The Problem Sheets are a tool to show first of all what should be mastered from the material in the lecture; these are the homework problems marked with “P”. Moreover, I will indicate certain aspects of the theory which are desirable to know but would lead too far if presented in the lecture; such areas are marked with “S” (for seminar). You are invited to present such an area (or part of it) in a seminar talk, possibly together with other participants.

You need not hand in written solutions, your admission to the examination will be based on your activity in the seminar part added to the lecture, which will normally take place on Tuesday (11:00-12:30, Rudower Chaussee 25, 2.009). The examination could either be an oral one of 30 minutes duration, or a “Hausarbeit”, i.e. a carefully written version of a seminar presentation or of another topic close to the material of the lecture on which we have to agree; your own suggestions are welcome.

Humboldt-Universität zu Berlin
Institut für Mathematik
Prof. Dr. Jochen Brüning
Lecture **The Atiyah-Singer Index Theorem**, WS 2011/12

Exercises no. 1

due Nov 9, 2011

1.P Consider a coordinate bundle $(M, \pi, B, F, \mathcal{A}_B, (\sigma_U)_{U\in\mathcal{A}_B})$ such that the corresponding $\mathcal{A}_B$-1-cocycle $g_{UV} := \sigma_U^{-1} \circ \sigma_V, \quad U, V \in \mathcal{A}_B$ with values in $\mathrm{Diff}(F)$ is trivial (or a coboundary), i.e. there are smooth maps

$$f_U : U \to \mathrm{Diff}(F), \quad U \in \mathcal{A}_B$$

such that

$$g_{UV} = f_U^{-1} \circ f_V.$$

Show that (M, π, B) is diffeomorphic to $B \times F$.

2.P Prove the following special case of the theorem on "integration along the fiber with compact vertical supports": The theorem holds for $M \times \mathbb{R}^k$, i.e. for trivial real vector bundles over arbitrary manifolds.

3.P Let $\pi : M \to B$ be a submersion with compact fibers. Is π locally trivial?

4.S **The topological theory of fibrations.**

Here we should study first the topological notion of a fibration (Hurewicz, Serre) and various applications (topologically, *every* map is homotopic to a fibration!). Then we could proceed to the necessary and sufficient conditions for a submersion to be a fibration as given in

> **G. Meigniez**, *Submersion, fibrations and bundles*, Transactions AMS, **354** (2002), 3771–3787.

We can then proceed to topological questions like the classification of bundles over a given base manifold with the same fiber; a beautiful source for this is the Habilitationsschrift of Friedrich Hirzebruch which includes in particular the definition of (Čech)-cohomology with coefficients in a sheaf ("Garbe"):

F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer 1962

Then the more interesting applications of topological considerations–from the point of view of index theory–are the techniques of spectral sequences (to compute the cohomology of M in terms of the cohomology of B and F) and the exact homotopy sequence associated with a fibration, with similar intentions.