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SOBOLEV–MORREY SPACES ASSOCIATED WITH EVOLUTION EQUATIONS

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Abstract. In this text we introduce new classes of Sobolev–Morrey spaces being adequate for the regularity theory of second-order parabolic boundary-value problems on Lipschitz domains of space dimension $n \geq 3$ with nonsmooth coefficients and mixed boundary conditions. We prove embedding and trace theorems as well as invariance properties of these spaces with respect to localization, Lipschitz transformation, and reflection. In the second part [11] of our presentation we show that the class of second-order parabolic systems with diagonal principal part generates isomorphisms between the above-mentioned Sobolev–Morrey spaces of solutions and right-hand sides.

Introduction

Many interesting evolutionary processes may be formulated in terms of second-order parabolic boundary-value problems of drift-diffusion type. Applications we have in mind are, for instance, transport processes of electrically charged particles in semiconductor heterostructures, phase separation processes of nonlocally interacting particles, or chemotactic aggregation of biological organisms in heterogeneous environments. The adequate description of these processes requires the treatment of problems with fully nonsmooth data, linear diffusion and nonlinear drift terms.

One way to solve these problems is to study the regularity of solutions to auxiliary linear problems, which is also of its own interest. In this first part of our presentation we introduce new classes of function spaces which allow a natural and satisfactory treatment of the regularity problem discussed in detail in the second part [11].

The preliminary part of this text is dedicated to both the functional analytic formulation and the unique solvability of initial-value problems in a

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more abstract context. As a starting point, in Section 1 we consider the Hilbert space

$$W_E(S;Y) = \{u \in L^2(S;Y) : (\mathcal{E}u)' \in L^2(S;Y^*)\},$$

which gives us freedom to model a great variety of evolutionary processes by choosing appropriate Hilbert spaces Y, H and linear operators $K \in \mathcal{L}(Y;H)$ and $E = K^* J_H K \in \mathcal{L}(Y;Y^*)$. Here, we assume that the range $K[Y]$ is dense in H and that $J_H \in \mathcal{L}(H;H^*)$ denotes the duality map between H and H^*. Moreover, $S = (t_0, t_1)$ is an open time interval, and the operators $\mathcal{K} : L^2(S;Y) \to L^2(S;H)$ and $\mathcal{E} : L^2(S;Y) \to L^2(S;Y^*)$ are associated with S, K, and E via $(\mathcal{K}u)(s) = Ku(s)$ and $(\mathcal{E}u)(s) = Eu(s)$ for $u \in L^2(S;Y)$ and $s \in S$.

These spaces were introduced by Gröger [15] as a natural and self-evident generalization of the well-established function space

$$W(S;Y) = L^2(S;Y) \cap H^1(S;Y^*),$$

for which we find a developed theory in the literature; see Lions [21, 22], Lions, Magenes [23], Gajewski, Gröger, Zacharias [6], Temam [31], Simon [28], and Dautray, Lions [4]. It turns out that all the basic facts known for $W(S;Y)$, like density of functions being smooth in time, integration by parts formulae, or embedding and trace theorems, carry over to the space $W_E(S;Y)$; see Gröger [15]. Section 2 is dedicated to the variational formulation and the solution of initial-value problems. Given a strongly monotone and Lipschitz continuous Volterra operator

$$\mathcal{M} : L^2(S;Y) \to L^2(S;Y^*),$$

for all $\alpha \in \mathbb{R}$ the problem

$$(\mathcal{E}u)' + \mathcal{M}u - \alpha\mathcal{E}u = f \in L^2(S;Y^*), \quad (\mathcal{K}u)(t_0) = w \in H,$$

is uniquely solvable and well-posed in $W_E(S;Y)$; see Gröger [15].

This class of problems is large enough to cover the case of linear second-order parabolic boundary-value problems on Lipschitz domains $\Omega \subset \mathbb{R}^n$ of space dimension $n \in \mathbb{N}$ with nonsmooth coefficients and mixed boundary conditions. We are mainly interested in the linear drift-diffusion problem

$$(\mathcal{E}u)' + \mathcal{A}u + \mathcal{B}u = f \in L^2(S;Y^*), \quad (\mathcal{K}u)(t_0) = 0, \tag{P}$$

where $H = L^2(\Omega)$ is equipped with the weighted scalar product

$$(v|w)_H = \int_\Omega avw \, d\lambda^n \quad \text{for } v,\, w \in H,$$

$H^1_0(\Omega) \subset Y \subset H^1(\Omega)$ is a closed subspace of $H^1(\Omega)$, and $K \in \mathcal{L}(Y;H)$ is simply the embedding operator. The nonsmooth capacity coefficient $a \in L^\infty(\Omega)$ is essentially bounded from below by some constant $\varepsilon > 0$.

We consider nonsmooth diffusivity coefficients $A \in L^\infty(S \times \Omega; \mathbb{S}^n)$ with values in the set $\mathbb{S}^n$ of symmetric $(n \times n)$-matrices, and we assume that the corresponding quadratic form is essentially bounded from below by $\varepsilon > 0$, too. With regard to problem (P) we are concerned with principal parts $\mathcal{A} : L^2(S;Y) \to L^2(S;Y^*)$ of the form

$$\langle \mathcal{A}u, w\rangle_{L^2(S;Y)} = \int_S \int_\Omega A\nabla u(s) \cdot \nabla w(s)\, d\lambda^n\, ds \quad \text{for } u,\, w \in L^2(S;Y).$$

Given lower-order coefficients

$$b \in L^\infty(S \times \Omega; \mathbb{R}^n), \quad b_0 \in L^\infty(S \times \Omega), \quad b_\Gamma \in L^\infty(S \times \Gamma),$$

modelling drift and damping phenomena, for u, $w \in L^2(S;Y)$ we define the map $\mathcal{B} : L^2(S;Y) \to L^2(S;Y^*)$ by

$$\begin{aligned}\langle \mathcal{B}u, w\rangle_{L^2(S;Y)} = &\int_S \int_\Omega \big(u(s) b \cdot \nabla w(s) + b_0 u(s) w(s)\big)\, d\lambda^n\, ds \\ &+ \int_S \int_\Gamma b_\Gamma K_\Gamma u(s) K_\Gamma w(s)\, d\lambda_\Gamma\, ds.\end{aligned}$$

Here, $\Gamma = \partial\Omega$ is the Lipschitz boundary of $\Omega \subset \mathbb{R}^n$, and $K_\Gamma : H^1(\Omega) \to L^2(\Gamma)$ denotes the corresponding trace map. Note that the bounded linear Volterra operator $\mathcal{M} = \mathcal{A} + \mathcal{B} + \alpha\mathcal{E} : L^2(S;Y) \to L^2(S;Y^*)$ is positively definite, whenever $\alpha > 0$ is large enough.

Our regularity problem can be formulated as follows: We are looking for Banach spaces $L^+ \subset L^2(S;Y)$ and $L^- \subset L^2(S;Y^*)$ such that

1. The space $W = \big\{u \in L^+ : (\mathcal{E}u)' \in L^-\big\}$ is embedded into a space of functions being Hölder continuous in time and space up to the boundary.

2. The parabolic operator corresponding to problem (P) is a linear isomorphism between $\big\{u \in W : u(t_0) = 0\big\}$ and L^-, and it has the maximal regularity property: $f \in L^- \longrightarrow \mathcal{A}u,\ \mathcal{B}u,\ (\mathcal{E}u)' \in L^-$.

Following, for instance, classical results of Ladyzhenskaya, Solonnikov, Uraltseva [19], there are well-known conditions on the right-hand side $f \in L^2(S;Y^*)$ in terms of usual Sobolev spaces for evolution equations which ensure the Hölder continuity of the solution. But in the case $n \geq 3$ these spaces are *not* the right choice for maximal regularity results without further assumptions on the smoothness of the domain or the coefficients. In the second part [11] of our presentation we fill this gap: There, we prove that the class of problems (P) generates isomorphisms between appropriate Sobolev–Morrey spaces meeting *all* the requirements of the above regularity problem; see also Lieberman [20], Hong-Ming Yin [33].

