

SYZYGIES OF CURVES AND THE EFFECTIVE CONE OF $\overline{\mathcal{M}}_g$

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1. INTRODUCTION

The aim of this paper is to describe a systematic way of constructing effective divisors on $\overline{\mathcal{M}}_g$ having exceptionally small slope. In particular, these divisors provide a string of counterexamples to the Harris-Morrison Slope Conjecture (cf. [HMo]). In a previous paper [FP], we showed that the divisor $\overline{\mathcal{K}}_{10}$ on $\overline{\mathcal{M}}_{10}$ consisting of sections of $K3$ surfaces contradicts the Slope Conjecture on $\overline{\mathcal{M}}_{10}$. Since the moduli spaces $\overline{\mathcal{M}}_g$ are known to behave erratically for small g and since the condition that a curve of genus g lie on a $K3$ surface is divisorial only for $g = 10$, the question remained whether $\overline{\mathcal{K}}_{10}$ is an isolated example or the first in a series of counterexamples. Here we prove that any effective divisor on $\overline{\mathcal{M}}_g$ consisting of curves satisfying a Green-Lazarsfeld syzygy type condition for a linear system residual to a pencil of minimal degree, violates the Slope Conjecture. A consequence of the existence of these effective divisors is that various moduli spaces $\overline{\mathcal{M}}_{g,n}$ with $g \leq 22$, are proved to be of general type.

We recall that the slope $s(D)$ of an effective divisor D on $\overline{\mathcal{M}}_g$ is defined as the smallest rational number $a/b \geq 0$ such that the divisor class $a\lambda - b(\delta_0 + \dots + \delta_{\lfloor g/2 \rfloor}) - [D]$ is an effective combination of boundary divisors. The Slope Conjecture predicts that $s(D) \geq 6 + 12/(g + 1)$ for all effective divisors D on $\overline{\mathcal{M}}_g$ (cf. [HMo]). More generally, the question of finding a good lower bound for the slope of $\overline{\mathcal{M}}_g$

$$s_g := \inf\{s(D) : D \in \text{Eff}(\overline{\mathcal{M}}_g)\}$$

is of great interest for a variety of reasons, for instance it would provide a new geometric solution to the Schottky problem. In a different direction, since $s(K_{\overline{\mathcal{M}}_g}) = 13/2$ (cf. [HM]), to prove that $\overline{\mathcal{M}}_g$ is of general type it suffices to exhibit a single effective divisor D on $\overline{\mathcal{M}}_g$ of slope $s(D) < 13/2$.

The Slope Conjecture is true for all $\overline{\mathcal{M}}_g$ with $g \leq 9$ but in [FP] we proved that on $\overline{\mathcal{M}}_{10}$, we have the equality $s(\overline{\mathcal{K}}_{10}) = 7 < 6 + 12/11$ (in fact $\overline{\mathcal{K}}_{10}$ is the only effective divisor on $\overline{\mathcal{M}}_{10}$ having slope $< 6 + 12/11$). In [FP] we also showed that \mathcal{K}_{10} has four incarnations as a geometric subvariety of \mathcal{M}_{10} . In particular, \mathcal{K}_{10} can be thought of as either

(1) the locus of curves $[C] \in \mathcal{M}_{10}$ for which the rank 2 Mukai type Brill-Noether locus

$$\text{SU}_2(C, K_C, 6) := \{E \in \text{SU}_2(C, K_C) : h^0(C, E) \geq 7\}$$

is not equal to \emptyset , or

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(2) the locus of curves $[C] \in \mathcal{M}_{10}$ carrying a pencil $A \in W_6^1(C)$ such that the multiplication map $\mu_A : \text{Sym}^2 H^0(C, K_C \otimes A^\vee) \rightarrow H^0(C, (K_C \otimes A^\vee)^{\otimes 2})$ is not surjective.

The geometric conditions (1) and (2), unlike the original definition of \mathcal{K}_{10} , can be extended to other genera. We fix $g = 2k - 2$ and denote by $\sigma : \mathfrak{G}_k^1 \rightarrow \mathcal{M}_g$ the Hurwitz scheme of k -sheeted coverings of \mathbf{P}^1 of genus g parametrizing pairs (C, A) with $A \in W_k^1(C)$. For each $i \geq 0$ we introduce the cycle $\mathcal{U}_{g,i}$ consisting of pairs $(C, A) \in \mathfrak{G}_k^1$ such that $K_C \otimes A^\vee$ fails the Green-Lazarsfeld property (N_i) . By setting $\mathcal{Z}_{g,i} := \sigma(\mathcal{U}_{g,i})$ we obtain an induced geometric stratification of \mathcal{M}_g

$$\mathcal{Z}_{g,0} \subset \mathcal{Z}_{g,1} \subset \dots \subset \mathcal{Z}_{g,i} \subset \dots \subset \mathcal{M}_g.$$

If for each $(C, A) \in \mathfrak{G}_k^1$ we consider the map $C \xrightarrow{|K_C \otimes A^\vee|} \mathbf{P}^{k-2}$ induced by the residual linear system, we can define two vector bundles \mathcal{A} and \mathcal{B} on \mathfrak{G}_k^1 such that

$$\mathcal{A}(C, A) = H^0(\Omega_{\mathbf{P}^{k-2}}^i(i+2)) \text{ and } \mathcal{B}(C, A) = H^0(\Omega_{\mathbf{P}^{k-2}}^i(i+2) \otimes \mathcal{O}_C).$$

For $g = 6i + 10$ (hence $k = 3i + 6$), it turns out that $\text{rank}(\mathcal{A}) = \text{rank}(\mathcal{B})$ and $\mathcal{U}_{g,i}$ is the degeneracy locus of the natural vector bundle map $\phi : \mathcal{A} \rightarrow \mathcal{B}$, that is, $\mathcal{Z}_{g,i}$ is a virtual divisor on \mathcal{M}_g , with virtual class $\sigma_* c_1(\mathcal{B} - \mathcal{A})$. The main result of this article is the computation of the compactification inside $\overline{\mathcal{M}}_g$ of this degeneracy locus (see Theorem 4.1 for a more precise statement):

Theorem A. *If $\sigma : \overline{\mathfrak{G}}_{3i+6}^1 \rightarrow \overline{\mathcal{M}}_{6i+10}$ is the compactification of the Hurwitz stack by limit linear series, then there is a natural extension of the vector bundle map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ over $\overline{\mathfrak{G}}_{3i+6}^1$ such that $\overline{\mathcal{Z}}_{g,i}$ is the degeneracy locus of ϕ . Moreover the class of the pushforward to $\overline{\mathcal{M}}_g$ of the virtual degeneracy locus of ϕ is given by*

$$\sigma_*(c_1(\mathcal{B}) - c_1(\mathcal{A})) \equiv a \lambda - b_0 \delta_0 - b_1 \delta_1 - \dots - b_{3i+5} \delta_{3i+5},$$

where a, b_0, \dots, b_{3i+5} are explicitly determined coefficients such that $b_j \geq b_0$ for $j \geq 1$ and

$$\frac{a}{b_0} = \frac{3(4i+7)(6i^2+19i+12)}{(i+2)(12i^2+31i+18)} < 6 + \frac{12}{g+1}.$$

Corollary. *If the degeneracy locus $\mathcal{Z}_{6i+10,i}$ is an actual divisor on \mathcal{M}_{6i+10} , then we have that*

$$s(\overline{\mathcal{Z}}_{6i+10,i}) = \frac{3(4i+7)(6i^2+19i+12)}{(i+2)(12i^2+31i+18)} < 6 + \frac{12}{g+1},$$

thus contradicting the Slope Conjecture on $\overline{\mathcal{M}}_{6i+10}$.

The idea of the proof is to define a whole host of vector bundles $\mathcal{G}_{a,b}$ and $\mathcal{H}_{a,b}$ over \mathfrak{G}_k^1 for $a, b \geq 0$, having fibres

$$\mathcal{G}_{a,b}(C, A) = H^0(\Omega_{\mathbf{P}^{k-2}}^a(a+b) \otimes \mathcal{O}_C) \text{ and } \mathcal{H}_{a,b}(C, A) = H^0(\Omega_{\mathbf{P}^{k-2}}^a(a+b)).$$

These bundles are related to one another by certain exact sequences (4) and (6) over \mathfrak{G}_k^1 (see Section 3). After an analysis over each boundary divisor $\sigma^{-1}(\Delta_j)$, where $0 \leq j \leq 3i+5$, we find in Section 3 a unique way of extending $\mathcal{G}_{a,b}$ and $\mathcal{H}_{a,b}$ to vector bundles over $\overline{\mathfrak{G}}_k^1$ such that (4) and (6) continue to make sense and be exact and then use this to compute the Chern numbers of $\mathcal{A} = \mathcal{H}_{i,2}$ and $\mathcal{B} = \mathcal{G}_{i,2}$.

We expect $\mathcal{Z}_{6i+10,i}$ to be always a divisor on \mathcal{M}_{6i+10} but we are able to check this only for small i . To verify this in general one would have to prove that if $[C] \in \mathcal{M}_{6i+10}$ is a general curve, then one (or equivalently all) of the finitely many linear systems $\mathfrak{g}_{9i+12}^{3i+4} = K_C(-\mathfrak{g}_{3i+6}^1)$ satisfies property (N_i) . This is a direct generalization of Green's Conjecture to the case of curves and line bundles L with $h^1(L) = 2$ (see Section 2 for more on this analogy).

For $g = 10$ we recover of course the results from [FP]. In the next two cases, $g = 16$ and 22 we have complete results:

Theorem 1.1. *The following subvariety of \mathcal{M}_{16}*

$$\mathcal{Z}_{16,1} := \{[C] \in \mathcal{M}_{16} : \exists L \in W_{21}^7(C) \text{ such that } C \xrightarrow{|L|} \mathbf{P}^7 \text{ is not cut out by quadrics}\},$$

is an effective divisor, and the class of its compactification is given by the formula

$$\overline{\mathcal{Z}}_{16,1} \equiv 286(407\lambda - 61\delta_0 - 325\delta_1 - b_2\delta_2 - \cdots - b_8\delta_8),$$

where $b_j \geq b_1$ for all $2 \leq j \leq 8$. In particular $s(\overline{\mathcal{Z}}_{16,1}) = 407/61 = 6.6721\dots < 6 + 12/17 = 6.705\dots$, hence $\overline{\mathcal{Z}}_{16,1}$ provides a counterexample to the Slope Conjecture on $\overline{\mathcal{M}}_{16}$.

In a similar manner we have the following example of a geometric divisor on $\overline{\mathcal{M}}_{22}$ of very small slope:

Theorem 1.2. *The following subvariety of \mathcal{M}_{22}*

$$\mathcal{Z}_{22,2} := \{[C] \in \mathcal{M}_{22} : \exists L \in W_{30}^{10}(C) \text{ such that } C \xrightarrow{|L|} \mathbf{P}^{10} \text{ fails property } (N_2)\},$$

is an effective divisor on \mathcal{M}_{22} and the class of its compactification is given by the formula

$$\overline{\mathcal{Z}}_{22,2} \equiv 25194(1665\lambda - 256\delta_0 - 1407\delta_1 - b_2\delta_2 - \cdots - b_{11}\delta_{11}),$$

where $b_j \geq b_1$ for all $2 \leq j \leq 11$. In particular $s(\overline{\mathcal{Z}}_{22,2}) = 1665/256 = 6.50390\dots < 6 + 12/23 = 6.52173\dots$, and $\overline{\mathcal{Z}}_{22,2}$ gives a counterexample to the Slope Conjecture on $\overline{\mathcal{M}}_{22}$.

We note that while Theorem A gives a sequence of (virtual) examples of divisors contradicting the Slope Conjecture, for any given g one can construct other (actual) divisors on \mathcal{M}_g of slope $< 6 + 12/(g+1)$ also defined in terms of syzygies. For instance, the locus consisting of curves $[C] \in \overline{\mathcal{M}}_{20}$ for which there exists $L \in W_{20}^4(C)$ such that $C \xrightarrow{|L|} \mathbf{P}^4$ is not cut out by *quartics* (or equivalently, the map $I_4(L) \otimes H^0(L) \rightarrow I_5(L)$ is not bijective), is a divisor on $\overline{\mathcal{M}}_{20}$ violating the Slope Conjecture.

We record the following consequence of Theorems 1.1 and 1.2:

Theorem 1.3. (1) *The moduli space of n -pointed curves $\overline{\mathcal{M}}_{22,n}$ is of general type for all $n \geq 2$.*

(2) *The moduli space $\overline{\mathcal{M}}_{21,n}$ is of general type for all $n \geq 4$.*

(3) *The moduli space $\overline{\mathcal{M}}_{16,n}$ is of general type for all $n \geq 9$.*

(4) *The moduli space $\overline{\mathcal{M}}_{20,n}$ is of general type for $n \geq 6$.*

It is also possible to give examples of divisors of small slope on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of n -pointed stable curves. We have only pursued this for genera $g \leq 22$ with the goal of proving that various moduli spaces $\overline{\mathcal{M}}_{g,n}$ are of general type (this is automatic for $g \geq 23$). One of our examples is the following:

Theorem 1.4. *The following subvariety of $\overline{\mathcal{M}}_{14,1}$*

$$\mathcal{Z}_{14,0}^1 := \{[C, p] \in \mathcal{M}_{14,1} : \exists L \in W_{18}^6(C) \text{ such that } C \xrightarrow{|L(-p)|} \mathbf{P}^5 \text{ fails property } (N_0)\}$$

is an effective divisor on $\mathcal{M}_{14,1}$. The class of its compactification $\overline{\mathcal{Z}}_{14,0}^1$ is given by the formula

$$\overline{\mathcal{Z}}_{14,0}^1 \equiv 33(237\lambda + 14\psi - 35\delta_0 - 169\delta_1 - 183\delta_{13} - b_2\delta_2 - \cdots - b_{12}\delta_{12}),$$

where $b_j \geq 15 + 27j$ for $3 \leq j \leq 12$, $j \neq 4$, $b_2 \geq 325$ and $b_4 \geq 271$. In particular $[\overline{\mathcal{Z}}_{14,0}^1]$ lies outside the cone of $\text{Pic}(\overline{\mathcal{M}}_{14,1})$ spanned by pullbacks of effective divisors from $\overline{\mathcal{M}}_{14}$, the boundary divisors $\delta_0, \dots, \delta_{13}$ and the Weierstrass divisor $\overline{\mathcal{W}}$ on $\overline{\mathcal{M}}_{14,1}$.

An equivalent formulation of the Slope Conjecture on $\overline{\mathcal{M}}_g$ for g such that $g+1$ is composite (which we learned from S. Keel), is to say that the Brill-Noether divisors $\overline{\mathcal{M}}_{g,d}^r$ consisting of curves with a \mathfrak{g}_d^r when $g - (r+1)(g-d+r) = -1$, lie on a face of the effective cone $\text{Eff}(\overline{\mathcal{M}}_g)$. It is well-known that $s(\overline{\mathcal{M}}_{g,d}^r) = 6 + 12/(g+1)$ (cf. [EH3]), so the Slope Conjecture singles out these divisors as being of minimal slope. One can ask a similar question on $\overline{\mathcal{M}}_{g,1}$. Are the pullbacks $\pi^*(\overline{\mathcal{M}}_{g,d}^r)$ of the Brill-Noether divisors from $\overline{\mathcal{M}}_g$ and the Weierstrass divisor $\overline{\mathcal{W}} := \{[C, p] \in \overline{\mathcal{M}}_{g,1} : p \in C \text{ is a Weierstrass point}\}$ on a face of the effective cone $\text{Eff}(\overline{\mathcal{M}}_{g,1})$? The question makes sense especially since in [EH2] it is proved that the class of any generalized Brill-Noether divisor on $\overline{\mathcal{M}}_{g,1}$ (that is, any codimension 1 locus of curves $(C, p) \in \overline{\mathcal{M}}_{g,1}$ having a linear series with special ramification at p) lies inside the cone of $\text{Pic}(\overline{\mathcal{M}}_{g,1})$ spanned by $[\pi^*(\overline{\mathcal{M}}_{g,d}^r)]$ and $[\overline{\mathcal{W}}]$. Theorem 1.4 shows that at least for $g \geq 14$ the answer to the question raised above is emphatically negative.

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2. SYZYGIES OF ALGEBRAIC CURVES

In this paragraph we review a few facts about the resolution of the ideal of a curve embedded in a projective space. As a general reference for syzygies and Koszul cohomology we recommend [L] or [GL].

Suppose that C is a smooth curve of genus g and L is a very ample line bundle on C giving an embedding $C \rightarrow \mathbf{P}^r = \mathbf{P}(V)$, where $V = H^0(L)$. We denote by I_{C/\mathbf{P}^r} the ideal of C in \mathbf{P}^r and consider its minimal resolution of free $S = \text{Sym}(V)$ -modules

$$0 \rightarrow F_{r+1} \rightarrow \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow I_{C/\mathbf{P}^r} \rightarrow 0.$$

Then one can write $F_j = \bigoplus_{l \in \mathbb{Z}} S(-j-l)^{b_{j,l}(C)}$, where $b_{j,l}(C) = \dim_{\mathbb{C}} \text{Tor}_j^S(S(C), \mathbb{C})_{j+l}$ is the graded Betti number of C that comes on the l -th row and j -th column in the Betti

diagram of C . Following Green and Lazarsfeld we say that the pair (C, L) satisfy the property (N_i) for some integer $i \geq 1$, if $F_j = \oplus S(-j-1)$ for all $j \leq i$ (or equivalently in terms of graded Betti numbers, $b_{i,l}(C) = 0$ for all $l \geq 2$). Using the computation of $b_{j,l}(C)$ in terms of Koszul cohomology, there is a well-known cohomological interpretation of property (N_i) : we denote by $M = \Omega_{\mathbb{P}^r}(1)$ and $M_L = M \otimes \mathcal{O}_C$, hence we have an exact sequence

$$0 \rightarrow M_L \rightarrow H^0(L) \otimes \mathcal{O}_C \rightarrow L \rightarrow 0.$$

By taking exterior powers, for each $i \geq 0$ we obtain the exact sequence:

$$(1) \quad 0 \rightarrow \wedge^{i+1} M_L \rightarrow \wedge^{i+1} H^0(L) \otimes \mathcal{O}_C \rightarrow \wedge^i M_L \otimes L \rightarrow 0.$$

If L is a normally generated line bundle on C , then (C, L) satisfies property (N_i) if and only if for all $j \geq 1$, the natural map

$$u_{i,j} : \wedge^{i+1} H^0(L) \otimes H^0(L^{\otimes j}) \rightarrow H^0(\wedge^i M_L \otimes L^{\otimes(j+1)})$$

obtained by tensoring the sequence (1) and taking global sections, is surjective (cf. e.g. [GL], Lemma 1.10).

