E.

We observe that the projective resonance $\mathbf{R}(X, E)$ of a vector bundle enjoys a nice geometric property. Recall the notation in Lemma 2.5, that is, $\mathbf{P} = \mathbf{P}(V^\vee)$.

Proposition 6.1. *Let E be a vector bundle over a smooth projective variety X. Then*

(1) *The projective resonance scheme $\mathbf{R}(X, E)$ comes equipped with a regular morphism $\chi\colon \mathbf{R}(X, E) \to \mathcal{Z}$, where*

$$\mathcal{Z} = \Big\{ A \hookrightarrow E : A \text{ is a saturated line subbundle of } E, \ \ h^0(X, A) \geqslant 2 \Big\}.$$

Moreover, all fibres of χ are positive dimensional projective spaces.

(2) *$\mathbf{R}(X, E)$ is linear if and only if $\mathcal{Z}$ is finite.*

Proof. Given $[s] \in \mathbf{R}(X, E)$, with $0 \neq s \in H^0(X, E)$, let A be the saturation of the rank one subsheaf of E generated by s. Since $[s] \in \mathbf{R}(X, E)$, there exists $s' \in H^0(X, E)$ which is not a scalar multiple of s, with $d(s \wedge s') = 0$. This means that s' is a multiple of s at the generic point, and therefore s' defines a section of A, in particular $h^0(X, A) \geqslant 2$. We set $\chi\big([s]\big) := [A \hookrightarrow E] \in \mathcal{Z}$. In order to establish (1) it suffices to prove that if ℓ_1 and ℓ_2 are two lines in $\mathbf{P}$ with $\ell_1 \cap \ell_2 = \{[s]\}$ and $\ell_1, \ell_2 \subseteq \mathbf{R}(X, E)$, then $\overline{\ell_1, \ell_2} \subseteq \mathbf{R}(X, E)$. Each line ℓ_i corresponds to a subspace $\langle s, s_i \rangle \subseteq H^0(X, E)$ that generates a rank-one subsheaf $A_i \hookrightarrow E$ with $h^0(X, A_i) \geqslant 2$. In a general fibre $E(x)$ of E the values $s(x), s_i(x)$ generate a subspace of dimension at most one. Therefore, the space spanned by the sections s, s_1, s_2 generates a rank-one subsheaf A of E, and $\ell_1, \ell_2 \subseteq \mathbf{P}H^0(X, A) \subseteq \mathbf{R}(X, E)$. The sheaf A being saturated, $\chi^{-1}(A) = \mathbf{P}H^0(X, A) \subseteq \mathbf{R}(X, E)$. Part (2) is a direct consequence of (1). $\square$

6.2. **Rank** 2 **vector bundles on curves.** We now specialize to the case X is a smooth curve of genus $g \geqslant 2$. For a line bundle $L \in \mathrm{Pic}^d(X)$, let $\mathcal{SU}_X(2, L)$ be the moduli space of S-equivalence classes of semistable rank 2 vector bundles E on X with $\det(E) \cong L$. We shall undertake a more systematic study of the geometry of the subpencils of given degree of a tank 2 vector bundle.

For $\frac{g+2}{2} \leqslant a \leqslant g + 1$, we denote by $W_a^1(X) := \left\{ A \in \mathrm{Pic}^a(X) : h^0(X, A) \geqslant 2 \right\}$ the Brill–Noether variety of pencils of degree a on X, see [6, Chapter 4]. If X is a general curve of genus g, then $W_a^1(X)$ is equidimensional of dimension $2a - g - 2$ and the Petri map

$$\mu_A \colon H^0(X, A) \otimes H^0(X, \omega_X \otimes A^\vee) \longrightarrow H^0(X, \omega_X) \tag{6.2}$$

obtained by multiplication of sections is injective for each pencil $A \in W_a^1(X)$. For a vector bundle $E \in \mathcal{SU}_X(2, L)$ and a line subbundle $A \hookrightarrow E$ (which implies $H^0(X, E \otimes A^\vee) \neq 0$), we introduce the *twisted Petri map*,

$$\beta = \beta_{E,A} \colon H^0(X, E \otimes A^\vee) \otimes H^0(X, E^\vee \otimes \omega_X \otimes A) \longrightarrow H^0(X, \omega_X), \tag{6.3}$$

obtained by composing the multiplication and the trace maps respectively.

We first describe in terms of extensions when the resonance $\mathcal{R}(X, E)$ is strongly isotropic.

Lemma 6.2. *Let E be a rank 2 vector bundle on a curve X expressed as an extension*

$$0 \longrightarrow A \longrightarrow E \xrightarrow{\ j\ } L \otimes A^\vee \longrightarrow 0, \tag{6.4}$$

where A is a pencil on X with $h^0(X, A) = 2$. Let $F := |A|$ be the base locus of A and let $e \in \mathrm{Ext}^1(L \otimes A^\vee, A) \cong H^0(X, \omega_X + L - 2A)^\vee$ be the extension class corresponding to E. Then $H^0(X, A) \subseteq H^0(X, E)$ is not strongly isotropic if and only if there exists a non-zero element $u \in H^0(X, L(-2A + F))$ such that

$$u \cdot \left(H^0(X, A) \otimes H^0(X, \omega_X \otimes A^\vee) \right) \subseteq \ker(e). \tag{6.5}$$

Note that in (6.5) the left hand side is viewed as a linear subspace of $H^0(X, \omega_X + L - 2A)$ via the multiplication map.

Proof. We apply Lemma 3.13, where with the notations introduced there, $V^\vee = H^0(X, E)$, $\overline{V}^\vee = H^0(X, A)$, $U^\vee = j(H^0(X, E)) \subseteq H^0(X, L \otimes A^\vee)$, and $K^\vee = \mathrm{im}(d)$, with d being the determinant map given by (6.1). The map

$$\mu \colon H^0(X, A) \otimes j(H^0(X, E)) \longrightarrow H^0(X, L)$$

described in Lemma 3.13 can be viewed as the restriction of the multiplication map of sections $H^0(X, A) \otimes H^0(X, L \otimes A^\vee) \to H^0(X, L)$. Observe also that $H^0(X, A) \subseteq H^0(X, E)$ is clearly isotropic, so it remains to determine when it is a separable subspace.

