Entropic Risk Measures on Wiener Space

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Abstract

An entropic risk measure $\rho_{H,\beta}$ on Wiener space is defined in terms of *relative entropy* $H(\cdot|P)$ with respect to Wiener measure P. Since Wiener measure satisfies a 0-1 law on the fine structure of paths, such a risk measure is not sensitive to risks in the fine structure. In order to capture such risks, and in particular the volatility risk generated by the quadratic variation of paths, we introduce a new rescaled version $\rho_{h,\beta}$ of the entropic risk measure. Here the relative entropy $H(\cdot|P)$ is replaced by the *specific relative entropy* $h(\cdot, P)$, introduced on Wiener space in [11]. We compute the resulting risk assessment for various functionals of the quadratic variation. We also show how the risk measure $\rho_{h,\beta}$ can be described in terms of large deviations in the quadratic variation of paths.

1 Introduction

On any probability space $(\Omega, \mathcal{F}, P)$, the entropic risk measure

$$\rho_{H,\beta}(X) = \frac{1}{\beta} \log E_P[\exp(-\beta X)] \tag{1}$$

is well defined for any $X \in L^1(P)$. The parameter β specifies the level of risk aversion; as it increases from 0 to ∞, the risk $\rho_{H,\beta}(X)$ ranges from the risk-neutral assessment $E_P[-X]$ to the worst-case assessment, given by the essential supremum of $-X$ under P. The risk measure $\rho_{H,\beta}$ has the dual representation

$$\rho_{H,\beta}(X) = \sup \big(E_Q[-X] - \frac{1}{\beta} H(Q|P) \big), \tag{2}$$

where $H(Q|P)$ denotes the *relative entropy* of Q with respect to P, and where the supremum is taken over all probability measures Q on $(\Omega, \mathcal{F})$ such that $H(Q|P)$ is finite.

In the sequel, $(\Omega, \mathcal{F}, P)$ will denote the Wiener space, that is, P is the Wiener measure on the canonical path space $\Omega = C_0[0,1]$. Wiener measure P

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satisfies a 0-1-law on the fine structure of paths, and in particular on the σ-field $\hat{\mathcal{F}}$ generated by the quadratic variation of paths. Thus, any functional $\hat{X}$ depending on the quadratic variation is, P-almost surely, equal to its expectation $E_P[\hat{X}]$. The same is true for any Q such that $H(Q|P) < \infty$, and so equation (2) as well as equation (1) yield the risk-neutral assessment

$$\rho_{H,\beta}(\hat{X}) = E_P[-\hat{X}].$$

In this sense, the classical risk measure $\rho_{H,\beta}$ is not sensitive to the volatility risk generated by the quadratic variation of paths.

In order to capture such risks, we are going to rescale the entropic risk measure, replacing the relative entropy $H(Q|P)$ in equation (2) by the *specific relative entropy* $h(\cdot|P)$, introduced on Wiener space by N. Gantert in her thesis [11]. It is defined as

$$h(Q|P) = \lim_{N\uparrow\infty} 2^{-N} H_N(Q|P),$$

where $H_N(Q|P)$ denotes the relative entropy of Q with respect to P on the σ-field generated by observing the path along the N-th dyadic partition of the unit interval. The corresponding risk measure $\rho_{H,\beta}$ is given by

$$\rho_{h,\beta}(X) = \sup_{Q\in\mathcal{Q}} \big(E_Q[-X] - \frac{1}{\beta} h(Q|P)\big); \tag{3}$$

here the supremum is taken over a general class $\mathcal{Q}$ of semimartingale measures Q on $(\Omega, \mathcal{F})$, to be specified below. The functional $\rho_{h,\beta}$ is indeed a convex risk measure in the sense of [10], Ch.4. But, in contrast to $\rho_{H,\beta}$, the new risk measure $\rho_{h,\beta}$ does not neglect the null-sets of Wiener measure, and it does recognise risks in the fine structure.

In this paper, we focus on how the risk measure $\rho_{h,\beta}$ quantifies the risk of positions $\hat{X}$ that depend on the quadratic variation of paths. More precisely, let us denote by $\mu_N(\omega)$ the finite measure on $[0,1]$ defined by the quadratic variation of the path ω along the N-th dyadic partition of the unit interval, that is,

$$\mu_N(\omega) = \sum_{k=1}^{2^N} \big(\Delta_{N,k} W\big)^2 \delta_{(k-1)2^{-N}},$$

where $\Delta_{N,k}W$ denotes the k-th increment of the coordinate process W along the N-th dyadic partition. Under Wiener measure P, these discrete measures converge to Lebesgue measure λ on $[0,1]$, that is,

$$\lim_{N\uparrow\infty} \mu_N(\cdot) = \lambda \qquad P - \text{a.}s., \tag{4}$$

where the limit is taken in the weak topology. Under any semimartigale measure $Q \in \mathcal{Q}$, λ will be replaced by a continuous finite random measure $\mu(\omega)$, defined by the quadratic variation of the coordinate process W under Q.

Now consider a position $\hat{X}$ of the form

$$\hat{X}(\cdot) = F\big(\mu(\cdot)\big),$$

where F is some functional on the space of non-negative finite measures on $[0,1]$. Let $\mathcal{M}$ denote the class of continuous finite measures on $[0,1]$. In section 3 we show that

$$\rho_{h,\beta}(\hat{X}) = \sup_{\nu \in \mathcal{M}} \big(-F(\nu) - \frac{1}{\beta} h(Q_\nu | P)\big), \tag{5}$$

where Q_ν denotes the law of $(W_{\nu[0,t]})_{0 \le t \le 1}$ under Wiener measure P. Thus, the class $\mathcal{Q}$ of semimartingale measures in (3) can be reduced to a much smaller class of martingale measures if we view $\rho_{h,\beta}$ as a convex risk measure on $(\Omega, \hat{\mathcal{F}})$. The reduced representation (5) will allow us to compute the risk $\rho_{h,\beta}(\hat{X})$ explicitly for various examples.

Equation (5) also allows us to connect the risk measure $\rho_{h,\beta}$ with large deviations of the quadratic variation from its convergence as stated in equation (4). Indeed, it was shown in [11] that the sequence $(\mu_N)_{N=1,2,\ldots}$ satisfies a large deviations principle

$$\frac{1}{2^N} \log P[\mu_N(\cdot) \in A] \asymp -\inf_{\nu \in A} I(\nu), \tag{6}$$

where the rate function I is given by the specific relative entropy $h(Q_\nu|P)$. Combined with (5), this will lead us to the representation

$$\rho_{h,\beta}(\hat{X}) = \lim_{N \uparrow \infty} \rho_{H,\beta 2^N}(\hat{X}_N), \tag{7}$$

where $\hat{X}(\cdot) = F(\mu(\cdot))$ and $\hat{X}_N(\cdot) = F(\mu_N(\cdot))$. Thus, the risk assessment provided by the entropic risk measure $\rho_{h,\beta}$ can be described as a combination of *zooming* and *rescaling*: We zoom into the fine structure by observing the quadratic variation along increasing dyadic partitions; at each stage we apply the classical entropic risk measure, but at the same time we rescale it by sending the parameter of risk aversion to ∞ at the rate $\beta 2^N$.