We will be interested in the vector bundle M_L in the case when C is a curve of genus $g = 2k - 2$ and $L = K_C \otimes A^\vee$ is residual to a base point free pencil $A \in W_k^1(C)$. We start by establishing a more general technical result used throughout the paper:

Proposition 2.1. *Let C be a curve of genus g and L a globally generated \mathfrak{g}_d^r on C . If $p \geq 1$ is an integer such that $g + \min\{r, p\} \leq d$, then for all $j \geq 2$ and $0 \leq i \leq p + 2 - j$, we have the vanishing $H^1(\wedge^i M_L \otimes L^{\otimes j}) = 0$.*

Proof. We use a filtration argument due to Lazarsfeld [L]. We choose general points $x_1, \dots, x_{r-1} \in C$ such that $h^0(C, L \otimes \mathcal{O}_C(-x_1 - \dots - x_{r-1})) = 2$ and then there is an exact sequence

$$0 \longrightarrow L^\vee(x_1 + \dots + x_{r-1}) \longrightarrow M_L \longrightarrow \bigoplus_{j=1}^{r-1} \mathcal{O}_C(-x_j) \longrightarrow 0.$$

Taking exterior powers, we can write the exact sequence

$$0 \rightarrow \wedge^{i-1} \left(\bigoplus_{j=1}^{r-1} \mathcal{O}_C(-x_j) \right) \otimes L^\vee(x_1 + \dots + x_{r-1}) \longrightarrow \wedge^i M_L \longrightarrow \wedge^i \left(\bigoplus_{j=1}^{r-1} \mathcal{O}_C(-x_j) \right) \longrightarrow 0.$$

In order to conclude that $H^0(\wedge^i M_L \otimes L^{\otimes j}) = 0$, it suffices to show that for all $1 \leq i \leq \min\{r-1, p\}$ we have the following:

- (1) $H^0(L^{\otimes j} \otimes \mathcal{O}_C(-D_i)) = 0$ for each effective divisor $D_i \in C_i$ with support in the set $\{x_1, \dots, x_{r-1}\}$, and
- (2) $H^0(L^{\otimes(j-1)} \otimes \mathcal{O}_C(D_{r-i})) = 0$, for any effective divisor $D_{r-i} \in C_{r-i}$ with support contained in $\{x_1, \dots, x_{r-1}\}$.

Statement (2) follows because of degree reasons. The only way (1) could fail is when $j = 2$ and $h^0(K_C \otimes L^\vee) \geq r - i + 1$, which by the Riemann-Roch theorem implies that $g + i \geq d + 1$ and this contradicts our numerical assumption.

□

Remark 2.2. (1) For $p \geq r$, the condition $g + \min\{p, r\} \leq d$ is equivalent with $H^1(L) = 0$. (2) Since we work in characteristic 0, the same filtration argument from the proof of Proposition 2.1 can be used to show that M_L is a stable vector bundle on C .

We fix $g = 2k - 2$ where $k \geq 3$. We denote by \mathcal{M}_g^0 the open substack of \mathcal{M}_g consisting of curves C of genus g which carry no \mathfrak{g}_{k-1}^1 's and which have no automorphisms (clearly the complement of \mathcal{M}_g^0 inside \mathcal{M}_g has codimension ≥ 2). Over \mathcal{M}_g^0 we have a universal curve $\pi : \mathcal{C} \rightarrow \mathcal{M}_g^0$ and we also consider the Hurwitz stack $\sigma : \mathfrak{G}_k^1 \rightarrow \mathcal{M}_g^0$ parametrising pairs (C, A) , where $[C] \in \mathcal{M}_g^0$ and $A \in W_k^1(C)$ is a (necessarily base point free) \mathfrak{g}_k^1 on C . It is a classical result that \mathfrak{G}_k^1 is a smooth irreducible stack which is finite over \mathcal{M}_g^0 . The isomorphism $\mathfrak{G}_k^1 \ni (C, A) \mapsto (C, K_C \otimes A^\vee) \in \mathfrak{G}_{3k-6}^{k-2}$ will be used throughout the paper.

Let us consider now an element $(C, A) \in \mathfrak{G}_k^1$ and set $L := K_C \otimes A^\vee$. The genericity assumption on C implies that L is very ample and gives an embedding $C \subset \mathbf{P}^{k-2}$ of degree $3k - 6$. We denote by \mathcal{I}_C the ideal sheaf of C in this embedding.

Proposition 2.3. *Fix integers $g = 2k - 2$ with $k \geq 3$ and $i \geq 0$ with $2k \geq 3i + 7$. If $(C, A) \in \mathfrak{G}_k^1$ and $L = K_C \otimes A^\vee$, then (C, L) fails to satisfy property (N_i) if and only if*

$$h^0(\mathbf{P}^{k-2}, \wedge^i M_{\mathbf{P}^{k-2}} \otimes \mathcal{I}_C(2)) \geq \binom{k-2}{i+1}(k-3i-6) + 2 \binom{k-1}{i+1} - \binom{k-1}{i+2} + 1.$$

Proof. We use the following diagram of exact sequences for the embedding $C \xrightarrow{|L|} \mathbf{P}^{k-2}$:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \wedge^{i+1} M_{\mathbf{P}^{k-2}} \otimes \mathcal{I}_C(1) & \longrightarrow & \wedge^{i+1} H^0(\mathcal{O}_{\mathbf{P}^{k-2}}(1)) \otimes \mathcal{I}_C(1) & \longrightarrow & \wedge^i M_{\mathbf{P}^{k-2}} \otimes \mathcal{I}_C(2) \\ \downarrow & & \downarrow & & \downarrow \\ \wedge^{i+1} M_{\mathbf{P}^{k-2}}(1) & \longrightarrow & \wedge^{i+1} H^0(\mathcal{O}_{\mathbf{P}^{k-2}}(1)) \otimes \mathcal{O}_{\mathbf{P}^{k-2}}(1) & \longrightarrow & \wedge^i M_{\mathbf{P}^{k-2}}(2) \\ \downarrow a & & \downarrow b & & \downarrow c \\ \wedge^{i+1} M_L \otimes L & \longrightarrow & \wedge^{i+1} H^0(L) \otimes L & \longrightarrow & \wedge^i M_L \otimes L^2 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

For simplicity we denote by the same symbol the map on global sections corresponding to a sheaf morphism and the morphism itself. Since $H^1(\mathbf{P}^{k-2}, \wedge^{i+1} M_{\mathbf{P}^{k-2}}(1)) = 0$ (because of Bott vanishing, see [OSS], pg. 8), we can apply the snake lemma to the diagram obtained by taking global sections in the two lower rows. The map induced by b is an isomorphism, which gives an isomorphism

$$H^0(\mathbf{P}^{k-2}, \wedge^i M_{\mathbf{P}^{k-2}} \otimes \mathcal{I}_C(2)) \cong H^1(\mathbf{P}^{k-2}, \wedge^{i+1} M_{\mathbf{P}^{k-2}} \otimes \mathcal{I}_C(1)).$$

From Proposition 2.1 we have that $H^1(\wedge^i M_L \otimes L^2) = 0$, and thus C satisfies (N_i) if and only if the map

$$H^1(\wedge^{i+1} M_L \otimes L) \rightarrow \wedge^{i+1} H^0(L) \otimes H^1(L)$$

is an isomorphism, or equivalently $h^1(\wedge^{i+1} M_L \otimes L) \leq h^1(\wedge^{i+1} H^0(L) \otimes L) = 2 \binom{k-1}{i+1}$.

From the Riemann-Roch theorem, this is the same thing as having

$$\dim(\text{Coker}(a)) \leq \binom{k-2}{i+1}(k-3i-6) + 2 \binom{k-1}{i+1} - \binom{k-1}{i+2}.$$

But $\text{Coker}(a) \cong H^0(\wedge^i M_{\mathbf{P}^{k-2}} \otimes \mathcal{I}_C(2))$, so failure of (N_i) is equivalent to the map c having a kernel of dimension at least

$$\begin{aligned} & \binom{k-2}{i+1}(k-3i-6) + 2\binom{k-1}{i+1} - \binom{k-1}{i+2} + 1 = \\ & = \binom{k-2}{i+1}(k-3i-6)\left(1 - \frac{1}{(k-i-2)(i+2)}\right) + 1. \end{aligned}$$

□

Remark 2.4. An immediate consequence of this proof is that when $k - 3i - 6 < 0$ (or equivalently $g \leq 6i + 8$), the pair (C, A) always fails property (N_i) . When $k = 3i + 6$ we have that (C, A) fails (N_i) if and only if $h^0(\mathbf{P}^{k-2}, \wedge^i M_{\mathbf{P}^{k-2}} \otimes \mathcal{I}_C(2)) \geq 1$, which we expect to be a divisorial condition on \mathfrak{G}_k^1 .

This proposition allows us to define a determinantal substack of \mathfrak{G}_k^1 consisting of pairs (C, A) for which L fails to satisfy property (N_i) . On the fibre product $\mathcal{C} \times_{\mathcal{M}_g^0} \mathfrak{G}_k^1$ there is a universal Poincaré bundle \mathcal{L} whose existence is guaranteed by the universal property of \mathfrak{G}_k^1 . If $p_1 : \mathcal{C} \times_{\mathcal{M}_g^0} \mathfrak{G}_k^1 \rightarrow \mathcal{C}$ and $p_2 : \mathcal{C} \times_{\mathcal{M}_g^0} \mathfrak{G}_k^1 \rightarrow \mathfrak{G}_k^1$ are the natural projections, then $\mathcal{E} := p_{2*}(p_1^* \omega_\pi \otimes \mathcal{L}^\vee)$ is a vector bundle of rank $k-1$ with fibre $\mathcal{E}(C, A) = H^0(C, K_C \otimes A^\vee)$ over each point $(C, A) \in \mathfrak{G}_k^1$. We have a tautological embedding of the pullback of the universal curve $\mathcal{C} \times_{\mathcal{M}_g^0} \mathfrak{G}_k^1$ into the projective bundle $u : \mathbf{P}(\mathcal{E}) \rightarrow \mathfrak{G}_k^1$ and we denote by $\mathcal{J} \subset \mathcal{O}_{\mathbf{P}(\mathcal{E})}$ the ideal sheaf of the image. Next, we define the vector bundle \mathcal{N} on $\mathbf{P}(\mathcal{E})$ by the sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow u^*(\mathcal{E}) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \longrightarrow 0,$$

and we further introduce two vector bundles \mathcal{A} and \mathcal{B} over \mathfrak{G}_k^1 by setting

$$\mathcal{A} := u_*(\wedge^i \mathcal{N} \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(2)), \text{ and } \mathcal{B} := u_*(\wedge^i \mathcal{N} \otimes \mathcal{O}_{\mathcal{C} \times_{\mathcal{M}_g^0} \mathfrak{G}_k^1}(2)).$$

If $C \subset \mathbf{P}^{k-2}$ is the embedding given by $L = K_C \otimes A^\vee$, then

$$\mathcal{A}(C, A) = H^0(\mathbf{P}^{k-2}, \wedge^i M_{\mathbf{P}^{k-2}}(2)) \text{ and } \mathcal{B}(C, A) = H^0(C, \wedge^i M_L \otimes L^{\otimes 2})$$

and there is a natural vector bundle morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$. From Grauert's Theorem we see that both \mathcal{A} and \mathcal{B} are vector bundles over \mathfrak{G}_k^1 (for \mathcal{B} use that $H^1(C, \wedge^i M_L \otimes L^2) = 0$ because of Proposition 2.1). Moreover from Bott's Theorem (see again [OSS]) and the Riemann-Roch Theorem respectively, we obtain that

$$\text{rank}(\mathcal{A}) = (i+1) \binom{k}{i+2} \text{ and } \text{rank}(\mathcal{B}) = \binom{k-2}{i} (4k-9-3i).$$

Then Proposition 2.3 can be restated as follows:

Proposition 2.5. *The cycle*

$$\mathcal{U}_{g,i} := \{(C, A) \in \mathfrak{G}_k^1 : (C, K_C \otimes A^\vee) \text{ fails property } (N_i)\},$$

can be realized as the degeneracy locus of rank $(i-1) \binom{k-1}{i+1} + (i+2) \binom{k-1}{i+2} - \binom{k-2}{i+1} (k-3i-6) - 1$ of the vector bundle map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ over \mathfrak{G}_k^1 .

In this way we obtain a stratification of \mathcal{M}_g^0 with strata $\mathcal{Z}_{g,i} := \sigma(\mathcal{U}_{g,i})$ consisting of those curves C for which there exists a pencil $A \in W_k^1(C)$ such that $(C, K_C \otimes A^\vee)$ fails property (N_i) . Note that when $g = 6i + 10$ then $\text{rank}(A) = \text{rank}(B)$ and $\mathcal{U}_{g,i}$ is simply the degeneracy locus of ϕ and we expect $\mathcal{Z}_{g,i}$ to be a divisor on \mathcal{M}_g^0 .

It is of course very natural to compare this newly defined stratification

$$\mathcal{Z}_{g,0} \subset \mathcal{Z}_{g,1} \subset \dots \subset \mathcal{Z}_{g,i} \subset \dots \subset \mathcal{M}_g^0$$

with the more classical stratification of \mathcal{M}_g defined in terms of syzygies of K_C . If for each $i \geq 0$ we set

$$\mathcal{N}_{g,i} := \{[C] \in \mathcal{M}_g : (C, K_C) \text{ fails property } (N_i)\},$$

then Green's Conjecture for generic curves of fixed gonality (cf. [V3], [V2]) can be read as saying that for all $0 \leq i \leq (g-2)/2$, the locus $\mathcal{N}_{g,i}$ coincides with the $(i+2)$ -gonal locus $\mathcal{M}_{g,i+2}^1$ of curves with a \mathfrak{g}_{i+2}^1 , while $\mathcal{N}_{g,i} = \mathcal{M}_g$ for all $i \geq (g-1)/2$. One of the morals of this paper is that the stratification $\mathcal{Z}_{g,i}$ is very different in nature from the one given by gonality: whereas $\mathcal{N}_{g,0}$ is the locus of hyperelliptic curves of genus g , the smallest stratum $\mathcal{Z}_{g,0}$ is intimately related to the locus of curves lying on $K3$ surfaces which is well-known to be transversal to any Brill-Noether locus (a curve of genus g lying on a general $K3$ surface satisfies the Brill-Noether Theorem, cf. [L]).

We have already seen that $\mathcal{Z}_{g,i} = \mathcal{M}_g^0$ when $g \leq 6i + 8$ (cf. Remark 2.4). A result due to Mukai and Voisin (cf. [V1], Proposition 3.2) states that if C is a smooth curve of genus $g = 2k - 2$ sitting on a $K3$ surface S with $\text{Pic}(S) = \mathbb{Z}[C]$, then for all $A \in W_k^1(C)$ the multiplication map $\text{Sym}^2 H^0(K_C \otimes A^\vee) \rightarrow H^0((K_C \otimes A^\vee)^{\otimes 2})$ is not surjective, in other words, $[C] \in \mathcal{Z}_{g,0}$. In a forthcoming paper we will show that the converse also holds, that is, the closure in \mathcal{M}_g of the smallest stratum $\mathcal{Z}_{g,0}$ coincides with the locus

$$\mathcal{K}_g := \{[C] \in \mathcal{M}_g : C \text{ lies on a } K3 \text{ surface}\}.$$

The possibility of such an equality of cycles has already been raised in Voisin's paper (cf. [V1], Remarques 4.13). Its main appeal lies in the fact that it gives an intrinsic characterization of a curve lying on a $K3$ surface which makes no reference to the $K3$ surface itself! We now make the following:

Conjecture 2.6. For an even genus $g \geq 6i + 10$, the stratum $\mathcal{Z}_{g,i}$ is a proper subvariety of \mathcal{M}_g^0 . In particular, when $g = 6i + 10$, the stratum $\mathcal{Z}_{g,i}$ is a divisor on \mathcal{M}_g^0 .

For $g \leq 8$ the conjecture is trivially true because $\mathcal{K}_g = \mathcal{M}_g$, that is, every curve of (even) genus $g \leq 8$ sits on a $K3$ surface. The first interesting case is $g = 10$ when the conjecture holds and the identification $\mathcal{K}_{10} = \mathcal{Z}_{10,0}$ is part of a more general picture (see [FP] for details). We have checked Conjecture 2.6 for all genera $g \leq 24$ sometimes using the program Macaulay. We will describe this in detail in the most interesting cases, $g = 16$ and $g = 22$, when the strata $\mathcal{Z}_{16,1}$ and $\mathcal{Z}_{22,2}$ are divisors on \mathcal{M}_{16} and \mathcal{M}_{22} respectively.

Theorem 2.7. Let C be a general curve of genus 16. Then every linear series $A \in W_9^1(C)$ gives an embedding $C \xrightarrow{|K_C \otimes A^\vee|} \mathbf{P}^7$ of degree 21 such that the ideal of C is generated by quadrics (that is, it satisfies property (N_1)). In particular $\mathcal{Z}_{16,1}$ is a divisor on \mathcal{M}_{16} .

Proof. From the irreducibility of the Hurwitz scheme of coverings of \mathbf{P}^1 it follows that the variety $\mathfrak{G}_9^1 = \mathfrak{G}_{16,9}^1$ parametrizing pairs (C, A) with $[C] \in \mathcal{M}_{16}$ and $A \in W_9^1(C)$ is irreducible. To prove that $\mathcal{Z}_{16,1}$ is a divisor it is enough to exhibit an element $(C, A) \in \mathfrak{G}_9^1$ with $K_C \otimes A^\vee$ very ample, such that the embedded curve $C \xrightarrow{|K_C \otimes A^\vee|} \mathbf{P}^7$ is cut out by quadrics.

We consider 13 general points in \mathbf{P}^2 denoted by $p_1, p_2, q_1, \dots, q_7$ and r_1, \dots, r_4 respectively, and define the linear system

$$H \equiv 8h - 3(E_{p_1} + E_{p_2}) - 2 \sum_{i=1}^7 E_{q_i} - \sum_{j=1}^4 E_{r_j}$$

on the blow-up $S = \text{Bl}_{13}(\mathbf{P}^2)$. Here h denotes the pullback of the line class from \mathbf{P}^2 .

Using the program Macaulay it is easy to check that $S \xrightarrow{|H|} \mathbf{P}^7$ is an embedding and the graded Betti diagram of S is

$$\begin{array}{cccc} 1 & - & - & - \\ - & 7 & - & - \\ - & - & 35 & 56 \end{array}$$

Thus S is cut out by quadrics. To carry out this calculation we chose the 13 points in \mathbf{P}^2 randomly using the Hilbert-Burch theorem so that they satisfy the Minimal Resolution Conjecture (see [SchT] for details on how to pick random points in \mathbf{P}^2 using Macaulay). Next we consider a curve $C \subset S$ in the linear system

$$(2) \quad C \equiv 14h - 5(E_{p_1} + E_{p_2}) - 4 \sum_{i=1}^6 E_{q_i} - 3E_{q_7} - 2 \sum_{j=1}^3 E_{r_j} - E_{r_4}.$$

By using Macaulay we pick C randomly in its linear system and then check that C is smooth, $g(C) = 16$ and $\deg(C) = 21$. To show that C is cut out by quadrics one can compute directly the Betti diagram of C . Otherwise, since S is cut out by quadrics, to conclude the same thing about C , it suffices to show that the map

$$m : H^0(S, H) \otimes H^0(S, 2H - C) \rightarrow H^0(S, 3H - C)$$

is surjective (or equivalently injective). Since $h^0(S, 2H - C) = 2$, from the base point free pencil trick we get that $\text{Ker}(m) = H^0(S, C - H) = 0$, because $C - H \equiv 6h - 2E_{p_1} - 2E_{p_2} - 2 \sum_{i=1}^6 E_{q_i} - E_{q_7} - \sum_{j=1}^3 E_{r_j}$ is clearly not effective for a general choice of the 13 points. \square

Remark 2.8. A possible objection to this proof could be that while we defined the Hurwitz space $\mathfrak{G}_9^1 \rightarrow \mathcal{M}_{16}^0$ over the open part of \mathcal{M}_{16} consisting of curves carrying no g_8^1 's, the curve C we constructed in the proof of Theorem 2.7 might lie outside \mathcal{M}_{16}^0 . However, this does not affect our conclusions because for every g and $l < (g + 3)/2$, the Hurwitz space of pairs (C, A) with $[C] \in \mathcal{M}_g$ and $A \in W_l^1(C)$, is an *irreducible* variety, that is, there can be no component of the Hurwitz space mapping to a proper subvariety of the l -gonal locus $\mathcal{M}_{g,l}^1$ (This statement can be easily proved with the methods from [AC1]).

