The Problem Sheets are a tool to show first of all what should be mastered from the material in the lecture; these are the homework problems marked with "P". Moreover, I will indicate certain aspects of the theory which are desirable to know but would lead too far if presented in the lecture; such areas are marked with "S" (for seminar). You are invited to present such an area (or part of it) in a seminar talk, possibly together with other participants.

You need not hand in written solutions, your admission to the examination will be based on your activity in the seminar part added to the lecture, which will normally take place on Tuesday (11:00-12:30, Rudower Chaussee 25, 2.009). The examination could either be an oral one of 30 minutes duration, or a "Hausarbeit", i.e. a carefully written version of a seminar presentation or of another topic close to the material of the lecture on which we have to agree; your own suggestions are welcome.

Humboldt-Universität zu Berlin
Institut für Mathematik
Prof. Dr. Jochen Brüning
Lecture **The Atiyah-Singer Index Theorem**, WS 2011/12

Exercises no. 4

due Nov 22, 2011

1.P Consider a $\mathbb{K}$-vector bundle $E \xrightarrow{\pi^E} M$ of rank ℓ which is orientable (i.e. admits an atlas of frames with positive determinant for all transition functions) and carries a metric h^E, and denote by $P_{\mathrm{SO}}(E)$ the principal $\mathrm{SO}(\ell)$-bundle of oriented orthonormal frames. Show that there is a $1-1$ correspondence between connection one-forms on $P_{\mathrm{SO}}(E)$ and covariant derivatives ∇^E on E with the additional property that for all $X \in \tau(M)$, $s_1, s_2 \in C^\infty(M, E)$,

$$X\big(h^E(s_1, s_2)\big) = h^E(\nabla^E_X s_1, s_2) + h^E(s_1, \nabla^E_X s_2)$$

(such covariant derivatives are called "metric".)

2.P Let $P_G \xrightarrow{\pi} M$ be a principal G-bundle, G a Lie group, and let $\rho : G \to \mathrm{SO}(n)$ be a representation of G on $\mathbb{R}^n$. Denote by $E_\rho := P_G \times_\rho \mathrm{SO}(n)$ the associated bundle.

1) Show that E_ρ carries a Riemannian metric g^{E_ρ}.

2) Show that $P_{\mathrm{SO}}(E_\rho) \simeq P_G \times_\rho \mathrm{SO}(n)$.

3) Show that any connection one-form ω on P_G induces a connection one-form ω_ρ on $P_{\mathrm{SO}}(E_\rho)$

4) Show that there is a canonical injection

$$i_\rho : P_G \hookrightarrow P_{\mathrm{SO}}(E_\rho).$$

5) Show, finally, that

$$i_\rho^* \omega_\rho = \rho_* \omega,$$

where $\rho_* : \mathfrak{g} \to \mathfrak{so}(n)$ is the Lie algebra homomorphism induced by ρ.