The paper is organised as follows. Section 2 collects some general notions and facts related to relative entropy. In Section 3 we review special properties of relative entropy and specific relative entropy on Wiener space. In addition, we discuss the impact of entropy bounds $H(Q|P) < \infty$ and $h(Q|P) < \infty$ on the structure of Q, and we prove two contraction results for the specific relative entropy that will be used in Section 5. In Section 4 we introduce the rescaled risk measure $\rho_{h,\beta}$ and derive some of its key properties. In particular we show how the representation (3) can be reduced if we restrict the risk measure to positions $\hat{X} = F(\mu)$ that only depend on the quadratic variation of the paths. In the final Section 5 we compute the risk $\rho_{h,\beta}(\hat{X})$ explicitly for various special choices of the functional F, including the volatility swap $F(\mu) = \sqrt{\mu[0,1]}$.

2 Preliminaries

Let $(S, \mathcal{S})$ be some measurable space; in the sequel, S will be either an Euclidean space or the space $C_0[0,1]$ of all continuous functions ω on $[0,1]$ with initial value $\omega(0) = 0$.

Definition 1. *For two probability measures μ and ν on $(S, \mathcal{S})$, the relative entropy of ν with respect to μ is defined as*

$$H(\nu|\mu) = \begin{cases} \int \log \frac{d\nu}{d\mu} d\nu & \textit{if} \nu \ll \mu, \\ +\infty & \textit{otherwise.} \end{cases}$$

For $\nu \ll \mu$ we can write

$$H(\nu|\mu) = \int h(\frac{d\nu}{d\mu}) d\mu,$$

denoting by h the strictly convex function $h(x) = x \log x$ on $[0, \infty)$. Thus, Jensen's inequality implies

$$H(\nu|\mu) \geq 0, \tag{8}$$

with equality if and only if $\mu = \nu$.

Remark 2. *We are going to use the same notation if μ is a non-negative finite measure with total mass $\mu(S) > 0$. In this case, we get the lower bound*

$$H(\nu|\mu) = H(\nu|\tilde{\mu}) - \log \mu(S) \geq -\log \mu(S),$$

applying inequality (8) *to the probability measure $\tilde{\mu} := \mu(S)^{-1}\mu$.*

In the sequel we are going to use the following facts.

2.1 Relative entropy satisfies the equation

$$H(\nu|\mu) = \sup_f \big(\int f d\nu - \log \int e^f d\mu \big),$$

where the supremum is taken over all bounded measurable functions f on S; see, e.g., [10], Th. C.5. If S is a metric space with Borel field $\mathcal{S}$, it is enough to take the supremum over all bounded continuous functions. This shows that $H(\nu|\mu)$ is lower-semicontinuous with respect to the weak topology, both in μ and in ν.

2.2 For a sequence $(\mathcal{S}_n)_{n=1,2...}$ of σ-fields increasing to $\mathcal{S}$, the relative entropy $H_n(Q|P)$, computed on the σ-field $\mathcal{S}_n$, increases with n, and we have

$$\lim_{n \uparrow \infty} H_n(\nu|\mu) = H(\nu|\mu). \tag{9}$$

2.3 We denote by $\Gamma(\mu, \nu)$ the class of all probability measures γ on the product space $S \times S$ with marginals μ and ν, and by $\Delta = \{(x,x)|x \in \mathcal{S}\}$ the diagonal in $S \times S$. The total variation distance between μ and ν is given by

$$||\mu - \nu||_{TV} = 2 \sup_{A \in \mathcal{S}} |\mu(A) - \nu(A)| = \inf_{\gamma \in \Gamma(\mu,\nu)} \gamma(\Delta^c),$$

and *Pinsker's inequality* compares it to the relative entropy $H(\nu|\mu)$:

$$||\mu-\nu||_{TV} \leq \sqrt{2H(\nu|\mu)}. \tag{10}$$

2.4 Consider a measurable cost function $C(\cdot,\cdot)$ on $S\times S$ with values in $[0,\infty]$; typically, $C(\cdot,\cdot)$ will be a metric on S. For such a C, we are going to use the following Wasserstein distance of order two:

Definition 3. *The* Wasserstein distance *between ν and μ is defined as*

$$W_C(\nu,\mu) = \inf_{\gamma\in\Gamma(\mu,\nu)} (\int C^2(x,y)\gamma(dx,dy))^{1/2}. \tag{11}$$

Equivalently, we can write

$$W_C(\nu,\mu) = \inf \tilde{E}[C^2(\tilde{X},\tilde{Y})]^{1/2}, \tag{12}$$

where the infimum is taken over all couples $(\tilde{X},\tilde{Y})$ of S-valued random variables on some probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ such that $\tilde{X}$ and $\tilde{Y}$ have distributions μ and ν, respectively. Such a couple, and also any measure $\gamma\in\Gamma(\mu,\nu)$, will be called a *coupling of μ and ν*.

2.5 For $n=1,2,\ldots,\infty$ let μ_n be the product of standard normal distributions $N(0,1)$ on the Euclidean space $S=R^n$, and let $W_n(\cdot,\cdot)$ denote the Wasserstein distance corresponding to the Euclidean norm on R^n, that is, to the cost function $C(x,y)=(\sum_{k=1}^n(x_i-y_i)^2)^{\frac{1}{2}}$. *Talagrand's transport inequality* states that

$$W_n(\nu,\mu_n) \leq \sqrt{2H(\nu|\mu_n)}; \tag{13}$$

cf. [16]. If we replace μ_n by $\mu_n^\sigma := \prod_{i=1}^n N(0,\sigma^2)$, we obtain

$$W_n(\nu,\mu_n^\sigma) \leq \sigma\sqrt{2H(\nu|\mu_n^\sigma)}. \tag{14}$$

Two analogous inequalities on Wiener space will appear in Section 3.

3 Entropies on Wiener space

From now on, the underlying measurable space will be the path space

$$\Omega = C_0[0,1]$$

of all continuous functions ω on $[0,1]$ with initial value $\omega(0)=0$. We denote by $(\mathcal{F}_t)_{0\leq t\leq 1}$ the right-continuous filtration on Ω generated by the coordinate process

$$W=(W_t)_{0\leq t\leq 1}$$

defined by $W_t(\omega) = \omega(t)$. We set $\mathcal{F} = \mathcal{F}_1$, and we denote by P the *Wiener measure* on $(\Omega, \mathcal{F})$. For any probability measure Q on $(\Omega, \mathcal{F})$, the relative entropy of Q with respect to Wiener measure P is defined as

$$H(Q|P) = \begin{cases} E_Q[\log \frac{dQ}{dP}] & \text{if}\, Q \ll P, \\ +\infty & \text{otherwise.} \end{cases}$$

Let us first recall how, for $Q \ll P$, the relative entropy $H(Q|P)$ can be computed in terms of the *intrinsic drift* of Q. To this end, we denote by $\mathcal{H}$ the *Cameron-Martin space* of all absolutely continuous functions $\omega \in \Omega$ such that the derivative $\dot{\omega}$ is square integrable on $[0, 1]$. For $\omega \in \Omega$ we write

$$||\omega||_{\mathcal{H}} = \begin{cases} (\int_0^1 \dot{\omega}^2(t)dt)^{1/2} & \text{if } \omega \in \mathcal{H} \\ +\infty & \text{otherwise.} \end{cases}$$