Remark 2.9. For later use we are going to record a slight generalization of Theorem 2.7. The conclusion of the theorem holds for the image in \mathbf{P}^7 of any plane curve C that is sufficiently general in the linear system given by (2). In particular one can choose C to have a cusp at a general extra point $y \in \mathbf{P}^2$, in which case we obtain that a general one-cuspidal curve $C \subset \mathbf{P}^7$ of arithmetic genus 16 and degree 21 is cut out by quadrics (this again is quite easily checked with Macaulay). Since the Hilbert scheme of these curves is irreducible (use again the Hurwitz scheme), it follows that for a *general* $(C, y) \in \mathcal{M}_{15,1}$ and for an *arbitrary* linear system $L \in W_{21}^7(C)$ having a cusp at y , the ideal of the cuspidal image curve $C \xrightarrow{|L|} \mathbf{P}^7$ is generated by quadrics.

In a somewhat similar manner we are going to show that for $g = 22$ the locus $\mathcal{Z}_{22,2}$ is a divisor.

Theorem 2.10. *Let C be a general curve of genus 22. Then for every linear series $A \in W_{12}^1(C)$ we have that $|K_C \otimes A^\vee|$ is very ample and the resulting embedding $A \xrightarrow{|K_C \otimes A^\vee|} \mathbf{P}^{10}$ satisfies property (N_2) . In particular, the locus of curves C of genus 22 carrying a linear series $\mathfrak{g}_{30}^{10} = K_C(-\mathfrak{g}_{12}^1)$ which fails to satisfy property (N_2) , is a divisor on \mathcal{M}_{22} .*

Proof. Ideally we would like to construct a rational surface $S \subset \mathbf{P}^{10}$ which satisfies property (N_2) and then consider a suitable curve $C \subset S$ with $g(C) = 22$ and $\deg(C) = 30$. For numerical reasons this will turn out not to be possible but we will be able to construct a smooth curve $C' \subset S$ with $g(C') = 20$ and $\deg(C') = 28$ which satisfies (N_2) . The desired curve C will be a smoothing in \mathbf{P}^{10} of $C \cup L \cup L'$, where L and L' are general chords of C' .

We start by choosing random points $p, p_1, \dots, p_7, q_1, \dots, q_5 \in \mathbf{P}^2$. Then the linear system

$$H = 11h - 4E_p - 3 \sum_{i=1}^7 E_{p_i} - 2 \sum_{j=1}^5 E_{q_j}$$

yields an embedding $S \subset \mathbf{P}^{10}$ of $\text{Bl}_{13}(\mathbf{P}^2)$. Using Macaulay we get that the upper left corner of the Betti diagram of S is

$$\begin{array}{cccccc} 1 & - & - & - & - & \\ - & 29 & 98 & 72 & - & \\ - & - & - & 264 & - & \end{array}$$

that is, S satisfies property (N_2) , hence $H^1(S, \wedge^3 M_S(1)) = 0$. If $\Gamma \equiv H$ is a general hyperplane section of S , then $g(\Gamma) = 13$, $\deg(\Gamma) = 22$ and an argument very similar to the one in Proposition 2.1 shows that the vector bundle M_Γ is stable. In particular $H^1(\Gamma, \wedge^2 M_\Gamma(2)) = 0$. Since $M_{S|\Gamma} \cong M_\Gamma \oplus \mathcal{O}_\Gamma$ one can write the exact sequence

$$0 \longrightarrow \wedge^i M_S(1) \longrightarrow \wedge^i M_S(2) \longrightarrow \wedge^i M_\Gamma(2) \oplus \wedge^{i-1} M_\Gamma(2) \longrightarrow 0,$$

which leads to the vanishing $H^2(S, \wedge^3 M_S(1)) = 0$. Suppose now that $C \subset S$ is a smooth non-degenerate curve. We have a commutative diagram:

$$\begin{array}{ccccc} H^1(C, \wedge^3 M_C(1)) & \longrightarrow & \wedge^3 H^0(\mathcal{O}_C(1)) \otimes H^1(\mathcal{O}_C(1)) & & \\ & & \downarrow = & & \downarrow = \\ H^1(\wedge^2 M_S(2H - C)) & \hookrightarrow & H^2(\wedge^3 M_S(H - C)) & \longrightarrow & \wedge^3 H^0(\mathcal{O}_S(1)) \otimes H^2(H - C), \end{array}$$

where the left vertical map is isomorphic because $H^1(S, \wedge^3 M_S(1)) = H^2(S, \wedge^3 M_S(1)) = 0$. To conclude that C satisfies (N_2) , it suffices to show that $H^1(S, \wedge^2 M_S(2H - C)) = 0$, or equivalently, that the map $f : \wedge^2 H^0(\mathcal{O}_S(1)) \otimes H^0(2H - C) \rightarrow H^0(M_S(3H - C))$ obtained from the Koszul complex, is surjective. If $(\deg(C), g(C)) = (30, 22)$, then one can check easily that $h^0(S, 2H - C) = 3$ and $h^0(S, M_S(3H - C)) = 207$, hence f cannot be surjective for dimensional reasons. The closest we can get to these numerical invariants is when $(\deg(C), g(C)) = (28, 20)$ and this is the type of curve on S we will be looking for. Take

$$C' \equiv 17h - 6E_p - 5 \sum_{i=1}^7 E_{p_i} - 3 \sum_{j=1}^5 E_{q_j},$$

and consider the embedding $C' \subset \mathbf{P}^{10}$ given by $|H|$. Using Macaulay we check that C' is a smooth curve of genus 20 and degree 28 with graded Betti diagram

$$\begin{array}{cccc} 1 & - & - & - \\ - & 27 & 80 & - \\ - & - & - & 432 \end{array}$$

Hence C' satisfies (N_2) . Now choose two general chords L and L' of C and define $C := C' \cup L \cup L'$. Clearly $g(C) = 22$, $\deg(C) = 30$ and one last check with Macaulay shows that $b_{2j}(C) = 0$ for $j \geq 2$, that is, C satisfies (N_2) . □

Remark 2.11. Just like in the case $g = 16$ we have a slight variation of the last Theorem. The same proof shows that for a *general* $(C, y) \in \mathcal{M}_{21,1}$ and for an *arbitrary* linear series $L \in W_{30}^{10}(C)$ that has a cusp at y , the one-cuspidal image curve $C \xrightarrow{|L|} \mathbf{P}^{10}$ satisfies property (N_2) .

3. INTERSECTION THEORY CALCULATIONS ON $\overline{\mathcal{M}}_g$

Recall that we have realized $\mathcal{Z}_{6i+10,i}$ as the image of the degeneracy locus $\mathcal{U}_{6i+10,i}$ of a vector bundle morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ over \mathfrak{G}_k^1 . To compute the class of the compactification $\overline{\mathcal{Z}}_{6i+10,i}$ we are going to extend the determinantal structure of $\mathcal{Z}_{6i+10,i}$ over the boundary divisors in $\overline{\mathcal{M}}_{6i+10}$.

We set $g := 6i + 10$, $k := 3i + 6$ and denote by $\widetilde{\mathcal{M}}_g := \mathcal{M}_g^0 \cup (\cup_{j=0}^{3i+5} \Delta_j^0)$ the locally closed substack of $\overline{\mathcal{M}}_g$ defined as the union of the locus \mathcal{M}_g^0 of smooth curves carrying no linear systems \mathfrak{g}_{k-1}^1 to which we add the open subsets $\Delta_j^0 \subset \Delta_j$ for $1 \leq j \leq 3i + 5$ consisting of 1-nodal genus g curves $C \cup_y D$, with $[C] \in \mathcal{M}_{g-j}$ and $[D, y] \in \mathcal{M}_{j,1}$ being Brill-Noether general curves, and the locus $\Delta_0^0 \subset \Delta_0$ containing 1-nodal irreducible genus g curves $C' = C/q \sim y$, where $[C, q] \in \mathcal{M}_{g-1}$ is a Brill-Noether general pointed

curve and $y \in C$, together with their degenerations consisting of unions of a smooth genus $g - 1$ curve and a nodal rational curve. One can then extend the finite covering $\sigma : \mathfrak{G}_{3k-6}^{k-2} \rightarrow \mathcal{M}_g^0$ to a proper, generically finite map

$$\sigma : \widetilde{\mathfrak{G}}_{3k-6}^{k-2} \rightarrow \widetilde{\mathcal{M}}_g$$

by letting $\widetilde{\mathfrak{G}}_{3k-6}^{k-2}$ be the variety of limit $\mathfrak{g}_{3k-6}^{k-2}$'s on the treelike curves from $\widetilde{\mathcal{M}}_g$ (see [EH1], Theorem 3.4 for the construction of the space of limit linear series).

We will be interested in intersecting the divisors $\overline{\mathcal{Z}}_{6i+10,i}$ on $\overline{\mathcal{M}}_g$ with test curves in the boundary of $\overline{\mathcal{M}}_{6i+10}$ which are defined as follows: we fix a Brill-Noether general curve C of genus $2k - 3 = 6i + 9$, a general point $q \in C$ and a general elliptic curve E . We define two 1-parameter families

$$(3) \ C^0 := \left\{ \frac{C}{y \sim q} : y \in C \right\} \subset \Delta_0 \subset \overline{\mathcal{M}}_{2k-2} \text{ and } C^1 := \{C \cup_y E : y \in C\} \subset \Delta_1 \subset \overline{\mathcal{M}}_{2k-2}.$$

It is well-known that these families intersect the generators of $\text{Pic}(\overline{\mathcal{M}}_{2k-2})$ as follows:

$$C^0 \cdot \lambda = 0, \ C^0 \cdot \delta_0 = -(4k - 6), \ C^0 \cdot \delta_1 = 1 \text{ and } C^0 \cdot \delta_a = 0 \text{ for } a \geq 2, \text{ and}$$

$$C^1 \cdot \lambda = 0, \ C^1 \cdot \delta_0 = 0, \ C^1 \cdot \delta_1 = -(4k - 8), \ C^1 \cdot \delta_a = 0 \text{ for } a \geq 2.$$

Next, we fix an integer $2 \leq j \leq k - 1$, a general curve C of genus $2k - 2 - j$ and a general curve pointed curve (D, y) of genus j . We define the 1-parameter family $C^j := \{C \cup_y D : y \in C\} \subset \Delta_j \subset \overline{\mathcal{M}}_{2k-2}$. We have that

$$C^j \cdot \lambda = 0, \ C^j \cdot \delta_a = 0 \text{ for } a \neq j \text{ and } C^j \cdot \delta_j = -(4k - 6 - 2j).$$

During our calculations we will often need the following enumerative result (cf. [HM], pg. 71):

Proposition 3.1. *Let C be a general curve of genus j , $y \in C$ a general fixed point and an integer $0 \leq \alpha \leq j/2$.*

- (1) *The number of pencils $A \in W_{j-\alpha+1}^1(Y)$ satisfying $h^0(Y, A(-(j+1-2\alpha)y)) \geq 1$ is equal to*

$$a(j, \alpha) = \frac{j+1-2\alpha}{j+1-\alpha} \binom{j}{\alpha}.$$

- (2) *The number of pencils $A \in W_{j-\alpha}^1(C)$ satisfying $h^0(Y, A(-(j-2\alpha)q)) \geq 1$ for some unspecified point $q \in Y$ is equal to*

$$b(j, \alpha) = (j-2\alpha-1)(j-2\alpha)(j-2\alpha+1) \binom{j}{\alpha}.$$

Definition 3.2. (1) For a fixed general curve C of genus $2k - 3$ we denote by $\delta(k, 1)$ the (finite) number of linear series $L \in W_{3k-6}^{k-2}(C)$ for which there exists a point $y \in C$ with $h^0(L(-2y)) \geq k - 2$ and $h^0(L(-ky)) \geq 1$.

(2) We fix $2 \leq j \leq k - 1$, $0 \leq \alpha \leq j/2$ and a general curve C of genus $2k - 2 - j$. We denote by $\delta(k, j, \alpha)$ the number of linear series $L_C \in W_{3k-j-4}^{k-2}(C)$ for which there exists a point $y \in C$ such that $h^0(L(-(\alpha+1)y)) \geq k - 1 - \alpha$, $h^0(L(-(j-\alpha+2)y)) \geq k - 1 + \alpha - j$ and $h^0(L(-(k+1)y)) \geq 1$.

The numbers $\delta(k, 1)$ and $\delta(k, j, \alpha)$ can be computed by degeneration. Since we do not need their actual values we skip this calculation. Next we describe $\tilde{\mathfrak{G}}_{3k-6}^{k-2}$ set theoretically. To set notation, if X is a treelike curve and l is a limit \mathfrak{g}_d^r on X , for a component Y of X we denote by $l_Y = (L_Y, V_Y \subset H^0(L_Y))$ the Y -aspect of l . For a point $y \in Y$ we denote by $\{a_s^{l_Y}(C)\}_{s=0 \dots r}$ the vanishing sequence of l at y and by $\rho(l_Y, y) := \rho(g, r, d) - \sum_{i=0}^r (a_i^{l_Y}(y) - i)$ the adjusted Brill-Noether number with respect to y

Proposition 3.3. (1) Let $C_y^1 = C \cup_y E$ be an element of Δ_1^0 . If (l_C, l_E) is a limit $\mathfrak{g}_{3k-6}^{k-2}$ on C_y^1 , then $V_C = H^0(L_C)$ and $L_C \in W_{3k-6}^{k-2}(C)$ has a cusp at y , that is, $h^0(C, L(-2y)) = k - 2$. If $y \in C$ is a general point, then $l_E = (\mathcal{O}_E((3k - 6)y), (2k - 5)y + |(k - 1)y|)$, that is, l_E is uniquely determined. If $y \in C$ is one of the finitely many points for which there exists $L_C \in W_{3k-6}^{k-2}(C)$ such that $\rho(L_C, y) = -1$, then $l_E(-2y)$ is a \mathfrak{g}_k^{k-2} with vanishing sequence at y being $\geq (0, 2, 3, \dots, k - 2, k)$. Moreover, at the level of 1-cycles we have the identification $\sigma^*(C^1) \equiv X + \delta(k, 1) T$, where

$$X := \{(y, D) \in C \times C_{k-2} : h^0(C, D + 2y) \geq 2\}$$

and T is the curve consisting of \mathfrak{g}_k^{k-2} 's on E with vanishing $\geq (0, 2, \dots, k - 2, k)$ at the fixed point $y \in E$.

(2) Let $C_y^0 = C/y \sim q$ be an element of Δ_0^0 . Then limit linear series of type $\mathfrak{g}_{3k-6}^{k-2}$ on C_y^0 are in 1:1 correspondence with complete linear series L on C of type $\mathfrak{g}_{3k-6}^{k-2}$ satisfying the condition $h^0(C, L \otimes \mathcal{O}_C(-y-q)) = h^0(C, L) - 1$. Thus there is an isomorphism between the cycle $\sigma^*(C^0)$ of $\mathfrak{g}_{3k-6}^{k-2}$ on all curves C_y^0 with $y \in C$ and the smooth curve

$$Y := \{(y, D) \in C \times C_{k-2} : h^0(C, D + y + q) \geq 2\}.$$

Proof. We only prove (1) the remaining case being similar. Suppose $l = (l_C, l_E)$ is a limit $\mathfrak{g}_{3k-6}^{k-2}$ on C_y^1 . Using the additivity of the Brill-Noether number (see e.g. [EH1], Lemma 3.6), we get that $0 = \rho(2k - 2, k - 2, 3k - 6) \geq \rho(l_C, y) + \rho(l_E, y)$. Since the vanishing sequence at y of the E -aspect of l is

$$a^{l_E}(y) \leq (2k - 5, 2k - 4, \dots, 3k - 8, 3k - 6),$$

it follows that we also have the inequality $a^{l_C}(y) \geq (0, 2, \dots, k - 1)$. If $y \in C$ is general, then $\rho(l_C, y) = 0$ and $a^{l_C}(y) = (0, 2, \dots, k - 1)$. Moreover, l_C corresponds to a complete linear series $|L_C|$ on C such that $h^1(C, L_C) = 1$ and $h^1(L_C \otimes \mathcal{O}_C(-2y)) = 2$. If $y \in C$ is one of the points such that $\rho(l_C, y) = -1$, then since C is Brill-Noether general we must have $a_0^{l_C}(y) = 0$, which implies that $a^{l_C}(y) = (0, 2, \dots, k - 2, k)$, hence $a^{l_E}(y) \geq (2k - 6, 2k - 4, \dots, 3k - 8, 3k - 6)$, or equivalently, $l_E = (2k - 6)y + |V_E|$, where $V_E \subset H^0(\mathcal{O}_E(ky))$ is a \mathfrak{g}_k^{k-2} with vanishing sequence $\geq (0, 2, 3, \dots, k - 2, k)$ at y . An easy argument shows that the variety Y of such V_E 's is isomorphic to $\mathbf{P}(\mathcal{O}_E(ky)|_{2y})$. \square

Proposition 3.4. Let $C_y^j := C \cup_y D$ be an element from Δ_j^0 , where $2 \leq j \leq k - 1$, $g(C) = 2k - 2 - j$ and $g(D) = j$. Suppose (l_C, l_D) is a limit $\mathfrak{g}_{3k-6}^{k-2}$ on C_y^j . If $y \in C$ is a general point, then l_C has the divisor $(j - 2)y$ as base locus and there exists an integer $0 \leq \alpha \leq [j/2]$ such that $l_C(-(j - 2)y) = |L_C|$, with $L_C \in W_{3k-j-4}^{k-2}(C)$ satisfying the inequality

$$a^{L_C}(y) \geq (0, 1, \dots, \alpha - 1, \alpha + 1, \alpha + 2, \dots, j - \alpha, j - \alpha + 2, j - \alpha + 3, \dots, k - 1, k).$$

Moreover l_D is one of the $a(j, \alpha)$ linear systems on D of type $(2k - j - 4)y + |L_D|$, with $L_D \in W_{k-2+j}^{k-2}(D)$ such that $h^0(L_D(-(k+1-\alpha)y)) = \alpha$ and $h^0(L_D(-(k+\alpha-j)y)) = j - \alpha$.

If for some integer $0 \leq \alpha \leq [j/2]$, $y \in C$ is one of the $b(2k - 2 - j, k - 1 + \alpha - j)$ points for which there exists $L_C \in \text{Pic}^{3k-j-3}(C)$ such that $h^0(L_C(-(\alpha + 2)y)) = k - 1 - \alpha$ and $h^0(L_C((j - \alpha + 2)y)) = k - j + \alpha$, then there are two possibilities:

(1) The aspect l_C is of the form $(j - 2)y + |L_C(-y)|$, whereas $l_D = (2k - j - 4)y + |L_D|$, where $L_D \in W_{k-2+j}^{k-2}(D)$ satisfies $h^0(L_D(-(k+1-\alpha)y)) \geq \alpha$ and $h^0(L_D(-(k+1+\alpha-j)y)) \geq j - \alpha - 1$.