Assume the map μ is not injective. If $H^0(X, A) = \langle t_1, t_2 \rangle$, via the Base Point Free Pencil Trick [6, p. 126], every element in $\ker(\mu)$ is of the form $t_1 \otimes (t_2 \cdot (-u)) + t_2 \otimes (t_1 \cdot u)$, where $0 \neq u \in H^0(X, L - 2A + F)$. If

$$m \colon H^0(X, L - A) \otimes H^0(X, \omega_X - A) \longrightarrow H^0(X, \omega_X + L - 2A)$$

denotes the multiplication map, then

$$\mathrm{im}\Big\{j\colon H^0(E) \to H^0(L \otimes A^\vee)\Big\} = \Big\{s \in H^0(L \otimes A^\vee) : m\big(s \otimes H^0(\omega_X \otimes A^\vee)\big) \subseteq \ker(e)\Big\},$$

where the extension class e is viewed as an element of $H^0(X, \omega_X + L - 2A)^\vee$. It follows that if $H^0(X, A)$ is not separable, then $\mathrm{im}(\mu_A) \cdot \langle u \rangle \subseteq \ker(e)$, and this completes the proof. $\quad\square$

Corollary 6.3. *Let E be a rank 2 vector bundle on X as before, such that $\mathcal{R}(X, E)$ is strongly isotropic. If $A \hookrightarrow E$ is a subpencil with $h^0(X, A) = 2$, then $h^0(X, E \otimes A^\vee) = 1$.*

Proof. We consider an extension such as the one in Lemma 6.2, with $h^0(X, A) = 2$; we may clearly assume that A is base point free. Assuming $h^0(X, E \otimes A^\vee) \geqslant 2$, we obtain the non-injectivity of the coboundary map $\partial\colon H^0(X, L - 2A) \to H^0(X, \omega_X)^\vee$ obtained by twisting the extension (6.4) by $A^\vee$ and taking cohomology in the corresponding exact sequence. It follows that there exists a non-zero element $u \in H^0(X, L - 2A)$ such that $u \cdot H^0(X, \omega_X) \subseteq \ker(e)$, where the left hand side is viewed as a subspace of $H^0(X, \omega_X + L - 2A)$ via the multiplication map. Applying Lemma 6.2, this contradicts the fact that $H^0(X, A)$ is a strongly isotropic component of $\mathcal{R}(X, E)$. $\quad\square$

Lemma 6.4. *Let X be a general curve of genus g and $L \in \mathrm{Pic}^d(X)$ be a line bundle of degree $d \leqslant 3g + 1$. Then a general vector bundle $E \in \mathcal{SU}_X(2, L)$ carries no line subbundles $A \hookrightarrow E$ satisfying either $\deg(A) > g + 1$ or $h^0(X, A) > 2$.*

Proof. Suppose we are given an extension $0 \to A \to E \to L \otimes A^\vee \to 0$, where A is a line bundle of degree a. Using results of Laumon [23] (see also [22]), the generic bundle $E \in \mathcal{SU}_X(2, L)$ is very stable in which case the maximal degree of a line subbundle of E equals $\lfloor \frac{d+g-1}{2} \rfloor - g + 1$. Since $d \leqslant 3g + 1$, we quickly obtain $a \leqslant g + 1$.

In order to deal with the second statement we perform a parameter count. Assume that for every $E \in \mathcal{SU}_X(2, L)$, we have an extension as above with $h^0(X, A) = r + 1 \geqslant 3$, that is, with $A \in W_a^r(X)$. Thus E corresponds to an extension class $e \in \mathbf{P}\,\mathrm{Ext}^1(L \otimes A^\vee, A)$. If $h^0(X, 2A - L) \leqslant 1$, then, by Riemann–Roch, $h^0(X, \omega_X + L - 2A) \leqslant g + d - 2a$. It follows that the number of parameters on which vector bundles E appearing as such extensions is bounded above by

$$\dim \mathbf{P}\,\mathrm{Ext}^1\big(L \otimes A^\vee, A\big) + \dim W_a^r(X) \leqslant g - 1 + d - 2a + g - (r + 1)(g - a + r)$$
$$= -r^2 - 1 + (a - g - 1)r + g - a + d$$
$$\leqslant 4g - a - r^2 - (g + 1 - a)r$$
$$\leqslant 2g + a - 6 \leqslant 3g - 5 < \dim \mathcal{SU}_X(2, L).$$

If, on the other hand, $h^0(X, 2A - L) \geqslant 2$, we can apply Clifford's inequality and write $h^0(X, \omega_X + L - 2A) \leqslant g - a + \frac{d}{2}$ and since $d \leqslant 3g + 1$, one again obtains that a general stable vector bundle $E \in \mathcal{SU}_X(2, L)$ does not appear in this way. $\quad\square$

6.3. **The variety of subpencils of a vector bundle.** Before stating the next result, recall that $\theta \in H^2\big(\mathrm{Pic}^d(X), \mathbb{Q}\big)$ is the class of the theta divisor, see [6, Chapter 7].

Proposition 6.5. *Let X be a general curve of genus g. Fix a positive integer d, a line bundle $L \in \text{Pic}^d(X)$ and a vector bundle $E \in \mathcal{SU}_X(2, L)$.*

(1) *For $\frac{d-2g+2}{2} \leqslant a \leqslant g+1$, each irreducible component of the variety of subpencils*

$$W_a^1(E) := \left\{ A \in W_a^1(X) : h^0(X, E \otimes A^\vee) \geqslant 2 \right\}$$

has dimension at least $d - 3g - 1$.

(2) *If $W_a^1(E)$ is of the expected dimension $d - 3g - 1$, its cohomology class equals*

$$[W_a^1(E)] = 2^{2g+2a-d-1} \frac{\theta^{4g-d+1}}{(2g+2a-d-1)! \cdot (g-a+1)! \cdot (g-a+2)!} \in H^* \big(\text{Pic}^d(X), \mathbb{Q}\big).$$

(3) *If E is very stable and $\mathcal{R}(X, E)$ is strongly isotropic, let $A \in W_a^1(E)$ be such that $h^0(X, A) = 2$. Then $W_a^1(E)$ is smooth of dimension $d - 3g - 1$ at $[A]$ if and only if there exists no element $0 \neq v \in H^0(X, A) \otimes H^0(X, \omega_X \otimes A^\vee)$ such that*

$$v \cdot H^0(X, L - 2A) \subseteq \ker(e). \tag{6.6}$$

Note that the left hand side in (6.6) is viewed via the multiplication map as a subspace of $H^0(X, \omega_X + L - 2A)$. Also note that Theorem 1.5 asserts that the hypotheses on the vector bundle E appearing in part (3) of Proposition 6.5 are satisfied for a general vector bundle $E \in \mathcal{SU}_X(2, L)$.

Proof. We denote by $\mathcal{P}$ the Poincaré bundle on $X \times W_a^1(X)$ and by $\pi_1 \colon X \times W_a^1(X) \to X$ and $\pi_2 \colon X \times W_a^1(X) \to W_a^1(X)$ the two projections. We fix an effective divisor $D := p_1 + \cdots + p_b$ of large degree $b := \deg(D) \gg 0$ and consider the following vector bundles over $W_a^1(X)$,

$$\mathcal{E} := (\pi_2)_* \left(\pi_1^* \big(E(D) \big) \otimes \mathcal{P}^\vee \right) \quad \text{and} \quad \mathcal{F} := (\pi_2)_* \left(\pi_1^* (E(D)) \otimes \mathcal{P}_{|D}^\vee \right).$$

Note that $\text{rank}(\mathcal{E}) = d - 2a + 2 - 2g + 2b$ and $\text{rank}(\mathcal{F}) = 2b$ and that our assumption on a amounts to the inequality $\text{rank}(\mathcal{E}) \leqslant \text{rank}(\mathcal{F})$. There exists a vector bundle morphism $\chi \colon \mathcal{E} \to \mathcal{F}$ which fibrewise corresponds to the evaluation map

$$H^0 \Big(X, E(D) \otimes A^\vee \Big) \longrightarrow H^0 \Big(X, E(D) \otimes A_{|D}^\vee \Big).$$

Then we can realize $W_a^1(E)$ as the locus where χ fails to be injective. Using the general theory of degeneracy loci, cf. [6, Chapter 3], we conclude that each component of $W_a^1(E)$ has dimension at least

$$\dim W_a^1(X) - \text{rank}(\mathcal{F}) + \text{rank}(\mathcal{E}) - 1 = 2a - g - 2 - 2b + (d - 2a + 2 - 2g + 2b) - 1$$
$$= d - 3g - 1.$$

This proves part (1).