Proposition 4. *Suppose that Q is absolutely continuous with respect to Wiener measure P. Then there exists an adapted process $B^Q = (B_t^Q(\omega))_{0 \le t \le 1}$ with paths in $\mathcal{H}$ such that $W^Q := W - B^Q$ is a Wiener process under Q, that is, W is a semimartingale under Q with Doob decomposition*

$$W = W^Q + B^Q. \tag{15}$$

The process $b^Q := \dot{B}^Q$ will be called the intrinsic drift *of Q, and the relative entropy of Q with respect to P is given by*

$$H(Q|P) = \frac{1}{2} E_Q\big[||B^Q||_{\mathcal{H}}^2\big] = \frac{1}{2} E_Q\big[\int_0^1 (b_t^Q)^2 dt\big]. \tag{16}$$

Proof. See [6] and [7] or, for the first part, Th. 7.11 in [15]. □

Remark 5. *As first observed by J. Lehec in* [14]*, the preceding proposition yields an immediate proof of Talagrand's transport inequality on Wiener space. Just note that the couple (W, W^Q), defined on the probability space $(\Omega, \mathcal{F}, Q)$, is a coupling of Q and P, and that equation* (16) *implies*

$$E_Q\big[||W - W^Q||_{\mathcal{H}}^2\big] = E_Q\big[||B^Q||_{\mathcal{H}}^2\big] = 2H(Q|P).$$

Thus, we obtain the transport inequality

$$W_{\mathcal{H}}(Q, P) \le \sqrt{2H(Q|P)}, \tag{17}$$

where the Wasserstein distance $W_{\mathcal{H}}$ is defined as in (11)*, using as cost function $C(\omega, \eta)$ the Cameron-Martin norm $||\omega - \eta||_{\mathcal{H}}$. On Wiener space, inequality* (17) *was first stated in [5]. However, using the Lévy-Ciesielski representation of Brownian motion in terms of Schauder functions, it can also be seen as a direct translation, for $n = \infty$, of Talagrand's original inequality* (13)*.*

Clearly, the relative entropy $H(Q|P)$, and in particular Talagrand's transport inequality (17), is of interest only if Q is absolutely continuous with respect to Wiener measure P. Let us now go beyond that case. For any $N \geq 1$, let D_N denote the N-th dyadic partition of the unit interval $[0,1]$, and put

$$\mathcal{F}_N = \sigma(\{W_t | t \in D_N\}) = \sigma(\{\Delta_{N,k} W | k = 1, \dots, 2^N\}),$$

where $\Delta_{N,k} W$ denotes the k-th increment of the coordinate process W along D_N. We denote by $H_N(Q|P)$ the relative entropy of Q with respect to P on the σ-field $\mathcal{F}_N$. Since the σ-fields $\mathcal{F}_N$ increase to $\mathcal{F}$, we have

$$H(Q|P) = \lim_{N \uparrow \infty} H_N(Q|P);$$

cf. subsection 2.2. We focus on the case $H(Q|P) = \infty$. Typically, the finite-dimensional marginals of Q will be such that $H_N(Q|P)$ is finite for each N. It is then natural to rescale the finite-dimensional entropies $H_N(Q|P)$ in order to obtain meaningful results.

The following concept of specific relative entropy on Wiener space was introduced by N. Gantert in her thesis [11], where it plays the role of a rate function for large deviations of the quadratic variation from its ergodic behaviour (4); cf. also [12]. In [9] and [8], specific relative entropy is used to extend Talagrand's inequality on Wiener space beyond the absolutely continuous case. In [1] it is shown to provide the solution to an optimisation problem posed by D. Aldous. In our present context, it will lead us to a new rescaled version of the entropic risk measure.

Definition 6. *For any probability measure Q on $(\Omega, \mathcal{F})$, the* specific relative entropy *of Q with respect to Wiener measure P is defined as*

$$h(Q|P) = \lim_{N \uparrow \infty} 2^{-N} H_N(Q|P), \tag{18}$$

whenever this limit exists. Otherwise, we denote by $\overline{h}(Q|P)$ the limit superior, and by $\underline{h}(Q|P)$ the limit inferior.

For a martingale measure Q, conditions for the existence of the limit $h(Q|P)$ are given in [2]; a special case appears in Proposition 15 below. See also [9] and [8].

The following proposition throws some light on the impact that an entropy bound of the form $\overline{h}(Q|P) < \infty$ or $\underline{h}(Q|P) < \infty$ has on the structure of the measure Q; see also Corollary 11 and Remark 13 below

Proposition 7. *For any probability measure Q on $(\Omega, \mathcal{F})$,*

$$\limsup_{N \uparrow \infty} E_Q\Big[\sum_{k=0}^{2^N} (\Delta_{N,k} W)^2\Big] \;\leq\; 2\big(1 + 2\overline{h}(Q|P)\big). \tag{19}$$

Thus, the condition $\overline{h}(Q|P) < \infty$ implies that W is a process of finite energy *under Q; cf.* [13].

Proof. The law of the vector $(\Delta_{N,k}W)_{k=1,\dots,2^N}$ under P coincides with the measure μ_n^σ defined in subsection 1.5 for $n = 2^N$ and $\sigma^2 = 2^{-N}$. We denote by ν its law under Q, and by $||\cdot||$ the Euclidean norm on R^n. For any coupling $\gamma \in \Gamma(\nu, \mu_n^\sigma)$, we obtain

$$\begin{aligned} E_Q\big[\sum_{k=0}^{2^N}(\Delta_{N,k}W)^2\big] &= \int ||x||^2\gamma(dx,dy) \\ &\leq 2\big(\int ||y||^2\gamma(dx,dy) + \int ||x-y||^2\gamma(dx,dy)\big), \end{aligned}$$

hence

$$E_Q\big[\sum_{k=0}^{2^N}(\Delta_{N,k}W)^2\big] \leq 2\big(1 + W_n^2(\nu, \mu_n^\sigma)\big).$$

Since $H(\nu|\mu_n^\sigma) = H_N(Q|P)$, Talagrand's inequality (14) yields

$$E_Q\big[\sum_{k=0}^{2^N}(\Delta_{N,k}W)^2\big] \leq 2\big(1 + 2\cdot 2^{-N}H_N(Q,P)\big), \tag{20}$$

and this implies (19). □

From now on we restrict the discussion to semimartingale measures Q that satisfy a condition of square-integrability, as suggested by Proposition 19. More precisely:

Definition 8. *We denote by* $\mathcal{Q}$ *the class of all probability measures* Q *on* $(\Omega, \mathcal{F})$ *such that the coordinate process* $W = (W_t)_{0\leq t\leq 1}$ *is a semimartingale under* Q *with Doob-Meyer decomposition*

$$W = M + A,$$

where

i) M *is a square-integrable continuous martingale with continuous quadratic variation* $< M > = (< M >_t)_{0\leq t\leq 1}$,

ii) the process A *has continuous paths of bounded variation with* $A_0 = 0$, *and the process* $|A|$ *of total variation satisfies* $|A|_1 \in L^2(Q)$.