The main goal of this work is to introduce these Sobolev–Morrey spaces and to discuss their properties in detail. In Section 3 we start with a collection of classical results concerning Morrey and Campanato spaces with parabolic metric; see Campanato [2], Da Prato [3]. Section 4 is dedicated to regular sets $G \subset \mathbb{R}^n$ with Lipschitz boundary and Sobolev spaces $H_0^1(G)$, which allow a proper functional analytic formulation of elliptic and parabolic problems with mixed boundary conditions in nonsmooth domains; see Gröger, Rehberg [16, 17, 18], and Griepentrog, Recke [10, 12]. Setting $Y = H_0^1(G)$ we consider Sobolev–Morrey spaces $L_2^\omega(S;Y) \subset L^2(S;Y)$ for Morrey exponents $\omega \in [0, n+2]$.

In Section 5 for $\omega \in [0, n+2]$ we introduce a new scale of Sobolev–Morrey spaces $L_2^\omega(S;Y^*) \subset L^2(S;Y^*)$ of functionals generalizing Rakotoson's approach to elliptic boundary-value problems; see [25, 26]. In Section 6 we make use of the function spaces introduced before to establish our class of Sobolev–Morrey spaces

$$W_E^\omega(S;Y) = \left\{ u \in L_2^\omega(S;Y) : (\mathcal{E}u)' \in L_2^\omega(S;Y^*) \right\} \subset W_E(S;Y).$$

Embedding and trace theorems for these spaces are based on invariance properties of the above Sobolev–Morrey spaces with respect to localization, Lipschitz transformation, and reflection, and on special variants of Poincaré inequalities; see Appendix A and Struwe [30]. Note that in the case of $\omega \in (n, n+2]$ the space $W_E^\omega(S;Y)$ is embedded into a space of Hölder-continuous functions.

1. Hilbert spaces for evolution equations

Let $(\mathbb{R}^n, \mathfrak{L}^n, \lambda^n)$ be the σ-finite measure space of n-dimensional Lebesgue-measurable subsets of $\mathbb{R}^n$. For $F \in \mathfrak{L}^n$ and $p \in [1, \infty)$ we denote by $L^p(F;X)$ the set of all Lebesgue p-integrable functions $u : F \to X$ with values in the Banach space $(X, \| \ \|_X)$. The class $L^\infty(F;X)$ consists of all Lebesgue-measurable functions $u : F \to X$ which are essentially bounded.

For every $G \subset \mathbb{R}^n$ we introduce the class $B(G;X)$ of bounded functions $u : G \to X$. We define the set $C(G;X)$ of continuous functions $u : G \to X$ and the subclass $BC(G;X) = B(G;X) \cap C(G;X)$. Moreover, for $\alpha \in (0,1]$ we consider the set $C^{0,\alpha}(G;X)$ of Hölder-continuous functions $u : G \to X$ and the subclass $BC^{0,\alpha}(G;X) = B(G;X) \cap C^{0,\alpha}(G;X)$.

For $k \in \mathbb{N} \cup \{\infty\}$ and open sets $U \subset \mathbb{R}^n$ we denote by $C^k(U;X)$ the set of functions $u : U \to X$ which have continuous derivatives up to the k-th order. The subclass of all these functions with bounded continuous derivatives up to the k-th order forms the set $BC^k(U;X)$. Finally, we introduce the subset $C_0^k(U;X)$ of functions $u \in C^k(U;X)$ with compact support $\operatorname{supp}(u)$ in U.

In this section we introduce function spaces which are suitable for the formulation of the class of evolution problems we are interested in. Our representation and terminology closely follows the ideas of Gröger [15]; see also Dautray, Lions [4]. Throughout the whole text we assume that

1. $S \subset \mathbb{R}$ is an open time interval,
2. Y, X, H are Hilbert spaces, Y is continuously embedded into X,
3. $K \in \mathcal{L}(X;H)$, the set $K[Y]$ is dense in H,
4. $J_H \in \mathcal{L}(H;H^*)$ denotes the duality map of H,
5. $E \in \mathcal{L}(X;Y^*)$ is defined by $E = (K|Y)^* J_H K$.

We denote by $(\ |\ )_H$ and $\langle\ ,\ \rangle_H$ the scalar product in H and the dual pairing between H and H^*, respectively. The duality map $J_H \in \mathcal{L}(H;H^*)$ of H is defined as usual by $\langle J_H u, v\rangle_H = (u|v)_H$ for u, $v \in H$. Note, that the restriction $E|Y = (K|Y)^* J_H K|Y \in \mathcal{L}(Y;Y^*)$ is symmetric and positively semidefinite.

Definition 1.1 (Sobolev space)**.** 1. The function $f \in L^2(S;Y^*)$ is called weakly differentiable if there exists some $f' \in L^2(S;Y^*)$ which satisfies

$$\int_S \langle f'(s), v\rangle_Y\, \vartheta(s)\, ds = -\int_S \langle f(s), v\rangle_Y\, \vartheta'(s)\, ds \quad \text{for all } \vartheta \in C_0^\infty(S),\ v \in Y.$$

We introduce the Sobolev space $H^1(S;Y^*)$ as usual by

$$H^1(S;Y^*) = \big\{ f \in L^2(S;Y^*) : f' \in L^2(S;Y^*) \big\}.$$

2. Corresponding to $\mathcal{E} : L^2(S;X) \to L^2(S;Y^*)$, which is associated with S and $E \in \mathcal{L}(X;Y^*)$ via $(\mathcal{E}u)(s) = Eu(s)$ for $s \in S$, $u \in L^2(S;X)$, we set

$$W_E(S;X) = \big\{ u \in L^2(S;X) : (\mathcal{E}u)' \in L^2(S;Y^*) \big\}.$$

Theorem 1.1. *$W_E(S;X)$ is a Hilbert space with the scalar product*

$$(u|v) = (u|v)_{L^2(S;X)} + ((\mathcal{E}u)'|(\mathcal{E}v)')_{L^2(S;Y^*)} \quad \textit{for } u,\ v \in W_E(S;X).$$

Proof. 1. The bilinear form is correctly defined for u, $v \in W_E(S;X)$, and it has the properties of a scalar product.

2. To prove the completeness of $W_E(S;X)$, let (u_k) be some Cauchy sequence in $W_E(S;X)$. Then the sequences (u_k) and $((\mathcal{E}u_k)')$ converge in

$L^2(S;X)$ and $L^2(S;Y^*)$ to functions $u \in L^2(S;X)$ and $f \in L^2(S;Y^*)$, respectively. Since $E \in \mathcal{L}(X;Y^*)$ the sequence $(\mathcal{E}u_k)$ converges in $L^2(S;Y^*)$ to $\mathcal{E}u \in L^2(S;Y^*)$. Hence, passing to the limit $k \to \infty$ in

$$\int_S \langle (\mathcal{E}u_k)'(s), v\rangle_Y \,\vartheta(s)\,ds = -\int_S \langle (\mathcal{E}u_k)(s), v\rangle_Y \,\vartheta'(s)\,ds,$$

for all $\vartheta \in C_0^\infty(S)$, $v \in Y$ we get the identity

$$\int_S \langle f(s), v\rangle_Y \,\vartheta(s)\,ds = -\int_S \langle (\mathcal{E}u)(s), v\rangle_Y \,\vartheta'(s)\,ds,$$

which proves $(\mathcal{E}u)' = f \in L^2(S;Y^*)$ and $u \in W_E(S;X)$. □

Lemma 1.2. *Let the map $C \in \mathcal{L}(X;Y)$ satisfy*

$$\langle Ew, Cv\rangle_Y = \langle Ev, Cw\rangle_Y \quad \textit{for all } v,\, w \in X. \tag{1.1}$$

Then the bounded linear operator $\mathcal{C} : L^2(S;X) \to L^2(S;Y)$, associated with S and C via $(\mathcal{C}u)(s) = Cu(s)$ for $s \in S$ and $u \in L^2(S;X)$, maps $W_E(S;X)$ continuously into $W_{E|Y}(S;Y)$.