Applying the Porteous formula, we compute the virtual class of $W_a^1(E)$ and we find that

$$\big[W_a^1(E)\big]^{\text{virt}} = c_{2g+2a-d-1}(\mathcal{F} - \mathcal{E}) = c_{2g+2a-d-1}(-\mathcal{E}),$$

where we used that $\mathcal{F}$ is algebraically equivalent to the trivial bundle, and thus $c(\mathcal{F}) = 1$, see [6, p. 309]. Furthermore, $c(-\mathcal{E}) = e^{\mathrm{rk}(E)\cdot\theta} = e^{2\theta}$, whereas it is well-known that

$$[W_a^1(X)] = \frac{1}{(g-a+1)! \cdot (g-a+2)!}\, \theta^{2g-2a+2} \in H^{2(2g-2a+2)}\big(\mathrm{Pic}^d(X), \mathbb{Q}\big),$$

see [6, Theorem 4.4]. Putting all these facts together, we obtain the equalities

$$
\begin{aligned}
\big[W_a^1(E)\big]^{\mathrm{virt}} &= c_{2a+2g-d-1}\big(e^{2\theta}\big)\big|_{W_a^1(X)} \\
&= \frac{2^{2a+2g-d-1}}{(2g+2a-d-1)!} \cdot \frac{\theta^{4g-d+1}}{(g-a+1)! \cdot (g-a+2)!}.
\end{aligned}
$$

This finishes the proof of part (2).

We now proceed to prove (3) and begin by describing the tangent space of $W_a^1(E)$ at a point $[A]$ using the identification $T_{[A]}\big(\mathrm{Pic}^d(X)\big) \cong H^0(X, \omega_X)^\vee$. From Brill–Noether theory we know that $T_{[A]}(W_a^1(X)) = \big(\mathrm{im}(\mu_A)\big)^\vee$, see [6, Chapter 4]. Then by Kodaira–Spencer theory, the tangent space of $W_a^1(E)$ will consist of those vectors $\varphi \in T_{[A]}\big(W_a^1(X)\big)$ such that $\varphi \cup s = 0 \in H^1(X, E \otimes A^\vee)$, for every $s \in H^0(X, E \otimes A^\vee)$. Via Serre duality, we have that $H^1(X, E \otimes A^\vee) \cong H^0\big(X, E^\vee \otimes \omega_X \otimes A\big)^\vee$; therefore, we obtain

$$T_{[A]}\big(W_a^1(E)\big) = \Big\{\varphi \in H^0(X, \omega_X)^\vee : \varphi_{|H^0(X,A)\otimes H^0(X,\omega_X\otimes A^\vee)+\mathrm{im}(\beta_{E,A})} = 0\Big\}, \tag{6.7}$$

where $\beta_{E,A}$ is the twisted Petri map introduced in (6.3).

We continue with the proof of (3) and assume $E \in \mathcal{SU}_X(2, L)$ is a very stable such that $\mathcal{R}(X, E)$ is strongly isotropic and choose $[A] \in W_a^1(E)$ be an element such that $h^0(X, A) = 2$. Using Corollary 6.3, we have that $h^0(X, E \otimes A^\vee) = 1$, whereas the very stability of E guarantees that the map $\beta := \beta_{E,A}$ is in fact injective; in particular,

$$\dim \mathrm{im}(\beta) \geqslant h^0(X, E^\vee \otimes \omega_X \otimes A) = 2a + 2g - d - 1.$$

Since X is general, the Petri map μ_A is injective, hence $\dim \mathrm{im}(\mu_A) = 2(g - a + 1)$ and

$$\dim \mathrm{im}(\beta) + \dim \mathrm{im}(\mu_A) = 4g - d + 1.$$

Thus $W_a^1(E)$ is smooth of the expected dimension $d - 3g - 1$ at the point $[A]$ if and only if

$$\mathrm{im}(\beta) \cap \mathrm{im}(\mu_A) = 0. \tag{6.8}$$

Assume (6.8) does not hold, in which case there are non-zero sections $s \in H^0(X, E \otimes A^\vee)$ and $t \in H^0\big(X, E^\vee \otimes \omega_X \otimes A\big)$ such that $\beta(s \otimes t) \in H^0(X, A) \otimes H^0(X, \omega_X \otimes A^\vee) \subseteq H^0(X, \omega_X)$.

In order to make this condition more explicit, we write down the exact sequence obtained by tensoring (6.4), using that $E \cong E^\vee(L)$ and then taking cohomology:

$$0 \longrightarrow H^0\big(X, E^\vee \otimes \omega_X \otimes A\big) \xrightarrow{\beta(s\otimes-)} H^0(X, \omega_X) \xrightarrow{\partial} H^0(X, L - 2A)^\vee$$

Assume $\beta(s \otimes t) \in H^0(X, A) \otimes H^0(X, \omega_X \otimes A^\vee)$ for $t \in H^0\big(X, E^\vee \otimes \omega_X \otimes A\big)$. We set $\upsilon := \beta(s \otimes t) \in H^0(X, A) \otimes H^0(X, \omega_X \otimes A^\vee)$ and denote by

$$\nu \colon \Big(H^0(X, A) \otimes H^0(X, \omega_X \otimes A^\vee)\Big) \otimes H^0(X, L - 2A) \longrightarrow H^0(X, \omega_X) \tag{6.9}$$

the multiplication map, where we identify $H^0(X, A) \otimes H^0(X, \omega_X \otimes A^\vee)$ with its image under the map μ_A. We then obtain that $W_a^1(E)$ is smooth of the expected dimension at the point $[A]$ if and only if there exist no element $0 \neq v \in H^0(X, A) \otimes H^0(X, \omega_X \otimes A^\vee)$ such that

$$\nu\big(v \cdot H^0(X, L - 2A)\big) \subseteq \ker(e), \tag{6.10}$$

where recall that $e \in \mathrm{Ext}^1(L \otimes A^\vee, A)$ is the extension class of the vector bundle E. □

Remark 6.6. The argument given in Proposition 6.5 also shows that when $a < \frac{d-2g+2}{2}$, the equality $W_a^1(E) = W_a^1(X)$ holds.

The following result shows that in the extremal case $d = 3g + 1$, the strong isotropicity of the resonance $\mathcal{R}(X, E)$ is equivalent to the smoothness of all determinantal loci $W_a^1(E)$.