(2) The aspect l_C is of the form $(j - 3)y + (L_C, V_C)$, where $V_C \subset H^0(L_C)$ is such that $H^0(L_C(-2y)) \subset V_C$, and $l_D = (2k - j - 4)y + |L_D|$ with $L_D \in W_{k-2+j}^{k-2}(D)$ and

$$a^{l_D}(y) = (0, 1, \dots, k - 1 - j + \alpha, k + 1 - j + \alpha, \dots, k - 1 - \alpha, k + 1 - \alpha, \dots, k - 1, k + 1).$$

Finally, if for some $0 \leq \alpha \leq [j/2]$, $y \in C$ is one of the $\delta(k, j, \alpha)$ points for which there exists $L_C \in \text{Pic}^{3k-4-j}(C)$ such that

$$a^{l_C}(y) = (0, 1, \dots, \alpha - 1, \alpha + 1, \dots, j - \alpha, j - \alpha + 2, \dots, k - 2, k - 1, k + 1),$$

then $l_C = (j - 2)y + |L_C|$ and $l_D = (2k - j - 5)y + l'_D$, where l'_D is a (non-complete) $\mathfrak{g}_{k-1+j}^{k-2}$ on D satisfying $a^{l'_D}(y) = (0, 2, \dots, k - 1 + \alpha - j, k + 1 + \alpha - j, \dots, k - \alpha, k + 2 - \alpha, \dots, k, k + 1)$.

Proof. Since by definition $[D, y] \in \mathcal{M}_{j,1}$ is Brill-Noether general, we have that $\rho(l_D, y) \geq 0$. The locus of those $(C, y) \in \mathcal{M}_{2k-2-j,1}$ carrying a $\mathfrak{g}_{3k-6}^{k-2}$ with $\rho(\mathfrak{g}_{3k-6}^{k-2}, y) \leq -2$ is of codimension ≥ 2 , hence we must also have that $\rho(l_C, y) \geq -1$. Using Proposition 1.2 from [EH3] we have that $a_{k-2}^{l_C}(y) \leq k - 2 + j$ and $a_0^{l_C}(y) \geq j - 3$. There are two cases to consider:

(a) If $\rho(l_C, y) = 0$, then $\rho(l_D, y) = 0$. Assume first that $a_0^{l_C}(y) = j - 2$. Since we have that $0 \leq a_s^{l_C}(y) - s - (j - 2) \leq 2$ for all $0 \leq s \leq k - 2$, there must exist an integer $0 \leq \alpha \leq [j/2]$, such that if L_C denotes the $\mathfrak{g}_{3k-4-j}^{k-2}$ obtained from l_C by removing $(j - 2)y$, then the vanishing at y is $a^{l_C}(y) = (0, 1, \dots, \alpha - 1, \alpha + 1, \alpha + 2, \dots, j - \alpha, j - \alpha + 2, \dots, k)$. By compatibility, l_D has $(2k - j - 4)y$ in its base locus and we write $l_D = (2k - j - 4)y + |L_D|$, where $a^{l_D}(y) = (0, 1, \dots, k - 2 + j - \alpha, k + j - \alpha, \dots, k - 1 - \alpha, k + 1 - \alpha, \dots, k)$. By Riemann-Roch, the number of such L_D 's equals the number of $\mathfrak{g}_{j-\alpha+1}^1$'s on D with a $(j - 2\alpha + 1)$ -fold point at y which is computed by $a(j, \alpha)$. If $a_0^{l_C}(y) = j - 3$, we denote by $L_C \in \text{Pic}^{3k-3-j}(C)$ the line bundle obtained from l_C by subtracting $(j - 3)y$, and again we see that there must exist $0 \leq \alpha \leq [j/2]$ such that $h^0(L_C(-(j - \alpha + 2)y)) = k - j + \alpha$ and $h^0(L_C(-(\alpha + 2)y)) = k - 1 - \alpha$. By duality, $K_C \otimes L_C^\vee \otimes \mathcal{O}_C((j - \alpha + 2)y)$ is one of the finitely many $\mathfrak{g}_{k-1-\alpha}^1$'s on C with a $(j - 2\alpha)$ -fold point at y . This number equals $b(2k - 2 - j, k - 1 + \alpha - j)$ and this explains case (2) of our statement.

(b) If $\rho(l_C, y) = -1$, then $\rho(l_D, y) = 1$. An argument similar to that used above gives two possibilities for $a^{l_C}(y)$ and the rest follows because l_C and l_D are compatible at y . \square

Retaining the notation from Proposition 3.4, for each integer $0 \leq \alpha \leq [j/2]$ we define the following 1-cycles:

$$X_{j,\alpha} := \{(y, L_C) \in C \times \text{Pic}^{3k-j-4}(C) : a_\alpha^{L_C}(y) \geq \alpha + 1, a_{j-\alpha}^{L_C}(y) \geq j - \alpha + 2\},$$

$$Y'_{j,\alpha} := \{(y, l) \in C \times G_{3k-j-3}^{k-2}(C) : a_1^l(y) \geq 2, a_\alpha^l(y) \geq \alpha + 2, a_{j-\alpha-1}^l(y) \geq j - \alpha + 2\},$$

$$Y''_{j,\alpha} := \{L_D \in \text{Pic}^{k-2+j}(D) : a_{k+\alpha-j}^{L_D}(y) \geq k + \alpha - j + 1, a_{k-\alpha-1}^{L_D}(y) \geq k - \alpha + 1\}$$

and

$$Y'''_{j,\alpha} := \{l \in G_{k-1+j}^{k-2}(D) : a_1^l(y) \geq 2, a_{k-1+\alpha-j}^l(y) \geq k + \alpha - j + 1, a_{k-1-\alpha}^l(y) \geq k - \alpha + 2\}.$$

Via Proposition 3.4 we view $X_{j,\alpha}, Y'_{j,\alpha}, Y''_{j,\alpha}$ and $Y'''_{j,\alpha}$ as 1-cycles on $\widetilde{\mathfrak{G}}_{3k-6}^{k-2}$. Note that $X_{j,\alpha}$ projects onto C while $Y'_{j,\alpha}$ is isomorphic to the union of $b(2k-2-j, k-1+\alpha-j)$ copies of \mathbf{P}^1 . We can then rephrase Proposition 3.4 as follows:

Proposition 3.5. *If $\sigma : \widetilde{\mathfrak{G}}_{3k-6}^{k-2} \rightarrow \widetilde{\mathcal{M}}_g$ is the natural projection and $[C] \in \mathcal{M}_{2k-2-j}$ where $2 \leq j \leq k-1$, then we have the following numerical equivalence relation between 1-cycles:*

$$\begin{aligned} \sigma^*(C^j) \equiv & \sum_{\alpha=0}^{\lfloor j/2 \rfloor} \left(a(j, \alpha) X_{j,\alpha} + \delta(k, 2k-2-j, k+\alpha-j) Y'_{j,\alpha} + \right. \\ & \left. + b(2k-2-j, k-1+\alpha-j) Y''_{j,\alpha} + \delta(j, \alpha) Y'''_{j,\alpha} \right). \end{aligned}$$

To compute the class of the cycles introduced in Propositions 3.3 and 3.5, we introduce some notation. Let C be a Brill-Noether general curve of genus g . We denote by C_d the d -th symmetric product of C , by $\mathcal{U} \subset C \times C_d$ the universal divisor, and by \mathcal{L} the Poincare bundle on $C \times \text{Pic}^d(C)$. We also introduce the projections $\pi_1 : C \times C_d \rightarrow C$ and $\pi_2 : C \times C_d \rightarrow C_d$ (and we use the same notation for the projections from $C \times \text{Pic}^d(C)$ onto the factors). We define the cohomology classes

$$\eta = \pi_1^*([point]) \in H^2(C \times C_d) \text{ and } x = \pi_2^*(D_{x_0}) \in H^2(C \times C_d),$$

where $x_0 \in C$ is an arbitrary point and $D_{x_0} := \{D \in C_d : x_0 \in D\}$ (the definition of x is obviously independent of the point x_0). Finally, if $\delta_1, \dots, \delta_{2g} \in H^1(C, \mathbb{Z}) \cong H^1(C_d, \mathbb{Z})$ is a symplectic basis, then we define the class

$$\gamma := - \sum_{\alpha=1}^g \left(\pi_1^*(\delta_\alpha) \pi_2^*(\delta_{g+\alpha}) - \pi_1^*(\delta_{g+\alpha}) \pi_2^*(\delta_\alpha) \right).$$

With these notations we have the formula (cf. [ACGH], p. 338) $[\mathcal{U}] \equiv d\eta + \gamma + x$, corresponding to the Hodge decomposition of $[\mathcal{U}]$. We also record the formulas $\gamma^3 = \gamma\eta = 0, \eta^2 = 0$ and $\gamma^2 = -2\eta\pi_2^*(\theta)$, where $\theta \in H^2(C_d, \mathbb{Z})$ is the pullback of the class of the theta divisor on $\text{Jac}(C)$. Similarly, $c_1(\mathcal{L}) = d\eta + \gamma$, where $\eta, \gamma \in H^2(C \times \text{Pic}^d(C))$ pull back to $\eta, \gamma \in H^2(C \times C_d)$. When $d = k-2$ and $g = 2k-3$, we set $\mathcal{M} := \pi_1^*(K_C) \otimes \mathcal{O}_{C \times C_{k-2}}(-\mathcal{U})$, hence $c_1(\mathcal{M}) = (3k-6)\eta - \gamma - x$.

We now compute the class of the curves X and Y defined in Proposition 3.3:

Proposition 3.6. *Let C be a Brill-Noether general curve of genus $g = 2k - 3$ and $q \in C$ a general point.*

(1) *The class of the curve $X = \{(y, D) \in C \times C_{k-2} : h^0(C, D + 2y) \geq 2\}$ is given by*

$$[X] \equiv \frac{\theta^{k-2}}{(k-2)!} - \frac{\theta^{k-3}x}{(k-3)!} - \frac{2\theta^{k-3}\gamma}{(k-3)!} + \\ + \frac{2x\theta^{k-4}\gamma}{(k-4)!} - \frac{4(k+1)}{(k-4)!}x\theta^{k-4}\eta + \frac{4k-2}{(k-3)!}\theta^{k-3}\eta.$$

(2) *The class of the curve $Y = \{(y, D) \in C \times C_{k-2} : h^0(C, D + y + q) \geq 2\}$ is given by*

$$[Y] \equiv \frac{\theta^{k-2}}{(k-2)!} - \frac{\theta^{k-3}x}{(k-3)!} - \frac{\theta^{k-3}\gamma}{(k-3)!} + \\ + \frac{x\theta^{k-4}\gamma}{(k-4)!} - \frac{k+1}{(k-4)!}x\theta^{k-4}\eta + \frac{k-1}{(k-3)!}\theta^{k-3}\eta.$$

Proof. We will realize both X and Y as pullbacks of degeneracy loci and compute their classes using the Thom-Porteous formula. We consider the map $\epsilon : C \times C_{k-2} \rightarrow C_k$ given by $\epsilon(y, D) := 2y + D$. It is easy to check that

$$\epsilon^*(x) = 2\eta + x \text{ and } \epsilon^*(\theta) = 4g\eta + \theta - 2\gamma.$$

Then $X = \epsilon^*(C_k^1)$ and to compute the class of $C_k^1 = \{E \in C_k : h^0(E) \geq 2\}$ we introduce the rank k vector bundle $\mathcal{P} := (\pi_2)_*(\pi_1^*K_C \otimes \mathcal{O}_U)$ on C_k having total Chern class $c_t(\mathcal{P}) = (1-x)^{-k+4}e^{\frac{\theta}{1-x}}$ (cf. [ACGH], pg. 240). There is a natural bundle map $\phi^\vee : \epsilon^*(\mathcal{P}^\vee) \rightarrow H^0(K_C)^\vee \otimes \mathcal{O}_{C \times C_{k-2}}$, and $X = Z_{k-1}(\phi^\vee)$. Then by Thom-Porteous, we can write

$$[X] = \left[\frac{1}{c_t(\epsilon^*(\mathcal{P}^\vee))} \right]_{k-2} = \epsilon^* \left(\frac{\theta^{k-2}}{(k-2)!} - \frac{x\theta^{k-3}}{(k-3)!} \right) = \\ = \frac{(4g\eta + \theta - 2\gamma)^{k-2}}{(k-2)!} - \frac{(4g\eta + \theta - 2\gamma)^{k-3}(2\eta + x)}{(k-3)!},$$

which quickly leads to the desired expression for $[X]$.

To compute the class of Y we proceed in a similar manner: we consider the map $\chi : C \times C_{k-2} \rightarrow C_k$ given by $\chi(y, D) := y + q + D$. Then $\chi^*(x) = x + \eta$ and $\chi^*(\theta) = g\eta + \theta - \gamma$. For each $(y, D) \in C \times C_{k-2}$ we have a natural map

$$H^0(K_C|_{y+q+D})^\vee \rightarrow H^0(K_C)^\vee$$

which globalizes to a vector bundle map $\psi^\vee : \chi^*(\mathcal{P}^\vee) \rightarrow H^0(K_C)^\vee \otimes \mathcal{O}_{C \times C_{k-2}}$ and then it is clear that $Y = Z_{k-1}(\psi^\vee)$. Applying Thom-Porteous again, we obtain

$$[Y] = \left[\frac{1}{c_t(\chi^*(\mathcal{P}^\vee))} \right]_{k-2} = \chi^* \left(\frac{\theta^{k-2}}{(k-2)!} - \frac{x\theta^{k-3}}{(k-3)!} \right) = \\ = \frac{(g\eta + \theta - \gamma)^{k-2}}{(k-2)!} - \frac{(g\eta + \theta - \gamma)^{k-3}(\eta + x)}{(k-3)!},$$

which after a straightforward calculation gives the class of Y . \square

We also need the following intersection theoretic result:

Lemma 3.7. *For $j \geq 1$ we denote by $\mathcal{F}_j := (\pi_2)_*(\mathcal{M}^{\otimes j})$ the vector bundle on C_{k-2} with fibre $\mathcal{F}_j(D) = H^0(C, jK_C - jD)$ over each $D \in C_{k-2}$. Then $c_1(\mathcal{F}_j) = -j^2\theta - j(3j-2)(k-2)x$.*

Proof. We apply the Grothendieck-Riemann-Roch Theorem for the map $\pi_2 : C \times C_{k-2} \rightarrow C_{k-2}$. We obtain that

$$\begin{aligned} c_1(\mathcal{F}_j) &= \left[(\pi_2)_* \left((1 + j((3k-6)\eta - \gamma - x) + \frac{j^2}{2}((3k-6)\eta - \gamma - x)^2) \cdot (1 - (2k-4)\eta) \right) \right]_1 = \\ &= -j^2\theta - j(3j-2)(k-2)x. \end{aligned}$$

□

Next we compute the class of $X_{j,\alpha}$ when $2 \leq j \leq k-1$. It is convenient to state our result as follows:

Proposition 3.8. *Let C be a general curve of genus $2k-2-j$ such that $2 \leq j \leq k-1$. We have the following relation in $H^{4k-4-2j}(C \times \text{Pic}^{3k-j-4}(C))$:*

$$\begin{aligned} \sum_{\alpha=0}^{\lfloor j/2 \rfloor} a(j, \alpha) X_{j,\alpha} &\equiv \frac{2(2k-3)!}{k!(k-2)!(2k-3-j)!} \left(\frac{\theta^{2k-2-j}}{2k-2-j} + \frac{(3kj+12k-9j-18)\theta^{2k-3-j}\gamma}{2k-3} + \right. \\ &\left. \frac{\theta^{2k-3-j}\eta}{(2k-5)(2k-3)} \left(60 - 64k + 16k^2 + j(60 - \frac{67}{2}k - 10k^2 + 6k^3) + j^2(15 + 2k - 5k^2) + \frac{3}{2}kj^3 \right) \right). \end{aligned}$$

Proof. We fix an integer $0 \leq \alpha \leq \lfloor j/2 \rfloor$ and a divisor $D \in C_e$ of degree $e \geq k-j-1$. We set $d := 3k-4-j$, $\Gamma := D \times \text{Pic}^d(C)$ and denote by $u, v : C \times C \times \text{Pic}^d(C) \rightarrow C \times \text{Pic}^d(C)$ the two projections and by $J_\alpha(\mathcal{L})$ the α -th jet bundle of the Poincare bundle on $C \times \text{Pic}^d(C)$. Then $\mathcal{P}(D) := (\pi_2)_*(\mathcal{L} \otimes_{C \times \text{Pic}^d(C)}(\Gamma))$ is a vector bundle of rank $e+k-1$. For each $\alpha \geq 0$ we define the vector bundle $J_\alpha(\mathcal{L}, D) := u_*(v^*(\mathcal{L}) \otimes \mathcal{O}_{(\alpha+1)\Delta + v^*(\Gamma)})$, where $\Delta \subset C \times C \times \text{Pic}^d(C)$ is the diagonal, and consider the vector bundle map

$$\phi_\alpha : J_\alpha(\mathcal{L}, D)^\vee \rightarrow (\pi_2)^*(\mathcal{P}(D))^\vee,$$

which on fibres is the dual of the map $H^0(L_C \otimes \mathcal{O}(D)) \rightarrow H^0(L_C \otimes \mathcal{O}_{(\alpha+1)y+D}(D))$ for each pair $(y, L_C) \in C \times \text{Pic}^d(C)$.

The cycle $X_{j,\alpha}$ consists of pairs (y, L_C) such that $\dim \text{Ker}(\phi_\alpha(y, L_C)) \geq 1$ and $\dim \text{Ker}(\phi_{j-\alpha+1}(y, L_C)) \geq 2$ and its class is given by the formula (cf. [Fu], Theorem 14.3):

$$X_{j,\alpha} \equiv c_{j+1-\alpha}(\alpha) c_{k-1+\alpha-j}(j-\alpha+1) - c_{k-\alpha}(\alpha) c_{k-2+\alpha-j}(j-\alpha+1),$$

where $c_l(\beta) := c_l((\pi_2)^*(\mathcal{P}(D))^\vee - J_\beta(\mathcal{L}, D)^\vee)$. Since $c_{tot}((\pi_2)^*(\mathcal{P}(D))) = e^\theta$ and $c_1(\mathcal{L}) = d\eta + \gamma$ (independent of D !), after a short calculation we obtain that

$$c_l(\beta) = \frac{\theta^l}{l!} + \frac{\theta^{l-1}}{(l-1)!} ((\beta+1)\gamma + (\beta+1)(d + \beta(2k-3-j))\eta) - \frac{\theta^{l-1}}{(l-2)!} (\beta+1)(\beta+2)\eta.$$

A straightforward computation now leads to the stated formula. □

For each integers $0 \leq a \leq k-2$ and $b \geq 2$ we shall define vector bundles $\mathcal{G}_{a,b}$ over $\tilde{\mathfrak{G}}_{3k-6}^{k-2}$ with fibre

$$\mathcal{G}_{a,b}(C, L) = H^0(\wedge^a M_L \otimes L^{\otimes b})$$

over every point $(C, L) \in \mathfrak{G}_{3k-6}^{k-2}$. Note that $\mathcal{G}_{i,2|\mathfrak{G}_{3k-6}^{k-2}} = \mathcal{B}$, where \mathcal{B} is the vector bundle we introduced in Proposition 2.5. Of course, the question is how to extend this description over the divisors Δ_j^0 . First we will extend $\mathcal{G}_{a,b}$ over $\sigma^{-1}(\mathcal{M}_g^0 \cup \Delta_0^0 \cup \Delta_1^0)$ and we start by constructing the vector bundles $\mathcal{G}_{0,b}$:

Proposition 3.9. *For each $b \geq 2$ there exists a vector bundle $\mathcal{G}_{0,b}$ over $\sigma^{-1}(\mathcal{M}_g^0 \cup \Delta_1^0 \cup \Delta_0^0) \subset \tilde{\mathfrak{G}}_{3k-6}^{k-2}$ of rank $k(3b-2) - 6b + 3$ whose fibres admit the following description:*

- For $(C, L) \in \mathfrak{G}_{3k-6}^{k-2}$, we have that $\mathcal{G}_{0,b}(C, L) = H^0(C, L^{\otimes b})$.
- For $t = (C \cup_y E, L) \in \sigma^{-1}(\Delta_1^0)$, where L is the linear series $\mathfrak{g}_{3k-6}^{k-2}$ on C determining a limit $\mathfrak{g}_{3k-6}^{k-2}$ on $C \cup_y E$, we have that

$$\mathcal{G}_{0,b}(t) = H^0(C, L^{\otimes b}(-2y)) + \mathbb{C} \cdot u^b \subset H^0(C, L^{\otimes b}),$$

where $u \in H^0(C, L)$ is any section such that $\text{ord}_y(u) = 0$.