Proof. We define bounded linear maps $\mathcal{C}_0 : L^2(S;Y) \to L^2(S;Y)$ and $\mathcal{E}_0 : L^2(S;Y) \to L^2(S;Y^*)$ associated with S, $C|Y \in \mathcal{L}(Y;Y)$, and $E|Y \in \mathcal{L}(Y;Y^*)$, by

$$(\mathcal{C}_0 u)(s) = Cu(s), \quad (\mathcal{E}_0 u)(s) = Eu(s) \quad \text{for } s \in S,\ u \in L^2(S;Y).$$

Due to (1.1) and the symmetry of $E|Y$ for every $u \in W_E(S;X)$, $\vartheta \in C_0^\infty(S)$, and $v \in Y$ we obtain

$$\begin{aligned}\int_S \langle (\mathcal{C}_0^*(\mathcal{E}u)')(s), v\rangle_Y \,\vartheta(s)\,ds &= \int_S \langle (\mathcal{E}u)'(s), Cv\rangle_Y \,\vartheta(s)\,ds \\ &= -\int_S \langle Eu(s), Cv\rangle_Y \,\vartheta'(s)\,ds = -\int_S \langle (\mathcal{E}_0\mathcal{C}u)(s), v\rangle_Y \,\vartheta'(s)\,ds,\end{aligned}$$

which yields $(\mathcal{E}_0\mathcal{C}u)' = \mathcal{C}_0^*(\mathcal{E}u)' \in L^2(S;Y^*)$ and $\mathcal{C}u \in W_{E|Y}(S;Y)$. Hence, $\mathcal{C}$ maps $W_E(S;X)$ continuously into $W_{E|Y}(S;Y)$. □

Density of smooth functions. Every function from $W_E(S;X)$ can be approximated by functions with values in X being smooth in time. We start with the case $S = \mathbb{R}$.

Lemma 1.3 (Density)**.** *The set $C_0^\infty(\mathbb{R};X)$ is dense in $W_E(\mathbb{R};X)$.*

Proof. 1. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a nonnegative function satisfying $\int_{\mathbb{R}} \varphi(s)\, ds = 1$ and $\operatorname{supp}(\varphi) \subset (-1,1)$, and define $\varphi_k \subset C_0^\infty(\mathbb{R})$ by $\varphi_k(s) = k\varphi(ks)$ for $k \in \mathbb{N}$, $s \in \mathbb{R}$. For every $u \in L^2(\mathbb{R};X)$ the sequence (u_k) of convolutions $u_k = \varphi_k * u \in BC^\infty(\mathbb{R};X) \cap L^2(\mathbb{R};X)$ converges to u in $L^2(\mathbb{R};X)$.

Due to $E \in \mathcal{L}(X;Y^*)$ for all $u \in W_E(\mathbb{R};X)$ we obtain $\mathcal{E}u \in H^1(\mathbb{R};Y^*)$. The sequence of convolutions $(\mathcal{E}u)_k = \varphi_k * (\mathcal{E}u) \in BC^\infty(\mathbb{R};Y^*) \cap H^1(\mathbb{R};Y^*)$ converges to $\mathcal{E}u$ in $H^1(\mathbb{R};Y^*)$. This yields $\lim_{k\to\infty} \|\mathcal{E}u_k - \mathcal{E}u\|_{H^1(\mathbb{R};Y^*)} = 0$ since $\mathcal{E}u_k = (\mathcal{E}u)_k$ holds true for all $k \in \mathbb{N}$.

2. We choose cut-off functions $\vartheta_k \in C_0^\infty(\mathbb{R})$ satisfying

$$\vartheta_k|(-k,k) = 1, \quad 0 \le \vartheta_k(s) \le 1, \quad |\vartheta_k'(s)| \le 1 \quad \text{for all } s \in \mathbb{R},\, k \in \mathbb{N}.$$

Now, we prove that the sequence of functions $v_k = \vartheta_k u_k \in C_0^\infty(\mathbb{R};X)$ converges to u in $W_E(\mathbb{R};X)$: From the estimate

$$\begin{aligned}
\int_{\mathbb{R}} \|v_k(s) - u(s)\|_X^2\, ds &\le 2\int_{\mathbb{R}} \left(|\vartheta_k(s) - 1|^2 \|u_k(s)\|_X^2 + \|u_k(s) - u(s)\|_X^2\right) ds \\
&\le 2\int_{\mathbb{R}\setminus(-k,k)} \|u_k(s)\|_X^2\, ds + 2\int_{\mathbb{R}} \|u_k(s) - u(s)\|_X^2\, ds \\
&\le 4\int_{\mathbb{R}\setminus(-k,k)} \|u(s)\|_X^2\, ds + 6\int_{\mathbb{R}} \|u_k(s) - u(s)\|_X^2\, ds,
\end{aligned}$$

it follows $\lim_{k\to\infty} \|v_k - u\|_{L^2(\mathbb{R};X)} = 0$. Because of the properties of the cut-off functions and $(\mathcal{E}v_k)' = (\vartheta_k \mathcal{E}u_k)' = \vartheta_k' \mathcal{E}u_k + (\mathcal{E}u_k)'\vartheta_k$, we get

$$\begin{aligned}
&\int_{\mathbb{R}} \|(\mathcal{E}v_k)'(s) - (\mathcal{E}u_k)'(s)\|_{Y^*}^2\, ds \\
&\qquad \le 2\int_{\mathbb{R}} \left(|\vartheta_k'(s)|^2 \|(\mathcal{E}u_k)(s)\|_{Y^*}^2 + |\vartheta_k(s) - 1|^2 \|(\mathcal{E}u_k)'(s)\|_{Y^*}^2\right) ds
\end{aligned}$$

for all $k \in \mathbb{N}$, and, hence,

$$\begin{aligned}
&\int_{\mathbb{R}} \|(\mathcal{E}v_k)'(s) - (\mathcal{E}u_k)'(s)\|_{Y^*}^2\, ds \\
&\quad \le 4\int_{\mathbb{R}\setminus(-k,k)} \left(\|(\mathcal{E}u)(s)\|_{Y^*}^2 + \|(\mathcal{E}u_k)(s) - (\mathcal{E}u)(s)\|_{Y^*}^2\right) ds \\
&\qquad + 4\int_{\mathbb{R}\setminus(-k,k)} \left(\|(\mathcal{E}u)'(s)\|_{Y^*}^2 + \|(\mathcal{E}u_k)'(s) - (\mathcal{E}u)'(s)\|_{Y^*}^2\right) ds,
\end{aligned}$$

which yields $\lim_{k\to\infty} \|(\mathcal{E}v_k)' - (\mathcal{E}u_k)'\|_{L^2(S;Y^*)} = 0$. Together with the fact that the sequence $((\mathcal{E}u_k)')$ converges to $(\mathcal{E}u)'$ in $L^2(\mathbb{R};Y^*)$ (see Step 1) we obtain $\lim_{k\to\infty} \|(\mathcal{E}v_k)' - (\mathcal{E}u)'\|_{L^2(\mathbb{R};Y^*)} = 0$. $\square$

To prove a corresponding statement for arbitrary open intervals $S \subset \mathbb{R}$ we need some preparation:

Lemma 1.4. *Let $S \subset S_0 \subset \mathbb{R}$ be two open intervals and $\vartheta \in BC^\infty(\mathbb{R})$ some function with $\operatorname{supp}(\vartheta) \cap S_0 \subset S$. Then, for $u \in W_E(S;X)$ the function $u_0 : S_0 \to X$ defined by $u_0|S = \vartheta u$ and $u_0|(S_0 \setminus S) = 0$ belongs to $W_E(S_0;X)$.*