Theorem 6.7. *Let X be a general curve of genus g and let $E \in \mathcal{SU}_X(2, L)$ be a very stable vector bundle of degree $3g + 1$.*

    (1) If $\mathcal{R}(X, E)$ is strongly isotropic, then $W_a^1(E)$ is smooth and zero-dimensional for each $\frac{g+2}{2} \leqslant a \leqslant g + 1$.

    (2) The variety $W_{g+1}^1(E)$ is smooth, zero-dimensional, and consists of 2^g reduced points.

Proof. Assume $\mathcal{R}(X, E)$ is strongly isotropic. Since E is very stable $H^1(X, L - 2A) = 0$, for every subpencil $A \in W_a^1(E)$, therefore by Riemann–Roch we obtain $h^0(X, L - 2A) = 2(g - a + 1)$. Moreover, the argument in Lemma 6.4 shows that $h^0(X, A) = 2$, for every such A. Set $W := H^0(X, A) \otimes H^0(X, \omega_X \otimes A^\vee)$ and $U := H^0(X, L - 2A)$, therefore $\dim(U) = \dim(W) = 2(g - a + 1)$. We consider the following diagram,

$$\begin{array}{ccc}
\mathbf{P}(W) \times \mathbf{P}(U) \cong \mathbf{P}^{2g-2a+1} \times \mathbf{P}^{2g-2a+1} & \xrightarrow{\;\nu\;} & \mathbf{P}\big(H^0(X, \omega_X + L - 2A)\big). \\
\quad\Big\downarrow{\scriptstyle\pi_1} \qquad\qquad\qquad {\scriptstyle\pi_2} & & \\
\mathbf{P}(W) & \longrightarrow & \mathbf{P}(U)
\end{array} \tag{6.11}$$

Here ν is the projection of the Segre embedding induced by the multiplication map, and thus

$$\nu^*\big(\mathcal{O}_{\mathbf{P}(H^0(\omega_X + L - 2A))}(1)\big) = \pi_1^*\big(\mathcal{O}_{\mathbf{P}(W)}(1)\big) \otimes \pi_2^*\big(\mathcal{O}_{\mathbf{P}(U)}(1)\big).$$

Assume by contradiction that $W_a^1(E)$ is not smooth and of dimension zero at a point $[A]$. Then by applying Proposition 6.5, there exists an element $[v] \in \mathbf{P}(W)$ such that $v \cdot H^0(X, L - 2A) \subseteq \ker(e)$, where recall that $\ker(e)$ is regarded as a hyperplane inside $H^0(X, \omega_X + L - 2A)$. In other words, if H is the pull-back of $\ker(e)$ under the natural map $\mathbf{P}(W \otimes U) \dashrightarrow \mathbf{P}\big(H^0(X, \omega_X + L - 2A)\big)$, writing the equation of H as

$$\sum_{1 \leqslant i,j \leqslant 2(g-a+1)} c_{ij} z_{ij} = 0,$$

it follows that the coefficient matrix (c_{ij}) is singular. Therefore, there exists an element $[u] \in \mathbf{P}(U)$ such that $\nu\big(\mathbf{P}(W) \times \{[u]\}\big) \subseteq \mathbf{P} \ker(e)$. Using Lemma 6.2, this amounts to $\mathcal{R}(X, E)$ not being strongly isotropic. This contradicts our hypothesis, and thus proves part (1).

Part (2) follows immediately from part (1), once we consider the tangent space description (6.7) when $d = 3g + 1$. Since E is very stable, the map β is surjective, hence condition (6.8) is automatically satisfied. This completes the proof. $\qquad\square$

Remark 6.8. Assuming $E \in \mathcal{SU}_X(2, L)$ is a vector bundle as above (of degree $d = 3g + 1$) which contains no line subbundles $A \hookrightarrow E$ with $h^0(X, A) > 2$ (an assumption satisfied outside a subset of codimension 3 in $\mathcal{SU}_X(2, L)$, see Lemma 6.4), the proof of Theorem 6.7 can be reversed and we obtain that $\mathcal{R}(X, E)$ is strongly isotropic if and only if $W_a^1(E)$ is smooth and zero-dimensional for each $\frac{g+2}{2} \leqslant a \leqslant g + 1$.

6.4. **The resonance variety of a general vector bundle.** We are now in a position to complete the proof of Theorem 1.5 from the Introduction.

Proof of Theorem 1.5. We fix a general vector bundle $E \in \mathcal{SU}_X(2, L)$ of degree $d \leqslant 3g + 1$. Assume by contradiction that $\mathcal{R}(X, E)$ is not strongly separable, which amounts to its non-separability. After invoking Lemma 6.4, we can express E as an extension

$$0 \longrightarrow A \longrightarrow E \longrightarrow L \otimes A^\vee \longrightarrow 0$$

as in Lemma 6.2, where $A \in W_a^1(X)$ is such that $h^0(X, A) = 2$ and the base locus F of $|A|$ is an effective divisor of degree b on X and the subspace $H^0(X, A) \subseteq H^0(X, E) =: V^\vee$ is a non-separable component of $\mathcal{R}(X, E)$. Therefore, the conclusion of Lemma 6.2 holds. Via a parameter count, we shall argue that this is not possible for a general choice of E.

For given values $\frac{g+2}{2} \leqslant a \leqslant g + 1$ and $0 \leqslant b \leqslant a$, we denote by $\mathcal{T}_{a,b}$ the subvariety of $W_a^1(X) \times X_b$ consisting of pairs (A, F), where F is an effective divisor of degree b on X and $A \in W_a^1(X)$ is a pencil having F in its base locus. Clearly $\mathcal{T}_{a,b}$ is irreducible of dimension $\dim W_{a-b}^1(X) + b = 2a - g - 2 - b$. We now introduce the parameter space

$$\Sigma_{a,b} := \left\{ \bigl(A, F, [u], [e]\bigr) : \begin{array}{c} A \in W_{a,F}^1(X), \ F \in X_b \\ [u] \in \mathbf{P}H^0(X, L - 2A + F), \ [e] \in \mathbf{P}H^0(X, \omega_X + L - 2A)^\vee \\ H^0(X, A) \otimes H^0(X, \omega_X \otimes A^\vee) \cdot \langle u \rangle \subseteq \ker(e) \end{array} \right\},$$

together with the projections

$$\mathcal{T}_{a,b} \xleftarrow{\ \mathrm{pr}_1\ } \Sigma_{a,b} \xrightarrow{\ \mathrm{pr}_2\ } \mathcal{SU}_X(2, L), \tag{6.12}$$

where pr_2 associates to $\bigl(A, F, [u], [e]\bigr)$ the vector bundle E corresponding to the extension class $e \in \mathrm{Ext}^1(L \otimes A^\vee, A)$, whereas $\mathrm{pr}_1\bigl(A, F, [u], [e]\bigr) := (A, F)$. We now estimate the general fibre dimension of pr_1. By the generality hypothesis, E may be assumed to be very stable, therefore $h^1(X, \omega_X + 2A - L) = 0$, hence $h^0(X, L - 2A + F) = d - 2a + b + 1 - g$. Having fixed A, F and $[u] \in \mathbf{P}H^0(X, L - 2A + F)$, the parameter space for extension classes $[e] \in \mathbf{P}\mathrm{Ext}^1(L \otimes A^\vee, A)$ such that $\bigl(A, F, [u], [e]\bigr) \in \Sigma_{a,b}$ is the projective space of dimension

$$h^0(X, \omega_X + L - 2A) - h^0(X, A) \cdot h^0(X, \omega_X \otimes A^\vee) - 1 = d - g - 4.$$

We obtain the estimate

$$\begin{aligned} \dim \Sigma_{a,b} &\leqslant \dim \mathcal{T}_{a,b} + (d - 2a + b - g) + (d - g - 4) \\ &= 2d - 3g - 6 \leqslant 3g - 4 < \dim \mathcal{SU}_X(2, L), \end{aligned}$$

which shows that the resonance of a general vector bundle $E \in \mathcal{SU}_X(2, L)$ is strongly isotropic.