- For $t = (C/y \sim q, L) \in \sigma^{-1}(\Delta_0^0)$, where $q, y \in C$ and L is a $\mathfrak{g}_{3k-6}^{k-2}$ on C , we have that

$$\mathcal{G}_{0,b}(t) = H^0(C, L^{\otimes b}(-y-q)) \oplus \mathbb{C} \cdot u^b \subset H^0(C, L^{\otimes b}),$$

where $u \in H^0(C, L)$ is a section such that $\text{ord}_y(u) = \text{ord}_q(u) = 0$.

Proof. Recalling that $\sigma : \tilde{\mathfrak{G}}_d^r \rightarrow \tilde{\mathcal{M}}_g$ denotes the variety of limit $\mathfrak{g}_d^{r'}$'s over all treelike curves of genus g , the morphism $\mathfrak{G}_{3k-6}^{k-2} \rightarrow \mathfrak{G}_{b(3k-6)}^{k(3b-2)-6b+2}$ given by $(C, L) \mapsto (C, L^{\otimes b})$ can be extended to a morphism $\nu_b : \tilde{\mathfrak{G}}_{3k-6}^{k-2} \rightarrow \tilde{\mathfrak{G}}_{b(3k-6)}^{k(3b-2)-6b+2}$ over $\mathcal{M}_g \cup \Delta_0^0 \cup \Delta_1^0$ as follows. We fix a point $t = (C \cup_y E, l = (l_C, l_E))$ a limit $\mathfrak{g}_{3k-6}^{k-2}$, where $l_C = (L_C, H^0(L_C)) \in G_{3k-6}^{k-2}(C)$ is such that $H^0(L_C) = H^0(L_C(-2y)) \oplus \mathbb{C} \cdot u$, for a certain $u \in H^0(L_C)$ with $\text{ord}_y(u) = 0$, while $l_E = (\mathcal{O}_E((3k-6)y), (2k-6)p + |V_E|)$ (any limit $\mathfrak{g}_{3k-6}^{k-2}$ on $C \cup_y E$ is of this form). Then we set $\nu_b(t) := (l_C^b, l_E^b)$, where $l_C^b = (L_C^{\otimes b}, H^0(L_C^{\otimes b}(-2y)) \oplus \mathbb{C} \cdot u^b \subset H^0(L_C^{\otimes b}))$ and $l_E^b = (\mathcal{O}_E(b(3k-6)p), (2k-3)p + |(k(3b-2) - 6b + 3)p|)$. In other words, (l_C^b, l_E^b) is the b -th power of the original limit linear series $\mathfrak{g}_{3k-6}^{k-2}$.

For a point $z = (C/y \sim q, L_C) \in \Delta_0^0$ with L_C being a $\mathfrak{g}_{3k-6}^{k-2}$ such that $H^0(L_C) = H^0(L_C(-y-q)) \oplus \mathbb{C} \cdot u$, we define $\nu_b(z) := (L_C^{\otimes b}, H^0(L_C^{\otimes b}(-y-q)) \oplus \mathbb{C} \cdot u^b \subset H^0(L_C^{\otimes b}))$. The fact that ν_b can be constructed algebraically follows easily from the equations of the scheme $\tilde{\mathfrak{G}}_{3k-6}^{k-2}$ described in [EH1], pg. 358. The variety $\tilde{\mathfrak{G}}_{b(3k-6)}^{k(3b-2)-6b+2}$ carries a tautological vector bundle $\mathcal{T} = \mathcal{T}_C$ with fibre over each point corresponding to a curve from $\mathcal{M}_g^0 \cup \Delta_0^0$ being the space of global sections of the linear series, while the fibre over a point corresponding to a curve from Δ_1^0 is the space of sections of the aspect of the limit linear series corresponding to the curve of genus $g-1$. We define $\mathcal{G}_{0,b} := \nu_b^*(\mathcal{T})$. The description of the fibres of the vector bundle $\mathcal{G}_{0,b}$ is now immediate. \square

Having defined the vector bundles $\mathcal{G}_{0,b}$ we now define inductively all vector bundles $\mathcal{G}_{a,b}$: first we define $\mathcal{G}_{1,b}$ as the kernel of the multiplication map $\mathcal{G}_{0,1} \otimes \mathcal{G}_{0,b} \rightarrow \mathcal{G}_{0,b+1}$,

that is, by the exact sequence

$$0 \longrightarrow \mathcal{G}_{1,b} \longrightarrow \mathcal{G}_{0,1} \otimes \mathcal{G}_{0,b} \longrightarrow \mathcal{G}_{0,b+1} \longrightarrow 0.$$

Having defined $\mathcal{G}_{l,b}$ for all $l \leq a-1$, we define the vector bundle $\mathcal{G}_{a,b}$ by the exact sequence

$$(4) \quad 0 \longrightarrow \mathcal{G}_{a,b} \longrightarrow \wedge^a \mathcal{G}_{0,1} \otimes \mathcal{G}_{0,b} \xrightarrow{d_{a,b}} \mathcal{G}_{a-1,b+1} \longrightarrow 0.$$

We now prove the right-exactness of the sequence (4) which will ensure the correctness of the definition of $\mathcal{G}_{a,b}$:

Proposition 3.10. *For integers $b \geq 2$ and $1 \leq a \leq i$, the Koszul multiplication map $d_{a,b} : \wedge^a \mathcal{G}_{0,1} \otimes \mathcal{G}_{0,b} \rightarrow \mathcal{G}_{a-1,b+1}$ is well-defined and surjective. In particular one can define the vector bundle $\mathcal{G}_{a,b} := \text{Ker}(d_{a,b})$ over $\sigma^{-1}(\mathcal{M}_g^0 \cup \Delta_0^0 \cup \Delta_1^0)$ and the sequence (4) makes sense.*

Proof. We do induction on a and start with the case $a = 1$. We only check the surjectivity of the map $\mathcal{G}_{0,1} \otimes \mathcal{G}_{0,b} \rightarrow \mathcal{G}_{0,b+1}$ over the locus $\sigma^{-1}(\Delta_1^0)$, the other cases, namely $\sigma^{-1}(\mathcal{M}_g^0)$ and $\sigma^{-1}(\Delta_0^0)$ being quite similar. It is enough to show that if C is a sufficiently general curve of genus $2k-3$, $y \in C$ is a point and $L \in W_{3k-6}^{k-2}(C)$ is a linear system with a cusp at y , then the map $H^0(L) \otimes H^0(L^{\otimes b}(-2y)) \rightarrow H^0(L^{\otimes(b+1)}(-2y))$ is onto. This is equivalent to $H^1(M_L \otimes L^{\otimes b}(-2y)) = 0$ which follows because of Proposition 2.1.

We now treat the general case $1 \leq a \leq i$ and we want to show that the Koszul map $d_{a,b} : \wedge^a H^0(L) \otimes \mathcal{G}_{0,b}(L) \rightarrow \mathcal{G}_{a-1,b+1}(L)$ is surjective for each $b \geq 2$ and for each (C, L) as above. For simplicity we have denoted $\mathcal{G}_{a,b}(L) = \mathcal{G}_{a,b}(t)$, where $t \in \sigma^{-1}(\Delta_1^0)$ is the point $(C \cup_y E, l)$, E is an arbitrary elliptic tail and l is a limit linear series having L as its C -aspect. We can then write $H^0(L)/H^0(L(-2y)) = \mathbb{C} \cdot u$, where $u \in H^0(L)$ is uniquely determined up to translation by an element from $H^0(L(-2y))$. We first claim that the restriction of the Koszul differential

$$d_{a,b}^0 : \wedge^a H^0(L) \otimes H^0(L^{\otimes b}(-2y)) \rightarrow H^0(\wedge^{a-1} M_L \otimes L^{\otimes(b+1)}(-2y))$$

is surjective. This is so because $\text{Coker}(d_{a,b}^0) = H^1(\wedge^a M_L \otimes L^{\otimes b}(-2y)) = 0$ (use again that M_L is a stable bundle). But then since

$$H^0(\wedge^{a-1} M_L \otimes L^{\otimes(b+1)}(-2y)) \subset \mathcal{G}_{a-1,b+1}(L)$$

is a linear subspace of codimension $\binom{k-2}{a-1}$, to prove that $d_{a,b}$ itself is surjective it suffices to notice that the image of the injective map $\wedge^{a-1} H^0(L(-2y)) \rightarrow \wedge^{a-1} H^0(L) \otimes \mathcal{G}_{0,b+1}(L)$ given by

$$f_1 \wedge \dots \wedge f_{a-1} \mapsto f_1 \wedge \dots \wedge f_{a-1} \otimes u^{b+1} - \sum_{l=1}^{a-1} (-1)^{l-1} u \wedge f_1 \wedge \dots \wedge \check{f}_l \wedge \dots \wedge f_{a-1} \otimes f_l u^b$$

is entirely contained in $\text{Im}(d_{a,b})$ and is clearly disjoint from $H^0(\wedge^{a-1} M_L \otimes L^{\otimes(b+1)}(-2y))$. \square

To compute the intersection numbers of the divisors $\overline{\mathcal{Z}}_{g,i}$ with the test curves C^0 and C^1 we need to understand the restriction of the vector bundle $\mathcal{G}_{i,2}$ to X and Y .

Proposition 3.11. *Let C be a general curve of genus $2k - 3$ and $k = 3i + 6$ with $i \geq 0$. If C^1 is the test curve in Δ_1 obtained by attaching to C a fixed elliptic tail at a varying point of C and $X = \sigma^{-1}(C^1)$, then we have the following formula:*

$$c_1(\mathcal{G}_{2,i|X}) = \binom{3i+4}{i} (c_x x + c_\eta \eta + c_\gamma \gamma + c_\theta \theta),$$

where

$$c_\eta = -27i + 40, \quad c_x = -\frac{27i^4 + 153i^3 + 331i^2 + 323i + 120}{(3i+4)(i+1)}$$

$$c_\gamma = \frac{5i+8}{3i+4}, \quad c_\theta = -\frac{27i^3 + 101i^2 + 124i + 48}{(3i+3)(3i+4)}.$$

Proof. For $j \geq 2$ we define the jet bundle $J_1(\mathcal{M}^{\otimes j}) := (u_*)(v^*(\mathcal{O}_{C \times C_{k-2}}(\mathcal{M}^{\otimes j})) \otimes \mathcal{O}_{2\Delta})$, where $u, v : C \times C \times C_{k-2} \rightarrow C \times C_{k-2}$ are the projections. We have two exact sequences of vector bundles on X :

$$0 \longrightarrow u_*(v^*(\mathcal{M}^{\otimes j}) \otimes \mathcal{O}(-2\Delta))|_X \longrightarrow \mathcal{G}_{0,j|X} \longrightarrow \mathcal{M}|_X^{\otimes j} \longrightarrow 0$$

and

$$0 \longrightarrow u_*(v^*(\mathcal{M}^{\otimes j}) \otimes \mathcal{O}(-2\Delta))|_X \longrightarrow \pi_2^*(\mathcal{F}_j)|_X \longrightarrow J_1(\mathcal{M}^{\otimes j})|_X \longrightarrow 0,$$

from which we can write that

$$c_1(\mathcal{G}_{0,j|X}) = c_1(\pi_2^*(\mathcal{F}_j)|_X) - c_1((\pi_1^*K_C \otimes \mathcal{M}^{\otimes j})|_X) =$$

$$= -j^2\theta - (3j+4)(k-2)\eta + j\gamma - j((3j-2)(k-2)-1)x.$$

From the exact sequences defining $\mathcal{G}_{i,2}$ we then obtain that

$$c_1(\mathcal{G}_{i,2|X}) = \sum_{l=0}^i (-1)^l c_1(\wedge^{i-l} \mathcal{G}_{0,1|X} \otimes \mathcal{G}_{0,l+2|X}).$$

Note that Riemann-Roch gives that $\text{rk}(\mathcal{G}_{0,l+2}) = (3k-6)l+4k-9$, while $c_1(\wedge^{i-l} \mathcal{G}_{0,1|X}) = \binom{k-2}{i-l-1}(-\theta - (k-2)x)$. To obtain a closed formula for $c_1(\mathcal{G}_{i,2|X})$ we now specialize to the case $k = 3i + 6$ and we write that

$$c_1(\mathcal{G}_{i,2|X}) = \sum_{l=0}^i (-1)^l \left[((9i+12)l + (12i+15)) \binom{3i+4}{i-l-1} (-\theta - (3i+4)x) + \right.$$

$$\left. + \binom{3i+5}{i-l} \left(-(l+2)^2\theta + (l+2)\gamma - (3i+4)(3l+10)\eta - (l+2)((3l+4)(3i+4)-1)x \right) \right].$$

A long but elementary calculation leads to the stated formula. □

A similar calculation yields the first Chern class of $\mathcal{G}_{i,2|Y}$:

Proposition 3.12. *Let (C, q) be a general pointed curve of genus $2k - 3$ and C^0 the test curve in Δ_0 obtained by identifying the fixed point q with a varying point y on C . If $Y = \sigma^{-1}(C^0)$, then we have the formula*

$$c_1(\mathcal{G}_{i,2|Y}) = \binom{3i+4}{i} (d_x x + \eta + d_\theta \theta),$$

where

$$d_x = -\frac{27i^4 + 153i^3 + 331i^2 + 323i + 120}{(3i+4)(i+1)}, \quad d_\theta = -\frac{27i^3 + 101i^2 + 124i + 48}{(3i+3)(3i+4)}.$$

Proof. We define $\Gamma_q := C \times \{q\} \times C_{k-2}$ and denote by $\iota : C \times C_{k-2} \hookrightarrow \Gamma_q$ the inclusion map. Then we have the exact sequences on Y

$$0 \longrightarrow (u_*)(v^*(\mathcal{M}^{\otimes j}) \otimes \mathcal{O}(-\Delta - \Gamma_q))|_Y \longrightarrow \mathcal{G}_{0,j|Y} \longrightarrow \mathcal{M}|_Y^{\otimes j} \longrightarrow 0,$$

and

$$0 \longrightarrow \iota^*(v^*(\mathcal{M}^{\otimes j}) \otimes \mathcal{O}(-\Delta))|_Y \longrightarrow (u_*)(v^*(\mathcal{M}^{\otimes j}) \otimes \mathcal{O}_{\Delta+\Gamma_q})|_Y \longrightarrow \pi_2^*(\mathcal{F}_j)|_Y \longrightarrow 0.$$

Using that $c_1(\mathcal{M}) = (3k-6)\eta - \gamma - x$, it is now straightforward to check that

$$c_1(\iota^*(v^*(\mathcal{M}^{\otimes j}) \otimes \mathcal{O}(-\Delta))) = -jx - \eta,$$

hence $c_1(\mathcal{G}_{0,j|Y}) = c_1(\pi_2^*(\mathcal{F}_j))|_Y + jx + \eta = -j^2\theta + \eta - j((3j-2)(k-2)-1)x$. But then as in the previous proposition we have that

$$c_1(\mathcal{G}_{i,2|Y}) = \sum_{l=0}^i (-1)^l c_1(\wedge^{i-l} \mathcal{G}_{0,1|Y} \otimes \mathcal{G}_{0,l+2|Y}),$$

which after some calculations yields the stated formula in the special case $k = 3i+6$. \square

In what follows we extend the vector bundles $\mathcal{G}_{a,b}$ over the whole $\tilde{\mathfrak{G}}_{3k-6}^{k-2}$. Roughly speaking, the fibre of $\mathcal{G}_{a,b}$ over a point corresponding to a singular curve $C \cup_y D$ will be the space $H^0(C \cup D, \wedge^a M_L \otimes L^{\otimes b})$, where L will be a line bundle on $C \cup D$ obtained by twisting appropriately a limit $\mathfrak{g}_{3k-6}^{k-2}$ on $C \cup D$. The twisting is chosen such that, the vector bundles $\mathcal{G}_{a,b}$ sit in the exact sequences (4). In the next statement we use the notation introduced in Proposition 3.5:

Theorem 3.13. *For integers $0 \leq a \leq i$, $b \geq 2$, or $(a, b) = (0, 1)$, there exists a vector bundle $\mathcal{G}_{a,b}$ over $\tilde{\mathfrak{G}}_{3k-6}^{k-2}$ having the following properties:*

(1) *If $(C, L) \in \mathfrak{G}_{3k-6}^{k-2}$, then $\mathcal{G}_{a,b}(C, L) = H^0(\wedge^a M_L \otimes L^{\otimes b})$, that is, $\mathcal{G}_{a,b}|_{\mathfrak{G}_{3k-6}^{k-2}} = \mathcal{G}_{a,b}$.*

If $t = (C \cup_y D, L_C, L_D) \in \sigma^{-1}(\Delta_j^0) \subset \tilde{\mathfrak{G}}_{3k-6}^{k-2}$, with $2 \leq g(D) = j \leq k-1$, $g(C) = 2k-2-j$, $L_C \in W_{3k-4-j}^{k-2}(C)$ and $L_D \in W_{k-2+j}^{k-2}(D)$, then there are two situations:

(2) *If $2 \leq j \leq 2i+2$, then $\mathcal{G}_{a,b}(t) = H^0(C \cup D, \wedge^a M_{L_{C \cup D}} \otimes L_{C \cup D}^{\otimes b})$, where $L_{C \cup D} = (L_C(-(j+2)y), L_D(-(k-2-j)y)) \in \text{Pic}^{3k-6-2j}(C) \times \text{Pic}^{2j}(D)$ is a globally generated line bundle on $C \cup D$.*

(3) *If $2i+3 \leq j \leq 3i+5$ and $c := [(j+2)/2]$, then $\mathcal{G}_{a,b}(t) = H^0(C \cup D, \wedge^a M_{L_{C \cup D}} \otimes L_{C \cup D}^{\otimes b})$, where $L_{C \cup D} = (L_C(-cy), L_D(-(k-c)y)) \in \text{Pic}^{3k-4-j-c}(C) \times \text{Pic}^{j+c-2}(D)$ is a globally generated line bundle on $C \cup D$.*

(4) *There are exact sequences over $\tilde{\mathfrak{G}}_{3k-6}^{k-2}$: $0 \longrightarrow \mathcal{G}_{a,b} \longrightarrow \wedge^a \mathcal{G}^{0,1} \otimes \mathcal{G}_{0,b} \longrightarrow \mathcal{G}_{a-1,b+1} \longrightarrow 0$.*

