HODGE CLASSES ON THE MODULI SPACE OF $W(E_6)$-COVERS AND THE GEOMETRY OF $\mathcal{A}_6$

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To Herb, with friendship and admiration.

Abstract. In previous work we showed that the Hurwitz space of $W(E_6)$-covers of the projective line branched over 24 points dominates via the Prym-Tyurin map the moduli space $\mathcal{A}_6$ of principally polarized abelian 6-folds. Here we determine the 25 Hodge classes on the Hurwitz space of $W(E_6)$-covers corresponding to the 25 irreducible representations of the Weyl group $W(E_6)$. This result has direct implications to the intersection theory of the toroidal compactification $\overline{\mathcal{A}}_6$. In the final part of the paper, we present an alternative, elementary proof of our uniformization result on $\mathcal{A}_6$ via Prym-Tyurin varieties of type $W(E_6)$.

1. Introduction

It is well known that the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension $g \leq 5$ can be uniformized via Prym varieties associated to unramified double covers of curves. This amounts to the fact that the Prym map $P\colon \mathcal{R}_{g+1} \to \mathcal{A}_g$ is dominant in this range. This explicit parametrization of the moduli space has important applications, for instance it implies that $\mathcal{A}_g$ is unirational for $g \leq 5$, see [D1, MM, C, V1]. Note also that $\mathcal{A}_g$ is a variety of general type for $g \geq 7$, see [M, T]. Using advances in automorphic forms, it has been recently proven [DSS] that the Kodaira dimension of $\mathcal{A}_6$ is non-negative.

There is a well documented history going back at least to [D3] showing the importance of the symmetries of the 27 lines on a cubic surface in the study of the Galois group of the Prym map $P\colon \mathcal{R}_6 \to \mathcal{A}_5$. Conversely, Clemens and Groffiths [CG] famously associated to a smooth cubic threefold its intermediate Jacobian in order to study rationality questions. For recent developments in moduli theory or in hyperkähler geometry related to this circle of ideas we refer to [CMGHL, LSV, V2].

In our previous paper [ADFIO] we found an explicit parametrization of $\mathcal{A}_6$ by means of one-dimensional objects. Recalling that $W(E_6)$ is the group of symmetries of the 27 lines on a smooth cubic surface, we proved that the general ppav $[A, \Theta] \in \mathcal{A}_6$ can be represented as the Prym-Tyurin variety of exponent 6 associated to an $W(E_6)$-cover $\pi\colon C \to \mathbb{P}^1$ branched over 24 points. Precisely, let Hur denote the Hurwitz space of covers $[\pi\colon C \to \mathbb{P}^1, p_1 + \cdots + p_{24}]$ having monodromy group $W(E_6) \subseteq S_{27}$ and branched over the points $p_1, \ldots, p_{24} \in \mathbb{P}^1$ such that the local monodromy of π at p_i is given by a reflection in a root of E_6. For each such cover $\pi\colon C \to \mathbb{P}^1$ we can identify the points in a general fiber with the lines on a smooth cubic surface. The curve C has genus 46 and is equipped with an *incidence correspondence* $D \subseteq C \times C$ first considered by Kanev [K2]. The correspondence D gives rise to an endomorphism $D\colon JC \to JC$ and to a *Prym-Tyurin-Kanev map*

$$PT\colon \mathrm{Hur} \to \mathcal{A}_6, \quad [\pi\colon C \to \mathbb{P}^1] \mapsto PT(C, D) := \mathrm{Im}(D-1) \subseteq JC.$$

Since $(D-1)(D+5) = 0$, one has $PT(C, D) = \mathrm{Ker}(D+5)^0$. Our main result from [ADFIO] is that the map PT is generically finite, in particular dominant. This parametrization opens the way to a study of $\mathcal{A}_6$ via the theory of curves and their correspondences. The main goal of this paper is to understand the intersection theory associated to this uniformization of $\mathcal{A}_6$, in particular to determine the 25 Hodge classes associated to the irreducible representations of the group $W(E_6)$.

The moduli space $\mathcal{A}_g$ has a partial compactification $\mathcal{A}_g^*$ obtained by adding rank 1 degenerations and contained in the toroidal compactification $\overline{\mathcal{A}}_g = \overline{\mathcal{A}}_g^{\mathrm{perf}}$ for the fan of perfect forms, with the complement $\overline{\mathcal{A}}_g \setminus \mathcal{A}_g^*$ having codimension 2. The Hurwitz space Hur has a modular compactification $\overline{\mathrm{Hur}}$ by means of $W(E_6)$-admissible covers. The Prym-Tyurin map PT extends to a rational map

$$PT\colon \overline{\mathrm{Hur}} \dashrightarrow \overline{\mathcal{A}}_6$$

with indeterminacy locus of codimension at least 2. Although the Hurwitz space $\overline{\mathrm{Hur}}$ has an intricate divisor theory, with boundary divisors associated to complicated discrete data, it is one of the important results of [ADFIO] that only three explicitly described boundary divisors $D_0, D_{\mathrm{azy}}, D_{\mathrm{syz}}$ of $\overline{\mathrm{Hur}}$ are not contracted under the map PT. Here D_{azy} and D_{syz} denote the boundary divisors of *azygetic* (respectively *syzygetic*) $W(E_6)$-admissible covers, having as general element a cover

$$[\pi\colon C = C_1 \cup C_2 \to R_1 \cup_q R_2,\ p_1 + \cdots + p_{24}],$$

with $\pi^{-1}(R_i) = C_i$ for $i = 1, 2$, where R_1 and R_2 are smooth rational curves meeting at the point q, precisely two branch points, say p_{23} and p_{24}, lie on R_2 and the *distinct* roots $r_{23}, r_{24} \in E_6$ determining the local monodromy at the corresponding points satisfy $r_{23} \cdot r_{24} \neq 0$ (respectively $r_{23} \cdot r_{24} = 0$). The divisor D_0 corresponds to the situation when the roots r_{23} and r_{24} are equal. In order to study $\overline{\mathcal{A}}_6$, it suffices therefore to restrict our attention to the partial compactification of the Hurwitz space

$$\widetilde{\mathrm{Hur}} := \mathrm{Hur} \cup \mathrm{D}_0 \cup \mathrm{D}_{\mathrm{azy}} \cup \mathrm{D}_{\mathrm{syz}} \subseteq \overline{\mathrm{Hur}}.$$

The divisor D_0 is mapped onto the the boundary divisor $D_6 := \overline{\mathcal{A}}_6 \setminus \mathcal{A}_6$, whereas D_{syz} and D_{azy} are mapped onto divisors of $\overline{\mathcal{A}}_6$ not contained in the boundary.

The Kanev correspondence $D \subseteq C \times C$ can be extended for any point $[\pi\colon C \to R, p_1 + \cdots + p_{24}] \in \overline{\mathrm{Hur}}$. In particular, it induces a decomposition

$$H^0(C, \omega_C) = H^0(C, \omega_C)^{(+1)} \oplus H^0(C, \omega_C)^{(-5)} \tag{1A}$$

into $(+1)$ and (-5) eigenspaces with respect to D and having dimensions 40 and 6 respectively. We denote by $\lambda, \lambda^{(+1)}$ and $\lambda^{(-5)}$ the Hodge eigenbundles on $\overline{\mathrm{Hur}}$ globalizing the decomposition (1A) over the entire moduli space. If $\lambda_1 \in CH^1(\overline{\mathcal{A}}_6)$ denotes the Hodge class, since $PT^*(\lambda_1) = \lambda^{(-5)}$ and $K_{\overline{\mathcal{A}}_6} = 7\lambda_1 - [D_6]$, where D_6 is the boundary divisor of $\overline{\mathcal{A}}_6$ of rank 1 degenerations, determining the class $\lambda^{(-5)}$ is essential to any further investigation of the birational geometry of $\overline{\mathcal{A}}_6$. One of the main results of this paper is that $\lambda^{(-5)}$ has a remarkably simple expression:

Theorem 1.1. *The class of the (-5)-Hodge eigenbundle on $\widetilde{\mathrm{Hur}}$ is given by the following formula:*

$$6\lambda^{(-5)} = \lambda - \frac{1}{2}[D_{\mathrm{syz}}].$$

Since it has been shown in [ADFIO, Theorem 6.17] that the Hodge class λ on $\overline{\mathrm{Hur}}$ can be expressed in terms of boundary divisors, Theorem 1.1 can be rewritten using only D_0, D_{syz} and D_{azy} and one has the following identity on $\widetilde{\mathrm{Hur}}$:

$$\lambda^{(-5)} = \frac{11}{92}[D_0] - \frac{1}{46}[D_{\mathrm{syz}}] + \frac{7}{276}[D_{\mathrm{azy}}]. \tag{1B}$$

Our approach to proving Theorem 1.1 is representation-theoretic: The Weyl group $W(E_6)$ has 25 irreducible representations $\rho_1, \ldots, \rho_{25}$. Each of these determines a variant $\mathbb{E}_i$ of the Hodge vector bundle over $\overline{\mathrm{Hur}}$. At a point given by the 27-sheeted cover $[\pi\colon C \to R, p_1 + \cdots + p_{24}] \in \overline{\mathrm{Hur}}$ with Galois closure $\widetilde{\pi}\colon \widetilde{C} \to R$, the fiber of $\mathbb{E}_i$ is defined to be $\mathrm{Hom}_{W(E_6)}\big(\rho_i, H^0(\widetilde{C}, \omega_{\widetilde{C}})\big)$. The Hodge classes in question are defined as $\lambda_i := c_1(\mathbb{E}_i)$, for $i = 1, \ldots, 25$. The Prym-Hodge bundles $\lambda^{(+1)}$ and $\lambda^{(-5)}$ are two special cases of this construction, obtained from the two non trivial representations of $W(E_6)$ that occur in the standard 27-dimensional permutation representation of $W(E_6)$. This gives the relation $\lambda^{(+1)} + \lambda^{(-5)} = \lambda$. Every representation ρ_i occurs in some permutation representation and every permutation representation gives rise to an associated cover, and the Hodge bundle arising from such a cover decomposes into contributions coming from the various classes λ_i. We calculate the Hodge bundles corresponding to a sufficiently large collection of such permutation representations, and use representation theory to extract from these formulas the formulas for the Hodge bundles λ_i corresponding to all 25 irreducible representations of $W(E_6)$. The permutation representations we use are quotients of the Galois cover $\widetilde{C}$ by cyclic subgroups W_α generated by representatives of the 25 conjugacy classes in $W(E_6)$. The list for the expression of the Hodge classes $\lambda_1, \ldots, \lambda_{25} \in CH^1(\widetilde{\mathrm{Hur}})$ can be found in the statement of Theorem 3.9.

Another important result of this paper concerns the class of the *Weyl-Petri divisor* on $\overline{\mathrm{Hur}}$. For a smooth $W(E_6)$-cover $\pi\colon C \to \mathbb{P}^1$ the Weyl-Petri map is the multiplication map

$$\mu(L)\colon H^0(C, L) \otimes H^0(C, \omega_C \otimes L^\vee) \to H^0(C, \omega_C),$$

where $L = \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \in W^1_{27}(C)$. By [ADFIO, Theorem 9.2], the map $\mu(L)$ is injective for a general point of Hur. Furthermore, it factors through the $(+1)$-eigenspace, that is, one has a map

$$\mu(L)\colon H^0(C, L) \otimes H^0(C, \omega_C \otimes L^\vee) \to H^0(C, \omega_C)^{(+1)}. \tag{1C}$$

Therefore, since its source and target have the same rank, its degeneracy locus is a divisor $\mathfrak{N}$ on the space of admissible $W(E_6)$-covers (see Section 4 for a more precise definition and a discussion of what happens when $h^0(C, L)$ jumps). Our next result determines the class of $\mathfrak{N}$ on $\widetilde{\mathrm{Hur}}$:

Theorem 1.2. *The class of the Weyl-Petri divisor on* $\widetilde{\mathrm{Hur}}$ *is given by the following formula:*

$$[\mathfrak{N}] = \frac{59}{42}\lambda - \frac{12}{7}[D_0] - \frac{29}{84}[D_{\mathrm{syz}}]. \tag{1D}$$

The proof of Theorem 1.2 involves passing to an alternative partial compactification $\widetilde{\mathcal{G}}_{E_6}$ of Hur over which the multiplication map (1C) can be defined globally, then reinterpreting the obtained result on $\widetilde{\mathrm{Hur}}$.

In [ADFIO, Theorem 0.4] we showed that if $[\pi\colon C \to \mathbb{P}^1] \in \mathrm{Hur}$ does not lie in the Weyl-Petri divisor $\mathfrak{N}$ then it lies in the ramification locus of the Prym-Tyurin map $PT\colon \mathrm{Hur} \to \mathcal{A}_6$ if and only if the *Prym-Tyurin canonical curve* $\varphi_{(-5)}(C) \subseteq \mathbb{P}H^0(C, \omega_C)^{(-5)} \cong \mathbb{P}^5$ induced by the sublinear system $\big|H^0(C, \omega_C)^{(-5)}\big|$ lies on a quadric, that is, the multiplication map

$$\mathrm{Sym}^2 H^0(C, \omega_C)^{(-5)} \to H^0(C, \omega_C^{\otimes 2})$$

in not injective. We clarify the set-theoretic description of the ramification divisor of PT:

Theorem 1.3. *The ramification divisor of the Prym-Tyurin map* $PT\colon \mathrm{Hur} \to \mathcal{A}_6$ *is contained in the union of the Weyl-Petri divisor* $\mathfrak{N}$ *and the effective divisor* $\mathfrak{M}$ *parametrising* $W(E_6)$*-covers* $[\pi\colon C \to \mathbb{P}^1]$ *such that* $h^0\big(C, \pi^*(\mathcal{O}_{\mathbb{P}^1}(1)\big) \geq 3$.

The fact that the condition $h^0(C, L) \geq 3$ for $L = \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$ defines a divisor $\mathfrak{M}$ on Hur comes to us as a surprise, for general Brill-Noether theory would predict that such curves depend on considerably fewer moduli. For the precise definition of the divisor $\mathfrak{M}$, we refer to (4.3).

By analysing directly the differential of the map PT at a general point of the boundary divisor D_0, we give a second, more elementary proof of the main result from [ADFIO].

Theorem 1.4. *The Prym-Tyurin map PT is generically unramified along the boundary divisor D_0 of $\overline{\mathrm{Hur}}$. It follows once more that $PT\colon \overline{\mathrm{Hur}} \dashrightarrow \overline{\mathcal{A}}_6$ is generically finite.*

We recall that the original proof of the dominance of PT amounted to the *tropicalization* of the Prym-Tyurin map. Precisely, we studied the principal term of the Prym-Tyurin map by expanding the monomial coordinates near the neighborhood of a maximally degenerate cover and then used the theory of degenerations of Prym-Tyurin varieties. This time, the proof, which we complete in Section 6 is more direct. The element of D_0 for which Theorem 1.4 is verified is obtained by choosing judiciously 12 points $q_1, \ldots, q_{12} \in \mathbb{P}^1$ together with roots $r_1, \ldots, r_{12} \in E_6$, determining a degree 27 stable map $\pi\colon C \to \mathbb{P}^1$, where C is the curve obtained from the disjoint union of 27 copies of $\mathbb{P}^1$ labeled by the 27 lines on a smooth cubic surface and then gluing over each point q_i the components labeled by the double-six corresponding to the root r_i. The verification that the $W(E_6)$-admissible cover associated to π verifies all required properties is completed in Theorem 5.6.

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2. THE WEYL GROUP OF E_6 AND THE UNIFORMIZATION OF $\mathcal{A}_6$.

We give a summary of some group theoretic facts and the results established in [ADFIO] that are used in this paper.

2.1. The group $W(E_6)$ and its representations.

Let $W(E_6)$ be the Weyl group of the root lattice E_6. It is the subgroup of the orthogonal group $\mathbb{O}(E_6)$ generated by reflections $r_\alpha\colon x \mapsto x + (x, \alpha)\alpha$ in a root α of E_6. One has $|W(E_6)| = 51840$ and $W(E_6)$ has 25 irreducible representations. The dimensions of these representations are 1, 1, 6, 6, 10, 15, 15, 15, 15, 20, 20, 20, 24, 24, 30, 30, 60, 60, 64, 64, 80, 81, 81, 90. In order to refer to the characters and conjugacy classes of $W(E_6)$ we use the notation from the character table from the Atlas [CCNPW, p.27] for the group $U_4(2).2 = W(E_6)$. It is obtained from the character table of $U_4(2)$ by the splitting and fusion rules. It can be reproduced in GAP [GAP] by using the command `Display(CharacterTable("W(E6)"))`.

In addition to the numbers 1, ..., 25 for the characters of $W(E_6)$, we use convenient names, as in Table 2. They start with the dimension of the representation and add attributes *a*, *b*, and so on, if there are several irreducible representations of the same dimension. We also group characters in pairs χ and $\overline{\chi} = \chi \otimes \overline{1}$ whenever these are different. Here, $\overline{1}$ is the 1-dimensional character of $W(E_6)$ sending an element $u \in W(E_6)$ to $(-1)^n$ if u is a product of n reflections.

Notation 2.1. We use repeatedly the geometric realization $E_6 \cong K_S^\perp \subseteq \mathrm{Pic}(S)$, where S is a smooth cubic surface. We use the classical notation $a_1, \ldots, a_6$, $b_1, \ldots, b_6$ and c_{ij}, for $1 \le i < j \le 6$ for the 27 lines on S. A system of fundamental roots of E_6 is then given by $\omega_i := a_i - a_{i+1}$ for $i = 1, \ldots, 5$ and $\omega_6 := h - a_1 - a_2 - a_3$, where $h := -K_S$ is the hyperplane class.

Notation 2.2. We record three important conjugacy classes in the Weyl group $W(E_6)$, namely the class 2c containing reflections $w \in W(E_6)$, the class 2b containing products $w_1 \cdot w_2$ of two commuting (syzygetic) reflections $w_1, w_2 \in W(E_6)$, and 3b containing products $w_1 \cdot w_2$ of two non-commuting (azygetic) reflections.

The character table of $W(E_6)$, playing a significant role in several of our calculations is reproduced in the appendix of this paper as Table 2. We fix representatives w_i of the 25 conjugacy classes in $W(E_6)$, labeled so that: $w_{1a} = 1$, w_{2c} is a reflection, that is, a representative of the class 2c in the notation of the character table of $W(E_6)$, then w_{2b} is the product of two syzygetic reflections and so on.

Notation 2.3. For an element $u \in W(E_6)$, we denote by Z_u its centralizer in $W(E_6)$ and by c_u its conjugacy class in $W(E_6)$.

Assume now that G is a subgroup of $W(E_6)$ of index d and let $u \in W(E_6)$ be a fixed element. The assignment $xG \mapsto uxG$ induces a bijection on the sets $W(E_6)/G$ of left cosets and can thus be regarded as a permutation from S_d. We shall need the following simple group-theoretic fact.

Lemma 2.4. *Let $u \in W(E_6)$ be an element of prime order p. Then its cycle type in S_d is $p^a 1^b$, where*

$$b = \frac{|G \cap c_u| \cdot \ |Z_u|}{|G|}, \qquad a = \frac{d-b}{p}. \tag{2A}$$

Proof. We consider the bijection $W(E_6)/G \to W(E_6)/G$ on the set of G-cosets induced by multiplication with u. Since $u \in W(E_6)$ has prime order p, there are only two possibilities for a coset xG. It is either fixed, or its orbit consists of exactly p cosets. We first count the number of elements $x \in W(E_6)$ such that $uxG = xG$. In this case $x^{-1}ux =: u' \in G \cap c_u$. We consider the surjective map $\chi_u \colon W(E_6) \to c_u$ given by $\chi_u(x) := x^{-1}ux$. Each fibre of χ_u consists of $|Z_u|$ elements, thus the number of elements x with $uxG = xG \in W(E_6)/G$ equals $|G \cap c_u| \cdot |Z_u|$. In order to obtain the number of u-fixed G-cosets we have to divide this number by $|G|$, which gives the stated formula for b. Then a is computed from the equality $pa + b = d$. □

The quantities a and b computed in Lemma 2.4 clearly depend only on the conjugacy class c_u of u. In particular, when the subgroup G is fixed, we obtain a vector of positive integers

$$\big(a_{2c}, b_{2c}, a_{2b}, b_{2b}, a_{3b}, b_{3b}\big). \tag{2B}$$

Since the order of the representatives w_{2c} and w_{2b} is equal to 2, whereas $\operatorname{ord}(w_{3b}) = 3$, one has

$$2a_{2c} + b_{2c} = 2a_{2b} + b_{2b} = 3a_{3b} + b_{2b} = [W(E_6) : G] = d.$$

2.2. **Maximal subgroups of $W(E_6)$.** Up to conjugation, the group $W(E_6)$ has five maximal subgroups, see [Do, Theorem 9.2.2].