We denote by $\mu(\omega, \cdot)$ *the finite random measure on* $[0,1]$ *defined,* Q*-almost surely, by*

$$\mu(\omega, [0,t]) = <M>_t(\omega) \ = <W>_t(\omega) \qquad (0 \leq t \leq 1).$$

Remark 9. *Any probability measure* Q *on* $(\Omega, \mathcal{F})$ *such that* $H(Q[P) < \infty$ *belongs to* $\mathcal{Q}$, *since* W *is a semimartingale under* Q *with Doob-Meyer decomposition* $W = W^Q + B^Q$; *cf. Proposition 4. Since the quadratic variation satisfies*

$$< W >_t = < W^Q >_t = t \qquad Q - a.s.,$$

we obtain

$$Q[\,\mu(\cdot)=\lambda\,] = P[\,\mu(\cdot)=\lambda\,] = 1,$$

where λ *denotes Lebesgue measure on the unit interval* $[0,1]$*.For a general* $Q \in \mathcal{Q}$*, note that*

$$Q[\,\mu(\cdot)\in\mathcal{M}\,] = 1,$$

where we denote by $\mathcal{M}$ *the class of continuous finite measures on* $[0,1]$*.*

The following theorem was first proved in [11] in the special case where $\mu(\cdot)$ is absolutely continuous under Q; see [9] for our present version and [8] for an extension beyond the class of semimartingale measures.

Theorem 10. *For any* $Q \in \mathcal{Q}$*, the specific relative entropy satisfies the inequality*

$$\underline{h}(Q|P) \geq \frac{1}{2}E_Q\big[\mu(\cdot,[0,1]) - 1 + H(\lambda|\mu(\cdot))\big]. \tag{21}$$

Corollary 11. *For any* $Q \in \mathcal{Q}$*, the condition* $\underline{h}(Q|P) < \infty$ *implies*

$$Q[\,\mu(\cdot)\in\mathcal{M}_0\,] = 1,$$

where we define

$$\mathcal{M}_0 := \{\nu\in\mathcal{M} \,|\, H(\lambda|\nu) < \infty\}.$$

Remark 12. *For any* $\nu \in \mathcal{M}$*, consider the Lebesgue decomposition*

$$\nu(dt) = \nu_s(dt) + \sigma^2(t)dt$$

of ν *with respect to* λ*, where* ν_s *denotes the singular part and* $\sigma^2(\cdot)$ *is the density of the absolutely continuous part. If* $\lambda \ll \nu$ *then we can write*

$$H(\lambda|\nu) = -\int_0^1 \log(\sigma^2(t))dt\big). \tag{22}$$

Thus,

$$\nu\in\mathcal{M}_0 \implies \lambda(\{\sigma^2(\cdot)>0\}) = 1. \tag{23}$$

Remark 13. *In [9] it is shown that the condition* $\underline{h}(Q|P) < \infty$ *implies the existence of a Wiener process* W^Q *under* Q *such that*

$$E_Q\big[< W - W^Q >_1 \big] \leq 2\,\underline{h}(Q|P).$$

Since the couple (W, W^Q) *on* $(\Omega,\mathcal{F},Q)$ *is a coupling of* Q *and* P*, we obtain the following analogue to Talagrand's inequality on Wiener space:*

$$W_{<\cdot>}(Q,P) \leq \sqrt{2\,\underline{h}(Q|P)}, \tag{24}$$

where $W_{<\cdot>}$ *denotes the Wasserstein distance defined in terms of quadratic variaiton, that is, the cost function is given by* $C(\omega,\eta) = \sqrt{<\omega-\eta>_1}$*. See [9] and [8] for further variants that go beyond absolute continuity, and also beyond the class of semimartingale measures.*

Let us now introduce a class of martingale measures contained in $\mathcal{Q}$ that will play a key role in the following section.

Definition 14. *For any $\nu \in \mathcal{M}$ we denote by Q_ν the law of $(W_{\nu[0,t]})_{0\leq t\leq 1}$ unter Wiener measure P.*

Note that W is a square-integrable martingale under Q_ν, and that its quadratic variation satisfies

$$Q_\nu[\mu(\cdot) = \nu] = 1.$$

In particular, we have $Q_\nu \in \mathcal{Q}$ for any $\nu \in \mathcal{M}$.

The following proposition was shown in [11]; see also [9] and [8].

Proposition 15. *For any $\nu \in \mathcal{M}$, the specific relative entropy $h(Q_\nu|P)$ exists as a limit and is given by*

$$\begin{aligned} h(Q_\nu|P) &= \lim_{N\uparrow\infty} 2^{-N} H_N(Q|P) \\ &= \frac{1}{2}\big(\nu[0,1]) - 1 + H(\lambda|\nu)\big). \end{aligned} \tag{25}$$

In particular, $h(Q_\nu|P)$ is finite if and only if $H(\lambda|\nu) < \infty$, that is,

$$h(Q_\nu|P) < \infty \iff \nu \in \mathcal{M}_0.$$

In many cases, the computation of the entropic risk measure $\rho_{h,\beta}$ defined in (3) will involve a contraction argument for the specific relative entropy. We are going to use the following two versions.

For any $Q \in \mathcal{Q}$ we define the measure $Q \otimes \mu$ on the product space $\Omega \times [0,1]$ by

$$(Q \otimes \mu)(A \times B) = \int_A \mu(\omega, B) Q(d\omega),$$

Since

$$(Q \otimes \mu)(S \times S) = E_Q[\mu(\cdot,[0,1])] = E_Q[<M>_1] = E_Q[M_1^2] < \infty,$$

the measure $Q \otimes \mu$ is finite. We denote by $Q\mu$ its marginal distribution on $[0,1]$, that is,

$$Q\mu(B) = \int \mu(\omega, B) Q(d\omega).$$

Proposition 16. *For any $\nu \in \mathcal{M}$,*

$$\begin{aligned} \inf_{Q:Q\mu=\nu} \underline{h}(Q|P) &= h(Q_\nu|P) \\ &= \frac{1}{2}\big(\nu[0,1] - 1 + H(\lambda|\nu)\big). \end{aligned} \tag{26}$$

Proof. Take any Q such that $Q\mu = \nu$. Due to (21), we have

$$\begin{aligned} \underline{h}(Q|P) &\geq \frac{1}{2}E_Q\big[\mu[(\cdot,[0,1]) - 1 + H(\lambda|\mu(\cdot))\big] \\ &= \frac{1}{2}\big(\nu[0,1] - 1 + E_Q\big[H(\lambda|\mu(\cdot))\big]. \end{aligned}$$

Since relative entropy decreases if the σ-field shrinks (cf. subsection 2.2), we obtain

$$\begin{aligned} E_Q\big[H(\lambda|\mu(\cdot))\big] &= H(Q\otimes\lambda|Q\otimes\mu) \\ &\geq H(\lambda|Q\mu) = H(\lambda|\nu), \end{aligned}$$

hence

$$\underline{h}(Q|P) \geq \frac{1}{2}\big(\nu[0,1] - 1 + H(\lambda|\nu)\big).$$

But Proposition 16 shows that this lower bound is attained by $Q = Q_\nu$, and so we obtain equation (26). ☐

Consider the functional I on the space $\mathcal{M}_+[0,1]$ of positive finite measures on $[0,1]$, defined by

$$I(\nu) = \begin{cases} h(Q_\nu|P) & \text{if } \nu\in\mathcal{M}, \\ +\infty & \text{otherwise.} \end{cases} \tag{27}$$

In Section 4, I will serve as a rate function for the large deviations principle (6).