In the case $d \leqslant 3g$, this parameter count also establishes that $\mathcal{R}(X, E) = 0$, for a general $E \in \mathcal{SU}_X(2, L)$. Indeed, assuming the general vector bundle E appears in an extension (6.4), we obtain that vector bundles appearing in this way depend on at most

$$\dim W_a^1(X) + h^0(X, \omega_X + L - 2A) - 1 = d - 4 \leqslant 3g - 4 < \dim \mathcal{SU}_X(2, L)$$

parameters, thus finishing the proof. $\qquad\square$

Remark 6.9. In degree $3g + 2 \leqslant d \leqslant 4g$, the projective resonance $\mathbf{R}_a(X, E)$ is not linear. Indeed, it follows from Propositions 6.1 and 6.5 that $\mathbf{R}_a(X, E)$ admits a regular fibration over $W_a^1(E)$. Each component of $W_a^1(E)$ is a positive dimensional variety of general type.

7. Kähler groups and Kodaira fibrations

7.1. Resonance varieties of Kähler manifolds.
We now discuss the case of Kähler groups, when the resonance varieties in question are *not* isotropic (unless they vanish). For a compact Kähler manifold X, let

$$\cup_X \colon \bigwedge^2 H^1(X, \mathbb{C}) \longrightarrow H^2(X, \mathbb{C})$$

be the cup product map. We consider the resonance variety $\mathcal{R}(X) := \mathcal{R}\big(\pi_1(X)\big)$ of the fundamental group of X. As is well-known, X is formal, and thus $\pi_1(X)$ is 1-formal; thus, $\mathcal{R}(X)$ is linear and projectively disjoint. On the other hand, if $\mathcal{R}(X) \neq \{0\}$, then, as shown in [14], all irreducible components of $\mathcal{R}(X)$ are 1-isotropic, that is, the restriction of $\cup_X$ to each such component has 1-dimensional image.

The first notable case is that of surface groups. Let Σ_g be a smooth algebraic curve of genus $g \geqslant 2$, and let $\Pi_g := \pi_1(\Sigma_g)$ be its fundamental group. It is well-known that Σ_g is formal, and thus Π_g is 1-formal. The cohomology ring has the form $H^\bullet(\Sigma_g) = E/I$, where $E = \bigwedge\big(e_1, \ldots, e_g, \bar{e}_1, \ldots, \bar{e}_g\big)$ and I is the ideal generated by the quadrics $e_i \wedge e_j, \bar{e}_i \wedge \bar{e}_j$ for $1 \leqslant i < j \leqslant g$, $e_i \wedge \bar{e}_j$ for $i \neq j$, respectively $e_1 \wedge \bar{e}_1 - e_i \wedge \bar{e}_i$ for $1 < i \leqslant g$.

It is readily seen that $\mathcal{R}(\Pi_g) = H^1(\Pi_g, \mathbb{C}) = \mathbb{C}^{2g}$. Clearly, this linear space is 1-isotropic and separable. Moreover, as shown in [29], the infinitesimal Alexander invariant $\mathfrak{B}(\Pi_g)$ has the following Hilbert series,

$$\mathrm{Hilb}(\mathfrak{B}(\Pi_g), t) = 1 - \frac{1 - 2gt + t^2}{(1 - t)^{2g}}.$$

Therefore, the Chen ranks are given by $\theta_1(\Pi_g) = 2g$, $\theta_2(\Pi_g) = 2g^2 - g - 1$, and

$$\theta_q(\Pi_g) = (q - 1)\binom{2g + q - 2}{q} - \binom{2g + q - 3}{q - 2} \quad \text{for } q \geqslant 3. \tag{7.1}$$

We discuss next the case of irregular fibrations. A fibration from a compact complex manifold X onto a smooth complex curve B is a surjective holomorphic map with connected fibers. For smooth projective varieties, and, more generally, for compact Kähler manifolds, all components of the resonance variety $\mathcal{R}(X)$ are of the form $V_f := f^* H^1(B, \mathbb{C})$, where

$f \colon X \to B$ runs through the (finite) set $\mathcal{E}(X)$ of equivalence classes of fibrations with base B being a smooth curve of genus $g \geqslant 2$ and with no multiple fibers (see e.g. [14, 13, 37]). We sometimes write these maps as $f \colon X \to B_f$ to emphasize the dependence of the base of the fibration on f. The resulting components V_f are linear (of dimension $2g(B) \geqslant 4$), projectively disjoint, and 1-isotropic. As shown by Catanese [7], the existence of such a fibration $f \colon X \to B$ is equivalent to the existence of an epimorphism $\pi_1(X) \twoheadrightarrow \pi_1(B)$ whose kernel is finitely generated.

Example 7.1. To illustrate the phenomenon that the resonance *does not* detect fibrations with multiple fibres, we mention the Catanese–Ciliberto–Mendes Lopes surface X [18, Example 2], which admits an elliptic fibration, $f \colon X \to B$ with base a smooth curve of genus 2. It follows from [18, 37] that the resonance variety $\mathcal{R}(X)$ consists of a single, 4-dimensional linear subspace in $H^1(X, \mathbb{C}) = \mathbb{C}^6$, which is equal to $f^*(H^1(B, \mathbb{C}))$. Indeed, direct computation shows that $H^*(X, \mathbb{C}) = E/I$, where $E = \bigwedge(e_1, \bar{e}_1, e_2, \bar{e}_2, e_3, \bar{e}_3)$ and $I = \langle e_1 \wedge e_2, \bar{e}_1 \wedge \bar{e}_2, e_1 \wedge \bar{e}_2, e_2 \wedge \bar{e}_1, e_1 \wedge \bar{e}_1 - e_2 \wedge \bar{e}_2 \rangle$. Setting $V^\vee = H^1(X, \mathbb{C})$ and $K^\perp = I_2 \subseteq \bigwedge^2 V^\vee$, we have that $\mathcal{R}(X) = \mathcal{R}(V, K) = \overline{V}^\vee = \langle e_1, \bar{e}_1, e_2, \bar{e}_2 \rangle$. Clearly, $K^\perp \subset \bigwedge^2 \overline{V}^\vee$ and $\overline{K} = \langle v_1 \wedge \bar{v}_1 \rangle$; thus, $\overline{V}^\vee$ is 1-isotropic. Moreover, $\overline{V}^\vee$ is separable, and thus, by Theorem 4.5 it defines an isolated component of $\mathbf{R}(V, K)$. To sum up, the resonance of X is linear, projectively disjoint, and reduced. The surface X also admits a fibration with base an elliptic curve and with 2 singular fibers, each of multiplicity 2. This (singular) fibration defines by pullback a 2-dimensional translated torus component of the characteristic variety $\mathcal{V}(X)$, but this component is not detected by $\mathcal{R}(X)$, see again [37].