- A subgroup $G_{27} \subsetneq W(E_6)$ of index 27, which can be viewed as the stabilizer of a line of the cubic surface S under the identification $E_6 \cong K_S^{\perp}$. One has $G_{27} \cong W(D_5)$. In this paper we constantly make the choice $G_{27} := \mathrm{Stab}_{W(E_6)}(a_6) = \langle \omega_1, \omega_2, \omega_3, \omega_4, \omega_6 \rangle$.
- A subgroup $G_{36} \subsetneq W(E_6)$ of index 36, viewed as the stabilizer of a double six on S.
- A subgroup $G_{45} \subsetneq W(E_6)$ of index 45, regarded as the stabilizer of a tritangent plane of S. Note that $G_{45} \cong W(F_4)$.
- Two subgroups G_{40} and G'_{40} of index 40.

For instance, for the subgroup G_{27} the vector described in (2B) is equal to

$$\big(a_{2c}, b_{2c}, a_{2b}, b_{2b}, a_{3b}, b_{3b}\big) = (6, 15, 10, 7, 6, 9).$$

2.3. **Three versions of compactified Hurwitz spaces of $W(E_6)$-covers.** We denote by $\mathcal{H}$ the Hurwitz space of smooth $W(E_6)$-covers $[\pi\colon C\to\mathbb{P}^1,p_1,\ldots,p_{24}]$ together with a labeling of its branch points. The map π is of degree 27. The global monodromy of π equals $W(E_6)$ and the local monodromy around each branch point $p_i\in\mathbb{P}^1$ is a reflection in a root of E_6, that is, an element in the conjugacy class 2c in the notation of the character table of $W(E_6)$. The curve C is smooth of genus 46 and the cover $\pi\colon C\to\mathbb{P}^1$ is not Galois.

Let $\overline{\mathcal{H}}$ be the compactification of $\mathcal{H}$ by admissible $W(E_6)$-covers. This can be regarded as the stack of *balanced twisted stable* maps into the classifying stack $\mathcal{B}W(E_6)$ of $W(E_6)$, that is,

$$\overline{\mathcal{H}}:=\overline{\mathcal{M}}_{0,24}\Big(\mathcal{B}W(E_6)\Big).$$

The map $\mathfrak{b}\colon\overline{\mathcal{H}}\to\overline{\mathcal{M}}_{0,24}$ forgetting the monodromy data is finite, so $\dim(\overline{\mathcal{H}})=21$. The symmetric group S_{24} acts on both $\overline{\mathcal{M}}_{0,24}$ and $\overline{\mathcal{H}}$ by permuting the marked (respectively branch) points, and we denote the corresponding quotients by

$$\overline{\mathrm{Hur}}:=\overline{\mathcal{H}}/S_{24}\quad\text{and}\quad\widetilde{\mathcal{M}}_{0,24}:=\overline{\mathcal{M}}_{0,24}/S_{24}.$$

Let $q\colon\overline{\mathcal{H}}\to\overline{\mathrm{Hur}}$ denote the quotient map. The space $\overline{\mathrm{Hur}}$ is the main object of study both in [ADFIO] and in the present paper, on which most of the intersection-theoretic formulas are written.

We have regular maps

$$\mathfrak{br}\colon\overline{\mathrm{Hur}}\to\widetilde{\mathcal{M}}_{0,24}\quad\text{and}\quad\widetilde{\varphi}\colon\overline{\mathrm{Hur}}\to\overline{\mathcal{M}}_{46}$$

associating to an admissible cover $[\pi\colon C\to R,p_1+\cdots+p_{24}]\in\overline{\mathrm{Hur}}$ the branch locus $[R,p_1+\cdots+p_{24}]$ and the stable model of its source curve C respectively.

The third version of a compactified space of $W(E_6)$-covers is the one that admits a universal $W(E_6)$-line bundle of degree 27, which is something both $\overline{\mathcal{H}}$ and $\overline{\mathrm{Hur}}$ lack. Following Section 9 of [ADFIO] we denote by $\widetilde{\mathcal{G}}_{E_6}$ the (normalization of the) moduli space parametrizing finite maps $[\pi:C\to R]$ with monodromy $W(E_6)$, where C is an irreducible stable curve of genus 46 and R is a smooth rational curve. For such a map, $L:=\pi^*\mathcal{O}_R(1)$ is a base point free line bundle of degree 27 on C with $h^0(C,L)\geq 2$. The spaces $\overline{\mathrm{Hur}}$ and $\widetilde{\mathcal{G}}_{E_6}$ share the open subspace Hur on which the source curve C is smooth. We denote by

$$\tilde{f}\colon\widetilde{\mathcal{C}}_{E_6}\to\widetilde{\mathcal{G}}_{E_6}$$

the universal genus 46 curve. The fibres of $\tilde{f}$ are irreducible curves of genus 46.

Following [ADFIO, 9.5], we denote by $\widetilde{\beta}\colon\overline{\mathrm{Hur}}\dashrightarrow\widetilde{\mathcal{G}}_{E_6}$ the birational map assigning to a point $[\pi\colon C\to\mathbb{P}^1,p_1+\cdots+p_{24}]\in\mathrm{Hur}$ the map $[\pi:C\to R]\in\widetilde{\mathcal{G}}_{E_6}$. Since $\overline{\mathrm{Hur}}$ is normal, $\widetilde{\beta}$ can be extended to a regular map outside a subvariety of codimension at least 2 in $\overline{\mathrm{Hur}}$.

2.4. **The dominance of the Prym-Tyurin map.** A fiber of the cover $\pi\colon C\to\mathbb{P}^1$ corresponding to an element of $\mathcal{H}$ has the combinatorial structure of the 27 lines on a cubic surface, and the $W(E_6)$-action on each of its fibres preserves the incidence relation. The correspondence sending a line ℓ to the 10 lines incident to it can be thus regarded as a correspondence on C and it induces an endomorphism D on the Jacobian $JC:=\mathrm{Pic}^0(C)$, satisfying the quadratic relation $(D-1)(D+5)=0$. By Kanev [K1, K2] the (-5)-eigenspace of this endomorphism

$$PT(C,D):=\mathrm{Ker}(D+5)^0=\mathrm{Im}(D-1)\subseteq JC$$

is a principally polarized abelian variety of dimension 6 and exponent 6, which we call the *Prym-Tyurin variety* of the pair $[C,D]$. This assignment defines the map $PT_{\mathcal{H}}\colon\mathcal{H}\to\mathcal{A}_6$ which factors through the Prym-Tyurin map $PT\colon\mathrm{Hur}\to\mathcal{A}_6$. By [ADFIO, Theorem 0.1] these maps are dominant and generically finite.

2.5. **Boundary divisors on the Hurwitz space.** The boundary divisors on the moduli space $\overline{\mathcal{M}}_{0,24}$ of stable 24-pointed rational curves are of the form $\Delta_{0:I}$, with $I \subseteq \{1, \dots, 24\}$ being a subset such that $|I| \geq 2$ and $|I^c| \geq 2$. A general point of $\Delta_{0:I}$ corresponds to a 24-pointed stable rational curve $[R, p_1, \dots, p_{24}]$ consisting of two smooth components R_1 and R_2 meeting at a single point, with the marked points $\{p_i\}_{i\in I}$ (respectively $\{p_j\}_{j\in I^c}$) lying on R_1 (respectively on R_2). For $i = 2, \dots, 12$, we have the S_{24}-invariant boundary divisor

$$B_i := \sum_{|I|=i} \Delta_{0:I}.$$

The boundary divisors of $\overline{\mathcal{H}}$ correspond to the components of the pull-back $\mathfrak{b}^*(B_i)$ under the map

$$\mathfrak{b} : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{0,24}. \tag{2C}$$

In order to keep track of these divisors, we need further combinatorial data. In addition to the partition $I \sqcup I^c = \{1, \dots, 24\}$, we also have the data of reflections $\{w_i\}_{i\in I}$ and $\{w_j\}_{j\in I^c}$ in $W(E_6)$ such that $\prod_{i\in I} w_i = u$, $\prod_{j\in I^c} w_j = u^{-1}$. The products are taken in order, and the sequence $w_1, \dots, w_{24}$ is defined up to conjugation by the same element $g \in W(E_6)$.

Let $\mu := (\mu_1, \dots, \mu_\ell)$ be the cycle type of the element $u \in W(E_6)$ considered as a permutation in S_{27}. Set

$$\frac{1}{\mu} := \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_\ell} \quad \text{and} \quad \mathrm{lcm}(\mu) := \mathrm{lcm}(\mu_1, \dots, \mu_\ell). \tag{2D}$$

We denote by $\mathcal{P}_i$ the set of partitions μ of 27 appearing as products of i reflections in $W(E_6)$. The possibilities for $\mu \in \mathcal{P}_i$ are listed in [ADFIO, Table 1]. For $\mu \in \mathcal{P}_i$, let $E_{i:\mu}$ denote the sum of all the divisors of $\overline{\mathcal{H}}$ whose general point corresponds to an $W(E_6)$-cover

$$t := \big[\pi\colon C \to R,\ p_1, \dots, p_{24}\big] \in \overline{\mathcal{H}},$$

where $[R = R_1 \cup_q R_2, p_1, \dots, p_{24}] \in B_i \subseteq \overline{\mathcal{M}}_{0,24}$ is a pointed union of two smooth rational curves R_1 and R_2 meeting at the point q. Over $q \in R_{\mathrm{sing}}$, the map π is ramified according to u, that is, the points in $\pi^{-1}(q)$ correspond to cycles in the permutation μ associated to the element $u \in W(E_6)$.

Next, we focus on three special divisors on $\overline{\mathcal{H}}$, see also [ADFIO, 6.8, 6.9]:

1. $E_0 := E_{2:(1^{27})}$
2. The *syzygetic divisor* $E_{\mathrm{syz}} := E_{2:(2^{10},1^7)}$.
3. The *azygetic divisor* $E_{\mathrm{azy}} := E_{2:(3^6,1^9)}$.

These three divisors correspond to the boundary divisors where there are exactly two branch points lying on the first irreducible component R_1 and having local monodromy $w_1, w_2 \in W(E_6)$. For E_0 the reflections w_1 and w_2 are equal, thus the partition associated to $w_1 \cdot w_2$ equals $\mu = (1^{27})$. For E_{syz} the local monodromies w_1 and w_2 are different and commuting and the associated partition is $\mu = (2^{10}, 1^7)$, whereas for E_{azy} the reflections w_1 and w_2 do not commute, in which case the partition describing the cycle type of $w_1 \cdot w_2$ is $(3^6, 1^9)$. As explained in [ADFIO, 6.6], we have the following relation:

$$\mathfrak{b}^*(B_i) = \sum_{\mu\in\mathcal{P}_i} \mathrm{lcm}(\mu) E_{i:\mu}. \tag{2E}$$

On the space $\overline{\mathrm{Hur}}$ we define the *reduced* divisors $D_{i:\mu}$ which are the set-theoretic images of $E_{i:\mu}$. In particular, we have the three key divisors D_0, D_{syz}, D_{azy}. By [ADFIO, 6.13] the pullbacks of the key divisors under the quotient map $q\colon \overline{\mathcal{H}} \to \overline{\mathrm{Hur}}$ are

$$E_0 = q^*\big(\frac{1}{2} D_0\big), \quad E_{\mathrm{syz}} = q^*(D_{\mathrm{syz}}), \quad E_{\mathrm{azy}} = q^*\big(\frac{1}{2} D_{\mathrm{azy}}\big). \tag{2F}$$

Furthermore, $q^*(D_{i:\mu}) = E_{i:\mu}$, for $i = 3, \dots, 12$ and $\mu \in \mathcal{P}_i$.

At the level of the partial compactification $\widetilde{\mathcal{G}}_{E_6}$ [ADFIO, 9.5] the pullbacks under $\tilde{\beta}\colon \overline{\mathrm{Hur}} \dashrightarrow \widetilde{\mathcal{G}}_{E_6}$ are

$$\tilde{\beta}^*(D_{E_6}) = D_0, \quad \tilde{\beta}^*(D_{\mathrm{syz}}) = D_{\mathrm{syz}}, \quad \tilde{\beta}^*(D_{\mathrm{azy}}) = D_{\mathrm{azy}}. \tag{2G}$$

For further details regarding the local description of the morphism $\tilde{\beta}$ we refer to Section 4.1. When carrying out divisor class calculations we will not distinguish between the spaces

$$\widetilde{\mathrm{Hur}} := \mathrm{Hur} \cup \mathrm{D}_0 \cup \mathrm{D}_{\mathrm{syz}} \cup \mathrm{D}_{\mathrm{azy}} \subseteq \overline{\mathrm{Hur}}$$

and $\widetilde{\mathcal{G}}_{E_6}$ and we will accordingly identify the divisors D_0, D_{syz} and D_{azy} on the two spaces.

2.6. **Properties of the rational map *PT*.** The Prym-Tyurin map $PT\colon \mathrm{Hur} \to \mathcal{A}_6$ extends to a rational map $PT\colon \overline{\mathrm{Hur}} \dashrightarrow \overline{\mathcal{A}}_6$ for which we use the same symbol. We denote by $U_{\overline{\mathrm{Hur}}}$ the domain of definition of this rational map. Since $\overline{\mathrm{Hur}}$ is normal, the complement $\overline{\mathrm{Hur}} \setminus U_{\overline{\mathrm{Hur}}}$ has codimension at least 2.

In Section 5.2. of [ADFIO], we assigned to any point $[\pi\colon C \to R, p_1, \dots, p_{24}] \in \overline{\mathcal{H}}$ a group Prym-Tyurin variety $PT(C, D) = \mathrm{Im}(D - 1)$ for the induced endomorphism D of $JC = \mathrm{Pic}^0(C)$. It is a semiabelian variety of dimension 6, that is, an extension

$$0 \longrightarrow T \longrightarrow PT(C, D) \longrightarrow A \longrightarrow 0$$

of an abelian variety A by a torus T.

The toric rank $\mathrm{tor.rk} := \dim T$ of the semiabelian variety $PT(C, D)$ is an upper semicontinuous function on $\overline{\mathrm{Hur}}$. By [ADFIO, Thm. 5.9], the domain of definition $U_{\overline{\mathrm{Hur}}}$ contains the open set $\{\mathrm{tor.rk} \leq 1\}$.

Lemma 2.5. *The rational map* $PT\colon \overline{\mathrm{Hur}} \dashrightarrow \overline{\mathcal{A}}_6$ *does not create new divisors. In other words, for any resolution of singularities*

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g"] & \\ \overline{\mathrm{Hur}} \arrow[r, dashed, "PT"] & \overline{\mathcal{A}}_6 \end{tikzcd}$$

and for any closed subset $Z \subseteq \overline{\mathrm{Hur}}$ *such that* $\operatorname{codim} Z \geq 2$, *one has* $\operatorname{codim} g(f^{-1}(Z)) \geq 2$.

Proof. We have to show that, for every irreducible subset $Z \subseteq \overline{\mathrm{Hur}} \setminus U_{\overline{\mathrm{Hur}}}$, one has $\operatorname{codim} g(f^{-1}(Z)) \geq 2$. By the previous paragraph, we know that $Z \subseteq \{\mathrm{tor.rk} \geq 2\}$.

By the Borel theorem [B, Thm. A] applied to a smooth cover of $\overline{\mathrm{Hur}}$, the map $PT\colon \mathrm{Hur} \to \mathcal{A}_6$ extends to a regular map to the Satake-Baily-Borel compactification $\overline{\mathrm{Hur}} \to \overline{\mathcal{A}}_6^{\mathrm{sat}} = \mathcal{A}_6 \sqcup \mathcal{A}_5 \sqcup \ldots \sqcup \mathcal{A}_0$. Thus, $g(f^{-1}(Z))$ is contained in the preimage of $\mathcal{A}_4 \sqcup \ldots \sqcup \mathcal{A}_0$ under the map $\overline{\mathcal{A}}_6 \to \overline{\mathcal{A}}_6^{\mathrm{sat}}$. It has codimension at least 2 in $\overline{\mathcal{A}}_6$. ☐

Corollary 2.6. *The divisorial pushforward map* $PT_*\colon \mathrm{Div}(\overline{\mathrm{Hur}}) \to \mathrm{Div}(\overline{\mathcal{A}}_6)$ *is well defined.*

By [ADFIO, Thm. 7.17], the divisors D_0, D_{syz}, D_{azy} are the only boundary divisors *not contracted by* the morphism $PT\colon U_{\overline{\mathrm{Hur}}} \to \overline{\mathcal{A}}_6$. The divisor D_0 maps to the boundary D_6 of $\overline{\mathcal{A}}_6 \setminus \mathcal{A}_6$, while D_{syz} and D_{azy} map onto divisors not supported on the boundary.

We have a bijection between divisors on $\overline{\mathrm{Hur}}$ and the divisors on the domain of definition $U_{\overline{\mathrm{Hur}}}$ of PT. Thus, for a divisor D on $\overline{\mathcal{A}}_6$ we have the rational pullback divisor $PT^*(D)$ on $\overline{\mathrm{Hur}}$ which is the closure of the corresponding regular pullback divisor on $U_{\overline{\mathrm{Hur}}}$.

Definition 2.7. Denote by $(\star)$ the subgroup of $\mathrm{Pic}(\overline{\mathrm{Hur}}) \otimes \mathbb{Q}$ generated by the boundary divisors on $\overline{\mathrm{Hur}}$ different from D_0, D_{syz}, D_{azy}.

2.7. **The Hodge classes $\boldsymbol{\lambda}, \boldsymbol{\lambda}^{(-5)}, \boldsymbol{\lambda}^{(+1)}$.** A point of $\overline{\mathrm{Hur}}$ represents a cover $t := [\pi\colon C \to R, p_1 + \cdots + p_{24}]$ with $W(E_6)$-monodromy. The Kanev correspondence D on C induces an eigenspace decomposition

$$H^0(C, \omega_C) = H^0(C, \omega_C)^{(-5)} \oplus H^0(C, \omega_C)^{(+1)}$$

into subspaces of dimension 6 and 40 respectively. We denote by $\mathbb{E}$ the Hodge bundle over $\overline{\mathrm{Hur}}$ with fiber $H^0(C, \omega_C)$ over a point $t \in \overline{\mathrm{Hur}}$ and by $\mathbb{E}^{(-5)}$ and $\mathbb{E}^{(+1)}$ the Hodge eigenbundles globalizing the decomposition (2.7), that is, having fibres $H^0(C, \omega_C)^{(-5)}$ and $H^0(C, \omega_C)^{(+1)}$ over t. We denote by

$$\lambda^{(-5)} = c_1(\mathbb{E}^{(-5)}) \quad \text{and} \quad \lambda^{(+1)} := c_1(\mathbb{E}^{(+1)}) \tag{2H}$$

the corresponding Hodge eigenclasses. Since $\lambda^{(-5)} = PT^*(\lambda_1)$, determining $\lambda^{(-5)}$ explicitly is essential for any application concerning the birational geometry of $\overline{\mathcal{A}}_6$.

Theorem 6.17 and Remark 6.18 of [ADFIO] establish the following important formula for the Hodge class on $\overline{\mathrm{Hur}}$:

$$\lambda = \frac{33}{46} D_0 + \frac{17}{46} D_{\mathrm{syz}} + \frac{7}{46} D_{\mathrm{azy}} \mod (\star) \tag{2I}$$

3. TWENTY FIVE FUNDAMENTAL HODGE BUNDLES ON $\overline{\mathrm{Hur}}$.

The main purpose of this section is to determine the Hodge classes $\lambda_1, \ldots, \lambda_{25} \in CH^1(\widetilde{\mathrm{Hur}})$ associated to the irreducible representations of $W(E_6)$. In particular, we shall compute the class of the (-5)-Hodge eigenbundle $\lambda^{(-5)}$ and thus prove Theorem 1.1. We first describe our strategy. Theorem 6.17 of [ADFIO] has been used to compute the Hodge class $\lambda \in CH^1(\overline{\mathrm{Hur}})$ for the universal family of degree 27 covers, corresponding to the lines on a fixed cubic surface. In that case, $\lambda = \lambda^{(-5)} + \lambda^{(+1)}$, and the summands of $H^0(C, \omega_C) = H^0(C, \omega_C)^{(-5)} \oplus H^0(C, \omega_C)^{(+1)}$ are associated with irreducible representations of the Weyl group $W(E_6)$. Namely, the 27-dimensional representation of $W(E_6) \hookrightarrow S_{27}$ has character 1 + 6 + 20b, whose dimensions add up to 27. The Hodge eigenbundles $\mathbb{E}_1$, $\mathbb{E}_6 = \mathbb{E}^{(-5)}$, and $\mathbb{E}_{20b} = \mathbb{E}^{(+1)}$ associated with these characters have ranks $0 + 6 + 40 = 46 = g(C)$.