Lemma 17. *The functional I is lower-semicontinuous with respect to the weak topology, and its level sets are weakly compact. Moreover, if $(\nu_n)_{n=1,2,\ldots}$ is a sequence in $\mathcal{M}$ such that $\lim_{n\uparrow\infty} I(\nu_n) = 0$, then ν_n converges to Lebesgue measure λ in the total variation norm:*

$$\lim_{n\uparrow\infty} I(\nu_n) = 0 \implies \lim_{n\uparrow\infty} ||\nu_n - \lambda||_{TV} = 0\ . \tag{28}$$

Proof. In view of equation (25), lower semicontinuity follows as in subsection 1.3; in particular, the level sets of I are weakly closed. For any $\nu\in\mathcal{M}_0$ we have $\nu[0,1] > 0$, and so we can write $\nu = c(\nu)\tilde{\nu}$, where $c(\nu) = \nu[0,1]$ and $\tilde{\nu}$ is a probability measure on $[0,1]$. As in Remark 2 we obtain

$$\begin{aligned} I(\nu) &= \frac{1}{2}\big(\nu[0,1] - 1 + H(\lambda|\nu)\big) \\ &= \frac{1}{2}\big(c(\nu) - 1 - \log c(\nu)) + \frac{1}{2}H(\lambda|\tilde{\nu})\big). \end{aligned} \tag{29}$$

Since both parts are non-negative, any level set of I is contained in a weakly compact level set of the functional $c(\cdot)$, and so it is weakly compact.

Now consider a sequence $(\nu_n)_{n=1,2,\ldots}$ in $\mathcal{M}$ such that $\lim_n I(\nu_n) = 0$; clearly we may assume $\nu_n \in \mathcal{M}_0$. Due to equation (29), this implies both

$$\lim_{n\uparrow\infty} c(\nu_n) = 1 \quad \text{and} \quad \lim_{n\uparrow\infty} H(\lambda|\tilde{\nu}_n) = 0.$$

Since

$$||\nu_n - \lambda||_{TV} \leq |c(\nu_n) - 1| + ||\tilde{\nu}_n - \lambda||_{TV},$$

Pinsker's inequality (10), applied to each measure $\tilde{\nu}_n$, yields

$$\lim_{n\uparrow\infty} ||\nu_n - \lambda||_{TV} = 0.$$

□

Let us now look at an additional contraction. For $m > 0$ we denote by $\nu(m) \in \mathcal{M}_0$ the finite measure $m \cdot \lambda$ on $[0,1]$, and by $Q_m := Q_{\nu(m)} \in \mathcal{Q}$ the corresponding probability measure on $(\Omega, \mathcal{F})$. Since $H(\lambda|\nu_m) = -\log m$, equation (25) implies

$$h(Q_m|P) = \frac{1}{2}(m - 1 - \log m). \tag{30}$$

Proposition 18. *For any $m > 0$,*

$$\inf_{Q:\mu[0,1]=m} \underline{h}(Q|P) = h(Q_m|P). \tag{31}$$

Proof. Using equality (26), we obtain

$$\begin{aligned}
\inf_{Q:Q\mu[0,1]=m} \underline{h}(Q|P) &= \inf_{\nu:\nu[0,1]=m} \inf_{Q:Q\mu=\nu} \underline{h}(Q|P) \\
&= \inf_{\nu:\nu[0,1]=m} \frac{1}{2}\big(m - 1 + H(\lambda|\nu)\big) \\
&= \frac{1}{2}\big(m - 1 + \inf_{\nu:\nu[0,1]=m} H(\lambda|\nu)\big).
\end{aligned}$$

Take any $\nu \in \mathcal{M}$ such that $\nu[0,1] = m$ and denote by $d\nu = d\nu_s + \sigma^2(t)dt$ its Lebesgue decomposition with respect to λ. Applying Jensen's inequality, we obtain

$$\begin{aligned}
H(\lambda|\nu) &= -\int_0^1 \log \sigma^2(t) dt \\
&\geq -\log \int_0^1 \sigma^2(t) dt \geq -\log m,
\end{aligned}$$

hence

$$\inf_{Q\in\mathcal{Q}:Q\mu[0,1]=m} \underline{h}(Q|P) \geq \frac{1}{2}(m - 1 - \log m).$$

Equation (30) shows that this lower bound is attained by $Q = Q_m$, and so we have proved equation (31). □

4 A new entropic risk measure on Wiener space

As mentioned in the introduction, the classical entropic risk measure $\rho_{H,\beta}$ can be written as

$$\rho_{H,\beta}(X) = \sup_{Q:H(Q|P)<\infty} \big(E_Q[-X] - \frac{1}{\beta}H(Q|P)\big). \tag{32}$$

Remark 19. *By equation* (1)*, the risk* $\rho_{H,\beta}(X)$ *is well-defined in* $(-\infty,\infty]$ *for any real-valued measurable function* X *on* $(\Omega,\mathcal{F})$*. Its dual representation* (32) *is usually stated for bounded* X*. But it is actually valid for any* X *such that* $\rho_{H,\beta}(X) < \infty$*. Indeed, for any* Q *such that* $H(Q|P) < \infty$ *the expectation* $E_Q[-X]$ *is well-defined in* $[-\infty,\infty)$*, since the density* $\phi = dQ/dP$ *satisfies*

$$-\beta X\phi \leq \phi\log\phi + e^{-\beta X-1} \in L^1(P);$$

cf. [10], Corollary C.7. Thus, the right-hand side of (32) *is well-defined, and equation* (32) *follows as in the proof of [10], Corollary C.7.*

We are now going to look at the convex risk measure that arises if, in the representation (32) of the entropic risk measure $\rho_{H,\beta}$, we replace the relative entropy $H(\cdot|P)$ by the specific relative entropy $\underline{h}(\cdot|P)$.

Definition 20. *We denote by* $\rho_{h,\beta}$ *the convex risk measure defined by*

$$\rho_{h,\beta}(X) := \sup_{Q\in\mathcal{Q}} \big(E_Q[-X] - \frac{1}{\beta}\underline{h}(Q|P)\big), \tag{33}$$

where $\mathcal{Q}$ *is the class of semimartingale measures introduced in Definition 8.*

Note that $\rho_{h,\beta}(X)$ is well-defined for any measurable function X on $(\Omega,\mathcal{F})$ that is bounded from below. Clearly, it suffices to take the supremum over measures $Q \in \mathcal{Q}$ such that $\underline{h}(Q|P)$ is finite.

Remark 21. *Since* $H(Q|P) < \infty$ *implies* $Q \in \mathcal{Q}$ *and* $h(Q|P) = 0$*, we obtain*

$$\begin{aligned}\rho_{h,\beta}(X) &\geq \sup_{Q:H(Q|P)<\infty} \big(E_Q[-X] - \frac{1}{\beta}h(Q|P)\big)\\ &= \sup_{Q:H(Q|P)<\infty} E_Q[-X]\\ &= \mathrm{ess.sup_P(-X)},\end{aligned}$$

that is, the risk measure $\rho_{h,\beta}$ *dominates the worst-case risk measure defined in terms of* P*. Thus,* $\rho_{h,\beta}$ *does not seem to be very useful as long as we continue to neglect the null-sets of Wiener measure* P*.*

Let us now focus on how $\rho_{h,\beta}$ evaluates risks in the fine structure of paths, and in particular the risks generated by their quadratic variation.