7.2. **Kodaira fibrations.** Assume now that $f \colon X \to B$ is a Kodaira fibration over a smooth projective curve of genus $b \geqslant 2$ and denote by Σ_g a general fibre of f. A Kodaira fibration induces a *monodromy representation*, $\rho \colon \Pi_b \to \mathrm{Sp}_{2g}(\mathbb{Z}) = \mathrm{Aut}\big(H_1(\Sigma_g, \mathbb{Z})\big)$. The Leray–Serre spectral sequence of the fibration gives rise to the short exact sequence

$$0 \longrightarrow H^1(B, \mathbb{C}) \xrightarrow{\ f^*\ } H^1(X, \mathbb{C}) \longrightarrow H^1(\Sigma_g, \mathbb{C})^\rho \longrightarrow 0. \tag{7.2}$$

In particular, the *relative irregularity*, $q_f := q(X) - g(B)$ of the fibration can be regarded as the invariant part of the monodromy representation; that is, $q_f = \frac{1}{2} h^1\big(\Sigma_g, \mathbb{C}\big)^\rho$. There are essentially two known ways to construct Kodaira fibrations. One is by taking generic complete intersections in the moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g using that the Satake compactification of $\mathcal{M}_g$ has boundary of codimension 2; the other via ramified branched cover constructions over product of curves, see [5, 9, 21]. No examples of compact Kähler surfaces X having at least three non-equivalent Kodaira fibrations are known, see [8, Question 10]. On the other hand, Salter [34] has provided examples of closed (non-algebraic) 4-manifolds which admit a number of non-equivalent surface bundle structures that is an arbitrary power of 2.

The resonance of complete intersection Kodaira fibrations turns out to be quite simple.

Lemma 7.2. *Let $\Sigma_g \hookrightarrow X \xrightarrow{\ f\ } B$ be a complete intersection Kodaira fibration. Then $H^1(X, \mathbb{C}) \cong f^* H^1(B, \mathbb{C})$ and accordingly the resonance $\mathcal{R}(X) = f^* H^1(B, \mathbb{C})$ is separable.*

Proof. We let $\mathcal{M}_g^{(n)}$ be the moduli space of genus g curves with a level $n \geqslant 3$ structure and denote by $\overline{\mathcal{M}}_g^{(n)}$ the normalization of the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{M}_g^{(n)}$. Then using the theory of Satake compactifications, there exists a regular map, $\iota \colon \overline{\mathcal{M}}_g^{(n)} \to \mathbf{P}^N$ such that the boundary $\overline{\mathcal{M}}_g^{(n)} \backslash \mathcal{M}_g^{(n)}$ is contracted to a codimension 2 set of its image. It follows that the inverse image under ι of the intersection of $3g - 4$ general hyperplanes in $\mathbf{P}^N$ is a smooth projective curve $B \subseteq \mathcal{M}_g^{(n)}$. Since $\mathcal{M}_g^{(n)}$ is a fine moduli space, the pull-back to B of its universal family induces a Kodaira fibration $f \colon X \to B$ of smooth curves of genus g. By the Lefschetz hyperplane section theorem, since B is obtained by intersecting $\overline{\mathcal{M}}_g^{(n)}$ with ample divisors, the map $\iota_* \colon \pi_1(B) \to \pi_1\big(\mathcal{M}_g^{(n)}\big)$ is surjective, in particular the image under the monodromy map of $\pi_1(B)$ inside the mapping class group Mod_g is of finite index. Then $H^1(\Sigma_g, \mathbb{C})^\rho = 0$ and via (7.2), we have that $H^1(X, \mathbb{C}) \cong f^* H^1(B, \mathbb{C})$. Therefore, $\ker(\cup_X) \cong \ker(\cup_B)$ and accordingly $\mathcal{R}(X) \cong f^* \mathcal{R}(B)$. Applying Lemma 3.10, we conclude that $\mathcal{R}(X)$ is separable and formula (7.1) applies for the Chen ranks of X. $\qquad\square$

Assume now X admits two non-equivalent Kodaira fibration structures, $\Sigma_{g_1} \hookrightarrow X \xrightarrow{f_1} B_1$ and $\Sigma_{g_2} \hookrightarrow X \xrightarrow{f_2} B_2$. We consider the product map, $f = (f_1, f_2) \colon X \longrightarrow B_1 \times B_2$, and the homomorphisms induced in cohomology,

$$H^i(f) \colon H^i(B_1 \times B_2, \mathbb{C}) \longrightarrow H^i(X, \mathbb{C}) \tag{7.3}$$

for $i = 1, 2$.

Proof of Theorem 1.7. With the notation of (7.3), we are in the situation of a double Kodaira fibration for which $H^1(f)$ is an isomorphism, whereas $H^2(f)$ is injective. We are going to verify the separability of $\mathcal{R}(X)$ using Lemma 3.10. To that end, we write down the following commutative diagram:

$$\begin{array}{ccc} \bigwedge^2 H^1\big(B_1 \times B_2, \mathbb{C}\big) & \xrightarrow{\bigwedge^2 H^1(f)} & \bigwedge^2 H^1(X, \mathbb{C}) \\ \Big\downarrow{\cup_{B_1 \times B_2}} & & \Big\downarrow{\cup_X} \\ H^2\big(B_1 \times B_2, \mathbb{C}\big) & \xrightarrow{\;\;H^2(f)\;\;} & H^2(X, \mathbb{C}) \end{array} \tag{7.4}$$

From the Künneth formula, it follows that $\ker(\cup_X) \cong \ker(\cup_{B_1}) \oplus \ker(\cup_{B_2})$, and accordingly, $\mathcal{R}(X) = f_1^* H^1(B_1, \mathbb{C}) \cup f_2^* H^1(B_2, \mathbb{C})$. Setting $U_i := \ker\big\{(f_i)_* \colon H_1(X, \mathbb{C}) \to H_1(B_i, \mathbb{C})\big\}$, we obtain that

$$K = \big(U_1 \otimes U_2\big) \oplus \big(\textstyle\bigwedge^2 f_*\big)^{-1}\big(H_2(B_1, \mathbb{C})\big) \oplus \big(\textstyle\bigwedge^2 f_*\big)^{-1}\big(H_2(B_2, \mathbb{C})\big) \subseteq \textstyle\bigwedge^2 H_1(X, \mathbb{C}). \tag{7.5}$$