Proof. For $j \geq 2$ the divisors Δ_j^0 are mutually disjoint and we can carry out the construction of $\mathcal{G}_{a,b}$ over each open set $\sigma^{-1}(\mathcal{M}_g^0 \cup \Delta_j^0) \subset \tilde{\mathfrak{G}}_{3k-6}^{k-2}$ and glue the resulting bundles together. We denote by $\pi : \mathcal{C}_j \rightarrow \mathcal{M}_g^0 \cup \Delta_j^0$ the restriction of the universal curve and by $p : \mathcal{Y} = \mathcal{C}_j \times_{\mathcal{M}_g^0 \cup \Delta_j^0} \tilde{\mathfrak{G}}_{3k-6}^{k-2} \rightarrow \tilde{\mathfrak{G}}_{3k-6}^{k-2}$. Then $(\sigma p)^{-1}(\Delta_j^0) = D_j + D_{2k-2-j}$, where D_j (resp. D_{2k-2-j}) denotes the divisor in \mathcal{Y} corresponding to the marked point being on the genus j (resp. $2k-2-j$) component. If $\mathcal{L} = \mathcal{L}_{0,3k-6}$ is the Poincare bundle on \mathcal{Y} chosen such that it parametrizes bundles having bidegree $(0, 3k-6)$ on curves $[D \cup_y C] \in \Delta_j^0$, then for $b \geq 1$ we set

$$\mathcal{G}_{0,b} := p_*(\mathcal{L}^{\otimes b} \otimes \mathcal{O}_{\mathcal{Y}}(-2bj D_j)), \text{ when } 2 \leq j \leq 2i+2,$$

and $\mathcal{G}_{0,b} := p_*(\mathcal{L}^{\otimes b} \otimes \mathcal{O}_{\mathcal{Y}}(-b(j-2+c)D_j))$ when $2i+3 \leq j \leq 3i+5$. For each point $t = (R = C \cup_y D, L_C, L_D) \in \sigma^{-1}(\Delta_j^0)$, we have that $\mathcal{G}_{0,b}(t) = H^0(R, L_R^{\otimes b})$, where in the case $j \leq 2i+2$ we have $L_R = (L_C(-(j+2)y), L_D(-(k-2-j)y)) \in \text{Pic}^{3k-6}(R)$ and since $H^1(R, L_R^{\otimes b}) = 0$ for $b \geq 2$, it follows that $\mathcal{G}_{0,b}$ is a vector bundle for all $b \geq 1$. When $1 \leq a \leq i$, the sheaves $\mathcal{G}_{a,b}$ are defined inductively using the sequences (4). For this to make sense and in order to conclude that $\mathcal{G}_{a,b}$ are vector bundles, one has to show that

$$(5) \quad H^1(R, \wedge^a M_{L_R} \otimes L_R^{\otimes b}) = 0,$$

for all $b \geq 2$ and $0 \leq a \leq i$. To achieve this we use the Mayer-Vietoris sequence on R :

$$0 \rightarrow \wedge^a M_{L_R} \otimes L_R^{\otimes b} \rightarrow (\wedge^a M_{L_R} \otimes L_R^{\otimes b})|_C \oplus (\wedge^a M_{L_R} \otimes L_R^{\otimes b})|_D \rightarrow \wedge^a M_{L_R} \otimes L_R^{\otimes b} \otimes \mathbb{C}(y) \rightarrow 0,$$

together with the exact sequences

$$0 \rightarrow H^0(L_{R|C}(-y)) \otimes \mathcal{O}_D \rightarrow M_{L_R} \otimes \mathcal{O}_D \rightarrow M_{L_{R|D}} \rightarrow 0,$$

and

$$0 \rightarrow H^0(L_{R|D}(-y)) \otimes \mathcal{O}_C \rightarrow M_{L_R} \otimes \mathcal{O}_C \rightarrow M_{L_{R|C}} \rightarrow 0.$$

We obtain that (5) holds if we can show that

$$H^1(C, \wedge^a M_{L_{R|C}} \otimes L_{R|C}^{\otimes b}) = H^1(C, \wedge^a M_{L_{R|C}} \otimes L_{R|C}^{\otimes b}(-y)) = 0$$

and

$$H^1(D, \wedge^a M_{L_{R|D}} \otimes L_{R|D}^{\otimes b}) = H^1(D, \wedge^a M_{L_{R|D}} \otimes L_{R|D}^{\otimes b}(-y)) = 0.$$

We only check this when $j \leq 2i+2$ the remaining case being similar. Since $h^0(L_{R|C}) = k-1-j$ and $h^0(L_{R|D}) = j+1$, the inequalities $g(C) + \min\{h^0(L_{R|C}) - 1, i\} \leq \deg(L_{R|C})$ and $g(D) + \min\{h^0(L_{R|D}) - 1, i\} \leq \deg(L_{R|D})$ are satisfied precisely because $j \leq 2i+2$ and we can invoke Lemma 2.1. \square

Next, for $0 \leq a \leq i$ and $b \geq 1$ we define vector bundles $\mathcal{H}_{a,b}$ over $\tilde{\mathfrak{G}}_{3k-6}^{k-2}$ having fibre $\mathcal{H}_{i,j}(C, L) = H^0(\wedge^i M_{\mathbf{P}^{k-2}}(j))$ over each point $(C, L) \in \tilde{\mathfrak{G}}_{3k-6}^{k-2}$ corresponding to an embedding $C \xrightarrow{|L|} \mathbf{P}^{k-2}$. First we set $\mathcal{H}_{0,1} := \mathcal{G}_{0,1}$ and $\mathcal{H}_{0,b} := \text{Sym}^b \mathcal{H}_{0,1}$ for $b \geq 1$. Having already defined $\mathcal{H}_{a-1,b}$ for all $b \geq 1$, we define $\mathcal{H}_{a,b}$ via the exact sequence

$$(6) \quad 0 \rightarrow \mathcal{H}_{a,b} \rightarrow \wedge^a \mathcal{H}_{0,1} \otimes \text{Sym}^b \mathcal{H}_{0,1} \rightarrow \mathcal{H}_{a-1,b+1} \rightarrow 0.$$

The bundles $\mathcal{H}_{a,b}$ being defined entirely in terms of $\mathcal{H}_{0,1}$, the right exactness of the sequence (6) is obvious. There is a natural vector bundle morphism $\phi_{a,b} : \mathcal{H}_{a,b} \rightarrow \mathcal{G}_{a,b}$

over $\widetilde{\mathfrak{G}}_{3k-6}^{k-2}$; when $k = 3i + 6$, $a = i$ and $b = 2$, then $\text{rank}(\mathcal{H}_{i,b}) = \text{rank}(\mathcal{G}_{i,b})$ and the degeneracy locus of $\phi_{i,2}$ is the codimension 1 compactification over $\widetilde{\mathfrak{G}}_{3k-6}^{k-2}$ of the locus $\mathcal{U}_{g,i}$ defined in Proposition 2.5.

Next we compute the Chern classes of $\mathcal{H}_{a,b}$ along the curves X and Y :

Proposition 3.14. *When $k = 3i + 6$, we have the following formulas for the first Chern class of $\mathcal{H}_{i,2}$:*

$$c_1(\mathcal{H}_{i,2|X}) = c_1(\mathcal{H}_{i,2|Y}) = -(3i + 6) \binom{3i + 4}{i} ((3i + 4)x + \theta).$$

Proof. Using repeatedly the sequence (6) we obtain the formula

$$c_1(\mathcal{H}_{i,2|X}) = c_1(\mathcal{H}_{i,2|Y}) = \sum_{l=0}^i (-1)^l c_1(\wedge^{i-l} \mathcal{G}_{0,1|X} \otimes \text{Sym}^{l+2} \mathcal{G}_{0,1|X}).$$

Since $c_1(\mathcal{G}_{0,1}) = -\theta - (3i + 4)x$ and $c_1(\text{Sym}^j(\mathcal{G}_{0,1})) = \frac{j(3i+4+j)}{(3i+5)(3i+4)} \binom{3i+j+3}{j} c_1(\mathcal{G}_{0,1})$, the stated formula is obtained after a straightforward calculation. \square

In the remainder of this section we will compute $c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) \cdot \sigma^*(C^j)$, where $C^j \subset \Delta_j$ is the test curve associated to $[C] \in \mathcal{M}_{2k-2-j}$. We retain the notation introduced in Propositions 3.4 and 3.5. There are two definitions of the bundles $\mathcal{G}_{a,b|\sigma^*(\Delta_j^0)}$ depending whether $j \leq 2i + 2$ or $j \geq 2i + 3$ and we will explain in detail the calculations only in the first case, the second being similar. We start by describing the Chern number of the restriction of $\mathcal{G}_{0,1}$ to the 1-cycle $\sum_{\alpha=0}^{\lfloor j/2 \rfloor} a(j, \alpha) X_{j,\alpha}$ on $C \times \text{Pic}^{3k-j-4}(C)$:

Proposition 3.15. *Suppose C is a general curve of genus $2k - 2 - j$ where $2 \leq j \leq 2i + 2$ and let $C^j \subset \Delta_j \subset \overline{\mathcal{M}}_{2k-2}$ be the test curve associated to C . Then*

$$c_1(\mathcal{G}_{0,1|\sum_{\alpha=0}^{\lfloor j/2 \rfloor} a(j,\alpha) X_{j,\alpha}}) = -\theta - j((3k - j - 4)\eta + \gamma) - j(j + 1)(2k - 3 - j)\eta + \aleph,$$

$$\text{where } \aleph = 6(j + 1) \frac{\binom{6i+10-j}{3i+6} \binom{6i+8}{j}}{\binom{3i+4}{j}}.$$

Proof. For each $0 \leq \alpha \leq j/2$, we fix once and for all a linear series $L_D \in W_{k-2+j}^{k-2}(D)$ such that $h^0(L_D(-(k + 1 - \alpha)y)) = \alpha$ and $h^0(L_D(-(k + \alpha - j)y)) = j - \alpha$ (there are $a(j, \alpha)$ such W_{k-2+j}^{k-2} 's and for each of them $X_{j,\alpha} \times \{L_D\}$ is a component of $\sigma^*(C^j)$). Just like in Proposition 3.8, we introduce the Poincaré bundle \mathcal{L} on $X_{j,\alpha}$ and we define the sheaf on $X_{j,\alpha}$ given by $\mathcal{P} := \pi_2^*(\pi_{2*} \mathcal{L})$. Furthermore, for each $a \geq 0$ we consider the Taylor map $\nu_a : \mathcal{P} \rightarrow J_a(\mathcal{L})$ (the target being the a -th jet bundle of \mathcal{L}), and we set $\mathcal{F}_a := \text{Im}(\nu_a) \subset J_a(\mathcal{L})$ and $\mathcal{P}_{a+1} := \text{Ker}(\nu_a)$.

For each pair $(y, L_C) \in X_{j,\alpha}$ we have an exact sequence

$$0 \longrightarrow H^0(D, L_{R|D}(-y)) \longrightarrow H^0(C \cup_y D, L_R) \longrightarrow H^0(C, L_{R|C}) \longrightarrow 0,$$

where $L_R = (L_C(-(j+2)y), L_D(-(k-2-j)y))$ is the line bundle on $C \cup D$ whose global sections give the fibre of $\mathcal{G}_{0,1}$ at each point in $\widetilde{\mathfrak{G}}_{3k-6}^{k-2}$. This exact sequence globalizes to

an exact sequence of vector bundles on $X_{j,\alpha}$

$$0 \longrightarrow H^0(L_D(-(k-1-j)y)) \otimes \mathcal{O}_{X_{j,\alpha}} \longrightarrow \mathcal{G}_{0,1|X_{j,\alpha}} \longrightarrow \mathcal{P}_{j+2} \longrightarrow 0,$$

thus $c_1(\mathcal{G}_{0,1|X_{j,\alpha}}) = c_1(\mathcal{P}_{j+2})$. We now study how the bundles \mathcal{F}_a relate to one another via the exact sequences $0 \longrightarrow \pi_1^*(K_C^{\otimes a}) \otimes \mathcal{L} \longrightarrow J_a(\mathcal{L}) \longrightarrow J_{a-1}(\mathcal{L}) \longrightarrow 0$, linking successive jet bundles. Recalling that for a generic point $(y, L_C) \in X_{j,\alpha}$ we have that $a^{L_C}(y) = (0, \dots, \alpha-1, \alpha+1, \dots, j-\alpha, j-\alpha+2, \dots, k)$ (cf. Proposition 3.4), it follows that $\mathcal{F}_a = J_a(\mathcal{L})$ for $a \leq \alpha-2$, while $\mathcal{F}_{\alpha-1}$ is obtained from $J_{\alpha-1}(\mathcal{L})$ via an elementary transformation along the divisor $D_1 \subset X_{j,\alpha}$ consisting of the $b(2k-2-j, k-2+\alpha-j)$ points $(y, L_C) \in X_{j,\alpha}$ where $a_{\alpha-1}^{L_C}(y) = \alpha$. More precisely, one has the exact sequence

$$0 \longrightarrow \pi_1^*(K_C^{\otimes(\alpha-1)}) \otimes \mathcal{L} \otimes \mathcal{O}_{X_{j,\alpha}}(-D_1) \longrightarrow \mathcal{F}_{\alpha-1} \longrightarrow J_{\alpha-2}(\mathcal{L}) \longrightarrow 0.$$

Furthermore, we have exact sequences

$$0 \longrightarrow \pi_1^*(K_C^{\otimes(\alpha+1)}) \otimes \mathcal{L} \longrightarrow \mathcal{F}_{\alpha+1} \longrightarrow \mathcal{F}_{\alpha-1} \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow \pi_1^*(K_C^{\otimes a}) \longrightarrow \mathcal{F}_a \longrightarrow \mathcal{F}_{a-1} \longrightarrow 0, \text{ for all } \alpha+2 \leq a \leq j-\alpha-2.$$

If $D_2 \subset X_{j,\alpha}$ is the divisor consisting of the $b(2k-2-j, k-1+\alpha-j)$ points (y, L_C) satisfying the condition $a_{j-\alpha-1}^{L_C}(y) = j-\alpha+1$, then we also have the exact sequences

$$0 \longrightarrow \pi_1^*(K_C^{\otimes(j-\alpha)}) \otimes \mathcal{L} \otimes \mathcal{O}_{X_{j,\alpha}}(-D_2) \longrightarrow \mathcal{F}_{j-\alpha} \longrightarrow \mathcal{F}_{j-\alpha-1} \longrightarrow 0,$$

$$0 \longrightarrow \pi_1^*(K_C^{\otimes(j-\alpha+2)}) \otimes \mathcal{L} \longrightarrow \mathcal{F}_{j-\alpha+2} \longrightarrow \mathcal{F}_{j-\alpha} \longrightarrow 0,$$

and finally

$$0 \longrightarrow \pi_1^*(K_C^{\otimes a}) \otimes \mathcal{L} \longrightarrow \mathcal{F}_a \longrightarrow \mathcal{F}_{a-1} \longrightarrow 0, \text{ for } j-\alpha+2 \leq a \leq k-1.$$

We gather all the information contained in these sequences to obtain that

$$c_1(\mathcal{F}_{j+1}) = jc_1(\mathcal{L}) + j(j+1)(g(C)-1)\eta - \deg(D_1 + D_2), \text{ and then}$$

$$c_1(\mathcal{G}_{0,1|\sum_{\alpha} a(j,\alpha)X_{j,\alpha}}) = c_1(\mathcal{P}) - c_1(\mathcal{F}_{j+1}) = -\theta - j((3k-j-4)\eta + \gamma) - j(j+1)(2k-3-j)\eta + \aleph,$$

$$\text{where } \aleph := \sum_{\alpha=0}^{\lfloor j/2 \rfloor} a(j,\alpha)(b(2k-2-j, k-2+\alpha-j) + b(2k-2-j, k-1+\alpha-j)) =$$

$$= j!(6i+10-j)! \sum_{\alpha=0}^{j+1} \frac{(j-2\alpha-1)(j-2\alpha)(j-2\alpha+1)^2}{(3i+5-\alpha)!(3i+5+\alpha-j)!(j-\alpha+1)! \alpha!}.$$

This sum can be computed with Maple which finishes the proof. \square

Next we compute the Chern class of $\mathcal{G}_{0,b}$ with $b \geq 2$. The proof being similar to that in Proposition 3.15 (in fact simpler), we decided to omit it.

Proposition 3.16. *Let $b \geq 2$ and $2 \leq j \leq 2i+2$. Then for each $0 \leq \alpha \leq j/2$ we have that*

$$c_1(\mathcal{G}_{0,b|X_{j,\alpha}}) = -b^2\theta - b^2(j+2)(\gamma + (3k-4-j)\eta) - \binom{bj+2b}{2}(2k-3-j)\eta.$$

We turn our attention to the remaining components of $\sigma^*(C^j)$. First we show that $Y_{j,\alpha}'''$ does not appear in the computation of $c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) \cdot \sigma^*(C^j)$:

Proposition 3.17. *For $0 \leq \alpha \leq j/2, a \geq 0$ and $b \geq 1$, we have that $c_1(\mathcal{G}_{a,b}) \cdot Y'_{j,\alpha} = c_1(\mathcal{H}_{a,b}) \cdot Y''_{j,\alpha} = 0$.*

Proof. Clearly, it suffices to show that $c_1(\mathcal{G}_{0,b}) \cdot Y''_{j,\alpha} = 0$ for all $b \geq 1$ and then use the exact sequences (4) and (6). We carry this out only for $b = 1$ the case $b \geq 2$ being analogous. Fix $l = (L_D, V_D \subset H^0(L_D)) \in Y''_{j,\alpha}$. By duality $K_D \otimes L_D^\vee((k+2-\alpha)y)$ is one of the $a(j, \alpha)$ (that is, finitely many) linear systems $\mathfrak{g}_{j-\alpha+1}^1$ on D with a $(j-2\alpha+1)$ -fold point at y . The space of sections $V_D \subset H^0(L_D)$ is chosen such that $H^0(L_D(-2y)) \subset V_D$, hence $Y''_{j,\alpha}$ can be identified with the disjoint union of $a(j, \alpha)$ copies of the projective line $\mathbf{P}(H^0(L_D)/H^0(L_D(-2y)))$, one for each choice of L_D . Then $H^0(L_{C \cup D|D}) = H^0(L_D(-(k-1-j)y))$, for each $l \in Y''_{j,\alpha}$ (that is, $\mathcal{G}_{0,1}(l)$ is independent of V_D). Globalizing this we get that $\mathcal{G}_{0,1|Y''_{j,\alpha}}$ is trivial. \square

In a similar fashion we have the following:

Proposition 3.18. *For $0 \leq \alpha \leq j/2, a \geq 0$ and $b \geq 1$, we have the equalities $c_1(\mathcal{G}_{a,b}) \cdot Y'_{j,\alpha} = c_1(\mathcal{H}_{a,b}) \cdot Y'_{j,\alpha} = 0$.*

Proposition 3.19. *For $0 \leq \alpha \leq j/2$ and $b \geq 1$, we have the equality*

$$c_1(\mathcal{G}_{0,b}) \cdot Y''_{j,\alpha} = -\frac{b^2(j-2\alpha)j!}{\alpha!(j-\alpha)!}.$$

Proof. We start by recalling that $Y''_{j,\alpha}$ is the locus of line bundles $L_D \in \text{Pic}^{k-2+j}(D)$ such that $h^0(L_D(-(k+1+\alpha-j)y)) \geq j-\alpha-1$ and $h^0(L_D(-(k-\alpha+1)y)) \geq \alpha$. Using [Fu], Theorem 14.3, one finds that $Y''_{j,\alpha} \cong \frac{j-2\alpha}{\alpha!(j-\alpha)!} \theta^{j-1}$. Having fixed once and for all one of the finitely many linear series $L_C \in W_{3k-j-4}^{k-2}(C)$ such that $\{L_C\} \times Y'_{j,\alpha} \subset \widetilde{\mathfrak{G}}_{3k-6}^{k-2}$ there is an identification

$$\mathcal{G}_{0,1}(L_D) = H^0(L_D(-(k-2-j)y)),$$

for each $L_D \in Y''_{j,\alpha}$. Since the vector bundle $\text{Pic}^{k-2+j}(D) \ni L_D \mapsto H^0(L_D \otimes \mathcal{O}_{\beta y})$, is algebraically trivial for each $\beta \geq 0$ (remember that $y \in D$ is a fixed point), we obtain that $c_1(\mathcal{G}_{0,1|Y''_{j,\alpha}}) = -\theta$. Similarly, for $b \geq 2$ we get that $c_1(\mathcal{G}_{0,b|Y''_{j,\alpha}}) = -b^2\theta$ from which now the conclusion follows. \square

Propositions 3.15, 3.16, 3.17, 3.18, 3.19 will enable us to compute the intersection numbers $c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) \cdot \sigma^*(C^j)$ needed to determine the coefficient of δ_j in the expansion of $[\overline{\mathcal{Z}}_{6i+10,i}]$. Repeatedly using (6), one can write that

$$(7) \quad c_1(\mathcal{H}_{i,2}) = \sum_{l=0}^i (-1)^l c_1(\wedge^{i-l} \mathcal{G}_{0,1} \otimes \text{Sym}^{l+2} \mathcal{G}_{0,1}) = (3i+6) \binom{3i+4}{i} c_1(\mathcal{G}_{0,1})$$

(note the similarity with Proposition 3.14). Using (4) we also have that

$$(8) \quad c_1(\mathcal{G}_{i,2}) = \sum_{l=0}^i (-1)^l c_1(\wedge^{i-l} \mathcal{G}_{0,1} \otimes \mathcal{G}_{0,l+2}),$$

and a simple calculation yields that

$$(9) \quad c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) = -\frac{13i^2 + 35i + 24}{(3i+4)(i+1)} \binom{3i+4}{i} c_1(\mathcal{G}_{0,1}) + \sum_{l=0}^i (-1)^l \binom{3i+5}{i-l} c_1(\mathcal{G}_{0,l+2}).$$

4. THE SLOPE OF $\overline{\mathcal{Z}}_{6i+10,i}$

In this section we finish the calculation of the class of the virtual divisor $\overline{\mathcal{Z}}_{g,i}$. As before, we fix $i \geq 0$, $k = 3i + 6$ and $g = 6i + 10$, hence $\phi : \mathcal{H}_{i,2} \rightarrow \mathcal{G}_{i,2}$ is a morphism between two vector bundles of the same rank defined over $\tilde{\mathfrak{G}}_{9i+12}^{3i+4}$.