Theorem 22. *Consider a position* $\hat{X}$ *of the form*

$$\hat{X}(\cdot) = F(\mu(\cdot)),$$

where F *is some measurable functional on* $\mathcal{M}$ *that is bounded from below. Then we have*

$$\rho_{h,\beta}(\hat{X}) = \sup_{\nu \in \mathcal{M}_0} \big(- F(\nu) - \frac{1}{\beta} h(Q_\nu|P)\big), \tag{34}$$

where Q_ν *denotes the law of* $(W_{\nu[0,t]})_{0 \leq t \leq 1}$ *under Wiener measure* P¿

Proof. Take any $Q \in \mathcal{Q}$. Using inequality (21), we obtain

$$\begin{aligned} E_Q[-\hat{X}] - \frac{1}{\beta}\underline{h}(Q|P) &\leq E_Q\big[- F(\mu(\cdot)) - \frac{1}{2\beta}\big(\mu(\cdot,[0,1]) - 1 + H(\lambda|\mu(\cdot))\big)\big] \\ &\leq \sup_{\nu \in \mathcal{M}} \big(- F(\nu) - \frac{1}{2\beta}(\nu([0,1]) - 1 + H(\lambda|\nu))\big). \end{aligned}$$

Since $h(Q_\nu|P)$ satisfies equation (25), and since $h(Q_\nu|P)$ is finite only if $\nu \in \mathcal{M}_0$, this implies

$$\rho_{h,\beta}(\hat{X}) \leq \sup_{\nu \in \mathcal{M}_0} \big(- F(\nu) - \frac{1}{\beta} h(Q_\nu|P)\big). \tag{35}$$

To prove the converse inequality, recall that, for any $\nu \in \mathcal{M}$, the measure Q_ν belongs to the class $\mathcal{Q}$. This implies

$$\begin{aligned} \rho_{h,\beta}(\hat{X}) &\geq \sup_{\nu \in \mathcal{M}} \big(E_{Q_\nu}[-F(\mu(\cdot))] - \frac{1}{\beta} h(Q_\nu|P)\big) \\ &\geq \sup_{\nu \in \mathcal{M}_0} \big(E_{Q_\nu}[-F(\mu(\cdot))] - \frac{1}{\beta} h(Q_\nu|P)\big) \\ &= \sup_{\nu \in \mathcal{M}_0} \big(- F(\nu) - \frac{1}{\beta} h(Q_\nu|P)\big), \end{aligned}$$

since $Q_\nu[\mu(\cdot) = \nu] = 1$. □

Remark 23. *Since* $E_{Q_\nu}[\hat{X}] = F(\nu]$, *the theorem may be read as follows: Viewed as a risk measure on the smaller space* $(\Omega, \hat{\mathcal{F}})$, $\rho_{h,\beta}$ *admits the representation*

$$\rho_{h,\beta}(\hat{X}) = \sup_{Q \in \mathcal{Q}_0} \big(E_Q[-\hat{X}] - \frac{1}{\beta} h(Q_\nu|P)\big),$$

where the supremum is taken over the class

$$\mathcal{Q}_0 := \{Q_\nu | \nu \in \mathcal{M}_0\} \subset \mathcal{Q}.$$

.

The following corollary shows that the risk $\rho_{h,\beta}(\hat{X})$ ranges from the risk-neutral value $E_P[-\hat{X}] = -F(\lambda)$ to a worst-case assessment of the form

$$\sup\{-F(\nu) \,|\, \nu \in \mathcal{M}_0\},$$

as the parameter β of risk aversion increases from 0 to ∞.

Corollary 24. *Take* $\hat{X}(\cdot) = F(\mu(\cdot))$ *and assume that* F *is continuous and bounded from below on* $\mathcal{M}$*. Then we have*

$$-F(\lambda) \leq \; \rho_{h,\beta}(\hat{X}) \;\leq\; \sup\{-F(\nu)\,|\,\nu \in \mathcal{M}_0\}. \tag{36}$$

Moreover,

$$\lim_{\beta\downarrow 0} \rho_{h,\beta}(\hat{X}) \;=\; -F(\lambda), \tag{37}$$

and

$$\lim_{\beta\uparrow\infty} \rho_{h,\beta}(\hat{X}) \;=\; \sup\{-F(\nu)\,|\,\nu \in \mathcal{M}_0\}. \tag{38}$$

Proof. 1) Since $Q_\lambda = P$ and hence $h(Q_\lambda|P) = 0$, the lower bound for $\rho_{h,\beta}(\hat{X})$ follows from equation (34) by taking $\nu = \lambda$. Also the upper bound follows from (34), since $h(Q_\nu|P) \geq 0$.

2) To prove equation (37), we define

$$C_n(\beta) \;:=\; \sup\big\{-F(\nu) - \frac{1}{\beta}h(Q_\nu|P)\,\big|\,\nu \in \mathcal{M}_0,\, h(Q_\nu|P) > \frac{1}{n}\big\}$$

and

$$M_n \;:=\; \sup\big\{-F(\nu)\big|\,\nu \in \mathcal{M}_0,\, h(Q_\nu|P) \leq \frac{1}{n}\big\}$$

for $n = 1, 2, \ldots$. Equation (34) implies

$$\rho_{h,\beta}(\hat{X}) \;\leq\; \max\big(C_n(\beta), M_n\big),$$

and since

$$\lim_{\beta\downarrow 0} C_n(\beta) \;\leq\; \lim_{\beta\downarrow 0}\big(\sup\{-F(\nu)|\nu \in \mathcal{M}_0\} - \frac{1}{\beta n}\big) \;=\; -\infty,$$

we obtain

$$\lim_{\beta\downarrow 0} \rho_{h,\beta}(\hat{X}) \;\leq\; M_n \qquad (n = 1, 2, \ldots). \tag{39}$$

For each n choose $\nu_n \in \mathcal{M}_0$ such that

$$I(\nu) = h(Q_\nu|P) \leq \frac{1}{n} \quad \text{and} \quad -F(\nu_n) \geq M_n - \frac{1}{n}.$$

Then ν_n converges to λ, due to Lemma 17. Since $-F(\nu_n) \leq M_n$, and since F is continuous, we obtain

$$\lim_{n\uparrow\infty} M_n = \lim_{n\uparrow\infty}(-F(\nu_n)) = -F(\lambda).$$

Combined with (39) and the lower bound in (36), this implies equation (37).

3) To prove equation (38), take any $c < \sup\{-F(\nu)\,|\nu \in \mathcal{M}_0\}$ and $\nu \in \mathcal{M}_0$ such that $-F(\nu) \geq c$. Since $h(Q_\nu|P) < \infty$, we have

$$\lim_{\beta\uparrow\infty} \rho_{h,\beta}(\hat{X}) \;\geq\; \lim_{\beta\uparrow\infty}\big(-F(\nu) - \frac{1}{\beta}h(Q_\nu|P)\big) \;\geq\; c.$$

and this implies equation (38). ☐

Let us now connect Theorem 22 with the theory of large deviations applied to the sequence

$$\mu_N(\omega) = \sum_{k=1}^{2^N} (\Delta_{N,k} W)^2 \delta_{(k-1)2^{-N}}, \qquad N = 1, 2, \ldots$$

Under Wiener measure P, these discrete measures converge weakly to Lebesgue measure λ on $[0,1]$:

$$\lim_{N\uparrow\infty} \mu_N(\cdot) = \lambda \qquad P - a.s.$$

As shown in [11], large deviations from this convergence are governed by a large deviation principle with the rate function I defined in (27), that is,

$$\liminf_{N\uparrow\infty} 2^{-N} \log P[\mu_N(\cdot) \in A] \, \geq \, - \inf_{\nu\in A} I(\nu)$$

for any open set $A \subseteq \mathcal{M}_+[0,1]$, and

$$\limsup_{N\uparrow\infty} 2^{-N} \log P[\mu_N(\cdot) \in A] \, \leq \, - \inf_{\nu\in A} I(\nu)$$

if A is closed. Combining this large deviations principle with Varadhan's Integral Lemma, we obtain the following corollary.