Proof. Let $u \in W_E(S;X)$ be given. The above construction implies that $u_0 \in L^2(S_0;X)$. We define $f_0 : S_0 \to Y^*$ by

$$f_0|S = \vartheta' \mathcal{E} u + (\mathcal{E} u)' \vartheta, \quad f_0|(S_0 \setminus S) = 0.$$

Due to $E \in \mathcal{L}(X;Y^*)$ this yields $f_0 \in L^2(S_0;Y^*)$. Note that for $\vartheta_0 \in C_0^\infty(S_0)$ the inclusion $\operatorname{supp}(\vartheta\vartheta_0) \subset \operatorname{supp}(\vartheta) \cap S_0 \subset S$ holds true. Hence, for all $v \in Y$ we obtain

$$\int_{S_0} \langle f_0(s), v \rangle_Y \, \vartheta_0(s) \, ds$$
$$= \int_S \langle (\mathcal{E} u)(s), v \rangle_Y \, \vartheta'(s) \vartheta_0(s) \, ds + \int_S \langle (\mathcal{E} u)'(s), v \rangle_Y \, \vartheta(s) \vartheta_0(s) \, ds$$

and therefore,

$$\begin{aligned} \int_{S_0} \langle f_0(s), v \rangle_Y \, \vartheta_0(s) \, ds &= - \int_S \langle (\mathcal{E} u)(s), v \rangle_Y \, \vartheta(s) \vartheta_0'(s) \, ds \\ &= - \int_{S_0} \langle (\mathcal{E}_0 u_0)(s), v \rangle_Y \, \vartheta_0'(s) \, ds, \end{aligned}$$

where $\mathcal{E}_0 : L^2(S_0;X) \to L^2(S_0;Y^*)$ is associated with S_0 and E. Consequently, this yields $(\mathcal{E}_0 u_0)' = f_0 \in L^2(S_0;Y^*)$ and $u_0 \in W_E(S_0;X)$. $\square$

For every $h \in \mathbb{R}$ we introduce the shifted interval $S_h = \{s + h : s \in S\}$. Note that the open interval $S \subset \mathbb{R}$ is unbounded if and only if there exists some $h \neq 0$ with $S_{-h} \subset S \subset S_h$. In that case we define the translation $\tau_h u : S_h \to X$ of the function $u : S \to X$ by $(\tau_h u)(s) = u(s - h)$ for $s \in S_h$.

Lemma 1.5 (Translation)**.** *Let $S \subset \mathbb{R}$ be some unbounded open interval, and let $h \neq 0$ satisfy $S_{-h} \subset S \subset S_h$. Then for all $u \in W_E(S;X)$ we have $\tau_h u \in W_E(S_h;X)$ and $\lim_{\delta \downarrow 0} \|(\tau_{\delta h} u)|S - u\|_{W_E(S;X)} = 0$.*

Proof. For $u \in W_E(S;X)$ we define $u_0 \in L^2(\mathbb{R};X)$, $f_0 \in L^2(\mathbb{R};Y^*)$ by

$$u_0|S = u, \quad u_0|(\mathbb{R} \setminus S) = 0, \quad f_0|S = (\mathcal{E} u)', \quad f_0|(\mathbb{R} \setminus S) = 0.$$

Due to the continuity of the translation operator we get

$$\lim_{\delta \downarrow 0} \|\tau_{\delta h} u_0 - u_0\|_{L^2(\mathbb{R};X)} = 0, \quad \lim_{\delta \downarrow 0} \|\tau_{\delta h} f_0 - f_0\|_{L^2(\mathbb{R};Y^*)} = 0.$$

Because $(\tau_{\delta h}u_0)|S = (\tau_{\delta h}u)|S$ and $(\tau_{\delta h}f_0)|S = (\tau_{\delta h}(\mathcal{E}u)')|S$ holds true for all $\delta \in (0,1)$, this yields

$$\lim_{\delta\downarrow 0}\|(\tau_{\delta h}u)|S - u\|_{L^2(S;X)} = 0, \quad \lim_{\delta\downarrow 0}\|(\tau_{\delta h}(\mathcal{E}u)')|S - (\mathcal{E}u)'\|_{L^2(S;Y^*)} = 0.$$

Note that for all $\delta \in (0,1)$ and $\vartheta \in C_0^\infty(S_{\delta h})$ we have $\tau_{-\delta h}\vartheta \in C_0^\infty(S)$. Consequently, for all $v \in Y$ we obtain

$$\begin{aligned}\int_{S_{\delta h}} \langle(\tau_{\delta h}(\mathcal{E}u)')(s), v\rangle_Y\, \vartheta(s)\,ds &= \int_S \langle(\mathcal{E}u)'(s), v\rangle_Y\, (\tau_{-\delta h}\vartheta)(s)\,ds \\ = -\int_S \langle(\mathcal{E}u)(s), v\rangle_Y\, (\tau_{-\delta h}\vartheta')(s)\,ds &= -\int_{S_{\delta h}} \langle(\mathcal{E}_{\delta h}\tau_{\delta h}u)(s), v\rangle_Y\, \vartheta'(s)\,ds,\end{aligned}$$

where the operator $\mathcal{E}_{\delta h} : L^2(S_{\delta h}; X) \to L^2(S_{\delta h}; Y^*)$ is associated with $S_{\delta h}$ and E. Hence, we get $(\mathcal{E}_{\delta h}\tau_{\delta h}u)' = \tau_{\delta h}(\mathcal{E}u)' \in L^2(S_{\delta h}; Y^*)$ which leads to the relation $\lim_{\delta\downarrow 0}\|(\tau_{\delta h}u)|S - u\|_{W_E(S;X)} = 0$. □

Theorem 1.6 (Density)**.** *The set of restrictions* $\{u|S : u \in C_0^\infty(\mathbb{R}; X)\}$ *is dense in* $W_E(S;X)$*.*

Proof. 1. In the case $S = \mathbb{R}$ the assertion follows from Lemma 1.3.

2. Let $S \subset \mathbb{R}$, $S \neq \mathbb{R}$ be an unbounded open interval and $\varepsilon > 0$. Then we have $S_{-h} \subset S \subset S_h$ for some $h \neq 0$. Applying Lemma 1.5 to $u \in W_E(S;X)$, we find some $\delta \in (0,1)$ with $\|(\tau_{\delta h}u)|S - u\|_{W_E(S;X)} \le \varepsilon$.

Next, we choose some cut-off function $\vartheta \in BC^\infty(\mathbb{R})$ with $\operatorname{supp}(\vartheta) \subset S_{\delta h}$ and $\vartheta|S = 1$. Because of $\tau_{\delta h}u \in W_E(S_{\delta h}; X)$ and Lemma 1.4 the function $u_0 : \mathbb{R} \to Y$ defined by

$$u_0|S_{\delta h} = \vartheta\tau_{\delta h}u, \quad u_0|(\mathbb{R} \setminus S_{\delta h}) = 0,$$

belongs to $W_E(\mathbb{R}; X)$, and we obtain $u_0|S = (\tau_{\delta h}u)|S$. Using Lemma 1.3 we find some function $w \in C_0^\infty(\mathbb{R}; X)$ with $\|w - u_0\|_{W_E(\mathbb{R};X)} \le \varepsilon$. Due to

$$\|w|S - u\|_{W_E(S;X)} \le \|w|S - u_0|S\|_{W_E(S;X)} + \|(\tau_{\delta h}u)|S - u\|_{W_E(S;X)}$$

this yields $\|w|S - u\|_{W_E(S;X)} \le 2\varepsilon$.