Theorem 4.1. *We fix $i \geq 0$ and denote by $\sigma : \tilde{\mathfrak{G}}_{9i+12}^{3i+4} \rightarrow \widetilde{\mathcal{M}}_{6i+10}$ the natural projection map. Then we have the following formula for the class of the pushforward to $\widetilde{\mathcal{M}}_{6i+10}$ of the virtual degeneracy locus of the morphism $\phi : \mathcal{H}_{i,2} \rightarrow \mathcal{G}_{i,2}$:*

$$\sigma_*(c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})) = \frac{1}{3i+5} \binom{6i+7}{3i+4, 2i+1, i+2} (a\lambda - b_0 \delta_0 - \dots - b_{3i+5} \delta_{3i+5}),$$

where

$$a = \frac{(4i+7)(6i^2+19i+12)}{(2i+3)(i+2)}, \quad b_0 = \frac{12i^2+31i+18}{3(2i+3)}, \quad b_1 = \frac{12i^2+33i+20}{i+2} \text{ and}$$

$$b_j = \frac{jf(i,j)}{6(i+1)(i+2)(2i+3)(6i+9-j)}, \text{ for } 2 \leq j \leq 2i+2,$$

with

$$f(i,j) = 864i^5 - (240j - 5256)i^4 + (16j^2 - 998j + 11830)i^3 + (31j^2 - 1254j + 11585)i^2 + (-5j^2 - 286j + 3981)i - 24j^2 + 240j - 216,$$

and

$$b_j = \frac{2(13i^2+35i+24)(3i+5)}{(i+1)(i+2)\binom{6i+10}{j}} \left(\frac{3i+4}{(j-1)/2} \right)^2 + \frac{g(i,j)}{12(i+1)(i+2)(2i+3)(6i+9-j)}$$

for an odd j such that $2i+3 \leq j \leq 3i+5$, where

$$g(i,j) = 1728ji^5 - (576j^2 - 10512j + 540)i^4 + (48j^3 - 2592j^2 + 24120j - 3234)i^3 + (140j^3 - 3926j^2 + 25176j - 7278)i^2 + (109j^3 - 2107j^2 + 10875j - 7263)i + 12j^3 - 156j^2 + 972j - 2700$$

and

$$b_j = \frac{(13i^2+35i+24)((3i+5)(2j+1) - j^2)}{2(i+1)(i+2)(3i+5)\binom{6i+10}{j}} \left(\frac{3i+5}{j/2} \right)^2 + \frac{h(i,j)}{12(i+1)(i+2)(2i+3)(6i+9-j)},$$

for an even j such that $2i+3 \leq j \leq 3i+5$, where

$$h(i,j) = 1728ji^5 - (576j^2 - 10512j + 1020)i^4 + (48j^3 - 2592j^2 + 24280j - 6034)i^3 + (140j^3 - 3942j^2 + 25830j - 13400)i^2 + (109j^3 - 2145j^2 + 11774j - 13230)i + 12j^3 - 180j^2 + 1392j - 4896.$$

In particular, $b_j \geq b_0$ for all $1 \leq j \leq 3i+5$ and

$$s(\sigma_*(c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}))) = \frac{a}{b_0} = \frac{3(4i+7)(6i^2+19i+12)}{(12i^2+31i+18)(i+2)}.$$

Proof. Since $\text{codim}(\overline{\mathcal{M}}_g - \widetilde{\mathcal{M}}_g, \overline{\mathcal{M}}_g) \geq 2$, it makes no difference whether we compute the class $\sigma_*(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})$ on $\widetilde{\mathcal{M}}_g$ or on $\overline{\mathcal{M}}_g$ and we can write

$$\sigma_*(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) = A\lambda - B_0 \delta_0 - B_1 \delta_1 - \dots - B_{3i+5} \delta_{3i+5} \in \text{Pic}(\overline{\mathcal{M}}_g),$$

where $\lambda, \delta_0, \dots, \delta_{3i+5}$ are the generators of $\text{Pic}(\overline{\mathcal{M}}_g)$. We start with the following:

Claim: *One has the relation $A - 12B_0 + B_1 = 0$.*

We pick a general curve $[C, q] \in \mathcal{M}_{6i+9,1}$ and at the fixed point q we attach to C a Lefschetz pencil of plane cubics. If we denote by $R \subset \overline{\mathcal{M}}_g$ the resulting curve, then $R \cdot \lambda = 1$, $R \cdot \delta_0 = 12$, $R \cdot \delta_1 = -1$ and $R \cdot \delta_j = 0$ for $j \geq 2$. The relation $A - 12B_0 + B_1 = 0$ follows once we show that $\sigma^*(R) \cdot c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) = 0$. To achieve this we check that $\mathcal{G}_{0,b|\sigma^*(R)}$ is trivial and then use (4) and (6). We take $[C \cup_q E]$ to be an arbitrary curve from R , where E is an elliptic curve. Using that limit $\mathfrak{g}_{9i+12}^{3i+4}$ on $C \cup_q E$ are in 1 : 1 correspondence with linear series $L \in W_{9i+12}^{3i+4}(C)$ having a cusp at q (this being a statement that holds independent of E) and that $\mathcal{G}_{0,b|\sigma^*(\Delta_1^0)}$ consists on each fibre of sections of the genus $g - 1$ aspect of the limit $\mathfrak{g}_{9i+12}^{3i+4}$, the claim now follows.

Now we determine A, B_0 and B_1 explicitly. We fix a general pointed curve $(C, q) \in \mathcal{M}_{6i+9,1}$ and construct the test curves $C^1 \subset \Delta_1$ and $C^0 \subset \Delta_0$. Using the notation from Proposition 3.3, we get that $\sigma^*(C^0) \cdot c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) = c_1(\mathcal{G}_{i,2|Y}) - c_1(\mathcal{H}_{i,2|Y})$ and $\sigma^*(C^1) \cdot c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) = c_1(\mathcal{G}_{i,2|X}) - c_1(\mathcal{H}_{i,2|X})$ (the other component T of $\sigma^*(C^1)$ does not appear because $\mathcal{G}_{0,b|T}$ is trivial for all $b \geq 1$). On the other hand

$C^0 \cdot \sigma_*(c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})) = (12i + 18)B_0 - B_1$ and $C^1 \cdot \sigma_*(c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})) = (12i + 16)B_1$, while we already know that $A - 12B_0 + B_1 = 0$. Using Propositions 3.11, 3.12, 3.14, the expressions for $[X]$ and $[Y]$ as well as the well-known formula $\theta^j x^{k-2-j} = g!/(g-j)!$ on C_{k-2} , we get a linear system of 3 equations in A, B_0 and B_1 which leads to the stated formulas for the first three coefficients.

To compute B_j for $2 \leq j \leq 3i + 5$, we fix general curves $[C] \in \mathcal{M}_{6i+10-j}$ and $[D, y] \in \mathcal{M}_{j,1}$ and consider the test curve $C^j \subset \Delta_j$. Then on one hand we have that $2(6i + 9 - j)B_j = \sigma^*(C^j) \cdot c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})$, on the other hand

$$\sigma^*(C^j) \cdot c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) = c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) \cdot \sum_{\alpha=0}^{\lfloor j/2 \rfloor} \left((a(j, \alpha)X_{j,\alpha} + b(6i+10-j, 3i+5+\alpha-j)Y_{j,\alpha}'') \right).$$

Using (9) together with Proposition 3.19, we get that

$$\begin{aligned} & c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) \cdot \sum_{\alpha=0}^{\lfloor j/2 \rfloor} b(6i+10-j, 3i+5+\alpha-j)Y_{j,\alpha}'' = \\ & = \frac{(8i^2 + 19i + 12)j!(6i+10-j)!(3i+2)!}{i!(2i+4)!} \left(\sum_{\alpha=0}^j \frac{(j-2\alpha-1)(j-2\alpha)^2(j-2\alpha+1)}{\alpha!(j-\alpha)!(3i+5-\alpha)!(3i+5+\alpha-j)!} \right), \end{aligned}$$

which can be computed by Maple, whereas $c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) \cdot (\sum_{\alpha=0}^{\lfloor j/2 \rfloor} a(j, \alpha)X_{j,\alpha})$ can be computed using (9) and Propositions 3.15 and 3.16 (note in particular that \aleph has to be multiplied by the coefficient of $c_1(\mathcal{G}_{0,1})$ in (9)). The rest now follows and checking that $b_j \geq b_0$ for $j \geq 1$ is elementary. \square

Remark 4.2. If we retain the notation of Theorem 4.1, we see that if we fix $j \geq 1$ and let i vary, we have that $\lim_{i \rightarrow \infty} \frac{b_i}{b_0} = 6j$. This is the same asymptotical estimate as in the case of the classical Brill-Noether divisors where $\frac{b_j}{b_0} = \frac{6j(g-j)}{g+3}$ for all $j \geq 1$ (cf. [EH3]).

Theorem 4.3. *Assume that vector bundle morphism $\phi : \mathcal{H}_{i,2} \rightarrow \mathcal{G}_{i,2}$ is non-degenerate at a general point from $\sigma^{-1}(\Delta_1^0)$. Then $\overline{\mathcal{Z}}_{6i+10,i}$ is a divisor on $\overline{\mathcal{M}}_{6i+10}$ and*

$$6 < s(\overline{\mathcal{Z}}_{6i+10,i}) = \frac{3(4i+7)(6i^2+19i+12)}{(12i^2+31i+18)(i+2)} < 6 + \frac{12}{g+1}.$$

Remark 4.4. Since $\sigma^{-1}(\Delta_1^0)$ is irreducible, to verify the assumption made in Theorem 4.3 it suffices to show that if $C \subset \mathbf{P}^{3i+4}$ is a general 1-cuspidal curve with $p_a(C) = 6i+10$ and $\deg(C) = 9i+12$, then C satisfies property (N_i) . We checked this for $g = 16, 22$ (cf. Remarks 2.9 and 2.11), while the case $g = 10$ was treated in [FP]. The assumption in Theorem 4.3 implies that ϕ is non-degenerate over a general point from $\sigma^{-1}(\Delta_0^0)$ and also over a general point from $\sigma^{-1}(\mathcal{M}_g^0)$, which is equivalent to $\mathcal{Z}_{g,i}$ being a divisor on \mathcal{M}_g^0 . Therefore the assumption is slightly stronger than Conjecture 2.6.

Proof of Theorem 4.3. We denote by $Z(\phi)$ the degeneracy locus of ϕ , thus $\sigma_*(Z(\phi))$ is a divisor on $\overline{\mathcal{M}}_{6i+10,i}$ such that $\sigma_*(Z(\phi)) = \overline{\mathcal{Z}}_{g,i} + \sum_{j=0}^{3i+5} d_j \Delta_j$, for certain coefficients $d_j \geq 0$. Using the Remark above we obtain that $d_0 = d_1 = 0$, hence $s(\overline{\mathcal{Z}}_{6i+10,i}) = s(\sigma_*(c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2})))$ and the rest follows from Theorem 4.1. \square

Remark 4.5. In the simplest case $i = 0$, we can compare the formula for $[\overline{\mathcal{Z}}_{6i+10,i}]$ with our findings in [FP]. Theorem 4.1 gives the formula

$$\sigma_*(c_1(\mathcal{G}_{0,2} - \mathcal{H}_{0,2})) \equiv 42(7\lambda - \delta_0 - 5\delta_1 - \frac{4}{3}\delta_2 - 7\delta_3 - \frac{44}{7}\delta_4 - \frac{685}{42}\delta_5),$$

whereas Theorem 1.6 from [FP] says that if \mathcal{K}_{10} is the divisor on \mathcal{M}_{10} of curves lying on a $K3$ surface then

$$\overline{\mathcal{K}}_{10} = 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - b_5\delta_5,$$

where $b_5 \geq 6$. Since we have also established the set-theoretic equality $\mathcal{Z}_{10,0} = \mathcal{K}_{10}$ (cf. [FP], Theorem 1.7), it follows that there is a scheme-theoretic equality $\mathcal{Z}_{10,0} = 42\mathcal{K}_{10}$. Here 42 is the number of pencils \mathfrak{g}_6^1 on a general curve of genus 10, and its appearance in the formula of $[\overline{\mathcal{Z}}_{10,0}]$ has a clear geometric meaning: if a Brill-Noether general curve $[C] \in \mathcal{M}_{10}$ fails (N_0) for one linear system $\mathfrak{g}_{12}^4 = K_C(-\mathfrak{g}_6^1)$, then it fails (N_0) for all 42 linear systems \mathfrak{g}_{12}^4 it possesses, that is, the map $\sigma : \mathcal{U}_{10,0} \rightarrow \mathcal{K}_{10,0}$ is $42 : 1$. Moreover, the vector bundle morphism $\mathcal{H}_{0,2} \rightarrow \mathcal{G}_{0,2}$ is degenerate along each of the boundary divisors $\Delta_2, \dots, \Delta_5$. This situation presumably extends to higher genera as well hence the main thrust of Theorem 4.1 is that it computes $s(\overline{\mathcal{Z}}_{6i+10,i})$.

To finish the proof of Theorems 1.1 and 1.2 we specialize to the cases $g = 16$ and 22 and note that the assumption made in Theorem 4.3 is satisfied in these situations (cf. Theorems 2.7 and 2.10).

5. AN EFFECTIVE DIVISOR ON $\overline{\mathcal{M}}_{14,1}$

In this section we describe how to construct effective divisors on $\overline{\mathcal{M}}_{g,n}$ using syzygy type conditions for pointed curves. We treat only one example. We denote by $\overline{\mathcal{Z}}_{14,0}^1$ the closure in $\overline{\mathcal{M}}_{14,1}$ of the locus $\mathcal{Z}_{14,0}^1$ of smooth pointed curves $[C, p] \in \mathcal{M}_{14,1}$ for which there exists a linear series $L \in W_{18}^6(C)$ such that the map

$$\mu_{L,p} : \text{Sym}^2 H^0(C, L(-p)) \rightarrow H^0(C, L^{\otimes 2}(-2p))$$

is not an isomorphism. Just like the in the case of the loci $\mathcal{Z}_{g,i}$ on \mathcal{M}_g , the locus $\mathcal{Z}_{14,0}^1$ can be naturally viewed as the pushforward of the degeneracy locus of a morphism between two vector bundles of the same rank 21 over $\mathfrak{G}_{18}^6 \times_{\mathcal{M}_{14}} \mathcal{M}_{14,1}$.

We are going to show that $\overline{\mathcal{Z}}_{14,0}^1$ is a divisor and in order to compute its class we need some preparations. Recall that for $g \geq 3$, the group $\text{Pic}(\overline{\mathcal{M}}_{g,1})$ is freely generated by λ , the tautological class ψ and the boundary classes $\delta_i = [\Delta_i]$ with $0 \leq i \leq g-1$, where for $i \geq 1$, the generic point of Δ_i is a union of two smooth curves of genus i and $g-i$ meeting at a point, the marked point lying on the genus i component. We denote by $\pi : \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$ the natural forgetful map. We can then write the class of $\overline{\mathcal{Z}}_{14,0}^1$ on $\overline{\mathcal{M}}_{14,1}$ as

$$\overline{\mathcal{Z}}_{14,0}^1 \equiv a \lambda + c \psi - b_0 \delta_0 - b_1 \delta_1 - \cdots - b_{13} \delta_{13}.$$

As before, we will determine the relevant coefficients in the expression of $[\overline{\mathcal{Z}}_{14,0}^1]$ by intersecting our divisor with various test curves.

Proposition 5.1. *Let C be a general curve of genus 14 and $\tilde{C} = \pi^{-1}([C]) \subset \overline{\mathcal{M}}_{14,1}$, the test curve obtained by letting the marked point vary along C . Then $\tilde{C} \cdot \overline{\mathcal{Z}}_{14,0}^1 = 12012$, hence $c = \tilde{C} \cdot \overline{\mathcal{Z}}_{14,0}^1 / (2g-2) = 462$.*

Proof. Let us fix a linear series $L \in W_{18}^6(C)$. We count the number of points $p \in C$, for which the multiplication map $\mu_{L,p}$ not injective. If $p_1 : C \times C \rightarrow C$ and $p_2 : C \times C \rightarrow C$ are the two projections, we define the vector bundles

$$\mathcal{E} := (p_2)_*(p_1^*(L) \otimes \mathcal{O}_{C \times C}(-\Delta)) \text{ and } \mathcal{F} := (p_2)_*(p_1^*(L^{\otimes 2}) \otimes \mathcal{O}_{C \times C}(-2\Delta)).$$

There is a natural multiplication map $\mu_L : \text{Sym}^2(\mathcal{E}) \rightarrow \mathcal{F}$, and the cardinality of its degeneracy locus is just $c_1(\mathcal{F}) - c_1(\text{Sym}^2(\mathcal{E}))$. A simple calculation shows that $c_1(\mathcal{E}) = -4x - \theta = -18$, hence $c_1(\text{Sym}^2(\mathcal{E})) = -126$, while $c_1(\mathcal{F}) = -98$. We obtain that $\mu_{L,p}$ is not an isomorphism for precisely 28 points $p \in C$. Since C has 429 linear series \mathfrak{g}_{18}^6 (cf. [ACGH]), we obtain $c = (\tilde{C} \cdot \overline{\mathcal{Z}}_{14,0}^1) / 26 = 429 \cdot 28 / 26 = 462$. \square

For more relations among the coefficients of $[\overline{\mathcal{Z}}_{14,0}^1]$ we define the map $\nu : \overline{\mathcal{M}}_{1,2} \rightarrow \overline{\mathcal{M}}_{g,1}$ obtained by attaching to each 2-pointed elliptic curve $[E, q, p]$ a fixed general 1-pointed curve $[C, q] \in \mathcal{M}_{g-1}$ (the point of attachment being q). One has the pullback formulas

$$\nu^*(\lambda) = \lambda, \nu^*(\psi) = \psi_p, \nu^*(\delta_0) = \delta_0, \nu^*(\delta_1) = -\psi_q \text{ and } \nu^*(\delta_{g-1}) = \delta_{qp},$$

where ψ_p and ψ_q are the tautological classes corresponding to the marked points p and q , while δ_{qp} is the boundary component of curves having a rational tail containing both

q and p . On $\overline{\mathcal{M}}_{1,2}$ these classes are not independent and we have the relations $\psi_q = \psi_p$, $\lambda = \psi_p - \delta_{qp}$ and $\delta_0 = 12(\psi_p - \delta_{qp})$ (see e.g. [AC2], Proposition 1.9).

