The Problem Sheets are a tool to show first of all what should be mastered from the material in the lecture; these are the homework problems marked with “P”. Moreover, I will indicate certain aspects of the theory which are desirable to know but would lead too far if presented in the lecture; such areas are marked with “S” (for seminar). You are invited to present such an area (or part of it) in a seminar talk, possibly together with other participants.

You need not hand in written solutions, your admission to the examination will be based on your activity in the seminar part added to the lecture, which will normally take place on Tuesday (11:00-12:30, Rudower Chaussee 25, 2.009). The examination could either be an oral one of 30 minutes duration, or a “Hausarbeit”, i.e. a carefully written version of a seminar presentation or of another topic close to the material of the lecture on which we have to agree; your own suggestions are welcome.

Humboldt-Universität zu Berlin
Institut für Mathematik
Prof. Dr. Jochen Brüning
Lecture **The Atiyah-Singer Index Theorem**, WS 2011/12

Exercises no. 3

due Nov 16, 2011

1.P Consider a principal $G-$bundle $P_G \to M$, where M is a smooth manifold and G a Lie group. Show that the choice of a connection,

$$P \in C^\infty\big(P_G, \mathcal{L}(TP_G, T_V P_G)\big),$$

is equivalent to the choice of a connection one form,

$$\omega_P \in \lambda^1(P_G, \mathfrak{g})$$

with the property

$$\omega_P\{\tilde{E}\} = E$$

for any $E \in \mathfrak{g}$ and $\tilde{E}(q) := \frac{d}{dt}|_{t=0} q \exp(tE)$.

2.P Let M be a smooth manifold, $X, Y \in \tau_c(M)$, and ψ a diffeomorphism of M; denote the flow of a vector field by ϕ_t^X etc.

a) We have $\psi_* X = X \iff \psi \circ \phi_t^X = \phi_t^X \circ \psi$

b) We have $\phi_t^X \circ \phi_s^X = \phi_s^X \circ \phi_t^X$ for all $s, t \in \mathbf{R} \iff [X, Y] = 0$.

3.P Show as a consequence that with the notation of 1. and

$$\Omega_P\{Z_1, Z_2\} := -P\{[(1-p)Z_1, (\mathrm{Id} - P)Z_2]\}$$

we have

$$\Omega_P = d\omega_P + [\omega_P, \omega_P]_{\mathfrak{g}}$$

4.S A new topic arises where we can present proofs of the so far unproven facts on smooth G-spaces, in particular the slice and the (First) Structure Theorem.

References:

Katsuo Kawakubo: *The theory of transformation groups.* Oxford University Press 1991.

Klaus Jänich: *Differenzierbare G-Manngfaltigkeiten*, LNM 59, 1968