3.1. The 27:1 cover $\pi\colon C \to \mathbb{P}^1$ whose fibres correspond to lines on a cubic surface is merely one of many. Let $\widetilde{\pi}\colon \widetilde{C} \to \mathbb{P}^1$ be the Galois closure of π. Then $C = \widetilde{C}/G_{27}$, where the maximal index 27 subgroup G_{27} has been introduced in 2.2. We have further covers associated to subgroups of $W(E_6)$:

1. A maximal subgroup of index 36. The cover $C_{36} := \widetilde{C}/G_{36} \to \mathbb{P}^1$ is associated with the permutation representation $W(E_6) \hookrightarrow S_{36}$ with character 1 + 15b + 20b. The points of the fibers of $C_{36} \to \mathbb{P}^1$ correspond to the pairs of roots $\pm r$ of the $W(E_6)$ root lattice; equivalently, to the double sixers of lines on a cubic surface. The ranks of the respective vector bundles $\mathbb{E}_i$ are $0 + 45 + 40 = 85 = g(C_{36})$.
2. A maximal subgroup of index 45. The cover $C_{45} := \widetilde{C}/G_{45} \to \mathbb{P}^1$ is associated with the permutation representation $W(E_6) \hookrightarrow S_{45}$ with character 1 + 24 + 20b. The points of the fibers of $C_{45} \to \mathbb{P}^1$ correspond to the triangles $\{\ell_1, \ell_2, \ell_3\}$ of lines on a cubic surface. The ranks of the respective vector bundles $\mathbb{E}_i$ are $0 + 96 + 40 = 136 = g(C_{45})$.
3. More generally, for each fixed representative w_α of one of the 25 conjugacy classes in $W(E_6)$, labeled as described in 2.2, recalling that $Z_\alpha := Z_{w_\alpha}$ is the centralizer of w_α, we have the curve $A_\alpha := \widetilde{C}/Z_\alpha$.
4. Similarly, let $W_\alpha = \langle w_\alpha \rangle$ be the cyclic subgroup generated by w_α. This gives rise to 25 curves $B_\alpha := \widetilde{C}/W_\alpha$.

Each of these families gives a map to a certain moduli space of curves and has a Hodge bundle whose first Chern class we can compute as a linear combination of D_0, D_{syz}, D_{azy} modulo the other boundary divisors ($\star$). Each Hodge bundle is a direct sum of isotypical components for the 25 irreducible representations of $W(E_6)$, that is, a direct sum of the same basic 25 Hodge bundles (with appropriate multiplicities). The multiplicities of these isotypical components are easily computable. Thus, given 25 "linearly independent" families, we can compute the semi-ample Chern classes $\lambda_i = c_1(\mathbb{E}_i)$ of the 25 bundles $\mathbb{E}_i$ labeled by the characters of $W(E_6)$. It turns out that the relations obtained by considering universal versions of the curves B_α are linearly independent, so they work for this purpose.

In particular, this gives us a formula for $\lambda_6 = \lambda^{(-5)}$, that is, the first Chern class of the vector bundle we denoted $\mathbb{E}^{(-5)}$ in Section 2.7. We now put this program to practice.

3.2. $\overline{\mathrm{Hur}}$ **as a moduli space of Galois admissible covers.** In what follows we choose to view $\overline{\mathcal{H}}$ as the moduli space of $W(E_6)$-*Galois admissible covers*

$$[\widetilde{\pi}\colon \widetilde{C} \to R, p_1, \dots, p_{24}].$$

This means that $[R, p_1, \dots, p_{24}] \in \overline{\mathcal{M}}_{0,24}$, as usual, $\widetilde{\pi}^{-1}(R_{\mathrm{sing}}) = \widetilde{C}_{\mathrm{sing}}$ and that there is a $W(E_6)$-action on $\widetilde{C}$ compatible with $\widetilde{\pi}$ such that the restriction

$$\widetilde{\pi}\colon \widetilde{\pi}^{-1}\big(R_{\mathrm{reg}} \setminus \{p_1, \dots, p_{24}\}\big) \to R_{\mathrm{reg}} \setminus \{p_1, \dots, p_{24}\}$$

is a principal $W(E_6)$-bundle. At each node $q \in C_{\mathrm{sing}}$, the action of the stabilizer $\mathrm{Stab}_q\big(W(E_6)\big) \subseteq W(E_6)$ is *balanced*, that is, the eigenvalues of the actions on the tangent spaces on the two branches of the tangent spaces of $\widetilde{C}$ at q are multiplicative inverses to one another.

To recover the description of $\overline{\mathcal{H}}$ given in (2.3), we fix the subgroup $G_{27} = \mathrm{Stab}_{W(E_6)}(a_6) \subseteq W(E_6)$ and note that if $\widetilde{\pi}\colon \widetilde{C} \to R$ is a $W(E_6)$-Galois cover, then $\pi := \pi_{G_{27}}\colon \widetilde{C}/G_{27} \to R$ is a degree 27 cover with monodromy group equal to $W(E_6)$. The inverse operation is obtained by taking the Galois closure of each degree 27 cover $\pi\colon C \to R$ with $W(E_6)$-monodromy. Both of these operations can be carried out in families.

Notation 3.1. For a Galois $W(E_6)$-cover $\widetilde{\pi}\colon \widetilde{C} \to R$ and for a subgroup $G \subseteq W(E_6)$, we denote $C_G := \widetilde{C}/G$ and $\pi_G\colon C_G \to R$ the induced cover of degree $d = [W(E_6) : G]$. We further set $g_G := p_a(C_G)$.

Lemma 3.2. *The arithmetic genus g_G of the curve C_G is*

$$g_G = 12a_{2c} - d + 1 \tag{3A}$$

where $d = [W(E_6) : G]$ and a_{2c} is given by Equation (2A) *for u in the conjugacy class 2c containing the reflections of $W(E_6)$.*

Proof. The sheets of the cover $\pi_G\colon C_G \to \mathbb{P}^1$ over a general point from $\mathbb{P}^1$ are in bijection with the set of cosets $W(E_6)/G$. The monodromy action by an element $u \in W(E_6)$ is given by multiplication $xG \mapsto uxG$ on the set of cosets. If $[\pi_G\colon C_G \to \mathbb{P}^1, p_1, \dots, p_{24}]$ corresponds to a general element from $\overline{\mathcal{H}}$, then π_G is ramified over each of the 24 points p_i according to the ramification profile $2^{a_{2c}}1^{a_{2c}}$, where a_{2c} and b_{2c} have been defined in (2B). Applying the Hurwitz formula to π_G, we thus have $2g_G - 2 = d(-2) + 24a_{2c}$, which finishes the proof. $\square$

3.3. **Computation of Hodge classes on $\overline{\mathcal{H}}$.** Having fixed a subgroup $G \subseteq W(E_6)$ of index d, the assignment $[\widetilde{\pi}\colon \widetilde{C} \to \mathbb{P}^1,\ p_1, \dots, p_{24}] \mapsto [C_G]$ induces a regular map

$$\overline{\mathcal{H}} \to \overline{\mathcal{M}}_{g_G}$$

and accordingly a Hodge bundle $\mathbb{E}_G$ on $\overline{\mathcal{H}}$ of rank g_G obtained by pulling back the Hodge bundle from $\overline{\mathcal{M}}_{g_G}$. We aim to compute its determinant $\lambda_G := c_1(\mathbb{E}_G)$ on $\overline{\mathcal{H}}$. To that end we need some preparation:

The universal stable curve over $\overline{\mathcal{M}}_{0,24}$ is denoted by $\pi_{25}\colon \overline{\mathcal{M}}_{0,25} \to \overline{\mathcal{M}}_{0,24}$ and forgets the marked point labeled by 25. We recall the following standard formulas, see for instance [FG].

$$c_1(\omega_{\pi_{25}}) = \psi_{25} - \sum_{i=1}^{24} \delta_{0:i,25} \in CH^1(\overline{\mathcal{M}}_{0,25}). \tag{3B}$$

$$\sum_{i=1}^{24} \psi_i = \sum_{i=2}^{12} \frac{i(24-i)}{23}[B_i] \in CH^1(\overline{\mathcal{M}}_{0,24}); \quad \kappa_1 = \sum_{i=2}^{12} \frac{(i-1)(23-i)}{23}[B_i] \tag{3C}$$

Here ψ_i are the cotangent tautological classes corresponding to the marked points, whereas κ_1 is the usual κ-class.

Theorem 3.3. *Let G be a subgroup of $W(E_6)$ as before. Assume the ramification profile of the degree d cover $C_G \to \mathbb{P}^1$ corresponding to a general element $[\widetilde{C} \to \mathbb{P}^1, p_1, \ldots, p_{24}] \in \overline{\mathcal{H}}$ over each of the 24 branch points p_i is of the type 2^a1^b, where $2a+b=d$. Then the Hodge class λ_G on $\overline{\mathcal{H}}$ is given by*

$$\lambda_G = \sum_{i=2}^{12} \sum_{\mu \in \mathcal{P}_i} \frac{1}{12}\mathrm{lcm}(\mu)\Big(\frac{3a}{2}\frac{i(24-i)}{23} - d + \frac{1}{\mu}\Big)\,[E_{i:\mu}] \in CH^1(\overline{\mathcal{H}}).$$

Proof. The proof follows the lines of that of [ADFIO, Theorem 6.17], with appropriate changes we indicate below. Over the Hurwitz space $\overline{\mathcal{H}}$ we consider the universal $W(E_6)$-admissible cover $f\colon \mathcal{C}_G \to \mathcal{P}$ of degree d, where

$$\mathcal{P} := \overline{\mathcal{H}} \times_{\overline{\mathcal{M}}_{0,24}} \overline{\mathcal{M}}_{0,25}$$

is the universal degree d *orbicurve* of genus zero over $\overline{\mathcal{H}}$. We fix a general point

$$t = [\pi_G\colon C_G \to R, p_1, \ldots, p_{24}]$$

of a boundary divisor $E_{i:\mu}$, where $\mu = (\mu_1, \ldots, \mu_\ell) \in \mathcal{P}_i$. In particular, R is the union of two smooth rational curves R_1 and R_2 meeting at a point q. The local ring of the space of Harris-Mumford admissible covers has the the following local description at t:

$$\mathbb{C}[[t_1, \ldots, t_{21}, s_1, \ldots, s_\ell]]/s_1^{\mu_1} = \cdots = s_\ell^{\mu_\ell} = t_1, \tag{3D}$$

where t_1 is the local parameter on $\overline{\mathcal{M}}_{0,24}$ corresponding to smoothing the node $q \in R$. The space $\mathcal{P}$ has a singularity of type $A_{\mathrm{lcm}(\mu)-1}$, and accordingly $\mathcal{C}_G$ has singularities of type $A_{\mathrm{lcm}(\mu)/\mu_i-1}$ at the ℓ points corresponding to the inverse image of R_{sing}. Indeed, to determine the local ring of $\overline{\mathcal{H}}$ at the point t, one normalizes the ring (3D). To that end, we introduce a further parameter τ and choose primitive μ_j-th roots of unity ζ_j for $j = 1, \ldots, \ell$. These choices correspond to specifying the stack structure of the cover $C_G \to R$ at the points of C_G lying over the point $q \in R_{\mathrm{sing}}$. Thus

$$\widehat{\mathcal{O}}_{[t,\zeta_1,\ldots,\zeta_\ell],\ \overline{\mathcal{H}}} = \mathbb{C}[[t_1, \ldots, t_{21}, \tau]]$$

and $s_j = \zeta_j \tau^{\frac{\mathrm{lcm}(\mu)}{\mu_j}}$, for $j = 1, \ldots, \ell$. Accordingly, the map $\mathfrak{b}\colon \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{0,24}$ is branched with order $\mathrm{lcm}(\mu)$ at each such point $[t, \zeta_1, \ldots, \zeta_\ell]$. When the stack data $(\zeta_1, \ldots, \zeta_\ell)$ is clear from the context, we drop it and we write as before $t = [t, \zeta_1, \ldots, \zeta_\ell] \in \overline{\mathcal{H}}$ when referring to a point of $\overline{\mathcal{H}}$.

Let $\phi\colon \mathcal{P} \to \overline{\mathcal{H}}$ and $\overline{q}\colon \mathcal{P} \to \overline{\mathcal{M}}_{0,25}$ be the two projections and put $v := \phi \circ f\colon \mathcal{C}_G \to \overline{\mathcal{H}}$ respectively $\overline{f} := \overline{q} \circ f\colon \mathcal{C}_G \to \overline{\mathcal{M}}_{0,25}$. Note that v respectively $\overline{f}$ are viewed as the universal curve of genus g_G over $\overline{\mathcal{H}}$ and $\overline{\mathcal{M}}_{0,25}$ respectively. The ramification divisor of f decomposes as

$$\mathrm{Ram}(f) = R_1 + \cdots + R_{24} \subseteq \mathcal{C}_G,$$

where a general point of R_i is of the form $[\pi\colon C_G \to R,\ p_1, \ldots, p_{24}, x]$, with R being a nodal rational curve and $x \in C$ being one of the a ramification points lying over the branch point p_i. Since over each branch point lie a ramification points, we have $f_*([R_i]) = a[\mathfrak{B}_i]$, where $\mathfrak{B}_i \subseteq \mathcal{P}$ is the corresponding branch divisor.

We apply the Riemann-Hurwitz formula to the finite map $f\colon \mathcal{C}_G \to \mathcal{P}$. Accordingly, we can write $c_1(\omega_v) = f^*\overline{q}^*c_1(\omega_{\pi_{25}}) + [\mathrm{Ram}(f)]$, where we recall that $\pi_{25}\colon \overline{\mathcal{M}}_{0,25} \to \overline{\mathcal{M}}_{0,24}$ is the morphism forgetting the last marked point. We square this identity and then push it forward via v to obtain a relation in $CH^1(\overline{\mathcal{H}})$. We have that

$$v_*c_1^2(\omega_v) = v_*\Big(\overline{f}^*c_1^2(\omega_{\pi_{25}}) + 2\overline{f}^*c_1(\omega_{\pi_{25}})\cdot[\mathrm{Ram}(f)] + [\mathrm{Ram}(f)]^2\Big).$$

We evaluate each term, starting with the second one. We write $v_*\Big(\overline{f}^*c_1(\omega_{\pi_{25}})\cdot[\mathrm{Ram}(f)]\Big) =$

$$\sum_{i=1}^{24}\phi_*\Big(\overline{q}^*c_1\left(\omega_{\pi_{25}}\right)\cdot a[\mathfrak{B}_i]\Big) = a\sum_{i=1}^{24}\phi_*\overline{q}^*\Big(c_1(\omega_{\pi_{25}})\cdot[\Delta_{0:i,25}]\Big) = a\,\mathfrak{b}^*\Big(\sum_{i=1}^{24}\psi_i\Big).$$

Furthermore, we write $f^*(\mathfrak{B}_i) = 2R_i + A_i$, where the residual divisor A_i defined by the previous equality maps $b:1$ onto $\mathfrak{B}_i$. Note that A_i and R_i are disjoint, hence $f^*([\mathfrak{B}_{\mathrm{i}}])\cdot R_i = 2R_i^2$. Therefore

$$v_*([R_i]^2) = \frac{a}{2}\phi_*([\mathfrak{B}_i^2]) = \frac{a}{2}\phi_*(\overline{q}^*(\delta^2_{0:i,25})) = -\frac{a}{2}\mathfrak{b}^*(\psi_i).$$

Using Equation (3C), we compute that

$$v_*\big([\mathrm{Ram}(f)]^2\big) = v_*\Big(\sum_{i=1}^{24}[R_i]^2\Big) = -\frac{a}{2}\mathfrak{b}^*\Big(\sum_{i=1}^{24}\psi_i\Big) = -\frac{a}{2}\sum_{i=2}^{12}\frac{i(24-i)}{23}\mathfrak{b}^*([B_i]).$$

We use Equation (3B), and the relation $\pi_*(\delta^2_{0:i,25}) = -\psi_i$ for $i = 1, \ldots, 24$, to write:

$$v_*\overline{f}^*c_1^2(\omega_{\pi_{25}}) = \phi_*\Big(d\,\overline{q}^*c_1^2(\omega_{\pi_{25}})\Big) = d\,\mathfrak{b}^*\pi_*\Big(\psi_{25} - \sum_{i=1}^{24}\delta_{0:i,25}\Big)^2 =$$

$$= d\,\mathfrak{b}^*\Big(\kappa_1 - \sum_{i=1}^{24}\psi_i\Big) = -d\,\mathfrak{b}^*\Big(\sum_{i=2}^{12}[B_i]\Big),$$

where the last equation is again a consequence of (3C).

We find the following expression for the pull-back of the Mumford κ class to $\overline{\mathcal{H}}$:

$$v_*c_1^2(\omega_v) \equiv \sum_{i=2}^{12}\Big(\frac{3a}{2}\frac{i(24-i)}{23} - d\Big)\mathfrak{b}^*(B_i) \equiv \sum_{i=2}^{12}\sum_{\mu\in\mathcal{P}_i}\mathrm{lcm}(\mu)\Big(\frac{3a}{2}\frac{i(24-i)}{23} - d\Big)E_{i:\mu}. \tag{3E}$$

Via a Grothendieck-Riemann-Roch calculation in the case of the universal genus g_G curve $v\colon \mathcal{C}_G \to \overline{\mathcal{H}}$, coupled with the local analysis of the fibers of the branch map $\mathfrak{b}$, we find

$$12\lambda_G = v_*c_1^2(\omega_v) + \sum_{i=2}^{12}\sum_{\mu\in\mathcal{P}_i}\mathrm{lcm}(\mu)\cdot\frac{1}{\mu}\,[E_{i:\mu}].$$

Substituting in (3E), we finish the proof. □

We now make Theorem 3.3 more precise involving the monodromy vectors defined in (2B).

Corollary 3.4. *Let G be a subgroup of $W(E_6)$ of index d and let $W(E_6) \hookrightarrow S_d$ be the monodromy action for a generic cover $[\pi\colon C_G \to \mathbb{P}^1, p_1, \dots, p_{24}]$ in this family. Suppose that the cycle types of the elements $\alpha \in W(E_6)$ in the conjugacy classes 2c, 2b, 3b are $2^{a_{2c}}1^{b_{2c}}$, $2^{a_{2b}}1^{b_{2b}}$ and $3^{a_{3b}}1^{b_{3b}}$ respectively. Then the Hodge class λ_G on $\overline{\mathrm{Hur}}$ is:*

$$\lambda_G = \frac{11a_{2c}}{92}[D_0] + \frac{1}{6}\Big(\frac{66a_{2c}}{23} - \frac{3a_{2b}}{2}\Big)[D_{\mathrm{syz}}] + \frac{1}{8}\Big(\frac{66a_{2c}}{23} - \frac{8a_{3b}}{3}\Big)[D_{\mathrm{azy}}] \mod (\star). \tag{3F}$$

Proof. For the divisors $E_0, E_{\mathrm{syz}}, E_{\mathrm{azy}}$ one has $i = 2$. The classes 2c, 2b, 3b are the conjugacy classes respectively of a reflection w, a product of two commuting reflections $w_1 \cdot w_2$ and two non commuting reflections $w_1 \cdot w_2$. Over D_0, respectively D_{syz}, D_{azy}, we compute $d - \frac{1}{\mu}$ to be respectively 0, $\frac{3a_{2b}}{2}$, $\frac{8a_{3b}}{3}$, and $\mathrm{lcm}(\mu)$ to be 1, 2, 3. Finally, we use the relation between E's and D's from Equation 2F. ☐

Example 3.5. For the maximal subgroup $G_{27} \subseteq W(E_6)$, using $(a_{2c}, a_{2b}, a_{3c}) = (6, 10, 6)$ we recover the formula for $\lambda_{G_{27}} = \lambda$ given in Theorem [ADFIO, Theorem 6.17].