Proposition 5.2. *If $\nu : \overline{\mathcal{M}}_{1,2} \rightarrow \overline{\mathcal{M}}_{14,1}$ is as above, then $\nu^*(\overline{\mathcal{Z}}_{14,0}^1) = \emptyset$. It follows that we have the relations*

$$a - 12b_0 + b_{13} = 0 \text{ and } c + b_1 = b_{13}.$$

Proof. We assume that $[X = C \cup_q E, p \in E] \in \overline{\mathcal{Z}}_{14,0}^1$. Then there exists a limit linear series \mathfrak{g}_{18}^6 on X determined by its aspects $L_C \in W_{18}^6(C)$ and $L_E \in W_{18}^6(E)$, together with compatible elements

$\rho_C \in \text{Ker}\{\mu_{L_C} : \text{Sym}^2 H^0(L_C) \rightarrow H^0(L_C^{\otimes 2})\}$, $\rho_E \in \text{Ker}\{\mu_{L_E} : \text{Sym}^2 H^0(L_E) \rightarrow H^0(L_E^{\otimes 2})\}$, satisfying the inequality

$$\text{ord}_q(\rho_C) + \text{ord}_q(\rho_E) \geq \deg(L_C) + \deg(L_E) = 36$$

and such that $\rho_E \in \text{Sym}^2 H^0(L_E(-p))$ (see [FP], Section 4, for how to study multiplication maps in the context of limit linear series). Because $(C, q) \in \overline{\mathcal{M}}_{13,1}$ is general, the vanishing sequence of L_C at q equals $(0, 2, 3, 4, 5, 6, 7)$, the vanishing sequence of L_E at q is $(11, 12, 13, 14, 15, 16, 18)$ and finally, the vanishing sequence of L_E at p is either $(0, 1, 2, 3, 4, 5, 6)$ or $(0, 1, 2, 3, 4, 5, 7)$, depending on whether $p - q \in \text{Pic}^0(E)$ is a 7-torsion class or not.

We claim that $\text{ord}_q(\rho_E) \leq 29 (= 13 + 16 = 14 + 15)$. Indeed, otherwise $\text{ord}_q(\rho_E) \geq 30 (= 14 + 16 = 15 + 15)$, and since $\rho_E \in \text{Sym}^2 H^0(L_E(-p))$, after subtracting base points ρ_E becomes a $\neq 0$ element in the kernel of the map $\text{Sym}^2 H^0(N) \rightarrow H^0(N^{\otimes 2})$, where $N = L_E(-p - 14q) \in \text{Pic}^3(E)$. This is obviously impossible. Therefore $\text{ord}_q(\rho_E) \leq 29$, so by compatibility, $\text{ord}_q(\rho_C) \geq 7 (= 2 + 5 = 3 + 4)$. We now show that when $(C, p) \in \mathcal{M}_{13,1}$ is chosen generically, there can be no such element ρ_C which leads to a contradiction.

Claim: Suppose $\sigma_0, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7$ is a basis of $H^0(L_C)$ adapted to the point q in the sense that $\text{ord}_q(\sigma_i) = i$. If $W(q, L_C) \subset \text{Sym}^2 H^0(L_C)$ denotes the 17-dimensional subspace spanned by the elements $\sigma_i \cdot \sigma_j$ with $4 \leq i \leq j \leq 7$, $\sigma_3 \cdot \sigma_j$ for $j \geq 4$ and $\sigma_2 \cdot \sigma_j$ for $j \geq 5$, then the restriction of the multiplication map $W(q, L_C) \rightarrow H^0(L_C^2(-7q))$ is an isomorphism. (Note that $W(q, L_C)$ does not depend on the chosen basis $\{\sigma_i\}$).

The proof of this claim is similar to the proof of Theorem 5.1 in [FP]. It is enough to construct a single 1-cuspidal curve $X \subset \mathbf{P}^6$ with $p_a(X) = 14$ and $\deg(X) = 18$, such that if $\nu : C \rightarrow X$ is the normalization of X and $q \in C$ is the inverse image of the cusp, then X does not lie on any quadric contained in $W(q, \nu^*(\mathcal{O}_X(1)))$. We construct the following cuspidal curve: define $\Gamma \subset \mathbf{P}^6$ to be the image of the map $t \mapsto [1, t^2, t^3, t^4, t^5, t^6, t^7]$, then choose a general hyperplane $H \subset \mathbf{P}^6$ which intersects Γ in distinct points p_1, \dots, p_7 . Take $D \subset H$ to be a general smooth curve of genus $g(D) = 7$ and $\deg(D) = 11$ which passes through p_1, \dots, p_7 . Then $X := \Gamma \cup D$ is a curve of arithmetic genus 14 and degree 21 having a cusp at the point $q = f(0) \in \Gamma$. The quadrics in $W(q, \nu^*(\mathcal{O}_X(1)))$ can of course be written down explicitly and to show that D can be chosen such that it is not contained in any quadric from $W(q, \nu^*(\mathcal{O}_X(1)))$ amounts to a simple counting argument. \square

Remark 5.3. The claim we have just proved also shows that $\overline{\mathcal{Z}}_{14,0}^1$ is a divisor on $\overline{\mathcal{M}}_{14,1}$. Alternatively, this can be proved in the same way as Theorem 2.7.

We now determine the coefficient b_1 :

Proposition 5.4. *Consider general curves $[E, p, q] \in \mathcal{M}_{1,2}$ and $[C] \in \mathcal{M}_{13}$, and denote by $C^2 \subset \overline{\mathcal{M}}_{14,1}$ the test curve consisting of points $\{[C \cup_q E, p \in E]\}_{\{q \in C\}}$. Then $C^2 \cdot \overline{\mathcal{Z}}_{14,0}^1 = 133848$, therefore $b_1 = C^2 \cdot \overline{\mathcal{Z}}_{14,0}^1 / 24 = 5577$.*

Proof. Throughout the proof we will use the notations introduced in Proposition 3.11. From the proof of Proposition 5.2 it is clear that the intersection number $C^2 \cdot \overline{\mathcal{Z}}_{14,0}^1$ equals the number of pairs $(q, D) \in X \subset C \times C_6$ such that the map

$$W(q, K_C(-D)) \rightarrow H^0(K_C^{\otimes 2}(-2D - 7q))$$

is not an isomorphism. To compute this number we construct the rank 17 vector bundle W over X having fibre $W(q, D) = W(q, K_C(-D))$ over each point $(q, D) \in X$. The curve C being general, the vanishing sequence $a^{K_C(-D)}(q)$ will be generically equal to $(0, 2, 3, 4, 5, 6, 7)$, while at a finite number of points $(q, D) \in X$ we will have that $a^{K_C(-D)}(q) = (0, 2, 3, 4, 5, 6, 8)$. In order to compute $c_1(W)$ we note that W has a subbundle $W_1 \subset W$ which fits into two exact sequences:

$$0 \rightarrow \text{Sym}^2 u_*(v^*(\mathcal{M}) \otimes \mathcal{O}(-4\Delta))|_X \rightarrow W_1 \rightarrow u_*(v^*(\mathcal{M}) \otimes \mathcal{O}(-4\Delta))|_X \otimes \mathcal{P} \rightarrow 0,$$

where $\mathcal{P} = u_*(v^*(\mathcal{M}) \otimes \mathcal{I}_\Delta^3 / \mathcal{I}_\Delta^4)|_{X'}$, and

$$0 \rightarrow W_1 \rightarrow W \rightarrow u_*(v^*(\mathcal{M}) \otimes \mathcal{O}(-5\Delta))|_X \otimes u_*(v^*(\mathcal{M}) \otimes \mathcal{I}_\Delta^2 / \mathcal{I}_\Delta^3)|_X \rightarrow 0.$$

Next, we consider the multiplication map $W \rightarrow u_*(v^*(\mathcal{M}^{\otimes 2}) \otimes \mathcal{O}(-7\Delta))|_X$ whose degeneration locus we want to compute. The intersection number $C^2 \cdot \overline{\mathcal{Z}}_{14,0}^1$ is equal to

$$\left(c_1(u_*(v^*(\mathcal{M}^{\otimes 2}) \otimes \mathcal{O}(-7\Delta))) - c_1(W) \right) \cdot [X] = (3\theta - 7x - \gamma + 18\eta) \cdot [X] = 133848,$$

as it turns out after a short calculation. Here we have used that $c_1(W)$ can be computed from the two exact sequences involving W and W_1 , while

$$c_1(u_*(v^*(\mathcal{M}^{\otimes 2}) \otimes \mathcal{O}(-7\Delta))) = c_1(\mathcal{F}_2) - c_1(J_6(\mathcal{M}^{\otimes 2})) = -4\theta - 34x + 14\gamma - 756\eta.$$

□

Since $b_1 = 5577$ we now obtain that $b_{13} = b_1 + c = 6039$. To compute the coefficient b_0 (and thus the λ -coefficient a) we use our last test curve:

Proposition 5.5. *Let $(C, q, p) \in \mathcal{M}_{13,2}$ be a general 2-pointed curve. We denote by $C^3 \subset \overline{\mathcal{M}}_{14,1}$ the family consisting of curves $\{[C/y \sim q, p]\}_{\{y \in C\}}$. Then $C^3 \cdot \overline{\mathcal{Z}}_{14,0}^1 = c + 26b_0 - b_{13} = 24453$. It follows that $b_0 = 1155$ and $a = 7821$.*

Proof. We retain the notations introduced in Proposition 3.12. We construct a vector bundle map $\text{Sym}^2(\mathcal{E}) \rightarrow \mathcal{F}$ over the curve $Y \subset C \times C_6$, where $\mathcal{E} = u_*(v^*(\mathcal{M}) \otimes \mathcal{O}(-\Gamma_p))$ is the bundle with fibre $\mathcal{E}(y, D) = H^0(K_C(-D - p))$, while \mathcal{F} is the bundle with fibre

$$\mathcal{F}(y, D) = H^0(K_C^{\otimes 2}(-2D - 2p - y - q)) \oplus \mathbb{C} \cdot t^2 \subset H^0(K_C^{\otimes 2}(-2D - 2p)),$$

where $H^0(K_C(-D-p))/H^0(K_C(-D-p-y-q)) = \mathbb{C} \cdot t$, for every $(y, D) \in Y$.

It is easy to show that $c_1(\mathcal{E}) = -5x - \theta$, hence $c_1(\text{Sym}^2(\mathcal{E})) = -35x - 7\theta$. The class $c_1(\mathcal{F})$ can be computed from the exact sequence

$$0 \longrightarrow u_*(v^*(\mathcal{M}^{\otimes 2}) \otimes \mathcal{O}(-2\Gamma_p - \Gamma_q - \Delta))|_Y \longrightarrow \mathcal{F} \longrightarrow (\mathcal{M}^{\otimes 2} \otimes \mathcal{O}(-2\Gamma_p))|_Y(-A) \longrightarrow 0,$$

where $A = Y \cap \pi_1^{-1}(p)$ is the effective divisor on the curve Y consisting of all points (p, D) such that $h^0(p+q+D) \geq 2$. Using the formula for $[Y]$ (cf. Proposition 3.12), we get that $|A| = 429$. We also compute that $c_1(u_*(v^*(\mathcal{M}^{\otimes 2}) \otimes \mathcal{O}(-2\Gamma_p - \Gamma_q - \Delta))) = -4\theta - 40x - 33\eta + 2\gamma$ and of course $c_1(\mathcal{M}^{\otimes 2} \otimes \mathcal{O}(-2\Gamma_p)) = 2(17\eta - \gamma - x)$. Therefore we can write that

$$\mathbb{C}^3 \cdot \overline{\mathcal{Z}}_{14,0}^1 = (c_1(\mathcal{F}) - c_1(\text{Sym}^2(\mathcal{E}))) \cdot [Y] - |A| = (-7x + 3\theta + \eta) \cdot [Y] - |A| = 24453.$$

□

We have thus far determined the coefficients a, c, b_0, b_1, b_{13} in the expansion of $[\overline{\mathcal{Z}}_{14,0}^1]$. This is already enough to conclude that $[\overline{\mathcal{Z}}_{14,0}^1]$ lies outside the cone of $\text{Pic}(\overline{\mathcal{M}}_{14,1})$ spanned by pullbacks of effective divisors from $\overline{\mathcal{M}}_{14}$, Brill-Noether divisors on $\overline{\mathcal{M}}_{14,1}$ and boundary divisors (see the discussion after Theorem 1.4 for the relevance of this result). To get a bound on the remaining coefficients b_j for $2 \leq j \leq 12$, we use a variant of Theorem 1.1 from [FP]. The boundary divisor $\Delta_j \subset \overline{\mathcal{M}}_{14,1}$ with $3 \leq j \leq 12, j \neq 4$ is filled-up by pencils R_j obtained by attaching to a fixed 2-pointed curve $[B, p, q] \in \mathcal{M}_{j,2}$ a variable 1-pointed curve $[C, q] \in \overline{\mathcal{M}}_{14-j,1}$ moving in a Lefschetz pencils of curves of genus $14-j$ sitting on a fixed $K3$ surface. Deformations of R_j cover Δ_j for $j \neq 2, 4$, hence we have that $R_j \cdot \overline{\mathcal{Z}}_{14,0}^1 \geq 0$. Since one also has the relations (see [FP], Lemma 2.4)

$$R_j \cdot \lambda = 15 - j, R_j \cdot \delta_j = -1, R_j \cdot \delta_0 = 6(17 - j), R_j \cdot \psi = 0 \text{ and } R_j \cdot \delta_i = 0 \text{ for } i \neq 0, j,$$

we immediately get the estimate $b_j \geq 15 + 27j$ for all $3 \leq j \leq 12, j \neq 4$. To obtain the bounds $b_2 \geq 325$ and $b_4 \geq 271$ we use similar pencils filling up Δ_2 and Δ_4 respectively. We skip these details. This completes the proof of Theorem 1.4.

6. THE KODAIRA DIMENSION OF $\overline{\mathcal{M}}_{g,n}$

In this last section we use the effective divisors $\overline{\mathcal{Z}}_{16,1}$ and $\overline{\mathcal{Z}}_{22,2}$ to improve Logan's results about which moduli spaces $\overline{\mathcal{M}}_{g,n}$ are of general type. For a general reference about $\text{Pic}(\overline{\mathcal{M}}_{g,n})$ we refer to [AC2] and [Log]. For each $1 \leq i \leq n$ we denote by $\pi_i : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,1}$ the morphism forgetting all marked points except the one labelled by i and we also consider the map $\pi : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_g$ which forgets all marked points. We recall that the canonical class of $\overline{\mathcal{M}}_{g,n}$ is given by the formula

$$K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda - 2\delta_0 + \sum_{i=1}^n \psi_i - 2 \sum_{i \geq 0, S} \delta_{i:S} - \sum_S \delta_{1:S}.$$

Here ψ_i is the tautological class corresponding to the i -th marked point, while $\delta_{i:S}$ with $i \geq 0$ and $\emptyset \neq S \subset \{1, \dots, n\}$ denotes the class of the boundary divisor with generic point being a union of two curves of genus i and $g-i$ such that the marked points on the genus i component are precisely those labelled by S .

Proof of Theorem 1.3. We start with the case $g = 22$ and we show that $\overline{\mathcal{M}}_{22,2}$ is of general type. On $\overline{\mathcal{M}}_{22,2}$ we consider the averaged pullback of the Weierstrass divisor

$$W_{12} := \pi_1^*(\overline{\mathcal{W}}) + \pi_2^*(\overline{\mathcal{W}}) \equiv -2\lambda + \frac{g(g+1)}{2}(\psi_1 + \psi_2) - \sum_{i \geq 0, S \neq \emptyset} c_{i,S} \delta_{i,S},$$

where $c_{i,S} \geq 0$, and significantly, the coefficient of δ_0 is 0. By Theorem 1.2, we have another effective class, namely $[\pi^*(\overline{\mathcal{Z}}_{22,2})] = c(\frac{1665}{256}\lambda - \delta_0 - \dots)$, where $c > 0$. One can easily check that $K_{\overline{\mathcal{M}}_{22,2}}$ can be written as a positive combination of $[W_{12}]$, $[\pi^*(\overline{\mathcal{Z}}_{22,2})]$, $\psi_1 + \psi_2$ and some other boundary classes. Since $\psi_1 + \psi_2$ is big and nef, it follows that $\overline{\mathcal{M}}_{22,2}$ is of general type.

When $g = 21$ we use the maps $\chi_{i,j} : \overline{\mathcal{M}}_{21,5} \rightarrow \overline{\mathcal{M}}_{22}$ for $1 \leq i < j \leq 5$, where $\chi_{i,j}$ identifies the marked points labelled by i and j and forgets those labelled by $\{i, j\}^c$. The \mathbb{Q} -divisor class

$$\sum_{i < j} \chi_{i,j}^*(\overline{\mathcal{Z}}_{22,2}) \equiv c\left(\frac{1665}{256}\lambda - \delta_0 + \frac{2}{5} \sum_{i=1}^5 \psi_i - \dots\right), \text{ where } c > 0,$$

is obviously effective on $\overline{\mathcal{M}}_{21,5}$. Since the \mathbb{Q} -class $-\lambda + 11 \sum_{i=1}^5 \psi_i - 0 \cdot \delta_0 - \dots$, is also effective (cf. [Log], Theorem 5.4 - we have retained only the coefficients that play a role in our argument), once again we see that $K_{\overline{\mathcal{M}}_{21,5}}$ can be written as the sum of an effective divisor and a positive multiple of $\sum_{i=1}^5 \psi_i$.

Finally we settle the case $g = 16$: we adapt Theorem 5.4 from [Log] to conclude that the \mathbb{Q} -class $-\lambda + \frac{23}{9}(\sum_{i=1}^9 \psi_i) - 0 \cdot \delta_0 - \dots$, is effective on $\overline{\mathcal{M}}_{16,9}$ (precisely, this is the class of the S_9 -orbit of the closure in $\overline{\mathcal{M}}_{16,9}$ of the effective divisor D on $\mathcal{M}_{16,9}$ consisting of points $[C, p_1, \dots, p_9]$ such that $h^0(C, 2p_1 + \dots + 2p_7 + p_8 + p_9) \geq 2$ and we have explicitly indicated that the coefficient of δ_0 is 0 and retained only the coefficients that are significant for this calculation). The class $[\pi^*(\overline{\mathcal{Z}}_{16,1})] = c(\frac{407}{61}\lambda - \delta_0 - \dots)$, where $c > 0$, is also effective and one writes $K_{\overline{\mathcal{M}}_{16,9}}$ as a positive sum of these two effective classes, boundary classes and the big and nef class $\sum_{i=1}^9 \psi_i$. \square

Remark 6.1. Theorem 6.3 from [Log] claims that $\overline{\mathcal{M}}_{22,4}$ is of general type but the numerical argument used in the proof seems to be incorrect.

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