Corollary 25. *Let F be a continuous functional on $\mathcal{M}_+[0,1]$ that is bounded from below. For $\hat{X} = F(\mu(\cdot))$ and $\hat{X}_N = F(\mu_N(\cdot))$ we have*

$$\rho_{h,\beta}(\hat{X}) = \lim_{N\uparrow\infty} \rho_{H,\beta 2^N}(\hat{X}_N). \tag{40}$$

Proof. Since the sequence $(\mu_N)_{N=1,2,\ldots}$ satisfies the large deviations principle with rate function I, Varadhan's Integral Lemma, applied to the functional $-\beta F(\cdot)$, shows that

$$\lim_{N\uparrow\infty} 2^{-N} \log E_P[e^{-\beta F(\mu_N(\cdot))}] = \sup_{\nu\in\mathcal{M}} (-\beta F(\nu) - I(\nu));$$

cf. [4], Theorem 4.3.1. Divided by β and combined with Theorem 22, this yields equation (40). □

The corollary shows that the risk assessment provided by the entropic risk measure $\rho_{h,\beta}$ can be described as a combination of *zooming* and *rescaling*: We zoom into the fine structure by observing the quadratic variation along increasing dyadic partitions. At each step we apply the classical entropic risk measure based on $H(\cdot|P)$, but at the same time we rescale it by sending the parameter of risk aversion to ∞ at the rate $\beta 2^N$.

5 Some Examples

In this section we compute the risk $\rho_{h\beta}(\hat{X})$ for some special positions

$$\hat{X}(\cdot) = F(\mu(\cdot)) \tag{41}$$

depending on the quadratic variation of the paths. First we consider the case that the functional F on $\mathcal{M}_+[0,1]$ is given by the integral

$$F(\nu) = \int_0^1 f(t)\nu(dt), \tag{42}$$

where f is a continuous function on $[0,1]$.

Proposition 26. *For $f \in C[0,1]$ and $\hat{X}$ defined by (41) and (42), we have*

$$\rho_{h,\beta}(\hat{X}) = \begin{cases} -\frac{1}{2\beta}\int_0^1 \log\big(1 + 2\beta f(t)\big)dt & \textit{if } f \geq -(2\beta)^{-1} \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof. Due to Theorem 22 we can write

$$\rho_{h,\beta}(\hat{X}) = \sup_{\nu \in \mathcal{M}} G(\nu) = \sup_{\nu \in \mathcal{M}_0} G(\nu), \tag{43}$$

where we use the notation

$$\begin{aligned} G(\nu) &:= -F(\nu) - \frac{1}{\beta} h(Q_\nu|P) \\ &= -\int f d\nu - \frac{1}{2\beta}\big(\nu[0,1] - 1 - H(\lambda|\nu)\big). \end{aligned}$$

Alternatively, we could use Proposition 16. Indeed, since

$$E_Q[-\hat{X}] = -\int_0^1 f d(Q\mu),$$

the definition of $\rho_{h,\beta}$ yields

$$\begin{aligned} \rho_{h,\beta}(\hat{X}) &= \sup_{Q \in \mathcal{Q}} \Big(-\int f d(Q\mu) - \frac{1}{\beta} h(Q|P)\Big) \\ &= \sup_{\nu \in \mathcal{M}} \sup_{Q:Q\mu=\nu} \Big(-\int f d\nu - \frac{1}{\beta} h(Q|P)\Big) \\ &= \sup_{\nu \in \mathcal{M}} \Big(-\int f d\nu - \frac{1}{\beta} \inf_{Q:Q\mu=\nu} h(Q|P)\Big). \end{aligned}$$

Applying equation (26), we obtain equation (43).

For $\nu \in \mathcal{M}$ we denote by $\nu = \nu_s + \nu_a$ the Lebesgue decomposition with respect to λ, and by $\sigma^2(\cdot)$ the density of the absolutely continuous part ν_a. Thus, we can write

$$G(\nu) = -\int (f + \frac{1}{2\beta}) d\nu_s - \int_0^1 (f(t) + \frac{1}{2\beta})\sigma^2(t) + \frac{1}{2\beta}(1 + \log\sigma^2(t))dt.$$

Now suppose that $f \geq -2\beta^{-1}$. Then we have

$$G(\nu) \leq \int_0^1 -(f(t) + \frac{1}{2\beta})\sigma^2(t) + \frac{1}{2\beta}(1 + \log \sigma^2(t))dt.$$

For any $c \in R^1$, the concave function g_c on $[0, \infty)$ defined by

$$g_c(x) := -(c + \frac{1}{2\beta})x + \frac{1}{2\beta}(1 + \log x)$$

satisfies

$$\sup_{x \geq 0} g_c(x) = \begin{cases} -\frac{1}{2\beta} \log(1 + 2\beta c) & \text{if } c \geq -2\beta^{-1} \\ +\infty & \text{otherwise,} \end{cases}$$

and for $c > -2\beta^{-1}$ the supremum

$$g_c(x(c)) = -\frac{1}{2\beta} \log(1 + 2\beta c)$$

is attained in $x(c) = (1 + 2\beta c)^{-1}$. Applied to each $t \in [0, 1]$ with $c = f(t)$, this implies

$$\rho_{h,\beta}(\hat{X}) = \sup_{\nu \in \mathcal{M}} G(\nu) \leq -\frac{1}{2\beta} \int_0^1 \log\big(1 + 2\beta f(t)\big)dt. \tag{44}$$

The converse inequality will be shown in three steps.

1) Assume first that $\min f > -(2\beta)^{-1}$. Then the measure ν defined by $d\nu = (1 + 2\beta f(t))^{-1}dt$ belongs to $\mathcal{M}$. The corresponding measure Q_ν satisfies

$$E_{Q_\nu}[-\hat{X}] - \frac{1}{\beta} h(Q_\nu|P) = -\int_0^1 \log\big(1 + 2\beta f(t)\big)dt,$$

and this shows that equality holds in (44).

2) For $f \geq -(2\beta)^{-1}$ define $f_n := \max\big(f, -\frac{1}{2\beta} + \frac{1}{n}\big)$ and $\hat{X}_n(\cdot) = f_n(\mu(\cdot))$. Since $\hat{X}_n \geq \hat{X}$, we obtain

$$\begin{aligned} \rho_{h,\beta}(\hat{X}) &\geq \lim_n \rho_{h,\beta}(\hat{X}_n) \\ &= -\lim_n \frac{1}{2\beta} \int_0^1 \log\big(1 + 2\beta f_n(t)\big)dt \\ &= -\frac{1}{2\beta} \int_0^1 \log\big(1 + 2\beta f(t)\big)dt, \end{aligned}$$

applying part 1) to each f_n and using monotone integration.