3.4. **Prym-Tyurin varieties via Galois covers.** We now discuss a different representation-theoretic interpretation of the Prym-Tyurin variety $PT(C, D)$ associated to a $W(E_6)$-cover $\pi\colon C \to \mathbb{P}^1$. Recall that in 2.2 we fixed the maximal index 27 subgroup $G_{27} = \mathrm{Stab}_{W(E_6)}(a_6)$ of $W(E_6)$. For a $W(E_6)$-Galois cover $[\widetilde{\pi}\colon \widetilde{C} \to R, p_1 + \dots + p_{24}]$, we denote by $\pi\colon C = \widetilde{C}/G_{27} \to R$ the associated degree 27 cover with monodromy group $W(E_6)$. Let $(E_6)_{\mathbb{C}} := E_6 \otimes \mathbb{C}$. Notice that $(E_6)_{\mathbb{C}}$ is also generated by the elements of the orbit of a_6 (all weights of E_6). Following [D2, 5.1], we define the *Prym variety* associated to the lattice E_6 as the abelian variety parametrizing equivariant maps to $J\widetilde{C}$, that is,

$$\mathrm{Prym}_{E_6}(J\widetilde{C}) := \mathrm{Hom}_{W(E_6)}\big((E_6)_{\mathbb{C}}, J\widetilde{C}\big).$$

The evaluation at the element a_6 induces an injective morphism of abelian varieties ([LP, Lemma 5.4.] and [LP, Proposition 5.2.])

$$\mathrm{eval}_{a_6}\colon \mathrm{Hom}_{W(E_6)}\big(E_6, J\widetilde{C}\big) \hookrightarrow J\widetilde{C}, \quad [v\colon E_6 \to J\widetilde{C}] \mapsto v(a_6).$$

In this way $\mathrm{Prym}_{E_6}(J\widetilde{C})$ is endowed with a polarization. The image of the map eval_{a_6} above lands inside $JC = \big(J\widetilde{C}\big)^{G_{27}}$. We now summarize results from [D2, Section 12], see also [LP, Section 5]:

Theorem 3.6. *The evaluation induces an isomorphism of 6-dimensional ppav $\mathrm{Prym}_{E_6}(J\widetilde{C}) \cong PT(C, D)$.*

Since the proof given in [D2, Section 12] is representation-theoretical it works without modification in families. Passing to tangent spaces at the origin, Theorem 3.6 implies that one has a natural isomorphism of vector spaces

$$\mathrm{Hom}_{W(E_6)}\big(E_6, H^0(\widetilde{C}, \omega_{\widetilde{C}})\big) \cong H^0(C, \omega_C)^{(-5)}. \tag{3G}$$

3.5. **Computing the 25 fundamental Hodge classes.** We denote by $\rho_1, \dots, \rho_{25}$ the irreducible representations of $W(E_6)$. We also fix a subgroup $G \subseteq W(E_6)$ of index d. For each $W(E_6)$-Galois cover $[\tilde{\pi}\colon \widetilde{C} \to \mathbb{P}^1, p_1, \dots, p_{24}]$, the space of differentials $H^0(\widetilde{C}, \omega_{\widetilde{C}})$ is a $W(E_6)$-module and accordingly we have the following decompositions into sums of irreducible representations:

(3H)

$$H^0(\widetilde{C}, \omega_{\widetilde{C}}) = \bigoplus_{i=1}^{25} \rho_i \otimes \mathrm{Hom}_{W(E_6)}\Big(\rho_i, H^0(\widetilde{C}, \omega_{\widetilde{C}})\Big), \quad H^0(C_G, \omega_{C_G}) = \bigoplus_{i=1}^{25} \rho_i^G \otimes \mathrm{Hom}_{W(E_6)}\Big(\rho_i, H^0(\widetilde{C}, \omega_{\widetilde{C}})\Big).$$

Notation 3.7. We denote by $\widetilde{\mathbb{E}}$ the $W(E_6)$-Hodge bundle on $\overline{\mathcal{H}}$, that is, having fibre $H^0(\widetilde{C}, \omega_{\widetilde{C}})$ over a point $[\tilde{\pi}\colon \widetilde{C} \to R] \in \overline{\mathrm{Hur}}$.

We now define Hodge bundles corresponding to each irreducible representation of $W(E_6)$.

Definition 3.8. For each $i=1,\dots,25$, let $\mathbb{E}_i := \operatorname{Hom}_{W(E_6)}\big(\rho_i, \widetilde{\mathbb{E}}\big)$ regarded as a vector bundle on $\overline{\mathrm{Hur}}$. We let $\lambda_i := c_1(\mathbb{E}_i) \in CH^1(\overline{\mathrm{Hur}})$.

We have therefore the following identity in the K-group of $\overline{\mathrm{Hur}}$:

$$\widetilde{\mathbb{E}} = \bigoplus_{i=1}^{25} \rho_i \otimes \mathbb{E}_i. \tag{3I}$$

The dimensions of the invariant subspaces ρ_i^G as usual are given by the formula

$$\dim(\rho_i^G) = \frac{1}{|G|} \sum_{g\in G} \operatorname{Tr}_{\rho_i}(g). \tag{3J}$$

Here, for $g\in W(E_6)$ in the conjugacy class α, we have $\operatorname{Tr}_{\rho_i}(g) = \operatorname{Tr}_{\chi_i}(\alpha)$ in the character table of $W(E_6)$, see Table 2.

We now come to the first main result of this paper, the explicit computation of all the classes λ_i. This implies Theorem 1.1.

Theorem 3.9. *The ranks* $\operatorname{rk}(\mathbb{E}_i)$ *and the 25 fundamental Hodge classes* $\lambda_i = c_1(\mathbb{E}_i)$ *on* $\overline{\mathrm{Hur}}$ *in terms of the generators* $D_0, D_{\mathrm{syz}}, D_{\mathrm{azy}} \mod (\star)$ *are given as in Table 1.*

Proof. We apply the above formulas to the 25 cyclic groups $G = W_\alpha = \langle w_\alpha\rangle$ generated by 25 fixed representatives w_α of the conjugacy classes of $W(E_6)$. Precisely, we have

$$\lambda_G = \sum_{i=1}^{25} \dim(\rho_i^G)\lambda_i.$$

From (3J) we compute the 25×25 matrix of multiplicities $M = \dim(\rho_i^{W_\alpha})_{1\le i,\alpha\le 25}$ and find its determinant to be $400771988324352 \neq 0$, so it is invertible.

We compute the vector of genera of the curves $B_\alpha = \widetilde{C}/W_\alpha$ by (3A). Multiplying this vector by M^{-1} we find the ranks of $\mathbb{E}_i$. Next, for each of the curves B_α, we find the 6-tuple $(a_{2c}, b_{2c}; a_{2b}, b_{2b}; a_{3c}, b_{3c})$ by applying (2A) to the elements u lying in the conjugacy classes 2c, 2b, 3b. Then, using Corollary 3.4, we find the corresponding lambda class λ_{W_α} on $\overline{\mathrm{Hur}}$. Finally, we multiply the 3×25 matrix of these lambda classes by M^{-1} to get the expressions for λ_i in terms of $D_0, D_{\mathrm{syz}}, D_{\mathrm{azy}} \mod (\star)$. ☐

Remark 3.10. Since $\lambda = \lambda^{(-5)} + \lambda^{(+1)}$, Equation 1B and Theorem 1.1 are equivalent. There are similar identities to 1B for the universal covers of degree 36 and 45 from 3.1(1,2).

Remark 3.11. From Corollary 3.4 we see that the Hodge class λ_G is a linear function of the vector $\vec{a} = (a_{2c}, a_{2b}, a_{3b})$ given by an invertible matrix. It follows that $\vec{a}$ is a linear function of the vector λ_G. Associating to a cover $C_G = \widetilde{C}/G$ the element $\sum_i (\dim \rho_i^G)\chi_i$ in the character space of $W(E_6)$, we see that

$$a_\alpha(C_G) = \sum_{i=1}^{25} (\dim \rho_i^G) a_\alpha(\chi_i) \quad \text{for } \alpha = 2c, 2b, 3b.$$

Then $a_\alpha(\chi)$ can be computed using the same linear algebra, from Equations 3J and 2A. We list them in the last three columns of Table 1.

χ	name	$\operatorname{rk}\mathbb{E}_i$	D_0	D_{syz}	D_{azy}	a_{2c}	a_{2b}	a_{3b}
1	1	0	0	0	0	0	0	0
2	$\overline{1}$	11	$11/92$	$11/23$	$33/92$	1	0	0
3	10	50	$55/92$	$32/23$	$127/276$	5	4	4
4	6	6	$11/92$	$-1/46$	$7/276$	1	2	1
5	$\overline{6}$	54	$55/92$	$87/46$	$403/276$	5	2	1
6	20a	100	$55/46$	$41/23$	$73/46$	10	12	6
7	15a	45	$55/92$	$9/23$	$127/276$	5	8	4
8	$\overline{15a}$	105	$55/46$	$64/23$	$311/138$	10	8	4
9	15b	45	$55/92$	$41/46$	$35/276$	5	6	5
10	$\overline{15b}$	105	$55/46$	$151/46$	$265/138$	10	6	5
11	20b	40	$55/92$	$9/23$	$35/276$	5	8	5
12	$\overline{20b}$	160	$165/92$	$119/23$	$1025/276$	15	8	5
13	24	96	$55/46$	$41/23$	$127/138$	10	12	8
14	$\overline{24}$	144	$77/46$	$85/23$	$325/138$	14	12	8
15	30	90	$55/46$	$59/46$	$27/46$	10	14	9
16	$\overline{30}$	210	$55/23$	$279/46$	$96/23$	20	14	9
17	60a	300	$165/46$	$169/23$	$473/138$	30	28	22
18	80	400	$110/23$	$210/23$	$346/69$	40	40	28
19	90	450	$495/92$	$219/23$	$565/92$	45	48	30
20	60b	240	$275/92$	$114/23$	$181/92$	25	28	21
21	$\overline{60b}$	360	$385/92$	$224/23$	$511/92$	35	28	21
22	64	224	$66/23$	$80/23$	$134/69$	24	32	20
23	$\overline{64}$	416	$110/23$	$256/23$	$530/69$	40	32	20
24	81	351	$99/23$	$309/46$	$90/23$	36	42	27
25	$\overline{81}$	459	$495/92$	$507/46$	$657/92$	45	42	27

TABLE 1. χ_i, $\operatorname{rk}\mathbb{E}_i$, λ_i, and $(a_{2c}, a_{2b}, a_{3b})(\chi_i)$

The following is also easy to see, cf. (3A). For any character χ one has

$$g(\chi) := \operatorname{rank}\mathbb{E}(\chi) = 12a_{2c}(\chi) - \chi(1a) + \mathrm{mult}_1(\chi), \tag{3K}$$

where $\chi(1a) = \dim V_\chi$ is the dimension of the representation, and $\mathrm{mult}_1(\chi)$ is the multiplicity of the trivial representation 1 in χ. For example $g(C_{27}) = 12 \cdot 6 - 27 + 1 = 46$, and $\operatorname{rank}(\mathbb{E}_6) = 12 \cdot 1 - 6 = 6$.

Remark 3.12. From Table 1 one can observe that for any character χ one has

$$\lambda(\chi \otimes \overline{1}) = \lambda(\chi) + \chi(2c)\lambda(\overline{1}), \qquad \vec{a}(\chi \otimes \overline{1}) = \vec{a}(\chi) + \chi(2c)(1,0,0).$$

4. THE WEYL-PETRI DIVISOR AND THE RAMIFICATION OF THE PRYM-TYURIN MAP

In [ADFIO, Section 10], we showed that, if a smooth $W(E_6)$-cover $[\pi\colon C \to R, p_1 + \cdots + p_{24}] \in \mathrm{Hur}$ lies in the ramification locus of PT, the line bundle L associated to π satisfies $h^0(C, L) = 2$ and the Petri map

$$H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \longrightarrow H^0(C, \omega_C)^{(+1)}$$

is an isomorphism, then the Prym-Tyurin canonical image of C is contained in a quadric. In this section we refine the above result by showing that the ramification divisor of PT is contained in the union of two divisors $\mathfrak{M}$ and $\mathfrak{N}$ which we shall describe. In this section, we work on an alternative compactification $\widetilde{\mathcal{G}}_{E_6}$ of Hur which we first discuss in some detail.

4.1. **The parameter space $\mathcal{G}_{E_6}$.** In [ADFIO, 9.4] we introduced the stack $\mathcal{G}_{E_6}$ classifying $SL(2)$-equivalence classes of finite maps $[\pi\colon C \to \mathbb{P}^1]$ with $W(E_6)$ monodromy, where C is an irreducible curve of genus 46. To construct $\mathcal{G}_{E_6}$, we let $\mathcal{X}_{E_6}$ denote the substack of the moduli stack $\overline{\mathcal{M}}_{46}(\mathbb{P}^1, 27)$ parametrizing finite stable maps $\pi\colon C \to \mathbb{P}^1$, from an irreducible nodal curve C of genus 46 and having monodromy group M_π contained in $W(E_6)$. Then we set

$$\mathcal{G}_{E_6} := \big[\mathcal{X}_{E_6}/SL(2)\big],$$

where $SL(2)$ acts on the base by linear transformations.

Let $f_{E_6}\colon \mathcal{C}_{E_6} \to \mathcal{G}_{E_6}$ be the universal curve of genus 46. One has a birational map $\beta\colon \widetilde{\mathrm{Hur}} \dashrightarrow \mathcal{G}_{E_6}$. We recall the effect of this map on the boundary divisors D_0, D_{syz} and D_{azy} of $\widetilde{\mathrm{Hur}}$. We fix a point

$$t = [\pi\colon C = C_1 \cup C_2 \to R = R_1 \cup_q R_2,\ p_1 + \cdots + p_{24}] \in \widetilde{\mathrm{Hur}},$$

where we assume that R_1 and R_2 are smooth rational curves meeting at q and that $p_1, \ldots, p_{22} \in R_1 \setminus \{q\}$ whereas $p_{23}, p_{24} \in R_2 \setminus \{q\}$.

If t represents a general point of D_0, then C_1 is a smooth curve of genus 40. The curve C_2 consists of 21 components, of which 6 map with degree 2 onto R_2 and meet C_1 in two points, whereas the remaining 15 map isomorphically onto R_2 and meet C_1 in one point. Then $\beta(t) = [\overline{\pi} : \overline{C} \to R_1] \in \mathcal{G}_{E_6}$, where $\overline{C}$ is the 6-nodal curve obtained from C_1 by pairwise identifying the six pairs of points lying on the components of C_2 mapping 2-to-1 onto R_2, and $\overline{\pi}$ is induced by π. If $\nu\colon C_1 \to \overline{C}$ is the normalization map, then $\overline{L} := \overline{\pi}^*\mathcal{O}_{R_1}(1) \in W^1_{27}(\overline{C})$ is uniquely characterized by the property $\nu^*(\overline{L}) = \pi^*_{|R_1}(\mathcal{O}_{C_1}(1)) \in W^1_{27}(C_1)$.

If t represents a general point of D_{azy}, then C_1 is smooth of genus 46 and $\pi_{|C_1}\colon C_1 \to R_1$ is a map of degree 27 with 6 ramification points of index 3 over the point $q \in R_1$. Then

$$\beta(t) = [\pi_{|C_1}\colon C_1 \to R_1] \in \mathcal{G}_{E_6}$$

and $L_1 := \pi^*_{|C_1}(\mathcal{O}_{R_1}(1)) \in W^1_{27}(C_1)$.

The case when t corresponds to a general point of D_{syz} requires care. Then C_1 is a smooth curve of genus 45. The permutations in S_{27} corresponding to the roots w_{23} and w_{24} describing the local monodromy around p_{23} and p_{24} share four elements. For instance, using the standard notation for the lines on a cubic surface, we may assume $w_{23} = \alpha_{\max} = 2h - a_1 - \cdots - a_6$ and $w_{24} = \alpha_{12} = a_1 - a_2$:

$$\alpha_{\max} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix} \quad \text{and} \quad \alpha_{12} = \begin{pmatrix} a_1 & b_1 & c_{13} & c_{14} & c_{15} & c_{16} \\ a_2 & b_2 & c_{23} & c_{24} & c_{25} & c_{26} \end{pmatrix}.$$

The curve C_1 meets a smooth rational component of E of C_2 at two points p_1 and p_2 corresponding to the sheets labelled by the transpositions (a_1, b_2) and (b_1, a_2) corresponding to multiplying $\alpha_{\max}$ and α_{12}. The map $\pi_{|E}\colon E \to R_2$ is of degree 4 and $\pi^*_{|E}(q) = 2p_1 + 2p_2$. We have $\beta(t) = [\overline{\pi} : \overline{C} \to R_1]$, where $\overline{C}$ is obtained from C_1 by identifying the points p_1 and p_2 and $\overline{\pi}$ is induced by π. Therefore $\overline{C}$ is an irreducible 1-nodal curve of genus 46. The line bundle $\overline{L} := \overline{\pi}^*\mathcal{O}_{R_1}(1) \in W^1_{27}(\overline{C})$ is characterized by the fact that if $\nu\colon C_1 \to \overline{C}$ is the normalization map, then $\nu^*(\overline{L}) = L_1 := \pi^*_{|C_1}(\mathcal{O}_{R_1}(1))$. Moreover, if $\overline{C}_{\mathrm{sing}} = \{z\}$, that is, $\nu^{-1}(z) = \{p_1, p_2\}$, then

$$h^0\big(C_1, L_1(-2p_1 - 2p_2)\big) \geq 1.$$

Because the points p_1 and p_2 are *ramification* points of L_1, it follows that the local equations of $\mathcal{G}_{E_6}$ around $t \in D_{\mathrm{syz}}$ are

$$(u, v, t_1, t_2, \ldots, t_{21}),\ u^2 = v^2 = t_1,$$

see [Va, Corollary 4.16] for a similar discussion. The parameters $t_1, \ldots, t_{21}$ correspond to deforming the branch points of π and the divisor $D_{\text{syz}} \subseteq \mathcal{G}_{E_6}$ is locally given by $(t_1 = 0)$. Therefore $\mathcal{G}_{E_6}$ is not normal along D_{syz}.

Notation 4.1. We denote by $\widetilde{\mathcal{G}}_{E_6} \to \mathcal{G}_{E_6}$ the normalization map. Let

$$\tilde{f}\colon \widetilde{\mathcal{C}}_{E_6} \to \widetilde{\mathcal{G}}_{E_6}$$

be the universal curve over $\widetilde{\mathcal{G}}_{E_6}$.

Finally, we denote by $\widetilde{\beta}\colon \widetilde{\text{Hur}} \dashrightarrow \widetilde{\mathcal{G}}_{E_6}$ the map induced from β by the universal property of the normalization $\widetilde{\mathcal{G}}_{E_6} \to \mathcal{G}_{E_6}$. We still denote by D_0, D_{syz} and D_{azy} the reduced boundary divisors on $\widetilde{\mathcal{G}}_{E_6}$ corresponding to the same symbols under the map $\widetilde{\beta}$, that is, $\widetilde{\beta}^*(D_0) = D_0$, $\widetilde{\beta}^*(D_{\text{syz}}) = D_{\text{syz}}$ and $\widetilde{\beta}^*(D_{\text{azy}}) = D_{\text{azy}}$.

Along the divisor D_{syz}, the space $\widetilde{\mathcal{G}}_{E_6}$ consists of *two* sheets having local coordinates $(s, t_2, \ldots, t_{21})$, such that the map $\widetilde{\mathcal{G}}_{E_6} \to \mathcal{G}_{E_6}$ is given locally by

$$(u = s, v = s, t_1 = s^2) \ \text{ and } \ (u = -s, v = s, t_1 = s^2)$$

respectively. Accordingly, the fibre product $\mathcal{C}'_{E_6} := \mathcal{C}_{E_6} \times_{\mathcal{G}_{E_6}} \widetilde{\mathcal{G}}_{E_6}$ has A_1-singularities along the codimension 2 locus corresponding to nodes $\big([C \to R], z \in C_{\text{sing}}\big)$ over points in D_{syz}. Indeed, if $xy = t_1$ is the local equation of $\mathcal{C}_{E_6}$ in coordinates $(x, y, t_1, \ldots, t_{21})$, then the local equation of $\mathcal{C}'_{E_6}$ is $xy = s^2$. Observe that $\widetilde{\mathcal{C}}_{E_6}$ is obtained from $\mathcal{C}'_{E_6}$ by blowing-up the locus of nodes. It follows that over a point $[\overline{C} \to R] \in D_{\text{syz}}$, we have

$$\tilde{f}^{-1}\big([\overline{C} \to R]\big) = C_1 \cup_{\{p_1, p_2\}} E,$$

where E is a smooth rational curve meeting the smooth curve C_1 at p_1 and p_2.

