HUMBOLDT–UNIVERSITÄT ZU BERLIN

MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT II

INSTITUT FÜR MATHEMATIK

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Exercises for the lecture "Introduction to Computer Algebra"

Series 1. (Return on Monday, May 5th 2008)

Aufg. 1: Complexity bounds – Master theorem of computer science

(a) Assume that the sequence $(T_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ of natural numbers is increasing and satisfies a recursion inequality **12 pts.**

$$T_n \le \begin{cases} b & \text{if } n = 1, \\ aT_{n/k} + bn & \text{if } k\big|n,\ k \text{ divides } n \end{cases}$$

with some fixed integer $k > 1$ and reals $a, b > 0$. Show that this results in the "pointwise" inequalities

$$T_n \le \begin{cases} O(n) & \text{if } a < k \\ O(n\log(n)) & \text{if } a = k \\ O(n^{\log a/\log k}) & \text{if } a > k\,. \end{cases}$$

Here $x_n \le O(y_n)$ means that for the two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ there is a constant $C > 0$ with $x_n \le C\,y_n$ for all $n \in \mathbb{N}$.

(b) The Karatsuba algorithm for integer multiplication requires for integer factors of $2n$ words 3 multiplications and 5 additions/substractions of integer operands of n words. What is the complexity of the recursive application of this kind of multiplication? **2 pts.**

(c) The Toom-Cock 3-way algorithm for integer multiplication of integer factors of length $3n$ requires 5 multiplications and 8 additions/substractions of operands of length n. What is the complexity of the recursive application of this kind of multiplication? **2 pts.**

Aufg. 2: The Fibonacci sequence $(F_n)_{n\in\mathbb{N}}$ of integers is recursively defined by

$$F_0 = 0,\ F_1 = 1\ F_{n+2} = F_{n+1} + F_n\ .$$

(a) Determine the number of function calls in the naive recursive implementation **3 pts.**

```
Fib(n){
 if n<2 then return n;
 return Fib(n-1)+Fib(n-2);
}
```

(b) Indicate a more efficient iterative algorithm. **5 pts.**

(c) Show that for all $n \in \mathbb{N}$ **8 pts.**

$$\begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+1}$$

(d) Show that $F_{n+m+1} = F_{m+1}F_{n+1} + F_m F_n$ for all integers m, n. **2 pts.**

(e) Give an algorithm to compute the pair (F_{2n-1}, F_{2n}) from the pair (F_{n-1}, F_n). Use this recursive dependency to construct a recursive algorithm for the determination of any Fibonacci number. Give an upper bound depending on n for the number of recursions in this algorithm. **6 pts.**

Aufg. 3: (Optional task) Chose your favorite programming language that has integer types of single and double size and implement for one of the bases $B = 10$, $B = 1000$, $B = 2^{16}$ or $B = 2^{32}$ **10 pts.**

(a) a data type containing the length, sign and (variable length) word sequence of a big integer;

(b) procedures to read a big integer from and write to a string and subsequently a file (in decimal or hexadecimal notation);

(c) prodedures to add and substract big integers;

(d) a procedure to multiply a big integer with a one word integer;

(e) a procedure to multiply two big integers in the basic way as defined algorithm 3 of the lecture.

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