The Problem Sheets are a tool to show first of all what should be mastered from the material in the lecture; these are the homework problems marked with "P". Moreover, I will indicate certain aspects of the theory which are desirable to know but would lead too far if presented in the lecture; such areas are marked with "S" (for seminar). You are invited to present such an area (or part of it) in a seminar talk, possibly together with other participants.

You need not hand in written solutions, your admission to the examination will be based on your activity in the seminar part added to the lecture, which will normally take place on Tuesday (11:00-12:30, Rudower Chaussee 25, 2.009). The examination could either be an oral one of 30 minutes duration, or a "Hausarbeit", i.e. a carefully written version of a seminar presentation or of another topic close to the material of the lecture on which we have to agree; your own suggestions are welcome.

Problem Sheet no. 9
due 24.1.2012

Always let $E \to M$, $\dim M = n$, be a vector bundle with connection ∇ and

$$x : \mathbf{x} \mapsto \exp_{x_0} \mathbf{x}, \quad T_{x_0} M \to M$$

be normal coordinates around a point $x_0 \in M$. Then

$$\exp_{x_0}^* \mathrm{vol}_g = j(\mathbf{x}) d\mathbf{x}, \quad d\mathbf{x} := \mathrm{vol}_{T_{x_0} M}, \quad j(\mathbf{x}) = \sqrt{\det(g_{ij}(\mathbf{x})_{i,j}}.$$

Let $\mathscr{R}$ be the radial vector field $\mathscr{R} = \sum \mathbf{x}^i \partial_{\mathbf{x}^i}$.

1.P   Prove the remaining equalites from Lemma 1 from the lecture, i.e.

- $\Delta \|\mathbf{x}\|^2 = -2(n + d(\log j)(\mathscr{R}))$

- For $q_t(\mathbf{x}) := (4\pi t)^{-n/2} e^{-\|\mathbf{x}\|^2/4t} d\mathbf{x}$, show

$$\Big(\frac{\partial}{\partial t} + \Delta - j \circ \Delta(j^{-1})\Big) q_t = 0$$

(Compare with Proposition 2.11 in Berline/Getzler/Vergne: Heat kernels and Dirac operators).

2.P   Fill in the details in the proof of the Lemma

$$\Big(\frac{\partial}{\partial t} + H\Big)(s_t q_t) = \left[\Big(\frac{\partial}{\partial t} + t^{-1}\nabla_{\mathscr{R}} + j \circ H \circ j^{-1}\Big)s_t\right] q_t$$

for any time dependent section s_t of E.