Notation 4.2. We denote by $\mathcal{L}$ a universal line bundle over $\widetilde{\mathcal{C}}_{E_6}$. For a point $[\overline{C} = C_1 \cup E, \overline{L}] \in D_{\text{syz}}$ as above, we have $\mathcal{L}_{|C_1} = \nu^*(\overline{L}) \in W^1_{27}(C_1)$ and $\mathcal{L}_{|E} = \mathcal{O}_E$.

Theorem 4.3. *At the level of $\widetilde{\mathcal{G}}_{E_6}$ one has the following formula:*

$$\lambda = \frac{33}{46}[D_0] + \frac{7}{46}[D_{\text{azy}}] + \frac{17}{46}[D_{\text{syz}}] \in CH^1(\widetilde{\mathcal{G}}_{E_6}).$$

Proof. We study the map $\varphi := \widetilde{\beta} \circ q \colon \overline{\mathcal{H}} \dashrightarrow \widetilde{\mathcal{G}}_{E_6}$. At the level of $\overline{\mathcal{H}}$ we have the formula [ADFIO, Theorem 6.17]:

$$\lambda = \frac{7}{23}[E_{\text{azy}}] + \frac{17}{46}[E_{\text{syz}}] + \frac{33}{28}[E_0] + \cdots \in CH^1(\overline{\mathcal{H}}).$$

We claim that $\varphi^*([D_0]) = 2[E_0]$, $\varphi^*([D_{\text{azy}}]) = 2[E_{\text{azy}}]$ and $\varphi^*([D_{\text{syz}}]) = [E_{\text{syz}}]$ which explains the result.

We start with a family of $W(E_6)$-pencils $\big(f_t\colon C_t \to \mathbb{P}^1\big)_{t \in T}$ and assume that over a special point $t_0 \in T$, two branch points coalesce. Depending on the situation, the curve C_0 is smooth (in the azygetic case), or nodal (in the syzygetic, or the D_0-case). In order to separate the branch points one makes a base change of order 2 which justifies the multiplicity in front of both E_0 and E_{azy}. This base change is not needed in the case E_{syz} for, when we passed to the normalization, the two branches were separated. □

Remark 4.4. Observe that a formula identical to Theorem 4.3 has been established in [ADFIO, Remark 5.21] at the level of $\widetilde{\text{Hur}}$. The stacks $\widetilde{\text{Hur}}$ and $\widetilde{\mathcal{G}}_{E_6}$ are however not isomorphic over the divisors D_0, D_{azy} and D_{syz}. For instance, over a general point in D_{azy} the non-normalized Harris-Mumford space $\mathcal{HM}_{E_6}$ of admissible covers has local equations

$$s_1^3 = \cdots = s_6^3 = t_1,$$

in local coordinates $(s_1, \ldots, s_6, t_1, \ldots, t_{21})$, where D_{azy} is given by $(t_1 = 0)$. Accordingly, the local equation of $\widetilde{\mathrm{Hur}}$ (which locally is the normalization of $\mathcal{HM}_{E_6}$) in coordinates $(a, t_2, \ldots, t_{21})$ is given by $s_1 = \zeta_1 a, \ldots, s_6 = \zeta_6 a,\ t_1 = a^3$, where $\zeta_1, \ldots, \zeta_6$ are primitive cubic roots of unity and a is a local parameter. In particular, over a general point of D_{azy} in $\widetilde{\mathcal{G}}_{E_6}$ there lie $3^5 = \frac{1}{3} \times 3^6$ points in $\widetilde{\mathrm{Hur}}$.

Theorem 4.5. *We have the following formula:*

$$\kappa = 12\lambda - 6[D_0] - 2[D_{\mathrm{syz}}] \in CH^1(\widetilde{\mathcal{G}}_{E_6}).$$

Proof. By definition $\kappa = \tilde{f}_*\big(c_1^2(\omega_{\tilde{f}})\big)$. We apply Grothendieck-Riemann-Roch to the universal curve $\tilde{f} \colon \widetilde{\mathcal{C}}_{E_6} \to \widetilde{\mathcal{G}}_{E_6}$. The usual calculation of Mumford yields

$$\kappa = 12\lambda - \tilde{f}_*[\mathrm{Sing}(\tilde{f})].$$

The general point of D_0 has 6 singularities, thus explaining the factor $6[D_0]$. Similarly, the general point of D_{syz} corresponds to a curve with *two* singularities, namely the points of intersection $E \cap C_1$, keeping the notation above. This explains the factor $2[D_{\mathrm{syz}}]$. $\square$

4.2. **Tautological classes on $\widetilde{\mathcal{G}}_{E_6}$.**

In [ADFIO, 9.6], after having chosen a universal line bundle $\mathcal{L}$ on the universal curve $\widetilde{\mathcal{C}}_{E_6}$, the following tautological classes over $\widetilde{\mathcal{G}}_{E_6}$ were defined:

$$\mathfrak{A} := \tilde{f}_*\big(c_1^2(\mathcal{L})\big), \quad \mathfrak{B} := \tilde{f}_*\big(c_1(\mathcal{L}) \cdot c_1(\omega_{\tilde{f}})\big), \ \gamma := \mathfrak{B} - \frac{5}{3}\mathfrak{A} \in CH^1(\widetilde{\mathcal{G}}_{E_6}).$$

Whereas $\mathfrak{A}$ and $\mathfrak{B}$ depend on the choice of a universal line bundle $\mathcal{L}$ on $\widetilde{\mathcal{C}}_{E_6}$, the class γ is intrinsically defined and does not depend on such a choice. We define the *tautological part* of $CH^1(\widetilde{\mathcal{G}}_{E_6})$ to be the three dimensional subspace with the following three distinguished bases:

- $(D_{\mathrm{azy}}, D_{\mathrm{syz}}, D_0)$. All calculations on $\overline{\mathrm{Hur}}$ are carried out using it.
- (λ, γ, D_0). This basis is best suited for working with the space $\widetilde{\mathcal{G}}_{E_6}$.
- $(\lambda, \lambda^{(-5)}, D_0)$. This is the basis compatible with the Prym-Tyurin map PT.

In what follows we clarify the relation between these bases:

Theorem 4.6. *The following relation holds:*[1]

$$[D_{\mathrm{azy}}] = \gamma + 4\lambda - 3[D_0] - 2[D_{\mathrm{syz}}] \in CH^1(\widetilde{\mathcal{G}}_{E_6}).$$

Proof. We represent D_{azy} as the push-forward of the codimension two locus in the universal curve $\widetilde{\mathcal{C}}_{E_6}$ of the locus of pairs $[C \to R, p]$, where $p \in C$ is such that $h^0(C, L(-3p)) \geq 1$. We form the fibre product of the universal curve $\widetilde{\mathcal{C}}_{E_6}$ together with its projections:

$$\begin{tikzcd} \widetilde{\mathcal{C}}_{E_6} & \widetilde{\mathcal{C}}_{E_6} \times_{\widetilde{\mathcal{G}}_{E_6}} \widetilde{\mathcal{C}}_{E_6} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \widetilde{\mathcal{C}}_{E_6}. \end{tikzcd}$$

For each $k \geq 1$, we consider the *locally free* jet bundle $J_k(\mathcal{L})$ defined, e.g., in [E96], as a locally free replacement (that is, double dual) of the sheaf of principal parts $\mathcal{P}^k_{\tilde{f}}(\mathcal{L}) := (\pi_2)_*\Big(\pi_1^*(\mathcal{L}) \otimes \mathcal{I}_{(k+1)\Delta}\Big)$ on $\widetilde{\mathcal{C}}_{E_6}$. Note that $\mathcal{P}^k_{\tilde{f}}(\mathcal{L})$ is not locally free along the codimension two locus in $\widetilde{\mathcal{C}}_{E_6}$ where $\tilde{f}$ is not smooth.

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[1] Theorem 4.6 corrects Theorem 8.14 from [ADFIO], where the non-normality of $\mathcal{G}_{E_6}$ along D_{syz} was not accounted for.

To remedy this problem, we consider the *wronskian* locally free replacements $J^k_{\tilde{f}}(\mathcal{L})$, which are related by the following commutative diagram for each $k \geq 1$:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^k_{\tilde{f}} \otimes \mathcal{L} \arrow[r] \arrow[d] & \mathcal{P}^k_{\tilde{f}}(\mathcal{L}) \arrow[r] \arrow[d] & \mathcal{P}^{k-1}_{\tilde{f}}(\mathcal{L}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega^{\otimes k}_{\tilde{f}} \otimes \mathcal{L} \arrow[r] & J^k_{\tilde{f}}(\mathcal{L}) \arrow[r] & J^{k-1}_{\tilde{f}}(\mathcal{L}) \arrow[r] & 0.
\end{tikzcd}$$

Here $\Omega^k_{\tilde{f}}$ denotes the $\mathcal{O}_{\widetilde{\mathcal{G}}_{E_6}}$-module $\mathcal{I}_{k\Delta}/\mathcal{I}_{(k+1)\Delta}$. The first vertical row here is induced by the canonical map $\Omega^k_{\tilde{f}} \to \omega^{\otimes k}_{\tilde{f}}$ relating the sheaf of relative Kähler differentials to the relative dualizing sheaf of the family $\tilde{f}$. The sheaves $\mathcal{P}^k_{\tilde{f}}(\mathcal{L})$ and $J^k_{\tilde{f}}(\mathcal{L})$ differ only along the codimension two singular locus of $\tilde{f}$. Setting $\mathcal{V} := \tilde{f}_*\mathcal{L}$, there is, for each integer $k \geq 0$, a vector bundle morphism $\nu_k \colon \tilde{f}^*(\mathcal{V}) \to J^k_{\tilde{f}}(\mathcal{L})$, which for points $[C, L, p] \in \widetilde{\mathcal{G}}_{E_6}$ such that $p \in C_{\mathrm{reg}}$, is just the evaluation morphism $H^0(C, L) \to H^0(L|_{(k+1)p})$. We specialize now to the case $k = 2$ and consider the codimension two locus $Z \subseteq \widetilde{\mathcal{C}}_{E_6}$ where

$$\nu_2 \colon \tilde{f}^*(\mathcal{V}) \to J^2_{\tilde{f}}(\mathcal{L})$$

is not injective. Then, at least over the locus of smooth curves, D_{azy} is the set-theoretic image of Z. Furthermore, a local analysis shows that the morphism ν_2 is simply degenerate for each point $[C, L, p]$, where $p \in C_{\mathrm{sing}}$. Taking into account that a general point of D_{azy} corresponds to a pencil with *six* triple points aligned over one branch point, and that the stable model of a general element of the divisor D_{syz} corresponds to a curve with *one* node, whereas that of a general point of D_0 to a curve with *six* nodes, we obtain the formula:

$$6[D_{\mathrm{azy}}] = \tilde{f}_* c_2\left(\frac{J^2_{\tilde{f}}(\mathcal{L})}{\tilde{f}^*(\mathcal{V})}\right) - 6[D_0] - 8[D_{\mathrm{syz}}] \in CH^1(\widetilde{\mathcal{G}}_{E_6}).$$

The fact that D_{syz} appears with multiplicity 8 is a result of the fact that $\tilde{f}^{-1}([C, L]) = \widetilde{C} \cup_{\{p_1,p_2\}} E$, over a general point $[C, L] \in D_{\mathrm{syz}}$ has *two* singularities, and that, at each of the nodes, there is a local multiplicity equal to 4 as we shall explain.

We choose a family $F \colon \mathcal{X} \to B$ of curves of genus 46 over a smooth 1-dimensional base B, such that $\mathcal{X}$ is smooth, and there is a point $b_0 \in B$ such that $X_b := F^{-1}(b)$ is smooth for $b \in B \setminus \{b_0\}$, whereas X_{b_0} has precisely two nodes p_1 and p_2. Assume $L \in \mathrm{Pic}(\mathcal{X})$ is a line bundle such that $L_b := L_{|X_b}$ is a pencil with $W(E_6)$-monodromy on X_b for each $b \in B$, and furthermore $[X_{b_0}, L_{b_0}] \in D_{\mathrm{syz}}$. We have that $X_{b_0} = C \cup_{\{p_1,p_2\}} E$, where C is a smooth curve of genus 45 and E is a smooth rational curve, meeting C at the nodes p_1 and p_2.

Choose local parameters $t \in \mathcal{O}_{B,b_0}$ and $u, v \in \mathcal{O}_{\mathcal{X},p_1}$, such that $uv = t$ represents the local equation of $\mathcal{X}$ around the point p_1. Here u is the local parameter on C, whereas v is the local parameter on E. Then ω_F is locally generated at the point $p_1 \in \mathcal{X}$ by the meromorphic differential $\tau = \frac{du}{u} = -\frac{dv}{v}$. We choose two sections $s_1, s_2 \in H^0(\mathcal{X}, L)$, where s_1 does not vanish at p_1 or p_2 and s_2 vanishes with order 2 at p_1, p_2 along C, while being identically zero along E. Thus (after a local analytic change of coordinates) we can write a relation $s_{2,p_1} = u^2 s_{1,p_1}$ between the germs of the two sections s_1 and s_2 at p_1. We compute

$$d(s_2) - 2udu = d(s_2) - 2u^2\tau \in (u, v)\tau, \quad \text{and} \quad d^2(s_2) - 4udu = d^2(s_2) - 4u^2\tau \in (u, v)\tau.$$

In local coordinates, the map $H^0\big(X_{b_0}, L_{b_0}\big) \to H^0\big(X_{b_0}, L_{b_0}|_{3p_1}\big)$ is then given by the following matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ u^2 & 2u^2 + (u,v) & 4u^2 + (u,v) \end{pmatrix},$$

where the symbol $f + (u,v)$, indicates an element of $\mathcal{O}_{\mathcal{X},p_1}$ that differs from f by an element in the ideal (u,v). The local equations of the degeneracy locus Z are the two by two minors of the above matrix. This shows that the local multiplicity coming from the node $p_1 \in X_{b_0}$ of $[D_{\mathrm{syz}}]$ in Z is equal to 4, hence $[D_{\mathrm{syz}}]$ appears with multiplicity $8 = 4 + 4$ in the degeneracy locus.[2]

We compute: $c_1\big(J^2_{\tilde{f}}(\mathcal{L})\big) = 3c_1(\mathcal{L}) + 3c_1(\omega_{\tilde{f}})$ and $c_2\big(J^2_{\tilde{f}}(\mathcal{L})\big) = 3c_1^2(\mathcal{L}) + 6c_1(\mathcal{L}) \cdot c_1(\omega_{\tilde{f}}) + 2c_1^2(\omega_{\tilde{f}})$, hence

$$\tilde{f}_* c_2\left(\frac{J^2_{\tilde{f}}(\mathcal{L})}{\tilde{f}^*(\mathcal{V})}\right) = 3\mathfrak{A} + 6\mathfrak{B} - 3(d + 2g - 2)c_1(\mathcal{V}) + 2\kappa = 6\gamma + 2\kappa.$$

As explained in Theorem 4.5, we also have $\kappa = 12\lambda - 6[D_{E_6}] - 2[D_{\mathrm{syz}}]$, which finishes the proof. □

Recall that $\tilde{f} \colon \widetilde{\mathcal{C}}_{E_6} \to \widetilde{\mathcal{G}}_{E_6}$ denotes the universal curve and $\mathcal{L}$ is a universal line bundle of relative degree 27 over $\widetilde{\mathcal{C}}_{E_6}$. The push-forward sheaves $\tilde{f}_*(\mathcal{L})$ and $\tilde{f}_*\big(\omega_{\tilde{f}} \otimes \mathcal{L}^\vee\big)$ are reflexive sheaves, therefore using [Ha], both are locally free outside a subset of codimension at least 3 in $\widetilde{\mathcal{G}}_{E_6}$. By possibly removing this locus, for all divisor class calculations that follow, we may assume that both $\tilde{f}_*(\mathcal{L})$ and $\tilde{f}_*\big(\omega_{\tilde{f}} \otimes \mathcal{L}^\vee\big)$ are locally free. Using [ADFIO, Lemma 11.5], for a general point $[\pi \colon C \to \mathbb{P}^1] \in \widetilde{\mathcal{G}}_{E_6}$, if $L := \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$, we have $h^0(C, L) = 2$ and $h^0(C, \omega_C \otimes L^\vee) = 20$, therefore by Grauert's Theorem

$$\mathrm{rk}\big(\tilde{f}_*(\mathcal{L})\big) = 2 \ \text{ and } \ \mathrm{rk}\big(\tilde{f}_*\big(\omega_{\tilde{f}} \otimes \mathcal{L}^\vee\big)\big) = 20.$$

We fix a point $[\pi \colon C \to \mathbb{P}^1] = [C, L] \in \widetilde{\mathcal{G}}_{E_6}$ and a point $p \in \mathbb{P}^1$ such that $\pi^{-1}(p) \subseteq C_{\mathrm{reg}}$. We consider the usual cohomology exact sequence on C

$$\text{(4A)} \qquad 0 \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow H^0(C, L) \longrightarrow H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p)) \xrightarrow{\alpha_p} H^1(C, \mathcal{O}_C) \longrightarrow H^1(C, L) \longrightarrow 0,$$

where Γ_p is the divisor of $|L| = |\pi^*\mathcal{O}_{\mathbb{P}^1}(1)|$ above p. We identify $H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p))$ with the $\mathbb{C}$-vector space spanned by the 27 lines on a fixed cubic surface S. The incidence correspondence on the set of lines of S induces an endomorphism

$$\gamma_p \colon H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p)) \to H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p))$$

with eigenvalues $10, 1$ and -5, with eigenspaces $H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p))^{(10)}$, $H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p))^{(1)}$ and $H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p))^{(-5)}$ of dimensions 1, 20 and 6 respectively. Note that $H^0(\mathcal{O}_{\Gamma_p})^{(+10)}$ is spanned by the sum of all the 27 lines on S and, as in the proof of [ADFIO, Theorem 9.3], the space $H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p))^{(+10)}$ can be identified with the trivial representation of $W(E_6)$. Furthermore, if $D \colon H^0(C, \omega_C) \to H^0(C, \omega_C)$ is the endomorphism induced by the Kanev correspondence on C, the following diagram is commutative for each $p \in \mathbb{P}^1$:

$$\begin{tikzcd}
H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p)) \arrow[r, "\alpha_p"] \arrow[d, "\gamma_p"'] & H^0(C, \omega_C)^\vee \arrow[d, "D^\vee"'] \\
H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p)) \arrow[r, "\alpha_p"] & H^0(C, \omega_C)^\vee.
\end{tikzcd}$$

---

[2] In [ADFIO, Theorem 9.12] there is a mistake in a similar calculation: the multiplicity there is 4 and not 3.

Therefore, the decomposition into eigenspaces produces the exact sequences

$$0 \longrightarrow H^0(C,\mathcal{O}_C) \longrightarrow H^0(C,L)^{(+10)} \longrightarrow H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p))^{(+10)} \longrightarrow 0,$$

and

$$0 \longrightarrow H^0(C,L)^{(-5)} \longrightarrow H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p))^{(-5)} \xrightarrow{\alpha_p^{(-5)}} H^1(C,\mathcal{O}_C)^{(-5)} \longrightarrow H^1(C,L)^{(-5)} \longrightarrow 0. \tag{4B}$$

It follows from [ADFIO, Section 11] that $h^0(C,L)=2$ (hence $h^1(C,L)=20$) for a general $[C,L]\in\widetilde{\mathcal{G}}_{E_6}$, therefore in this case we also have $H^0(C,L)=H^0(C,L)^{(+10)}$ and $H^0(C,L)^{(-5)}=0$ and $H^1(C,L)=H^1(C,L)^{(+1)}$. It also follows that the space $H^0(C,L)^{(+10)}$ can be canonically identified with the subspace $\pi^*H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1))$ of $H^0(C,L)$ and it always has dimension 2.