3. Finally, we consider the case of bounded open intervals $S = (t_0, t_1)$ with t_0, $t_1 \in \mathbb{R}$ and $t_0 < t_1$. Let $\varepsilon > 0$ be fixed arbitrarily. We choose some cut-off function $\vartheta \in BC^\infty(\mathbb{R})$ with $\operatorname{supp}(\vartheta) \subset S_0 = (-\infty, t_1)$ and $\operatorname{supp}(1 - \vartheta) \subset S_1 = (t_0, \infty)$. Due to Lemma 1.4 we get two functions $w_0 \in W_E(S_0; X)$ and $w_1 \in W_E(S_1; X)$ if we set

$$w_0|(-\infty, t_0] = 0,\ \ w_0|(t_0, t_1) = (1-\vartheta)u,\ \ w_1|(t_0, t_1) = \vartheta u,\ \ w_1|[t_1, \infty) = 0.$$

Applying Step 2 of the proof we find functions u_0, $u_1 \in C_0^\infty(\mathbb{R};X)$ such that

$$\|u_0|S_0 - w_0\|_{W_E(S_0;X)} \le \varepsilon, \quad \|u_1|S_1 - w_1\|_{W_E(S_1;X)} \le \varepsilon,$$

which yields

$$\begin{aligned}\|(u_0+u_1)|S - u\|_{W_E(S;X)} & \\ \le \|u_0|S - (1-\vartheta)u\|_{W_E(S;X)} &+ \|u_1|S - \vartheta u\|_{W_E(S;X)} \\ = \|u_0|S - w_0|S\|_{W_E(S;X)} &+ \|u_1|S - w_1|S\|_{W_E(S;X)} \le 2\varepsilon.\end{aligned}$$

Hence, $u_0 + u_1$ admits the desired approximation property. □

Integration by parts and continuous embeddings. In this subsection we temporarily assume that $X = Y$ holds true. Otherwise we refer to the special case treated in Appendix B.

Theorem 1.7 (Integration by parts)**.** *For every $u \in W_E(S;Y)$ there exists a uniquely determined continuous representative $\mathcal{K}u : \overline{S} \to H$ such that $(\mathcal{K}u)(s) = Ku(s)$ for almost all $s \in S$. The operator $\mathcal{K}$ is a bounded linear map from $W_E(S;Y)$ to $BC(\overline{S};H)$. Moreover, for every $u \in W_E(S;Y)$ and all s, $t \in \overline{S}$ the following integration by parts formula holds true:*

$$\|(\mathcal{K}u)(t)\|_H^2 - \|(\mathcal{K}u)(s)\|_H^2 = 2\int_s^t \langle(\mathcal{E}u)'(\tau), u(\tau)\rangle_Y \, d\tau. \tag{1.2}$$

Proof. 1. In the first step we prove the statement for $u \in C_0^\infty(\mathbb{R};Y)$. Then, the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(\tau) = \langle Eu(\tau), u(\tau)\rangle_Y$ for $\tau \in \mathbb{R}$ belongs to $C_0^\infty(\mathbb{R})$. Due to the symmetry of the operator $E = K^* J_H K \in \mathcal{L}(Y;Y^*)$, for all $\tau \in \mathbb{R}$ we get

$$\varphi'(\tau) = \langle Eu'(\tau), u(\tau)\rangle_Y + \langle Eu(\tau), u'(\tau)\rangle_Y = 2\langle Eu'(\tau), u(\tau)\rangle_Y.$$

By integration for all s, $t \in \overline{S}$ this yields

$$2\int_s^t \langle(\mathcal{E}u)'(\tau), u(\tau)\rangle_Y \, d\tau = \langle J_H Ku(t), Ku(t)\rangle_H - \langle J_H Ku(s), Ku(s)\rangle_H$$

and, hence,

$$\|Ku(t)\|_H^2 \le \|Ku(s)\|_H^2 + \|u|S\|_{W_E(S;Y)}^2.$$

Integrating over some bounded subinterval $(s_0, s_1) \subset S$, for all $t \in \overline{S}$ we obtain the estimate

$$\begin{aligned}(s_1 - s_0)\|Ku(t)\|_H^2 &\le \int_S \|Ku(s)\|_H^2 \, ds + (s_1 - s_0)\|u|S\|_{W_E(S;Y)}^2 \\ &\le \big(\|K\|_{\mathcal{L}(Y;H)}^2 + (s_1 - s_0)\big)\|u|S\|_{W_E(S;Y)}^2.\end{aligned}$$

Hence, we find some constant $c = c(S, K) > 0$ such that

$$\sup_{t\in\overline{S}} \|Ku(t)\|_H \leq c\,\|u|S\|_{W_E(S;Y)} \quad \text{for all } u \in C_0^\infty(\mathbb{R}; Y).$$

2. Let $u \in W_E(S;Y)$. Due to Theorem 1.6 there exists some sequence $(u_k) \subset C_0^\infty(\mathbb{R};Y)$ such that $(u_k|S)$ converges to u in $W_E(S;Y)$. Applying the result of Step 1 to the differences $u_k - u_\ell$, we get

$$\sup_{t\in\overline{S}} \|Ku_k(t) - Ku_\ell(t)\|_H \leq c\,\|u_k|S - u_\ell|S\|_{W_E(S;Y)} \quad \text{for all } k,\, \ell \in \mathbb{N}.$$

In view of the completeness of $BC(\overline{S};H)$ there exists a limit function $w \in BC(\overline{S};H)$ which satisfies $\lim_{k\to\infty}\sup_{t\in\overline{S}}\|Ku_k(t) - w(t)\|_H = 0$. For all $k \in \mathbb{N}$ and all bounded open subintervals $(s,\tau) \subset S$ we have the estimate

$$\begin{aligned}\int_s^\tau \|Ku(t){-}w(t)\|_H^2\,dt &\leq 2\int_s^\tau \big(\|Ku(t){-}Ku_k(t)\|_H^2{+}\|Ku_k(t){-}w(t)\|_H^2\big)\,dt\\ &\leq 2\,\|K\|_{\mathcal{L}(Y;H)}^2 \int_S \|u_k(t) - u(t)\|_Y^2\,dt + 2(\tau - s)\sup_{t\in\overline{S}}\|Ku_k(t) - w(t)\|_H^2.\end{aligned}$$

Passing to the limit $k \to \infty$ we obtain $w(t) = Ku(t)$ for almost all $t \in S$. We define by $\mathcal{K}u = w \in BC(\overline{S};H)$ the uniquely determined representative $\mathcal{K}u : \overline{S} \to H$, which satisfies $(\mathcal{K}u)(t) = Ku(t)$ for almost all $t \in S$.

3. Due to Step 1, we have $\|\mathcal{K}u_k\|_{BC(\overline{S};H)} \leq c\,\|u_k|S\|_{W_E(S;Y)}$ for all $k \in \mathbb{N}$. Passing to the limit $k \to \infty$ and using Step 2 we obtain

$$\|\mathcal{K}u\|_{BC(\overline{S};H)} \leq c\,\|u\|_{W_E(S;Y)} \quad \text{for all } u \in W_E(S;Y).$$

Hence, $\mathcal{K}$ is a bounded linear operator from $W_E(S;Y)$ to $BC(\overline{S};H)$. Moreover, following Step 1, for all $k \in \mathbb{N}$ and $s,\, t \in \overline{S}$ we get

$$\|Ku_k(t)\|_H^2 - \|Ku_k(s)\|_H^2 = 2\int_s^t \langle(\mathcal{E}u_k)'(\tau), u_k(\tau)\rangle_Y\,d\tau.$$

Passing to the limit $k \to \infty$, Step 2 yields the desired identity (1.2). □

Theorem 1.8 (Extension). *Let $t \in S = (t_0, t_1)$, $S_0 = (t_0, t)$, and $S_1 = (t, t_1)$. If $u_0 \in W_E(S_0;Y)$ and $u_1 \in W_E(S_1;Y)$ satisfy $(\mathcal{K}_0u_0)(t) = (\mathcal{K}_1u_1)(t)$, then the function $u : S \to Y$ defined by $u|S_0 = u_0$ and $u|S_1 = u_1$ belongs to $W_E(S;Y)$.*

Proof. Due to the construction we have $u \in L^2(S;Y)$, and the function $f : S \to Y^*$ defined by $f|S_0 = (\mathcal{E}_0u_0)'$ and $f|S_1 = (\mathcal{E}_1u_1)'$ belongs to $f \in L^2(S;Y^*)$.