3) If $\min f < -(2\beta)^{-1}$ then $\tilde{f} := \max(f, -(2\beta)^{-1})$ satisfies $\lambda(\{1 + (2\beta)\tilde{f} = 0\}) > 0$. Applying part 2) to $\tilde{X} = \tilde{f}(\mu) \geq \hat{X}$, we obtain

$$\begin{aligned} \rho_{h,\beta}(\hat{X}) &\geq \rho_{h,\beta}(\tilde{X}) \\ &= -\frac{1}{2\beta} \int_0^1 \log\big(1 + 2\beta \tilde{f}\big)dt = +\infty, \end{aligned}$$

and this concludes the proof of the converse inequality. $\square$

In our present case we can check the description (40) of $\rho_{h,\beta}(\hat{X})$ in terms of zooming and rescaling by a direct computation:

Corollary 27. *For $\hat{X}$ defined by* (41) *and* (42) *and for*

$$\hat{X}_N = \int_0^1 f d\mu_N = \sum_{k=1}^{2^N} f\big((k-1)2^{-N}\big)(\Delta_{N,k}W)^2$$

we have

$$\rho_{h,\beta}(\hat{X}) = \lim_{N\uparrow\infty} \rho_{H,\beta 2^N}(\hat{X}_N).$$

Proof. Assume that $f \geq -(2\beta)^{-1}$. Along the N-th dyadic partition, the random variables $\Delta_{N,k}W$ are independent under Wiener measure P with distribution $N(0, 2^{-N})$. Thus,

$$\begin{aligned} E_P[e^{-\beta 2^N \hat{X}_N}] &= \prod_{k=1}^{2^N} E_P[e^{-\beta 2^N f((k-1)2^{-N})(\Delta_{N,k}W)^2}] \\ &= \prod_{k=1}^{2^N} \big(1 + 2\beta f((k-1)2^{-N})\big)^{-\frac{1}{2}}, \end{aligned}$$

hence

$$\begin{aligned} \rho_{H,\beta 2^N}(\hat{X}_N) &= \frac{1}{\beta 2^N} \log E_P[e^{-\beta 2^N \hat{X}_N}] \\ &= -\frac{1}{2\beta} 2^{-N} \sum_{k=1}^{2^N} \log(1 + 2\beta f\big((k-1)2^{-N}\big)). \end{aligned}$$

Combined with proposition 26, this implies

$$\lim_{N\uparrow\infty} \rho_{H,\beta 2^N}(\hat{X}_N) = -\frac{1}{2\beta} \int_0^1 \log(1 + 2\beta f(t))dt = \rho_{h,\beta}(\hat{X}).$$

□

As a second class of examples, we look at the case where the functional F is of the form $F(\nu) = f(\nu[0,1])$ for some continuous function f on $[0,\infty)$, and in particular for a power function $f(x) = x^p$. For $p = \frac{1}{2}$ this includes the *volatility swap*

$$\hat{X}(\cdot) = \sqrt{\mu(\cdot, [0,1])} = \sqrt{< W >_1};$$

cf., e.g., [3].

Proposition 28. *For $\hat{X}(\cdot) = f(\mu(\cdot, [0,1])$ with $f \in C[0.\infty)$ we have*

$$\rho_{h,\beta}(\hat{X}) = \sup_{m>0} \big(- f(m) - \frac{1}{2\beta}(m - 1 - \log m)\big).$$

Proof. Using Theorem 22 and Proposition 18, we obtain

$$\begin{aligned}\rho_{H,\beta}(\hat{X}) &= \sup_{m>0} \sup_{\nu:\nu[0,1]=m} \big(- f(m) - \frac{1}{\beta} h(Q_\nu|P)\big) \\ &= \sup_{m>0} \big(- f(m) - \inf_{\nu:\nu[0,1]=m} h(Q_\nu|P)\big) \\ &= \sup_{m>0} \big(- f(m) - \frac{1}{2\beta}(m - 1 - \log m)\big).\end{aligned}$$

□

As an illustration consider the functional

$$F(\nu) = \nu[0,1]^p$$

with $p > 0$. Viewed as an asset, the position is $\hat{X}(\cdot) = \mu(\cdot,[0,1])^p$, and we get

$$\rho_{h,\beta}(\hat{X}) = x_\beta^p(p-1) + \frac{1}{2\beta}\log x_\beta,$$

where $x_\beta \in (0,1)$ denotes the unique solution of the equation $2\beta x_\beta^p = 1 - x_\beta$. For $p = 1$ we are back to the special case $f = 1$ in Proposition 26, that is,

$$\rho_{h,\beta}(\hat{X}) = -\frac{1}{2\beta}\log(1+2\beta).$$

Viewed as a liability, the position becomes $\hat{X} = -\mu(\cdot,[0,1])^p$. For $p \in (0,1)$ we get

$$\rho_{h,\beta}(\hat{X}) = x_\beta^p(1-p) + \frac{1}{2\beta}\log x_\beta,$$

where $x_\beta \in (1,\infty)$ is the unique solution of the equation $2\beta p x_\beta^p = x_\beta - 1$. For $p > 1$ we obtain $\rho_{h,\beta}(\hat{X}) = +\infty$, and for $p = 1$ we are back to the special case $f = -1$ in Proposition 26, that is,

$$\rho_{h,\beta}(\hat{X}) = \begin{cases} -\frac{1}{2\beta}\log(1-2\beta) & \text{if } \beta \le \frac{1}{2} \\ +\infty & \text{otherwise.}\end{cases}$$

Remark 29. *Talagrand's inequality* (17) *and our rescaled version* (24) *suggest to compare our entropic risk measures* $\rho_{H,\beta}$ *and* $\rho_{h,\beta}$ *to the following risk measures defined in terms of the Wasserstein distances* $W_{\mathcal{H}}$ *and* $W_{<\cdot>}$*:*

$$\rho_{H,\beta}(X) \le \rho_{\mathcal{H},\beta}(X) := \sup\big(E_Q[-X] - \frac{1}{2\beta}W_{\mathcal{H}}^2(Q,P)\big)$$

and

$$\rho_{h,\beta}(X) \le \rho_{<\cdot>,\beta}(X) := \sup\big(E_Q[-X] - \frac{1}{2\beta}W_{<\cdot>}^2(Q,P)\big). \tag{45}$$

A systematic study of the risk measures $\rho_{\mathcal{H},\beta}$ and $\rho_{<\cdot>,\beta}$ is beyond the scope of this paper. We only mention that $\rho_{\mathcal{H},\beta}$ is again risk-neutral on the σ-field $\hat{\mathcal{F}}$ generated by the quadratic variation, that is,

$$\rho_{\mathcal{H},\beta}(\hat{X}) = \rho_{H,\beta}(\hat{X}) = -F(\lambda)$$

for $\hat{X}(\cdot) = F(\mu(\cdot))$, and we just illustrate inequality (45) by the following example at the intersection of Proposition 26 and Proposition 28:

$$\rho_{<\cdot>,\beta}(\hat{X}) = -\frac{1}{1+2\beta} > -\frac{1}{2\beta}\log(1+2\beta) = \rho_{h,\beta}(\hat{X})$$

for $\hat{X} = \mu(\cdot, [0,1]$.

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