4.3. The divisor $\mathfrak{M}$.

The locus of those triples $[C,L,p]\in\widetilde{\mathcal{C}}_{E_6}$ such that the map

$$\alpha_p^{(-5)}\colon H^0(\mathcal{O}_{\Gamma_p}(\Gamma_p))^{(-5)} \longrightarrow \big(H^0(C,\omega_C)^{\vee}\big)^{(-5)}$$

is not an isomorphism can be represented as the pullback $\tilde{f}^*(\mathfrak{M})$ of an effective divisor $\mathfrak{M}$ on $\widetilde{\mathcal{G}}_{E_6}$, for the degeneracy of the map $\alpha_p^{(-5)}$ is independent of the choice of a point $p\in\mathbb{P}^1$.

In what follows we characterize this divisor set-theoretically and observe that, surprisingly, the locus in $\widetilde{\mathcal{G}}_{E_6}$ of pairs $[C,L]$ such that $h^0(C,L)>2$ is of codimension one.

Proposition 4.7. *If $[C,L]\in\mathfrak{M}$, then $h^0(C,L)\geq 3$. Furthermore, if $[C,L]\in\widetilde{\mathcal{G}}_{E_6}\setminus\mathfrak{M}$, then*

$$\operatorname{Im}\big\{H^0(C,L)\otimes H^0(C,\omega_C\otimes L^{\vee})\to H^0(C,\omega_C)\big\}\subseteq H^0(C,\omega_C)^{(+1)}.$$

Proof. Assume $h^0(C,L)=2$, therefore $H^0(C,L)=H^0(C,L)^{(+10)}$. From the sequence (4B), it follows that $\alpha_p^{(-5)}$ is injective, hence by comparing dimensions, it is an isomorphism, that is, $[C,L]\notin\mathfrak{M}$.

In order to establish the second claim, we use the exactness of the second half of the sequence (4B). Since $\operatorname{Im}\big(\alpha_p^{(-5)}\big)=H^0(C,\omega_C)^{(-5)}$, in particular $\operatorname{Im}(\alpha_p)\supseteq\big(H^0(C,\omega_C)^{\vee}\big)^{(-5)}$. By dualising, if $s\in H^0(C,L)$ is the section defining the divisor Γ_p, we obtain that $s\cdot H^0(C,\omega_C\otimes L^{\vee})\subseteq H^0(C,\omega_C)^{(+1)}$, which establishes the claim, by varying the section $s\in H^0(C,L)$. ☐

4.4. The divisor $\mathfrak{N}$.

We define the *Weyl-Petri divisor* $\mathfrak{N}$ to be degeneracy locus of the map of vector bundles of rank 40

$$\mu\colon \tilde{f}_*(\mathcal{L})\otimes\tilde{f}_*(\omega_{\tilde{f}}\otimes\mathcal{L}^{\vee})\to\tilde{f}_*(\omega_{\tilde{f}})^{(+1)}$$

over $\widetilde{\mathcal{G}}_{E_6}$. Observe that away from the divisor $\mathfrak{M}$, the points in $\mathfrak{N}$ are precisely those for which the Petri map $\mu(L)\colon H^0(C,L)\otimes H^0(C,\omega_C\otimes L^{\vee})\to H^0(C,\omega_C)$ is not injective.

Lemma 4.8. *For each point $[\pi\colon C\to\mathbb{P}^1]\in\widetilde{\mathcal{G}}_{E_6}$, one has the identification $\tilde{f}_*(\mathcal{L})[\pi]\cong H^0(C,L)^{(+10)}$.*

Proof. Use that $\tilde{f}_*(\mathcal{L})$ is locally free, coupled with the sequence (4A). ☐

In what follows we shall determine the class of the divisor $\mathfrak{N}$.

Proposition 4.9. *The following formula holds at the level of $\widetilde{\mathcal{G}}_{E_6}$:*

$$[\mathfrak{N}]=\lambda^{(+1)}-2\lambda+\gamma=\lambda^{(-5)}=-\lambda-\lambda^{(-5)}+\gamma.$$

Proof. Using the description of $\mathfrak{N}$ as a degeneracy locus, we compute that

$$[\mathfrak{N}] = \lambda^{(+1)} - c_1\big(\tilde{f}_*(\mathcal{L}) \otimes \tilde{f}_*(\omega_{\tilde{f}} \otimes \mathcal{L}^{\vee})\big) = \lambda^{(+1)} + c_1\big(\tilde{f}_*(\mathcal{L}) \otimes R^1\tilde{f}_*(\mathcal{L})\big).$$

Using [ADFIO, Proposition 9.11], we have $\mathfrak{A} = 27c_1(\tilde{f}_*(\mathcal{L}))$. Applying Grothendieck-Riemann-Roch to the universal curve $\tilde{f}\colon \widetilde{\mathcal{C}}_{E_6} \to \widetilde{\mathcal{G}}_{E_6}$, we write

$$c_1\big(\tilde{f}_*(\mathcal{L})\big) - c_1\big(R^1\tilde{f}_*(\mathcal{L})\big) = \tilde{f}_*\Big[\frac{c_1^2(\mathcal{L})}{2} - \frac{c_1(\mathcal{L}) \cdot c_1(\omega_{\tilde{f}})}{2} + \frac{1}{12}\big(c_1^2(\omega_{\tilde{f}}) - [\mathrm{Sing}(\tilde{f})]\big)\Big] = \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{2} + \lambda,$$

which leads to the claimed formulas. ☐

Combining Theorem 4.6 and Proposition 4.9, we obtain the following relation:

Theorem 4.10. *In the* $(\lambda, D_{\mathrm{syz}}, D_0)$ *basis of* $CH^1(\widetilde{\mathcal{G}}_{E_6})$, *we have:*

$$[\mathfrak{N}] = \frac{59}{42}\lambda - \frac{12}{7}[D_0] - \frac{29}{84}[D_{\mathrm{syz}}],$$

and

$$\gamma = \frac{18}{7}\lambda - \frac{3}{7}[D_{\mathrm{syz}}] - \frac{12}{7}[D_0].$$

Proof. Put together Theorem, 4.6, Proposition 4.9, together with the relation $\lambda^{(-5)} = \frac{1}{6}\lambda - \frac{1}{12}[D_{\mathrm{syz}}]$. ☐

Remark 4.11. In the $(\lambda, \lambda^{(-5)}, [D_0])$-basis of the tautological part of $CH^1(\widetilde{\mathcal{G}}_{E_6})$, the previous formula can be written as

$$[\mathfrak{N}] = \frac{5}{7}\lambda - \frac{12}{7}[D_0] + \frac{29}{7}\lambda^{(-5)}.$$

4.5. **The ramification divisor of PT.** We now show that the ramification divisor of the Prym-Tyurin map $PT\colon \mathrm{Hur} \to \mathcal{A}_6$ is contained in the union of the divisors $\mathfrak{M}$ and $\mathfrak{N}$. This improves on our [ADFIO, Theorem 0.3]. Recall that each $W(E_6)$-cover $[\pi\colon C \to \mathbb{P}^1, p_1 + \cdots + p_{24}] \in \mathrm{Hur}$ induces an *Prym-Tyurin canonical map*

$$\varphi_{(-5)} = \varphi_{|H^0(C,\omega_C)^{(-5)}|}\colon C \to \mathbb{P}^5.$$

Theorem 4.12. *If the Prym-Tyurin canonical image of a smooth curve* $[C, L] \in \mathrm{Hur}$ *is contained in a quadric, then* $[C, L] \in \mathfrak{M}$, *in particular,* $h^0(C, L) \geq 3$.

Proof. Let $Q \subseteq \mathbb{P}^5$ be a quadric containing the Prym-Tyurin canonical image of C. Recall from [ADFIO, Section 10] that, for each branch point p_i of the map $\pi\colon C \to \mathbb{P}^1$, the ramification points $r_{i1}, \ldots, r_{i6}$ have the same image, say $\overline{p}_i \in \mathbb{P}^5$ in the Prym-Tyurin canonical space $\mathbb{P}^5 \cong \mathbb{P}\big(H^0(C, \omega_C)^{(-5)}\big)^{\vee}$.

Since the Prym-Tyurin canonical image $\varphi_{(-5)}(C)$ is non-degenerate, the quadric Q has rank at least 3, hence its singular locus is a linear subspace of $\mathbb{P}^5$ of codimension at least 3. In particular, Q can be singular at most 14 of the points $\overline{p}_i$: indeed, if for instance Q is singular at $\overline{p}_1, \ldots, \overline{p}_{15}$, this implies

$$h^0\Big(C, \omega_C\big(-\sum_{i=1}^{15}\sum_{j=1}^{6} r_{ij}\big)\Big) \geq 3,$$

which is not possible because $\omega_C\big(-\sum_{1 \leq i \leq 15}(r_{i1} + \cdots + r_{i6})\big)$ has degree 0.

Therefore, there exists a branch point p of π, such that Q is smooth at the image $\overline{p}$ of the six ramification points on π lying over p. Let $\Gamma_p := 2(r_1 + \cdots + r_6) + q_1 + \cdots + q_{15}$ be the divisor of $|L|$

above p. We write $H^0(C,\omega_C)^{(-5)} = \langle \eta_0, \eta_1, \ldots, \eta_5\rangle$, where $\langle \eta_1, \ldots, \eta_5\rangle = H^0\big(C,\omega_C\big)^{(-5)}(-r_1-\cdots-r_6)$, therefore $\mathrm{ord}_{r_i}(\eta_0)=0$. Assume the equation defining Q is given by

$$q = a\cdot \eta_0^2 + \eta_0\cdot(a_1\eta_1+\cdots+a_5\eta_5) + q_1(\eta_1,\ldots,\eta_5) \in \mathrm{Sym}^2 H^0(C,\omega_C)^{(-5)},$$

where $a\in\mathbb{C}$. Evaluating q at r_i, we obtain $a=0$. Then $\eta := a_1\eta_1+\cdots+a_5\eta_5 \in H^0(C,\omega_C)^{(-5)}$ satisfies $\mathrm{ord}_{r_i}(\eta)\geq 2$, for $i=1,\ldots,6$. Furthermore, $\eta\neq 0$, because $\overline{p}\in Q_{\mathrm{reg}}$, that is, hence

$$\eta \in H^0\big(C,\omega_C\big)^{(-5)}(-2r_1-\cdots-2r_6)\neq 0.$$

Note that η is the equation of the tangent hyperplane to Q at the point $\overline{p}$.

Assume now that $[C,L]\in \widetilde{\mathcal{G}}_{E_6}\setminus(\mathfrak{M}\cup\mathfrak{N})$, thus the map $\alpha_p^{(-5)}$ is an isomorphism. The dual map can be identified with the evaluation map

$$\big(\alpha_p^{(-5)}\big)^{\vee}\colon H^0(C,\omega_C)^{(-5)} \to H^0\big(\omega_{C|\Gamma_p}\big)^{(-5)},$$

hence we obtain that $H^0\big(\omega_{C|\Gamma_p}\big)^{(-5)}(-2r_1-\cdots-2r_6)\neq 0$. Identifying $H^0(\omega_{C|\Gamma_p})^{(-5)}$ with the primitive cohomology of a 1-nodal cubic surface, this fact implies in fact that

$$H^0\big(\omega_{C|\Gamma}\big)^{(-5)}(-2r_1-\cdots-2r_6-q_1-\cdots-q_{15})\neq 0,$$

which yields $0\neq\eta\in H^0(C,\omega_C)^{(-5)}(-\Gamma_p)$, that is, $\eta\in \mathrm{Im}\big\{H^0(C,L)\otimes H^0(C,\omega_C\otimes L^{\vee})\to H^0(C,\omega_C)\big\}$. We conclude $\eta\in H^0(C,\omega_C)^{(-5)}\cap H^0(C,\omega_C)^{(+1)}=\{0\}$, which is a contradiction.

□

Proof of Theorem 1.3. It suffices to combine Theorem 4.12 with [ADFIO, Theorems 0.3 and 9.3], asserting that a point $[C,L]\in\widetilde{\mathcal{G}}_{E_6}\setminus\mathfrak{N}$ lies in the ramification divisor of PT if and only the Prym-Tyurin canonical curve $\varphi_{(-5)}(C)$ lies on a quadric. □

5. A UNIVERSAL THETA DIVISOR ON THE MODULI SPACE OF $W(E_6)$-COVERS

In this section we discuss the geometry of a very natural effective divisor on $\widetilde{\mathrm{Hur}}$, which can be viewed as (a translate of) the universal theta divisor (not to be confused with the pull-back of the universal theta divisor from $\overline{\mathcal{A}}_6$). Since the geometric construction we are interested in is defined directly in terms of a $W(E_6)$-pencil, it is easier to work again with the parameter space $\widetilde{\mathcal{G}}_{E_6}$.

Definition 5.1. We consider the following locus inside $\widetilde{\mathcal{G}}_{E_6}$

$$\mathfrak{D}_1 := \Big\{[C,L]\in\widetilde{\mathcal{G}}_{E_6} : H^0\big(C,2\omega_C-5L\big)\neq 0\Big\}. \tag{5A}$$

Note that since $\deg(2\omega_C-5L)=g(C)-1=45$, points in $\mathfrak{D}_1$ are characterized by the condition that $2\omega_C-5L$ lies in the theta divisor $W_{45}(C)\subseteq \mathrm{Pic}^{45}(C)$. In particular, $\mathfrak{D}_1$ is a virtual divisor on $\widetilde{\mathcal{G}}_{E_6}$.

Theorem 5.2. *The virtual class of $\mathfrak{D}_1$ is given by the following formula:*

$$[\mathfrak{D}_1]^{\mathrm{vir}} = -\lambda-\kappa+\frac{15}{2}\gamma\in CH^1(\widetilde{\mathcal{G}}_{E_6}).$$

Proof. We reinterpret the defining property of points in $\mathfrak{D}_1$ via the Base Point Free Pencil Trick, as saying that the multiplication map

$$\mu_1(L)\colon H^0(C,L)\otimes H^0\big(C,2\omega_C-4L\big)\longrightarrow H^0\big(C,2\omega_C-3L\big)$$

is not bijective. Note that one has $h^0(2\omega_C-4L)=27$ and that $h^0(C,2\omega_C-3L)=54$. Furthermore, using the construction given in 4.1 of the birational isomorphism $\widetilde{\beta}\colon\widetilde{\mathrm{Hur}}\dashrightarrow\widetilde{\mathcal{G}}_{E_6}$, it follows that L is a base point free pencil for every point $[C,L]\in\widetilde{\mathcal{G}}_{E_6}$. The map $\mu_1(L)$ can be globalized to a morphism of vector bundles over $\widetilde{\mathcal{G}}_{E_6}$ having the same rank

$$\mu_1\colon \tilde{f}_*(\mathcal{L})\otimes \tilde{f}_*\big(\omega_{\tilde{f}}^{\otimes 2}\otimes \mathcal{L}^{\otimes(-4)}\big)\ \longrightarrow\ \tilde{f}_*\big(\omega_{\tilde{f}}^{\otimes 2}\otimes \mathcal{L}^{\otimes(-3)}\big),$$

where, as in the previous section, $\mathcal{L}$ is a universal pencil with $W(E_6)$-monodromy over the universal curve $\tilde{f}\colon \widetilde{\mathcal{C}}_{E_6}\to\widetilde{\mathcal{G}}_{E_6}$. Clearly, $\mathfrak{D}_1$ is the degeneracy locus of μ_1.

Since one has

$$R^1\tilde{f}_*\Big(\omega_{\tilde{f}}^{\otimes 2}\otimes\mathcal{L}^{\otimes(-4)}\Big)=0,\quad R^1\tilde{f}_*\Big(\omega_{\tilde{f}}^{\otimes 2}\otimes\mathcal{L}^{\otimes(-3)}\Big)=0,$$

the Chern classes of the sheaves appearing in the definition of the morphism μ_1 can be computed via a Grothendieck-Riemann-Roch calculation. For instance,

$$c_1\Big(\tilde{f}_*\big(\omega_{\tilde{f}}^{\otimes 2}\otimes\mathcal{L}^{\otimes(-4)}\big)\Big)=\lambda+\kappa+8\mathfrak{A}-12\mathfrak{B},$$

and after routine manipulations we obtain the claimed formula. □

Corollary 5.3. *The (virtual) class of* $[\mathfrak{D}_1]$ *in the* $(\lambda,[D_{\mathrm{syz}}],[D_0])$ *basis of* $\mathrm{Pic}(\widetilde{\mathcal{G}}_{E_6})$ *is given by:*

$$[\mathfrak{D}_1]^{\mathrm{virt}}=\frac{44}{7}\lambda-\frac{17}{14}[D_{\mathrm{syz}}]-\frac{48}{7}[D_0].$$

5.1. **A degenerate $W(E_6)$-cover.** It is crucial to establish that the virtual divisor $\mathfrak{D}_1$ is a genuine divisor on $\widetilde{\mathcal{G}}_{E_6}$. To that end we shall use degeneration and we first need some preparation. We start once more with a $W(E_6)$-cover $[\pi\colon C\to\mathbb{P}^1, p_1+\cdots+p_{24}]\in\mathrm{Hur}$. Recall that fibers of π over a generic point in $\mathbb{P}^1$ can be identified with the lines $\ell_1,\ldots,\ell_{27}$ on a fixed smooth cubic surface S, as well as with the (-1)-vectors in the orbit $W(E_6).\varpi_6$ of the coweight lattice $\Lambda^*_{W(E_6)}$. The reflections $w\in W(E_6)$ can be identified with the roots of the root lattice $\Lambda_{W(E_6)}$ modulo ±1: the roots $+r$ and $-r$ give the same reflection. For each root r there are exactly 6 coweights $a_{r,i}$ with $(r,a_{r,i})=1$ and 6 coweights $b_{r,i}$ with $(r,b_{r,i})=-1$ so that $b_{r,i}=a_{r,i}+r$. The switch from r to $-r$ exchanges $a_{r,i}$'s and $b_{r,i}$'s. Under the monodromy representation $W(E_6)\hookrightarrow S_{27}$ the reflection w is represented by a *double sixer* $(a_{r,1},b_{r,1})\cdots(a_{r,6},b_{r,6})$.

The following lemma describes the basic degeneration used to show that $\mathfrak{D}_1$ is a genuine divisor. This degeneration will also prove to be instrumental in the final step of the proof of Theorem 1.4.

Lemma 5.4. *Let* $\mathcal{C}:=\big(\pi_t\colon C_t\to\mathbb{P}^1,p_1(t),\ldots,p_{24}(t)\big)$ *be a 1-parameter family of* $W(E_6)$*-covers such that the local monodromies* w_i *of the points* p_i *are pairwise equal:* $w_{2i-1}=w_{2i}$ *for* $i=1,\ldots,12$. *Assume* $\lim p_{2i-1}(t)=\lim p_{2i}(t)=q_i\in\mathbb{P}^1$. *Then the family* $\mathcal{C}$ *can be flatly completed to a family of covers of* $\mathbb{P}^1$ *so that the central fiber* $C=C_0$ *is a nodal curve labeled by the lines* $\ell_1,\ldots,\ell_{27}$*, a union of* 27 *copies of* $\mathbb{P}^1$ *each mapping isomorphically down to the base* $\mathbb{P}^1$. *The sheets are glued as follows. For each point* $q_j\in\mathbb{P}^1$, $j=1,\ldots,12$ *with local monodromy* w_j*, glue the point above* q_j *on the sheet labelled by* a_{jk} *to the point above* q_j *on the sheet* b_{jk}*, for* $k=1,\ldots,6$.

Proof. For a generic point $t\in\mathbb{P}^1$, each ramification point over $p_i(t)$ is of the form $y^2=x$, with the 6 pairs (a_{ik},b_{ik}) coming together. It is immediate that when two branch points on the base come together, the limit points on C are nodes. Let $\coprod_{s=1}^m\widetilde{C}_s$ be the normalization of C. It first follows that all the components of C are rational, since the map $C\to\mathbb{P}^1$ induces étale maps $\widetilde{C}_s\to\mathbb{P}^1$. The dual graph $\Gamma:=(V(\Gamma),E(\Gamma))$ of C is connected since the reflections w_i are chosen so that they generate $W(E_6)$.