Let $\vartheta \in C_0^\infty(S)$, $v \in Y$ be fixed. Then we have $w = \vartheta v \in W_E(S;Y)$, and using the integration by parts formula (see Theorem 1.7) we get

$$((\mathcal{K}_0 u_0)(t)|Kv)_H\, \vartheta(t) = \int_{S_0} \big(\langle(\mathcal{E}_0 u_0)'(s), w(s)\rangle_Y + \langle Ev, u_0(s)\rangle_Y\, \vartheta'(s)\big)\, ds,$$

$$-((\mathcal{K}_1 u_1)(t)|Kv)_H\, \vartheta(t) = \int_{S_1} \big(\langle(\mathcal{E}_1 u_1)'(s), w(s)\rangle_Y + \langle Ev, u_1(s)\rangle_Y\, \vartheta'(s)\big)\, ds.$$

Because of $(\mathcal{K}_0 u_0)(t) = (\mathcal{K}_1 u_1)(t)$ and the symmetry of E this yields

$$\begin{aligned}\int_S \langle f(s), v\rangle_Y\, \vartheta(s)\, ds \\ &= \int_{S_0} \langle(\mathcal{E}_0 u_0)'(s), v\rangle_Y\, \vartheta(s)\, ds + \int_{S_1} \langle(\mathcal{E}_1 u_1)'(s), v\rangle_Y\, \vartheta(s)\, ds \\ &= -\int_S \langle Ev, u(s)\rangle_Y\, \vartheta'(s)\, ds = -\int_S \langle(\mathcal{E}u)(s), v\rangle_Y\, \vartheta'(s)\, ds.\end{aligned}$$

Hence, we get $(\mathcal{E}u)' = f \in L^2(S;Y^*)$ and $u \in W_E(S;Y)$. $\square$

Completely continuous embeddings. It turns out that in the case of complete continuity of the operator $K \in \mathcal{L}(X;H)$ this property carries over to the operator $\mathcal{K}$ from $W_E(S;X)$ into $L^2(S;H)$, whenever $S = (t_0, t_1)$ is bounded. The following proof generalizes an idea of Temam [31]; see also Lions [21, 22], Simon [28].

Lemma 1.9. *Let $K \in \mathcal{L}(X;H)$ be completely continuous. Then, for every $\delta > 0$ there exists some constant $c > 0$ such that*

$$\|Kw\|_H^2 \le \delta\, \|w\|_X^2 + c\, \|Ew\|_{Y^*}^2 \quad \text{for all } w \in X. \tag{1.3}$$

Proof. 1. Assume that there exists some $\delta > 0$ such that we can find some sequence $(w_k) \subset X$ that satisfies

$$\|w_k\|_X = 1, \quad \|Kw_k\|_H^2 > \delta + k\, \|Ew_k\|_{Y^*}^2 \quad \text{for all } k \in \mathbb{N}.$$

Because of $K \in \mathcal{L}(X;H)$ this yields $\lim_{k\to\infty} \|Ew_k\|_{Y^*} = 0$.

2. Due to the complete continuity of $K \in \mathcal{L}(X;H)$ there exists an increasing subsequence $(k_\ell) \subset \mathbb{N}$ and some limit function $h \in H$ such that $\lim_{\ell\to\infty} \|Kw_{k_\ell} - h\|_H = 0$. Because of $E = (K|Y)^* J_H K \in \mathcal{L}(X;Y^*)$ and Step 1 for all $v \in Y$ this yields

$$\langle J_H h, Kv\rangle_H = \lim_{\ell\to\infty} \langle J_H K w_{k_\ell}, Kv\rangle_H = \lim_{\ell\to\infty} \langle Ew_{k_\ell}, v\rangle_Y = 0.$$

In view of $K|Y \in \mathcal{L}(Y;H)$ and the density of $K[Y]$ in H we obtain $h = 0$, which contradicts to the fact that $\|Kw_k\|_H^2 > \delta$ holds true for all $k \in \mathbb{N}$;

see Step 1. Hence, the assumption was not true, which proves the desired estimate (1.3). □

Theorem 1.10 (Complete continuity)**.** *Let $S = (t_0, t_1)$ be some bounded open interval and $K \in \mathcal{L}(X; H)$ be completely continuous. Then $\mathcal{K}$ maps $W_E(S; X)$ completely continuous into $L^2(S; H)$.*

Proof. 1. Let $(u_k) \subset W_E(S; X)$ be a bounded sequence and $c_1 > 0$ some constant such that

$$\int_S \big(\|u_k(s)\|_X^2 + \|(\mathcal{E}u_k)'(s)\|_{Y^*}^2\big)\, ds \le c_1 \quad \text{for all } k \in \mathbb{N}. \tag{1.4}$$

We choose an increasing subsequence $(k_\ell) \subset \mathbb{N}$ such that (u_{k_ℓ}) converges weakly to some limit u in $W_E(S; X)$. Due to the lower semicontinuity of the norm this yields

$$\int_S \big(\|u(s)\|_X^2 + \|(\mathcal{E}u)'(s)\|_{Y^*}^2\big)\, ds \le c_1. \tag{1.5}$$

Consequently, the sequence $(v_\ell) \subset W_E(S; X)$, defined by $v_\ell = u_{k_\ell} - u$ for $\ell \in \mathbb{N}$, converges weakly to 0 in $W_E(S; X)$. Note that $(\mathcal{E}v_\ell)$ is bounded in $H^1(S; Y^*)$. Together with the continuous embedding of $H^1(S; Y^*)$ in $BC(\overline{S}; Y^*)$ this implies the existence of some constant $c_2 > 0$ such that

$$\|(\mathcal{E}v_\ell)(s)\|_{Y^*} \le c_2 \quad \text{for all } s \in \overline{S},\ \ell \in \mathbb{N}. \tag{1.6}$$

2. To prove that $\lim_{\ell\to\infty} \int_S \|(\mathcal{E}v_\ell)(s)\|_{Y^*}^2\, ds = 0$ holds true, we proceed as follows: Let $t \in S$ and $\delta > 0$ be fixed arbitrarily. Because of $\mathcal{E}v_\ell \in H^1(S; Y^*) \subset BC(\overline{S}; Y^*)$, for all $\ell \in \mathbb{N}$ and $s \in (t, t_1)$ we have

$$(\mathcal{E}v_\ell)(t) = (\mathcal{E}v_\ell)(s) - \int_t^s (\mathcal{E}v_\ell)'(\tau)\, d\tau.$$

Next, we choose some $\theta \in (t, t_1)$ which satisfies $c_1(\theta - t) \le \frac{\delta}{8}$. Integrating over the interval (t, θ), and defining $w_\ell \in X$ and $f_\ell \in Y^*$ by

$$w_\ell = \frac{1}{\theta - t} \int_t^\theta v_\ell(s)\, ds,$$

and

$$f_\ell = \frac{1}{\theta - t} \int_t^\theta \int_t^s (\mathcal{E}v_\ell)'(\tau)\, d\tau\, ds = \frac{1}{\theta - t} \int_t^\theta (\theta - s)(\mathcal{E}v_\ell)'(s)\, ds,$$

we get the identity

$$(\mathcal{E}v_\ell)(t) = Ew_\ell - f_\ell \quad \text{for all } \ell \in \mathbb{N}.$$

Due to the weak convergence of (v_ℓ) to 0 in $L^2(S;X)$ (see Step 1) we have

$$\lim_{\ell\to\infty}\langle f, w_\ell\rangle_X = \lim_{\ell\to\infty}\frac{1}{\theta-t}\int_t^\theta \langle f, v_\ell(s)\rangle_X\,ds = 0 \quad \text{for all } f\in X^*.$$