Using the notation (3.4) for the component $\overline{V}_1^\vee = f_1^*\big(H^1(B_1, \mathbb{C})\big)$ of $\mathcal{R}(X)$, we have that

$$H = \textstyle\bigwedge^2 U_1, \;\; M = U_1 \otimes U_2, \;\text{and}\;\; L = \textstyle\bigwedge^2 U_2.$$

Since $U_1 \otimes U_2 \subseteq K$, it follows with the notation of (3.6) that the map $p_M \colon K \cap \big(H \oplus M\big) \to M$ is surjective. Using Lemma 3.10, we conclude that the component $f_1^* H^1(B_1, \mathbb{C})$ of $\mathcal{R}(X)$ is separable. Same considerations apply for the component $f_2^* H^1(B_2, \mathbb{C})$ of the resonance. $\quad\square$

7.3. **The Atiyah–Kodaira construction.** The simplest examples of double Kodaira fibrations where the conditions of Theorem 1.7 are satisfied are provided by the Atiyah–Kodaira fibrations constructed in the 1960's in [5, 21]. These surfaces can be described as follows. Let $\tau\colon B_2 \to B_2$ be a fixed point free involution of a curve of genus $g(B_2) = 2g - 1$, and let $\varphi\colon B_1 \to B_2$ be the congruence unramified cover classified by the homomorphism $\pi_1(B_2) \twoheadrightarrow H_1(B_2, \mathbb{Z}) \twoheadrightarrow H_1(B_2, \mathbb{Z}/2\mathbb{Z})$, thus $\deg(\varphi) = 2^{2(2g-1)}$ and $g(B_1) = 1 + 2^{4g-1}(g-1)$. We let X be the 2-fold branched cover of $B_1 \times B_2$ ramified along the divisor $Y_1 + Y_2$, where

$$Y_1 := \big\{(z, \varphi(z)) : z \in B_1\big\} \quad \text{and} \quad Y_2 := \big\{\big(z, \tau(\varphi(z))\big) : z \in B_1\big\}.$$

Observe that $Y_1 \cap Y_2 = \varnothing$. The two independent Kodaira fibrations of X are of the form

$$\Sigma_{4g-2} \longrightarrow X \xrightarrow{\ f_1\ } B_1 \quad \text{and} \quad \Sigma_{1+2^{4g-2}(4g-3)} \longrightarrow X \xrightarrow{\ f_2\ } B_2. \tag{7.6}$$

Using [20, Theorem 2.2], we obtain that $f = (f_1, f_2)\colon X \longrightarrow B_1 \times B_2$ induces an isomorphism on $H^1(-, \mathbb{C})$ and an injection on $H^2(-, \mathbb{C})$, see also [10, Theorem 1.1], where only the case $g = 2$ of this construction is treated. Indeed, one applies the sequence (7.2) to the fibration $f_1\colon X \to B_1$ and use that its general fibre Σ_{4g-2} is a double cover of $B_2 = \Sigma_{2g-1}$, branched at two points conjugate under the involution τ. Then $H^1\big(\Sigma_{4g-2}, \mathbb{C}\big)^{\pi_1(B_1)} \cong H^1(B_2, \mathbb{C})$, therefore $b_1(X) = 4g + 2^{4g}(g-1)$ and $\mathcal{R}(X) = \overline{V}_1^{\vee} \cup \overline{V}_2^{\vee}$, where $\overline{V}_i^{\vee} = f_i^* H^1(B_i, \mathbb{C})$ for $i = 1, 2$. Therefore, Theorem 1.7 applies in this case.

Similar considerations apply to the example of the double Kodaira fibration given in [11]. This complex surface X has $b_1(X) = 38$ and fibers in two distinct ways, $\Sigma_4 \to X \to \Sigma_{17}$ and $\Sigma_{49} \to X \to \Sigma_2$. A similar argument shows that $\mathcal{R}(X)$ is separable. We have not tried to verify the hypothesis of Theorem 1.7 for other families of double Kodaira fibrations, for instance, those constructed in [9].

These considerations naturally raise the following question.

Question 7.3. Is the resonance variety of a Kähler group G always linear, projectively disjoint, and reduced?

Of course, the real question is whether $\mathbf{R}(G)$ is reduced. Again, it would be enough to show that the components $f^*(H^1(B_f, \mathbb{C}))$ of $\mathcal{R}(G)$ are all separable. If this question were to have a positive answer, the following *Chen ranks conjecture for Kähler groups* would be the natural next step to consider.

Conjecture 7.4. *Let X be a compact Kähler manifold. For each $g \geqslant 2$, we denote by $h_g(X) := \#\big\{f \in \mathcal{E}(X) : g(B_f) = g\big\}$ the number of components of $\mathcal{R}(X)$ of dimension $2g$. Then for all $q \gg 0$, the following Chen rank formula holds:*

$$\theta_q\big(\pi_1(X)\big) = \sum_{f \in \mathcal{E}(X)} \theta_q(\pi_1(B_f)) = \sum_{g \geqslant 2} h_g(X)\theta_q(\Pi_g)$$

$$= (q-1) \cdot \sum_{g \geqslant 2} h_g(X)\binom{2g+q-2}{q} - \sum_{g \geqslant 2} h_g(X)\binom{2g+q-3}{q-2}.$$

8. Resonance of right-angled Artin groups

We now apply the general theory to the case when the subspace $K \subseteq \bigwedge^2 V$ admits a *monomial basis*, that is, there exists a basis $\{v_1, \ldots, v_n\}$ for V so that K admits a basis whose elements are of the form $v_i \wedge v_j$.

Let $\binom{[n]}{k}$ be the set of ordered k-tuples from $[n] = \{1, \ldots, n\}$. The above information is conveniently encoded in a (simple) graph $\Gamma = (\mathsf{V}, \mathsf{E})$ on vertex set $\mathsf{V} = [n]$ and edge set

$$\mathsf{E} = \left\{ (i,j) \in \binom{[n]}{2} : v_i \wedge v_j \in K \right\}. \tag{8.1}$$

Dually, $V^\vee$ is spanned by $\{e_1, \ldots, e_n\}$ and $K^\perp$ is the linear subspace spanned by the elements $\{e_i \wedge e_j : (i,j) \in \overline{\mathsf{E}}\}$, where $\overline{\mathsf{E}} = \binom{[n]}{2} \backslash \mathsf{E}$ is the edge set of the complement graph $\overline{\Gamma} = (\mathsf{V}, \overline{\mathsf{E}})$. We also denote by T the set of complete triangles in Γ, and by $\overline{\mathsf{T}} = \binom{[n]}{3} \backslash \mathsf{T}$ the set of triangles with at least one missing edge. To the graph Γ one can associate the *right-angled Artin group* G_Γ having the following presentation

$$G_\Gamma := \left\langle v_i : v_i \cdot v_j = v_j \cdot v_i \quad \text{for} \quad v_i \wedge v_j \in K \right\rangle.$$

The cohomology ring of G_Γ can be identified with the Stanley–Reisner algebra $E/\langle K^\vee \rangle_E$, where $E = \bigwedge V^\vee$ is the exterior algebra generated by $e_1, \ldots, e_n$.