For the arithmetic genus of C one has

$$\big|E(\Gamma)\big| - \big|V(\Gamma)\big| + 1 + \sum_{s=1}^{m} p_a(\widetilde{C}_s) = \big|E(\Gamma)\big| - \big|V(\Gamma)\big| + 1 = 46.$$

Since there are $12 \times 6 = 72$ edges, it follows that the number of vertices, that is, that of the irreducible components C_s of C is 27. Thus, the normalization of C is a disjoint union of 27 copies of $\mathbb{P}^1$'s and the gluing is as described. $\square$

Remark 5.5. The switch from a root r to $-r$ representing the same reflection w changes the orientation of the 6 respective edges in the oriented dual graph Γ.

The glued curve $C = C_0$ comes with an ample line bundle $L = \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. It also comes with a Kanev correspondence sending a point over $x \in \mathbb{P}^1$ on the sheet labeled ℓ to the 10 points in the same fiber on the sheets labeled ℓ' such that ℓ and ℓ' intersect on the abstract cubic surface S. The induced endomorphism D on $H^0(C, \omega_C)$ satisfies $(D+5)(D-1) = 0$ and the corresponding eigenspaces have dimension 6 and 40, just as on a smooth curve. For more details, see [ADFIO, Sections 4 and 5].

Theorem 5.6. *There exists a choice of reflections $w_1 = w_2, \ldots, w_{23} = w_{24}$ generating $W(E_6)$ and of points $q_1, \ldots, q_{12} \in \mathbb{P}^1$ for which the central curve C and the cover $\pi\colon C \to \mathbb{P}^1$ as described above have the following properties:*

1. $h^0(C, L) = 2$.
2. *The image of the multiplication map $H^0(C, L) \otimes H^0(C, \omega_C \otimes L^\vee) \to H^0(C, \omega_C)$ has dimension 40.*
3. $h^0(C, \omega_C^{\otimes 2}(-5L)) = 0$.
4. *The 6-dimensional eigenspace $H^0(C, \omega_C)^{(-5)}$ is base point free.*
5. *The image of the Prym-Tyurin canonical curve $\varphi_{(-5)}(C)$ in $\mathbb{P}\big(H^0(\omega_C)^{(-5)}\big)^\vee$ does not lie on a quadric.*

Proof. The computation is reduced to linear algebra. A line bundle on C of multidegree $(d_1, \ldots, d_{27})$ is identified with a sheaf $\coprod_{i=1}^{27} \mathcal{O}_{\mathbb{P}^1}(d_i)$ with specified twists $c_{q,i,j}$ at the nodes where the sheets labelled by i and j are glued over a point $q \in \mathbb{P}^1$. If $q_1, \ldots, q_{12} \in \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$, then a section of this line bundle is identified with a collection of polynomials $P_i(t)$ of degrees d_i with the values at the nodes matching up to multiplication by the twist $c_{q,i,j}$.

For the sheaf $L \in W^1_{27}(C)$ we choose the multidegree to be $(1, \ldots, 1)$ and the twists are all equal to 1. For ω_C the corresponding degrees are $d_i = \big|C_i \cap \overline{C \setminus C_i}\big| - 2$. The restriction ω_i to C_i of a section of ω_C can be viewed as

$$\omega_i = \frac{P_i(t)dt}{\prod(t - q_{is})},$$

where $P_i(t)$ is a polynomial of degree d_i. Here, q_{is} are the nodes lying on the sheet labelled by i. The twist at a node over $q \in \mathbb{A}^1$ joining the sheets i and j is the negative of the ratio of residues:

$$c_{q,i,j} = -\operatorname{Res}_q \frac{dt}{\prod(t - q_{is})} \Big/ \operatorname{Res}_q \frac{dt}{\prod(t - q_{jt})}.$$

The twists for the line bundles $\omega_C^{\otimes m}(dL)$ are then the appropriate products of the above twists. We thus reduce the computation of the dimension of the spaces of sections $H^0\big(C, \omega_C^{\otimes m}(dL)\big)$ for any integers m and d to a concrete linear algebra question.

The eigenspace $H^0(C, \omega_C)^{(-5)}$ is the subspace of $H^0(C, \omega_C)$ where for every branch point $q_1, \ldots, q_{12}$ the residues over each of the sheets $a_{i1}, \ldots, a_{i6}$ are equal to each other. The subspace $H^0(C, \omega_C)^{(+1)}$ is the subspace where the sums of these residues are zero.

We performed the check for a concrete glued curve corresponding to the following choices:

- The points $q_i = i \in \mathbb{Z} \subseteq \mathbb{C}$.
- The following roots, in standard notation for the Minkowski space $I^{1,6}$: $\alpha_{135} = e_0 - e_1 - e_3 - e_5$, $\alpha_{12} = e_1 - e_2$, $\alpha_{23} = e_2 - e_3$, $\alpha_{34} = e_3 - e_4$, $\alpha_{45} = e_4 - e_5$, $\alpha_{56} = e_5 - e_6$, $\alpha_{16} = e_1 - e_6$, $\alpha_{456} = e_0 - e_4 - e_5 - e_6$, $\alpha_{123} = e_0 - e_1 - e_2 - e_3$, $\alpha_{346} = e_0 - e_3 - e_4 - e_6$, $\alpha_{234} = e_0 - e_2 - e_3 - e_4$, $\alpha_{156} = e_0 - e_1 - e_5 - e_6$.

All the computations were done in *Mathematica* and are available at [Al]. ☐

As discussed in [ADFIO, Section 11], a consequence of parts (1,2,3) of (5.6) is that the morphism μ defining the Weyl-Petri divisor (see 4.4) is generically non-degenerate, that is, $\mathfrak{N}$ is indeed a genuine divisor on $\overline{\mathrm{Hur}}$. A consequence of the other parts is:

Theorem 5.7. *For a generic cover* $[\pi\colon C \to \mathbb{P}^1] \in \overline{\mathrm{Hur}}$*, one has* $H^0(C, 2\omega_C - 5L) = 0$*. Thus* $\mathfrak{D}_1$ *is a genuine divisor on* $\overline{\mathrm{Hur}}$*.*

Proof. Indeed, we consider a flat family degenerating to the glued curve as in Theorem 5.6. In the central fiber the dimension of $H^0(C, 2\omega_C - 5L)$ can only increase, which the above argument shows not to be the case. ☐

6. The Prym-Tyurin map is unramified generically along the divisor D_0

In this Section we prove Theorem 1.4 by showing that the differential of the Prym-Tyurin map $PT\colon \overline{\mathrm{Hur}} \dashrightarrow \overline{\mathcal{A}}_6$ is bijective at a general point of the divisor D_0 of $\overline{\mathrm{Hur}}$. We fix throughout the section a suitably general $W(E_6)$-admissible cover

$$[\pi\colon C = C_1 \cup C_2 \to R := R_1 \cup_q R_2, p_1 + \cdots + p_{24}] \in D_0 \subseteq \overline{\mathrm{Hur}}. \tag{6A}$$

We shall assume that C_1 is a smooth curve of genus 40. The curve C_2 has 21 components, all rational, with 6 components mapping to R_2 with degree 2 and the other 15 mapping isomorphically to R_2. The degree 27 map $\pi_1 = \pi_{|C_1}\colon C_1 \to R_1$ has monodromy $W(E_6)$ and is branched precisely at the points $p_1, \ldots, p_{22} \in R_1 \setminus \{q\}$.

Definition 6.1. Let Hur_1 denote the Hurwitz space of $W(E_6)$-covers $[\pi_1\colon C_1 \to \mathbb{P}^1, p_1 + \cdots + p_{22}]$ of degree 27 with branch points at $p_1, \ldots, p_{22}$. The source C_1 is a smooth curve of genus 40 and the local monodromy of π_1 at each branch point $p_i \in \mathbb{P}^1$ is given by a reflection in a root of E_6. As in the case of covers with 24 branch points, the curve C_1 has a Kanev correspondence which we denote by D_1 and which induces an endomorphism $D_1\colon JC_1 \to JC_1$ and a 5-dimensional Prym-Tyurin variety $PT(C_1, D_1) := \mathrm{Im}(D_1 - 1) \subseteq JC_1$. Put $L_1 := \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \in W^1_{27}(C_1)$.

Let $\rho\colon C \to \overline{C}$ be the map contracting C_2. The curve $\overline{C}$ is the stabilization of C and it has 6 ordinary double points obtained by identifying two points of C_1 if they are connected by a component of C_2. We denote by $\overline{L} \in W^1_{27}(\overline{C})$ the line bundle characterized by the property $\left(\rho^*_{|C_1}(\overline{L}\right) \cong L_1$.

Given a reduced fiber Γ of the map $\pi_1\colon C_1 \to \mathbb{P}^1$, we consider the usual exact sequence, see also (4A)

$$0 \longrightarrow H^0(C_1, \mathcal{O}_{C_1}) \longrightarrow H^0(C_1, L_1) \longrightarrow H^0(\mathcal{O}_\Gamma(\Gamma)) \xrightarrow{\alpha_1} H^1(C_1, \mathcal{O}_{C_1}) \longrightarrow H^1(C_1, L_1) \longrightarrow 0. \tag{6B}$$

The map α_1 is equivariant for the action of the Kanev correspondence D_1, hence it maps the 6-dimensional (-5)-eigenspace of $H^0(\mathcal{O}_\Gamma(\Gamma))^{(-5)}$ into the 5-dimensional space $H^1(C, \mathcal{O}_C)^{(-5)}$. It follows that $h^0(C_1, L_1) \geq 3$, in particular $[C_1] \in \mathcal{M}_{40}$ is a Brill-Noether special curve.

Notation 6.2. Let $\mathfrak{M}_1 \subseteq \mathrm{Hur}_1$ denote the locus where $H^0(C_1, \omega_{C_1} \otimes L_1^{\otimes(-2)}) \neq 0$.

We denote by $PT_5\colon \mathrm{Hur}_1 \to \mathcal{A}_5$ the Prym-Tyurin map. The proof in [ADFIO, Section 10] carries through without changes to the case of 22 branch points so that we have the following result:

Theorem 6.3. *The Prym-Tyurin map* PT_5 *is ramified at a point* $[\pi_1\colon C_1 \to \mathbb{P}^1, p_1+\cdots+p_{22}] \in \mathrm{Hur}_1\backslash\mathfrak{M}_1$ *if and only if the Prym-Tyurin canonical image of* C_1 *is contained in a quadric.*

6.1. **The map** PT_5 **is dominant.** This follows for instance, from the fact that the ordinary Prym map $P\colon \mathcal{R}_6 \to \mathcal{A}_5$ is dominant, using the fact that 6-dimensional Prym-Tyurin varieties degenerate to Prym varieties, as was shown in [ADFIO, Theorem 5]. Therefore the codifferential of the map PT_5 is generically injective. The rest of this Section is devoted to the proof of the above result.

Theorem 6.4. *Assume* $[\pi_1\colon C_1 \to R_1, p_1+\cdots+p_{22}] \in \mathrm{Hur}_1$. *If the map* PT *is ramified at the point*

$$[C = C_1 \cup C_2 \to R_1 \cup R_2] \in \overline{\mathrm{Hur}},$$

then, either $h^0\big(C_1, \omega_{C_1} - 2L_1\big) > 0$, *or, the Prym-Tyurin canonical image of* C_1 *is contained in a quadric, in which case* $h^0(C_1, L_1) \geq 4$ *and* $h^0(\overline{C}, L) \geq 3$. *Generically on* D_0, *none of these cases occur.*

In what follows, we first recall the interpretation of the cotangent spaces to $\overline{\mathcal{A}}_6$, $\overline{\mathcal{A}}_{46}$, $\overline{\mathcal{M}}_{46}$ and $\overline{\mathrm{Hur}}$, then we describe the codifferential of PT.

6.2. Let $\overline{P}$ be the usual compactification of the semi-abelian variety $PT(C,D)$ obtained by first completing $PT(C,D)$ to a $\mathbb{P}^1$-bundle over the 5-dimensional ppav $B := PT(C_1, D_1)$, and then identifying the 0 and ∞-sections after translating by the extension datum of $PT(C,D)$ over B. We refer to [M] for details. The local to global spectral sequence induces the exact sequence

$$0 \longrightarrow H^0(\mathcal{E}xt^1_{\overline{P}}(\Omega^1_{\overline{P}}, \mathcal{O}_{\overline{P}}))^\vee \longrightarrow \Omega^1_{\overline{\mathcal{A}}_6,[PT(C,D)]} \longrightarrow \Omega^1_{D_6,[PT(C,D)]} \longrightarrow 0,$$

where $\Omega^1_{D_6,[PT(C,D)]}$ is the cotangent space to the boundary divisor D_6 of $\overline{\mathcal{A}}_6$. Note that $\Omega^1_{D_6,[PT(C,D)]}$ is the dual to the space of deformations of $PT(C,D)$ that stay singular. Let $\Omega^1_{\overline{\mathcal{A}}_6}(\log D_6)$ be the sheaf of 1-forms with at worst simple logarithmic poles along D_6. By [CF, IV Proposition 3.1(vi), p. 107], the fiber $\Omega^1_{\overline{\mathcal{A}}_6}(\log D_6)_{[PT(C,D)]}$ can be identified with $\mathrm{Sym}^2 H^0(C,\omega_C)^{(-5)}$, and this induces an identification

$$\Omega^1_{D_6,[PT(C,D)]} = H^0(C,\omega_C)^{(-5)} \odot H^0(C_1,\omega_{C_1})^{(-5)},$$

where

$$H^0(C,\omega_C)^{(-5)} \odot H^0(C,\omega_{C_1})^{(-5)} := \Big(H^0(C,\omega_C)^{(-5)} \otimes H^0(C_1,\omega_{C_1})^{(-5)}\Big) \bigcap \mathrm{Sym}^2 H^0(C,\omega_C)^{(-5)}.$$

Remark that in this description $H^0(C_1,\omega_{C_1})^{(-5)} \subseteq H^0(C,\omega_C)^{(-5)}$ is a codimension one subspace.

6.3. The cotangent space to $\overline{\mathcal{M}}_{46}$ at $[\overline{C}]$ is $H^0(\overline{C}, \Omega^1_{\overline{C}} \otimes \omega_{\overline{C}})$. We have the natural map

$$\Omega^1_{\overline{C}} \otimes \omega_{\overline{C}} \longrightarrow \rho_*(\Omega^1_C \otimes \omega_C),$$

obtained from $\rho^*(\Omega^1_{\overline{C}} \otimes \omega_{\overline{C}}) \to \Omega^1_C \otimes \omega_C$, which induces the map

$$H^0(\overline{C}, \Omega^1_{\overline{C}} \otimes \omega_{\overline{C}}) \longrightarrow H^0(C, \Omega^1_C \otimes \omega_C). \tag{6C}$$

A local computation shows that the natural map $\omega_{\overline{C}} \to \rho_*\omega_C$ is an isomorphism. Therefore it induces an isomorphism $H^0(\overline{C}, \omega_{\overline{C}}) \xrightarrow{\cong} H^0(C,\omega_C)$, which shows that $H^0(\overline{C}, \omega_{\overline{C}})$ is endowed with an endomorphism, which we still denote by D, that is induced by the Kanev correspondence.

6.4. Let $\overline{JC}$ denote the compactification of the Jacobian of $\overline{C}$ described as the scheme parametrizing torsion-free sheaves of degree 0 on $\overline{C}$, see [OS]. As above, we have the exact sequence

$$0 \longrightarrow H^0\big(\mathcal{E}xt^1_{\overline{JC}}(\Omega^1_{\overline{JC}}, \mathcal{O}_{\overline{JC}})\big)^\vee \longrightarrow \Omega^1_{\overline{\mathcal{A}}_{46},[\overline{JC}]} \longrightarrow H^0(\overline{C}, \omega_{\overline{C}}) \odot H^0(C_1, \omega_{C_1}) \longrightarrow 0,$$

where, again by [CF, IV Proposition 3.1(vi), p. 107], the space on the right classifies deformations of $\overline{JC}$ of toric rank 6. Here $H^0(C_1, \omega_{C_1}) \subseteq H^0(\overline{C}, \omega_{\overline{C}})$ is viewed as a subspace of codimension 6.

6.5. Consider the pull-back diagram

$$\begin{tikzcd}
\overline{\mathcal{H}} \arrow[r, "\mathfrak{b}"] \arrow[d, "q"] & \overline{\mathcal{M}}_{0,24} \arrow[d, "p"] \\
\overline{\mathrm{Hur}} \arrow[r, "\mathfrak{br}"] & \widetilde{\mathcal{M}}_{0,24}.
\end{tikzcd}$$

The ramification divisor of p is the divisor B_2, its ramification index being equal to 2. The ramification divisor of q is the divisor $E_0 + E_{\mathrm{azy}}$ [ADFIO, Paragraph 6.11]. Furthermore, $\mathfrak{b}^*(B_2) = E_0 + 3E_{\mathrm{azy}} + 2E_{\mathrm{syz}}$. It follows that the map $\mathfrak{br}$ is generically unramified along D_0 and we can identify the cotangent space $\Omega^1_{\overline{\mathrm{Hur}},[C,\pi]}$ with $H^0(R, \Omega^1_R \otimes \omega_R(B))$ which is the cotangent space to $\widetilde{\mathcal{M}}_{0,24}$.

Definition 6.5. Let M and A be the ramification and anti-ramification divisors of the $W(E_6)$-admissible cover $\pi\colon C \to R$. As M and A are supported on the smooth locus of C, we have the usual identities

$$\text{(6D)} \quad \pi^*(B) = 2M + A, \quad \Omega^1_C = \pi^*(\Omega^1_R)(M), \quad \omega_C = \pi^*(\omega_R)(M), \quad \Omega^1_C \otimes \omega_C(A) = \pi^*(\Omega^1_R \otimes \omega_R(B)),$$

and we can define the trace map as for smooth covers:

Definition 6.6. Let $\mathrm{tr}\colon \pi_*\mathcal{O}_C(-A) \to \mathcal{O}_R$ be the trace map on regular functions. For an open affine subset $U \subseteq \mathbb{P}^1$, a regular function $\varphi \in \Gamma(U, \mathcal{O}_C(-A))$, and a point $y \in U$, one has

$$\mathrm{tr}(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x),$$

counted with multiplicities. Note that tr is surjective. By (6D), the trace map induces the map $\pi_*(\Omega^1_C \otimes \omega_C) \to \Omega^1_R \otimes \omega_R(B)$. Let $\mathrm{Tr}\colon H^0(C, \Omega^1_C \otimes \omega_C) \to H^0\big(R, \Omega^1_R \otimes \omega_R(B)\big)$ be the induced map on global sections. The composition of Tr with the map (6C)

$$\overline{\mathrm{Tr}}\colon H^0(\overline{C}, \Omega^1_{\overline{C}} \otimes \omega_{\overline{C}}) \longrightarrow H^0(C, \Omega^1_C \otimes \omega_C) \longrightarrow H^0(R, \Omega^1_R \otimes \omega_R(B))$$

can be viewed as the codifferential of the forgetful map $\overline{\mathrm{Hur}} \to \overline{\mathcal{M}}_{46}$ at the point $[C, \pi]$.

Proposition 6.7. *The codifferential* $(dPT)^\vee_{[C,\pi]}\colon T^\vee_{[PT(C,D)]}(\overline{\mathcal{A}}_6) \to T^\vee_{[C,\pi]}\big(\overline{\mathrm{Hur}}\big)$ *is given by the following composition of maps:*

$$\text{(6E)} \qquad T^\vee_{[PT(C,D)]}\big(\overline{\mathcal{A}}_6\big) \hookrightarrow T^\vee_{[\overline{JC}]}\big(\overline{\mathcal{A}}_{46}\big) \xrightarrow{\mathrm{tor}} H^0(\overline{C}, \Omega^1_{\overline{C}} \otimes \omega_{\overline{C}}) \xrightarrow{\overline{\mathrm{Tr}}} H^0\big(R, \Omega^1_R \otimes \omega_R(B)\big),$$

where the second map is the codifferential of the Torelli map $\overline{\mathcal{M}}_{46} \to \overline{\mathcal{A}}_{46}$.

Proof. Follows along the lines of the proof of [ADFIO, Theorem 10.3] (which treats the same question in the case of a point $[C, \pi] \in \mathrm{Hur}$ corresponding to a smooth source curve) with obvious modifications. The first map in (6E) is the codifferential of the map from the perfect cone compactification of the moduli space of ppav of dimension 46 having an endomorphism D with eigenvalues $+1$ and -5 of eigenspaces of dimensions 40 and 6 respectively to $\overline{\mathcal{A}}_6$. □

6.6. We first study the codifferential $dPT^\vee$ on the conormal space to the boundary divisor D_6 of $\overline{\mathcal{A}}_6$. To that end, we first describe locally differentials on $C, \overline{C}$ and R near the node q of R corresponding to the point described in (6A).

