The Problem Sheets are a tool to show first of all what should be mastered from the material in the lecture; these are the homework problems marked with “P”. Moreover, I will indicate certain aspects of the theory which are desirable to know but would lead too far if presented in the lecture; such areas are marked with “S” (for seminar). You are invited to present such an area (or part of it) in a seminar talk, possibly together with other participants.

You need not hand in written solutions, your admission to the examination will be based on your activity in the seminar part added to the lecture, which will normally take place on Tuesday (11:00-12:30, Rudower Chaussee 25, 2.009). The examination could either be an oral one of 30 minutes duration, or a “Hausarbeit”, i.e. a carefully written version of a seminar presentation or of another topic close to the material of the lecture on which we have to agree; your own suggestions are welcome.

Humboldt-Universität zu Berlin
Institut für Mathematik
Prof. Dr. Jochen Brüning
Lecture **The Atiyah-Singer Index Theorem**, WS 2011/12

Problem Sheet no. 5

due Nov 29, 2011

1.P Let $E \to M$ be a smooth $\mathbb{K}$-vector bundle over the manifold M, and let $\eta_i \in \lambda(M)$, $i = 1, 2$, $s \in \lambda(M, E)$. Show that

$$\eta_1 \wedge (\eta_2 \wedge s) = (\eta_1 \wedge \eta_2) \wedge s.$$

2.P Prove that any covariant derivative ∇^E on E can be extended to $\lambda(M, E)$ by defining

for $s \in \lambda^k(M, E)$ and $X_i \in \tau(M)$, $i = 0, \ldots, k$

$$(\nabla^E s)\{X_0, \ldots, X_k\} = \sum_{i=0}^{k} (-1)^i \nabla^E_{X_i}(s\{X_0, \ldots, \widehat{X_i}, \ldots, X_k\}) + \sum_{i<j} (-1)^{i+j} s\{[X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k\}$$

For this extended operator we have, for $\eta \in \lambda^k(M), s \in \lambda(M, E)$

$$\nabla^E(\eta \wedge s) = d\eta \wedge s + (-1)^k \eta \wedge \nabla^E s, \tag{1}$$

and ∇^E is uniquely determined by $\nabla^E(\lambda^0(M, E))$ and (1).

3.P Show that the pair (E, ∇^E) is flat if and only if there is a locally trivial atlas $\mathcal{A}$ for E such that σ_U is constant for all $U \in \mathcal{A}$.

4.P Consider a smooth manifold with atlas $(\phi_U)_{U \in \mathcal{A}}, \phi_U : \mathbb{R}^m \to U$ diffeomorphisms, defining the smooth structure. Define a real line bundle (i.e. $\mathbb{K} = \mathbb{R}, \ell = 1$) by the $\mathbb{Z}_2$-cocycle

$$\left(\operatorname{sgn} \det d(\phi_V^{-1} \circ \phi_U)\right)_{U,V \in \mathcal{A}}$$

which is called $\mathrm{Or}(M)$, the *orientation line* of M.

Show that $\mathrm{Or}(M)$ has always a flat connection but is not trivial in general.