A graph $\Gamma' = (\mathsf{V}', \mathsf{E}')$ of Γ is a *full subgraph of* Γ (or, an induced subgraph on the vertex set $\mathsf{V}' \subseteq \mathsf{V}$) if $\mathsf{E}' = \mathsf{E} \cap \binom{\mathsf{V}'}{2}$. Given two graphs, $\Gamma' = (\mathsf{V}', \mathsf{E}')$ and $\Gamma'' = (\mathsf{V}'', \mathsf{E}'')$, their *join* $\Gamma = \Gamma' * \Gamma''$, is the graph with vertex set $\mathsf{V} = \mathsf{V}' \cup \mathsf{V}''$ and having the edge set $\mathsf{E} = \mathsf{E}' \cup \mathsf{E}'' \cup \{(i', i'') : i' \in \mathsf{V}', i'' \in \mathsf{V}''\}$.

Let $S = S_\Gamma$ be the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ and let $W_\Gamma = W(G_\Gamma) := W(V, K)$ be the Koszul module associated to Γ. This module has the following graph-theoretic description:

Lemma 8.1 ([31])**.** *The Koszul module of a graph Γ admits a presentation,*

$$W_\Gamma = \mathrm{coker}\left\{ \Theta \colon \mathrm{Span}(\overline{\mathsf{T}}) \otimes S \longrightarrow \mathrm{Span}(\overline{\mathsf{E}}) \otimes S \right\}, \tag{8.2}$$

where Θ is the matrix with entries

$$\Theta_{ijk,\ell m} = \begin{cases} x_k & \text{if } \ell = i, \, m = j \\ -x_j & \text{if } \ell = i, \, m = k \\ x_i & \text{if } \ell = j, \, m = k \\ 0 & \text{otherwise.} \end{cases} \tag{8.3}$$

Example 8.2. For the complete graph K_n on n vertices, we have $W_{K_n} = 0$. On the opposite end, if $\Gamma = \overline{K_n}$ is a discrete graph, then $W_\Gamma = \ker(\delta_2)$.

Let $\mathcal{R}_\Gamma = \mathcal{R}(G_\Gamma) := \mathcal{R}(V, K)$ be the resonance scheme associated to the Koszul module W_Γ. The underlying set $\mathcal{R}_\Gamma$ was described in [30, Theorem 5.5] as a union of coordinate subspaces of $V^\vee$. For a subgraph $\Gamma' \subseteq \Gamma$, we denote by $V_{\Gamma'}^\vee \subseteq V^\vee$ the coordinate subspace spanned by the vertices of Γ'. With this notation,

$$\mathcal{R}_\Gamma = \bigcup \left\{ V_{\Gamma'}^\vee : \Gamma' \text{ is a maximally disconnected full subgraph of } \Gamma \right\}. \tag{8.4}$$

The next result explains in combinatorial terms the isotropicity and separability conditions for resonance components introduced in this paper.

Proposition 8.3. *Let Γ be a connected graph, let Γ' be a maximally disconnected full subgraph, and let $V'^{\vee} = V_{\Gamma'}^{\vee}$ be the corresponding component of $\mathcal{R}_{\Gamma}$. Then,*

(1) $V'^{\vee}$ is isotropic if and only if Γ' is discrete.
(2) $V'^{\vee}$ is separable if and only if $\Gamma = \Gamma' * \Gamma''$.

In particular, isotropic implies separable for the resonance varieties of graphs.

Proof. By definition, the linear subspace $V'^{\vee}$ is isotropic if $\bigwedge^2 V'^{\vee} \subseteq K^{\perp}$, that is, the set $\{e_i \wedge e_j : (i,j) \in \mathsf{V}'\}$ is contained in $\{e_i \wedge e_j : (i,j) \in \overline{\mathsf{E}}\}$. This last condition amounts to $\binom{\mathsf{V}'}{2} \subseteq \overline{\mathsf{E}}$, which is equivalent to Γ' being discrete.

Finally, $V'^{\vee}$ is separable if $K^{\perp} \cap \left(\bigwedge^2 V'^{\vee} \oplus (V'^{\vee} \otimes V''^{\vee}) \right) \subseteq \bigwedge^2 V'^{\vee}$. This condition is equivalent to $\overline{\mathsf{E}} \cap \left(\binom{\mathsf{V}'}{2} \cup (\mathsf{V}', \mathsf{V}'') \right) \subseteq \binom{\mathsf{V}'}{2}$, that is, $\overline{\mathsf{E}} \cap (\mathsf{V}', \mathsf{V}'') = \varnothing$, which again means that $\Gamma = \Gamma' * \Gamma''$. $\qquad\square$

We now give two classes of graphs that satisfy the conditions of Proposition 8.3, and thus, of Theorem 1.1. In what follows, we denote by K_n the complete graph on n vertices.

Example 8.4. Let $\Gamma = \overline{K_{n_1}} * \cdots * \overline{K_{n_r}}$ be a complete multipartite graph, that is, an iterated join of discrete graphs. Then $\mathcal{R}_{\Gamma}$ is projectively disjoint and isotropic (and hence, separable).

As a concrete example, take the complete graph $K_n = K_1 * \cdots * K_1$; then $\mathbf{R}_{K_n} = \varnothing$. As another example, consider the square, $\Gamma = \overline{K_2} * \overline{K_2}$; then $\mathbf{R}_{\Gamma}$ is the disjoint union of two (isotropic) lines.

Example 8.5. Let Γ be a graph obtained from the square with one diagonal, by taking iterated cones. Then $\mathbf{R}_{\Gamma}$ consists of a single line, and therefore is projectively disjoint and isotropic (and hence, separable).

In general, though, the resonance varieties $\mathcal{R}_{\Gamma}$ are neither isotropic, nor separable, nor projectively disjoint. The simplest such example is as follows.

Example 8.6. Let Γ be a path on 4 vertices. Then $V^{\vee} = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ and $K^{\perp} = \mathrm{Span}\{e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_4\}$. Setting $S = \Bbbk[x_1, x_2, x_3, x_4]$, the S-module W_{Γ} is presented by the matrix

$$\Theta_{\Gamma} = \begin{array}{c} \\ 123 \\ 124 \\ 134 \\ 234 \end{array} \begin{array}{c} \begin{matrix} 13 & 14 & 24 \end{matrix} \\ \begin{pmatrix} -x_2 & 0 & 0 \\ 0 & -x_2 & x_1 \\ x_4 & -x_3 & 0 \\ 0 & 0 & -x_3 \end{pmatrix} \end{array}.$$

Note that $\mathrm{Fitt}_0(W_{\Gamma}) = (x_2) \cap (x_3) \cap (x_1, x_2^2, x_3^2, x_4)$ is not reduced, though the annihilator $\mathrm{Ann}(W_{\Gamma}) = (x_2) \cap (x_3)$ is reduced. It follows that $\mathcal{R}(V, K) = \{x_2 = 0\} \cup \{x_3 = 0\}$ is linear and reduced, but neither projectively disjoint, nor isotropic, nor separable.

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