That means (w_ℓ) converges weakly to 0 in X. Because of the complete continuity of $E=(K|Y)^*J_HK\in\mathcal{L}(X;Y^*)$ this yields $\lim_{\ell\to\infty}\|Ew_\ell\|_{Y^*}=0$. We choose $\ell_0=\ell_0(\delta)\in\mathbb{N}$ such that

$$\|Ew_\ell\|_{Y^*}^2 \le \frac{\delta}{4} \quad \text{for all } \ell\in\mathbb{N},\ \ell\ge\ell_0. \tag{1.7}$$

On the other hand, for all $\ell\in\mathbb{N}$ we get

$$\|f_\ell\|_{Y^*}^2 \le \frac{1}{(\theta-t)^2}\left(\int_t^\theta(\theta-s)^2\,ds\right)\left(\int_t^\theta\|(\mathcal{E}v_\ell)'(s)\|_{Y^*}^2\,ds\right).$$

Hence, using (1.4), (1.5), and $c_1(\theta-t)\le\frac{\delta}{8}$ we obtain $\|f_\ell\|_{Y^*}^2\le\frac{\delta}{6}$ for all $\ell\in\mathbb{N}$. Due to $(\mathcal{E}v_\ell)(t)=Ew_\ell-f_\ell$ and (1.7) this yields

$$\|(\mathcal{E}v_\ell)(t)\|_{Y^*}^2 \le 2\,\|Ew_\ell\|_{Y^*}^2+2\,\|f_\ell\|_{Y^*}^2\le\delta \quad \text{for all } \ell\in\mathbb{N},\ \ell\ge\ell_0.$$

Because we have fixed $\delta>0$ and $t\in S$ arbitrarily at the beginning, we get pointwise convergence, which means $\lim_{\ell\to\infty}\|(\mathcal{E}v_\ell)(t)\|_{Y^*}^2=0$ for all $t\in S$. In view of the uniform estimate (1.6) and the boundedness of the interval $S\subset\mathbb{R}$ the dominated convergence theorem yields

$$\lim_{\ell\to\infty}\int_S\|(\mathcal{E}v_\ell)(s)\|_{Y^*}^2\,ds=0. \tag{1.8}$$

3. Let $\delta>0$ be fixed. Applying Lemma 1.9 we find some constant $c_3>0$ such that for all $\ell\in\mathbb{N}$ we have

$$\int_S\|(\mathcal{K}v_\ell)(s)\|_H^2\,ds\le\delta\int_S\|v_\ell(s)\|_X^2\,ds+c_3\int_S\|(\mathcal{E}v_\ell)(s)\|_{Y^*}^2\,ds.$$

Due to (1.4) and (1.5) this yields

$$\int_S\|(\mathcal{K}v_\ell)(s)\|_H^2\,ds\le 4c_1\delta+c_3\int_S\|(\mathcal{E}v_\ell)(s)\|_{Y^*}^2\,ds \quad \text{for all } \ell\in\mathbb{N}.$$

Using (1.8) and passing to the limit $\ell\to\infty$ we end up with

$$\limsup_{\ell\to\infty}\int_S\|(\mathcal{K}v_\ell)(s)\|_H^2\,ds\le 4c_1\delta.$$

Because $\delta>0$ was fixed arbitrarily, we have found a subsequence $(\mathcal{K}u_{k_\ell})$ which converges to $\mathcal{K}u$ in $L^2(S;H)$. $\square$

Corollary 1.11 (Complete continuity)**.** *Let $S = (t_0, t_1)$ be some bounded open interval, and let both $K \in \mathcal{L}(X; H)$ and $K_0 \in \mathcal{L}(X; H_0)$ be completely continuous, where H_0 is some further Hilbert space. If for every $\delta > 0$ there exists some constant $c > 0$ such that*

$$\|K_0 w\|_{H_0}^2 \leq \delta \, \|w\|_X^2 + c \, \|Kw\|_H^2 \quad \textit{for all } w \in X,$$

then the operator $\mathcal{K}_0 : L^2(S; X) \to L^2(S; H_0)$, associated with S and K_0 via $(\mathcal{K}_0 u)(s) = K_0 u(s)$ for $s \in S$, maps $W_E(S; X)$ completely continuous into $L^2(S; H_0)$.

Proof. 1. Let (u_k) be a bounded sequence in $W_E(S; X)$. Then there exists an increasing subsequence $(k_\ell) \subset \mathbb{N}$ such that (u_{k_ℓ}) converges weakly to some limit u in $W_E(S; X)$. We take some constant $c_1 > 0$ such that

$$\int_S \|u_k(s)\|_X^2 \, ds \leq c_1 \quad \text{for all } k \in \mathbb{N}, \quad \int_S \|u(s)\|_X^2 \, ds \leq c_1. \tag{1.9}$$

Since $\mathcal{K}$ maps $W_E(S; X)$ completely continuous into $L^2(S; H)$ (see Theorem 1.10), the sequence $(\mathcal{K} u_{k_\ell})$ converges to $\mathcal{K} u$ in $L^2(S; H)$.

2. Let $\delta > 0$ be arbitrarily fixed. Due to the assumption we find some constant $c_2 > 0$ such that for all $\ell \in \mathbb{N}$ we have

$$\begin{aligned}\int_S &\|(\mathcal{K}_0 u_{k_\ell})(s) - (\mathcal{K}_0 u)(s)\|_{H_0}^2 \, ds \\ &\leq \delta \int_S \|u_{k_\ell}(s) - u(s)\|_X^2 \, ds + c_2 \int_S \|(\mathcal{K} u_{k_\ell})(s) - (\mathcal{K} u)(s)\|_H^2 \, ds.\end{aligned}$$

In view of (1.9) and the convergence of $(\mathcal{K} u_{k_\ell})$ to $\mathcal{K} u$ in $L^2(S; H)$ (see Step 1), we pass to the limit $\ell \to \infty$ to get

$$\limsup_{\ell \to \infty} \int_S \|(\mathcal{K}_0 u_{k_\ell})(s) - (\mathcal{K}_0 u)(s)\|_{H_0}^2 \, ds \leq 4 c_1 \delta;$$

in other words, $(\mathcal{K}_0 u_{k_\ell})$ converges to $\mathcal{K}_0 u$ in $L^2(S; H_0)$. □

2. Solvability of initial-value problems

Throughout this section we assume that $S = (t_0, t_1)$ is a bounded open interval and that $X = Y$ holds true. We provide the unique solvability and well-posedness for a broad class of evolution equations; see Gröger [15]:

Lemma 2.1. *The map $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset L^2(S; Y) \times H \to L^2(S; Y^*) \times H^*$ defined by*

$$\mathrm{dom}(\mathcal{D}) = \big\{ (u, (\mathcal{K} u)(t_0)) : u \in W_E(S; Y) \big\},$$

$$\mathcal{D}(u,w) = ((\mathcal{E}u)', J_H w) \quad \textit{for } (u,w) \in \operatorname{dom}(\mathcal{D}),$$

is maximal monotone.

Proof. 1. Integrating by parts, for all $(u,w) \in \operatorname{dom}(\mathcal{D})$ we get

$$\begin{aligned} 2\,\langle \mathcal{D}(u,w),(u,w)\rangle &= 2\int_S \langle (\mathcal{E}u)'(s), u(s)\rangle_Y \, ds + 2\,\|w\|_H^2 \\ &= \|(\mathcal{K}u)(t_1)\|_H^2 + \|w\|_H^2, \end{aligned}$$

which means the linear operator $\mathcal{D}$ is monotone.