Choose local coordinates t on R_1 and s on R_2 at the node q of R. These can be identified via π with local coordinates at the nodes $o_1, \dots, o_{27}$ of C above q. Then the stalks of the sheaves $\Omega^1_R, \omega_R, \Omega^1_C, \omega_C, \Omega^1_R \otimes \omega_R, \Omega^1_C \otimes \omega_C$ at their nodes have the following presentations

$$\begin{array}{ll}
\Omega^1_{R,q}, \Omega^1_{C,o_i}: & \mathcal{O}\langle ds, dt\rangle / (tds + sdt) \\
\omega_{R,q}, \omega_{C,o_i}: & \mathcal{O}\langle \frac{ds}{s}, \frac{dt}{t}\rangle / \left(\frac{ds}{s} + \frac{dt}{t}\right) \\
\Omega^1_{R,q} \otimes \omega_{R,q}, \Omega^1_{C,o_i} \otimes \omega_{C,o_i}: & \mathcal{O}\left\langle \frac{(ds)^2}{s}, \frac{(dt)^2}{t} \right\rangle \Big/ \left(t\frac{(ds)^2}{s} - s\frac{(dt)^2}{t} \right).
\end{array}$$

We have the natural exact sequence on R

$$0 \longrightarrow \operatorname{Tors}(\Omega^1_R) \longrightarrow \Omega^1_R \xrightarrow{\iota_R} \omega_R \longrightarrow \mathbb{C}_q \longrightarrow 0$$

where $\operatorname{Tors}(\Omega^1_R) \cong \mathbb{C}_q$ is a sky-scraper sheaf at q generated by the torsion differential $sdt = -tds$. From this, by tensoring with the locally free sheaf ω_R we obtain the exact sequence

$$0 \longrightarrow \mathbb{C}_q \longrightarrow \Omega^1_R \otimes \omega_R \xrightarrow{\kappa_R} \omega_R^{\otimes 2} \longrightarrow \mathbb{C}_q \longrightarrow 0$$

where the kernel of κ_R is generated by $ds\,dt = s\frac{(dt)^2}{t} = t\frac{(ds)^2}{s}$. One has a similar exact sequence for C at the points o_i. A torsion section $\gamma \in H^0(C, \Omega^1_C \otimes \omega_C)$ can be written as

$$\gamma = \lambda_i t \frac{(ds)^2}{s} = \lambda_i s \frac{(dt)^2}{t} \quad \text{near } o_i \in C.$$

6.7. **Local description at the nodes.** Assume the nodes $o_1, \dots, o_{27}$ of C are labeled in such a way that o_{2i-1} and o_{2i} map to the node u_i of $\overline{C}$ for $i = 1, \dots, 6$. Labeling by s_i, t_i the local coordinates on the two branches of C_2 and C_1 at the point o_i for $i = 1, \dots, 27$, then t_{2i-1}, t_{2i} are local coordinates at the point $u_i \in \overline{C}$ for $i = 1, \dots, 6$. We have the natural commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=1}^6 \mathbb{C}_{u_i} \arrow[r] \arrow[d, "\rho^*"] & \Omega^1_{\overline{C}} \otimes \omega_{\overline{C}} \arrow[r, "\kappa_{\overline{C}}"] \arrow[d, "\rho^*"] & \omega_{\overline{C}}^{\otimes 2} \arrow[d, "\rho^*"] \\
0 \arrow[r] & \rho_*\left(\bigoplus_{i=1}^{27} \mathbb{C}_{o_i}\right) \arrow[r] & \rho_*\left(\Omega^1_C \otimes \omega_C\right) \arrow[r, "\rho_*\kappa_C"] & \rho_*\omega_C^{\otimes 2}
\end{tikzcd}$$

where $\bigoplus_{i=1}^6 \mathbb{C}_{u_i}$ is the torsion subsheaf of $\Omega^1_{\overline{C}} \otimes \omega_{\overline{C}}$. The torsion part $\bigoplus_{i=1}^{27} \mathbb{C}_{o_i}$ of $\Omega^1_C \otimes \omega_C$ has an action of the correspondence D which leaves the image of $\bigoplus_{i=1}^6 \mathbb{C}_{u_i}$ invariant. The action of D on this subspace has two eigenspaces of dimensions 1 and 5 for the eigenvalues -5 and $+1$ respectively. The proof of this is analogous to [ADFIO, Lemma 10.8].

A torsion section $\overline{\gamma}$ of $\Omega^1_{\overline{C}} \otimes \omega_{\overline{C}}$ can be locally written near $u_i \in \overline{C}$ as

$$\overline{\gamma} = \mu_i t_{2i} \frac{(dt_{2i-1})^2}{t_{2i-1}} = \mu_i t_{2i-1} \frac{(dt_{2i})^2}{t_{2i}}, \quad \text{where } \mu_i \in \mathbb{C}.$$

Identifying the local coordinates on C with those on R as in the previous paragraph, a generator of the (-5)-eigenspace is the section $\overline{\gamma} \in \operatorname{Tors}(\Omega^1_{\overline{C}} \otimes \omega_{\overline{C}})$ with $\mu_i = 1$ for $i = 1, \dots, 6$.

6.8. **Injectivity in conormal directions.** By Proposition 6.7, the map PT is ramified at $[C,\pi] \in \overline{\mathrm{Hur}}$ if the kernel of the composition of maps (6E) is nonzero. Each of the above cotangent spaces has a natural subspace which is the conormal space to the equisingular deformations. Restricting the above sequence to each conormal space appearing in (6E), we obtain the exact sequence:
(6F)

$$H^0\big(\mathcal{E}xt^1_{\overline{P}}(\Omega^1_{\overline{P}},\mathcal{O}_{\overline{P}})\big)^\vee \hookrightarrow H^0\big(\mathcal{E}xt^1_{\overline{JC}}(\Omega^1_{\overline{JC}},\mathcal{O}_{\overline{JC}})\big)^\vee \xrightarrow{\mathrm{tor}} H^0\big(\mathcal{E}xt^1_{\overline{C}}(\Omega^1_{\overline{C}},\mathcal{O}_{\overline{C}})\big)^\vee \xrightarrow{\overline{\mathrm{Tr}}} H^0\big(\mathcal{E}xt^1_{R}(\Omega^1_{R}(B),\mathcal{O}_{R})\big)^\vee$$

Using e.g., [An, Corollary 15.4], the map tor in (6F) is an isomorphism. Identifying the second and third space in (6F), by Paragraph 6.7, the second space has an action of the correspondence D and the image of the first arrow is the 1-dimensional eigenspace for the eigenvalue -5. With our earlier choice of bases (see 6.7), a generator of the (-5)-eigenspace is the element $\sum_{i=1}^6 t_{2i}\frac{(dt_{2i-1})^2}{t_{2i-1}}$. The image of an element $\sum_{i=1}^6 \mu_i t_{2i}\frac{(dt_{2i-1})^2}{t_{2i-1}}$ in the last space is $\sum_{i=1}^6 \mu_i t\frac{(ds)^2}{s}$. It follows that the composition above is an isomorphism between two 1-dimensional spaces.

Note that, via push-forward to R_1, we have the following identification

$$H^0(R,\Omega^1_R\otimes\omega_R(B)) \cong \mathrm{Tors}_q\big(\Omega_R\otimes\omega_R(B)\big)\oplus H^0\big(R_1,\omega_{R_1}^{\otimes 2}(B_1+q)\big)\cong \mathbb{C}_q\oplus H^0\big(R_1,\omega_{R_1}^{\otimes 2}(B_1+q)\big),$$

where $B_1 = p_1+\cdots+p_{22}$ and the skyscraper sheaf $\mathbb{C}_q$ is generated by $ds\,dt = s\frac{(dt)^2}{t} = t\frac{(ds)^2}{s}$. The image of $H^0(\overline{C},\Omega^1_{\overline{C}}\otimes\omega_{\overline{C}})$ in $H^0(\overline{C},\omega_{\overline{C}}^{\otimes 2})$ is the space of sections vanishing at the nodes of $\overline{C}$. This image will be then identified with $H^0\big(C_1,\omega_{C_1}^{\otimes 2}(o_1+\cdots+o_{12})\big)\subseteq H^0(\overline{C},\omega_{\overline{C}}^{\otimes 2})\subseteq H^0\big(C_1,\omega_{C_1}^{\otimes 2}(2o_1+\cdots+2o_{12})\big)$.

6.9. Taking the quotient of the exact sequence (6E) by (6F), we obtain the commutative diagram

$$\begin{tikzcd}
T^\vee_{[PT(C,D)]}(\overline{\mathcal{A}}_6) \arrow[r] \arrow[d] & T^\vee_{[\overline{JC}]}(\overline{\mathcal{A}}_{46}) \arrow[r] \arrow[d] & {} \\
H^0(\overline{C},\omega_{\overline{C}})^{(-5)}\odot H^0(C_1,\omega_{C_1})^{(-5)} \arrow[r] & H^0(\overline{C},\omega_{\overline{C}})\odot H^0(C_1,\omega_{C_1}) \arrow[r] & {} \\
{} \arrow[r] & H^0(\overline{C},\Omega^1_{\overline{C}}\otimes\omega_{\overline{C}}) \arrow[r, "\overline{\mathrm{Tr}}"] \arrow[d] & H^0(R,\Omega^1_R\otimes\omega_R(B)) \arrow[d] \\
{} \arrow[r] & H^0\big(C_1,\omega_{C_1}^{\otimes 2}(o_1+\cdots+o_{12})\big) \arrow[r, "\overline{\mathrm{tr}}"] & H^0\big(R_1,\omega_{R_1}^{\otimes 2}(B_1+q)\big).
\end{tikzcd}$$

To summarize the discussion above, the injectivity of the codifferential of PT at the point $[C,\pi]\in D_0$ is equivalent to the injectivity of the composition in the bottom row above.

6.10. **The kernel of $\overline{\mathrm{tr}}$.** For each of the branch points $p_i\in R_1$ with $i=1,\dots,22$, let $\{r_{ij}\}_{j=1}^6\subseteq C_1$ be the ramification points lying over p_i. The formal neighborhoods of the points r_{ij} are naturally identified, so that we can choose a single local parameter x and write a section $\gamma\in H^0\big(C_1,\omega_{C_1}^{\otimes 2}(o_1+\cdots+o_{12})\big)$ as

$$\gamma=\varphi_{ij}(x)\cdot(dx)^2 \ \text{ near } r_{ij}\in C.$$

Choose a local parameter y at the point p_i, so that $\pi|_{C_1}$ is given locally by the map $y=x^2$. We can use the same local parameter at the remaining 15 antiramification points $\{q_{ik}\}_{k=1}^{15}$ over p_i at which π is unramified, and write $\gamma=\psi_{ik}(y)\cdot(dy)^2$ near $q_{ik}\in C$, for $k=1,\dots,15$.

At the point q, we similarly choose a local parameter x and identify it with the local parameters at the points $o_1,\dots,o_{27}$. Write $\gamma=\rho_i(x)\frac{(dx)^2}{x}$ near o_i for $i=1,\dots,12$.

Lemma 6.8. *The kernel of the trace map* $\overline{\mathrm{tr}}\colon H^0\big(C_1,\omega_{C_1}^{\otimes 2}(o_1+\cdots+o_{12})\big)\to H^0\big(R_1,\omega_{R_1}^{\otimes 2}(B_1+q)\big)$ *consists of those quadratic differentials* γ *which, using the previous notation, satisfy*

$$\sum_{j=1}^{6}\varphi_{ij}(r_{ij})=0,\quad \textit{for}\ \ i=1,\ldots,22,\quad \textit{and}\quad \sum_{j=1}^{12}\rho_j(o_j)=0.$$

Proof. Local calculation, very similar to the proof of [ADFIO, Lemma 10.5]. □

We are now in a position to describe set-theoretically the ramification of the map $PT\colon \mathrm{Hur}\to\mathcal{A}_6$ along D_0, which then quickly leads to an alternative proof of the dominance of PT.

Proof of Theorem 6.4. The global sections of $\omega_{\overline{C}}$ can be identified with the sections of $\omega_{C_1}(o_1+\cdots+o_{12})$ whose residues at o_{2i-1} and o_{2i} are opposite for $i=1,\ldots,6$. A proof analogous to that of [ADFIO, Lemma 10.8] shows that, under this identification, the elements of $H^0(\overline{C},\omega_{\overline{C}})^{(-5)}$ correspond to sections having *the same residue* at o_{2i-1} and o_{2i} for $i=1,\ldots,6$ (in addition to opposite residues at o_{2i-1} and o_{2i}). This first implies that the points $o_1,\ldots,o_{12}$ have the same image, say $\overline{o}$, in the Prym-Tyurin canonical space $\mathbb{P}\big(H^0(\overline{C},\omega_{\overline{C}})^{(-5)}\big)^\vee\cong\mathbb{P}^5$. Next, using Lemma 6.8, we deduce that if an element

$$\beta\in H^0(\overline{C},\omega_{\overline{C}})^{(-5)}\odot H^0(C_1,\omega_{C_1})^{(-5)}$$

belongs to the kernel of the composition on the bottom row of the diagram in paragraph 6.9, then its image in $H^0\big(C_1,\omega_{C_1}^2(o_1+\cdots+o_{12})\big)$ belongs to the subspace

$$H^0\Big(C_1,\omega_{C_1}^{\otimes 2}\big(-\sum_{i=1}^{22}\sum_{j=1}^{6}r_{ij}\big)\Big)=H^0(C_1,\omega_{C_1}\otimes L_1^{\otimes(-2)}).$$

Assuming $H^0(C_1,\omega_{C_1}\otimes L_1^{\otimes(-2)})=0$, and regarding β as an element of $\mathrm{Sym}^2H^0(\overline{C},\omega_{\overline{C}})^{(-5)}$, we obtain that β is the equation of a quadric containing the image of $\overline{C}$ in the Prym-Tyurin canonical space $\mathbb{P}\big(H^0(\overline{C},\omega_{\overline{C}})^{(-5)}\big)^\vee$.

Since, as explained, $PT_1\colon \mathrm{Hur}_1\to\mathcal{A}_5$ is dominant, we may assume via Theorem 6.3 that the Prym-Tyurin canonical image of C_1 in $\mathbb{P}^4$ is not contained in a quadric. It follows that the quadric defined by β is not a pull-back from $\mathbb{P}\big(H^0(C_1,\omega_{C_1})^{(-5)}\big)^\vee$ via the projection from $\overline{o}$. Therefore this quadric is *not* singular at $\overline{o}$ and its tangent hyperplane at $\overline{o}$ contains the lines tangent to the Prym-Tyurin canonical image of $\overline{C}$. The image of this tangent hyperplane in $\mathbb{P}\big(H^0(C_1,\omega_{C_1})^{(-5)}\big)^\vee$ contains the images of $o_1,\ldots,o_{12}$. In other words, the image of $H^0\big(\mathcal{O}_{o_1+\cdots+o_{12}}(\Gamma)\big)$ by the map α_1 in the sequence (6B) is contained in a hyperplane. This first implies that $h^0(C_1,L_1)\geq 4$. Next, since the (-5)-eigenspace in $H^0(\mathcal{O}_\Gamma(\Gamma))$ can be identified with the primitive Picard group of a smooth cubic surface, having the same value at each pair of points o_{2i-1},o_{2i} for $i=1,\ldots,6$ imposes *only one* condition on the sections of L_1. Hence we always have $h^0(\overline{C},L)\geq h^0(C_1,L_1)-1$, and, in this case, $h^0(\overline{C},L)\geq 3$.

The fact that these situations do not occur for a general choice of a point of D_0 is a consequence of Theorem 5.6, for the $W(E_6)$-admissible cover constructed there lies in D_0. □

Corollary 6.9. *The Prym-Tyurin map* $PT\colon\overline{\mathrm{Hur}}\dashrightarrow\mathcal{A}_6$ *is generically finite.*

Proof. Indeed, the above shows that the differential of PT on tangent spaces is generically an isomorphism. □

APPENDIX: THE CHARACTER TABLE OF $W(E_6)$

At several points in this paper we have used the character table of $W(E_6)$. We record it in the form presented by GAP [GAP] by applying the command `Display(CharacterTable("W(E6)"))`. It is also the same as the table in Atlas [CCNPW, p.27] for the group $U_4(2).2 = W(E_6)$, obtained from the character table of $U_4(2)$ by the splitting and fusion rules. As usual, rows are for characters (we added convenient names in column 2), and columns are for conjugacy classes.

χ	name	1a	2a	2b	3a	3b	3c	4a	4b	5a	6a	6b	6c	6d	9a	12a	2c	2d	4c	4d	6e	6f	6g	8a	10a	12b
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	$\overline{1}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
3	10	10	-6	2	1	-2	4	2	-2	.	-3	.	.	2	1	-1	.	.	.	.	.	.	.	.	.	.
4	6	6	-2	2	-3	3	.	2	.	1	1	1	-2	-1	.	-1	4	.	-2	2	1	-2	.	.	-1	1
5	$\overline{6}$	6	-2	2	-3	3	.	2	.	1	1	1	-2	-1	.	-1	-4	.	2	-2	-1	2	.	.	1	-1
6	20a	20	4	-4	-7	2	2	4	.	.	1	-2	-2	2	-1	1	.	.	.	.	.	.	.	.	.	.
7	15a	15	-1	-1	6	3	.	3	-1	.	2	-1	2	-1	.	.	5	-3	1	1	-1	2	.	-1	.	1
8	$\overline{15a}$	15	-1	-1	6	3	.	3	-1	.	2	-1	2	-1	.	.	-5	3	-1	-1	1	-2	.	1	.	-1
9	15b	15	7	3	-3	.	3	-1	1	.	1	-2	1	.	.	-1	5	1	3	-1	2	-1	1	-1	.	.
10	$\overline{15b}$	15	7	3	-3	.	3	-1	1	.	1	-2	1	.	.	-1	-5	-1	-3	1	-2	1	-1	1	.	.
11	20b	20	4	4	2	5	-1	.	.	.	-2	1	1	1	-1	.	10	2	2	2	1	1	-1	.	.	-1
12	$\overline{20b}$	20	4	4	2	5	-1	.	.	.	-2	1	1	1	-1	.	-10	-2	-2	-2	-1	-1	1	.	.	1
13	24	24	8	.	6	.	3	.	.	-1	2	2	-1	.	.	.	4	4	.	.	-2	1	1	.	-1	.
14	$\overline{24}$	24	8	.	6	.	3	.	.	-1	2	2	-1	.	.	.	-4	-4	.	.	2	-1	-1	.	1	.
15	30	30	-10	2	3	3	3	-2	.	.	-1	-1	-1	-1	.	1	10	-2	-4	.	1	1	1	.	.	-1
16	$\overline{30}$	30	-10	2	3	3	3	-2	.	.	-1	-1	-1	-1	.	1	-10	2	4	.	-1	-1	-1	.	.	1
17	60a	60	12	4	-3	-6	.	4	.	.	-3	.	.	-2	.	1	.	.	.	.	.	.	.	.	.	.
18	80	80	-16	.	-10	-4	2	.	.	.	2	2	2	.	-1	.	.	.	.	.	.	.	.	.	.	.
19	90	90	-6	-6	9	.	.	2	2	.	-3	.	.	.	.	-1	.	.	.	.	.	.	.	.	.	.
20	60b	60	-4	4	6	-3	-3	.	.	.	2	-1	-1	1	.	.	10	2	-2	-2	1	1	-1	.	.	1
21	$\overline{60b}$	60	-4	4	6	-3	-3	.	.	.	2	-1	-1	1	.	.	-10	-2	2	2	-1	-1	1	.	.	-1
22	64	64	.	.	-8	4	-2	.	.	-1	.	.	.	.	1	.	16	.	.	.	-2	-2	.	.	1	.
23	$\overline{64}$	64	.	.	-8	4	-2	.	.	-1	.	.	.	.	1	.	-16	.	.	.	2	2	.	.	-1	.
24	81	81	9	-3	.	.	.	-3	-1	1	.	.	.	.	.	.	9	-3	3	-1	.	.	.	1	-1	.
25	$\overline{81}$	81	9	-3	.	.	.	-3	-1	1	.	.	.	.	.	.	-9	3	-3	1	.	.	.	-1	1	.

TABLE 2. The character table of $W(E_6)$

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