HUMBOLDT–UNIVERSITÄT ZU BERLIN
MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT II
INSTITUT FÜR MATHEMATIK
PROF. ANDREAS GRIEWANK PHD,
DR. LUTZ LEHMANN

Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, D-10099 Berlin

Berlin, den June 9, 2008

Exercises for the lecture "Introduction to Computer Algebra"

Series 4. (Return on Monday, June 23th 2008)

Task 1: Let the points $x_1, \ldots, x_n \in \mathbb{R}$ be pairwise distinct in their absolute value. Let values $y_1, \ldots, y_n \in \mathbb{R}$ be fixed. Show that the interpolating polynomial $p(x)$ of minimal degree satisfying $p(x_k) = p(-x_k) = y_k$, $k = 1, \ldots, n$ is always even, that is, symmetric to the y-axis. What is the relation of p to the minimal interpolating polynomial q that satisfies $q(x_k^2) = y_k$. **6 pts.**

Task 2: Find interpolating polynomials $p(x)$ and the approximation of $\sqrt{2}$ from samples $(x_k, f(x_k))$, $k = 1, \ldots, 4$ **6/8 pts.**

(a) $f(x) = 2^x$, $(x_k)_{k=1,\ldots,4} = (-1, 0, 1, 2)$, evaluate at $x = 0.5$ for the approximation.

(b) $f(x) = \sqrt{x}$, $(x_k)_{k=1,\ldots,4} = (1, \frac{16}{9}, \frac{9}{4}, \frac{25}{9})$, evaluate at $x = 2$, compare the approximations with the true value of $\sqrt{2}$.

Task 3: Let $x_1, \ldots, x_r \in \mathbb{K}$ be distinct elements of a field $\mathbb{K}$ with characteristic 0 and $n_1, \ldots, n_n \in \mathbb{N}_+$ with $n_1 + \cdots + n_r = n$. Let be given further numbers $y_{i,k} \in \mathbb{K}$, $1 \le i \le r$, $1 \le k \le n_i$. **6/10/12* pts.**

(a) Prove that there exists a unique polynomial $p \in \mathbb{K}[X]$, the Hermite interpolation polynomial, with $\deg p < n$ and

$$p^{(k-1)}(x_i) = y_{i,k}, \qquad 1 \le i \le r, \quad 1 \le k \le n_i\,.$$

(b) Find the Hermite interpolation polynomials to the sine and cosine functions that are tangent at the supporting points $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$.

(c) (Optional) Find the odd resp. even hermite polynomials that are tangent to the same points. Hint: Try to find polynomials S, C with $\sin(x) \approx x \cdot S(x^2)$ and $\cos(x) \approx C(x^2)$.

Task 4: Let $a_0 = m, a_1 = p$ be two elements of an euclidean ring. By the extended euclidean algorithm, relations **8/12/6* pts.**

$$a_k = u_k\, m + v_k\, p$$

are obtained. If v_k is relatively prime to m, this leads to the relation

$$p \equiv a_k\,(p_k)^{-1} \pmod{m}\ .$$

This is called a Cauchy approximant of p if $m = (X - x_1)\dots(X - x_r)$, a Padé approximant if $m = (X - x_0)^r$ and a rational Hermite interpolant in the mixed case $m = (X - x_1)^{\mu_1}\dots(X - x_r)^{\mu_r}$.

(a) Find the Hermite interpolant for $f(x) = \tan x$ that is tangent of order 3 at $x = 0$ and interpolating at $x = \pm\frac{\pi}{4}$.

(b) Compute all rational Hermite interpolants modulo $m = x^4(x^2 - \frac{\pi^2}{16})$ for the obtained Hermite interpolant.

(c) (Optional) Produce the graphs of the rational interpolants and give (estimates of) error bounds.

Task 5: Newton's method finds its next approximation by solving $f(x_{k+1}) \equiv 0 \bmod (x - x_k)^2$. To solve $f(x_k + h) \equiv 0 \bmod h^k$, one may compute the Padé approximand of $f(x_k + h) \bmod h^k$ with linear numerator and set h to be the root of the numerator. **6/4/6* pts.**

(a) Let $a > 0$ be fixed. Determine the iteration equation for the Newton's method of order $k = 3$ for $f(x) = x^3 - a$.

(b) Tabulate the iteration for $a = 5$ starting with $x_0 = 1$.

(c) (Optional) Prove that the order of convergence is indeed 3.

phone: 030/2093-5820 fax: 030/2093-5848
e-mail: griewank@math.hu-berlin.de llehmann@math.hu-berlin.de
Internet-Seite: http://www.math.hu-berlin.de/~gaggle/S08/COMPALG