2. To prove the maximality of $\mathcal{D}$, let $(u,w) \in L^2(S;Y) \times H$ and $(f,g) \in L^2(S;Y^*) \times H^*$ be pairs satisfying

$$\int_S \langle f(s) - (\mathcal{E}\hat{u})'(s), u(s) - \hat{u}(s)\rangle_Y \, ds + \langle g - J_H\hat{w}, w - \hat{w}\rangle_H \geq 0 \qquad (2.1)$$

for all $(\hat{u},\hat{w}) \in \operatorname{dom}(\mathcal{D})$. Let $v \in Y$, $\vartheta \in C_0^\infty(S)$. Choosing $(\hat{u},\hat{w}) = (\vartheta v, 0) \in \operatorname{dom}(\mathcal{D})$ in (2.1) and integrating by parts we obtain

$$\int_S \langle f(s), v\rangle_Y \, \vartheta(s)\, ds + \int_S \langle Ev, u(s)\rangle_Y \, \vartheta'(s)\, ds \leq \int_S \langle f(s), u(s)\rangle_Y \, ds + \langle g, w\rangle_H.$$

Since the left-hand side is linear with respect to v and the right-hand side does not depend on v, this inequality can be true, only, if the left-hand side vanishes for all $v \in Y$. Hence, for all $v \in Y$ and $\vartheta \in C_0^\infty(S)$ we get

$$\int_S \langle f(s), v\rangle_Y \, \vartheta(s)\, ds + \int_S \langle (\mathcal{E}u)(s), v\rangle_Y \, \vartheta'(s)\, ds = 0,$$

which proves $(\mathcal{E}u)' = f \in L^2(S;Y^*)$ and $u \in W_E(S;Y)$.

3. Due to $(\mathcal{E}u)' = f$ and (2.1) by partial integration we get

$$\|(\mathcal{K}u)(t_1) - (\mathcal{K}\hat{u})(t_1)\|_H^2 - \|(\mathcal{K}u)(t_0) - \hat{w}\|_H^2 + 2\,\langle g - J_H\hat{w}, w - \hat{w}\rangle_H \geq 0 \quad (2.2)$$

for all $(\hat{u},\hat{w}) \in \operatorname{dom}(\mathcal{D})$. Because $K[Y]$ is dense in H, we can choose two sequences (v_k°) and (v_k) in Y which satisfy

$$\lim_{k\to\infty} \|Kv_k^\circ - w\|_H = 0, \quad \lim_{k\to\infty} \|K(v_k^\circ + (t_1 - t_0)v_k) - (\mathcal{K}u)(t_1)\|_H = 0.$$

Setting $\hat{u}(s) = v_k^\circ + (s - t_0)v_k$ for $s \in S$ and $\hat{w} = Kv_k^\circ$ we obtain $(\hat{u},\hat{w}) \in \operatorname{dom}(\mathcal{D})$, and from (2.2) it follows that

$$\begin{aligned} \|(\mathcal{K}u)(t_1) - K(v_k^\circ + (t_1 - t_0)v_k)\|_H^2 - \|(\mathcal{K}u)(t_0) - Kv_k^\circ\|_H^2 \\ + 2\,\langle g - J_H K v_k^\circ, w - Kv_k^\circ\rangle_H \geq 0. \end{aligned}$$

Passing to the limit $k \to \infty$ we get $\|(\mathcal{K}u)(t_0) - w\|_H^2 \leq 0$, which means $(\mathcal{K}u)(t_0) = w$ and $(u,w) \in \operatorname{dom}(\mathcal{D})$.

4. Let $v \in Y$ and $\tau > 0$ be fixed. Setting $\hat{u}(s) = u(s) + \tau v$ for $s \in S$ and $\hat{w} = w + \tau K v$, we get $(\hat{u}, \hat{w}) \in \mathrm{dom}(\mathcal{D})$. Then, inequality (2.2) yields that

$$\langle g - J_H(w + \tau K v), \tau K v\rangle_H \le 0.$$

Dividing by $\tau > 0$ and passing to the limit $\tau \downarrow 0$ we find

$$\langle g - J_H w, K v\rangle_H \le 0 \quad \text{for all } v \in Y.$$

Since $K \in \mathcal{L}(Y; H)$ and $K[Y]$ is dense in H, we arrive at $g = J_H w$. In other words, we have shown that the operator $\mathcal{D}$ is maximal monotone. $\square$

Theorem 2.2 (Unique solvability)**.** *Assume that* $\mathcal{M} : L^2(S;Y) \to L^2(S;Y^*)$ *is a strongly monotone and Lipschitz-continuous operator with the domain* $\mathrm{dom}(\mathcal{M}) = L^2(S;Y)$*. Then, under the general assumptions mentioned above, for any* $f \in L^2(S;Y^*)$ *and* $w \in H$*, the initial-value problem*

$$(\mathcal{E}u)' + \mathcal{M}u = f, \quad (\mathcal{K}u)(t_0) = w, \tag{2.3}$$

has a uniquely determined solution $u \in W_E(S;Y)$*. Moreover, the assignment* $(f, w) \mapsto u$ *is Lipschitz continuous from* $L^2(S;Y^*) \times H$ *into* $W_E(S;Y)$*.*

Proof. 1. Due to the assumptions, $\mathcal{M} : L^2(S;Y) \to L^2(S;Y^*)$ is a maximal monotone operator. We define $\mathcal{M}_0 : L^2(S;Y) \times H \to L^2(S;Y^*) \times H^*$ by

$$\mathcal{M}_0(u, w) = (\mathcal{M}u, 0) \quad \text{for } (u, w) \in L^2(S;Y) \times H.$$

Elementary arguments show that the maximal monotonicity of $\mathcal{M}$ carries over to $\mathcal{M}_0$, where $\mathrm{dom}(\mathcal{M}_0) = L^2(S;Y) \times H$.

Let $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset L^2(S;Y) \times H \to L^2(S;Y^*) \times H^*$ be the maximal monotone operator of Lemma 2.1. Because of $\mathrm{dom}(\mathcal{D}) \subset L^2(S;Y) \times H$ we have $\mathrm{dom}(\mathcal{M}_0 + \mathcal{D}) = \mathrm{dom}(\mathcal{D})$, and Rockafellar's sum theorem yields the maximal monotonicity of $\mathcal{M}_0 + \mathcal{D}$; see [27]. Moreover, the strong monotonicity of $\mathcal{M}$ implies that $\mathcal{M}_0 + \mathcal{D}$ is strongly monotone, too, since for all (u, w), $(\hat{u}, \hat{w}) \in \mathrm{dom}(\mathcal{D})$ we have

$$\begin{aligned} &2\,\langle (\mathcal{M}_0 + \mathcal{D})(u, w) - (\mathcal{M}_0 + \mathcal{D})(\hat{u}, \hat{w}), (u, w) - (\hat{u}, \hat{w})\rangle \\ &\quad = 2\,\langle \mathcal{M}u - \mathcal{M}\hat{u}, u - \hat{u}\rangle_{L^2(S;Y)} + \|(\mathcal{K}u)(t_1) - (\mathcal{K}\hat{u})(t_1)\|_H^2 + \|w - \hat{w}\|_H^2. \end{aligned}$$

Applying Browder's theorem, for any $(f, w) \in L^2(S;Y^*) \times H$ the problem

$$(\mathcal{M}_0 + \mathcal{D})(u, w) = (f, J_H w)$$

has a solution $(u, w) \in \mathrm{dom}(\mathcal{D})$; see [1]. By construction, from this it follows that $u \in W_E(S;Y)$ solves the initial-value problem (2.3).

2. Let w, $\hat{w} \in H$ and f, $\hat{f} \in L^2(S;Y^*)$ be given data. Using Step 1 of the proof we find solutions u, $\hat{u} \in W_E(S;Y)$ of the problems

$$(\mathcal{E}u)' + \mathcal{M}u = f, \quad (\mathcal{K}u)(t_0) = w,$$
$$(\mathcal{E}\hat{u})' + \mathcal{M}\hat{u} = \hat{f}, \quad (\mathcal{K}\hat{u})(t_0) = \hat{w}.$$

Let M, $L > 0$ be the monotonicity and the Lipschitz constant of $\mathcal{M}$, respectively. Applying Young's inequality and the strong monotonicity of $\mathcal{M}$ we get the estimate

$$\begin{aligned} 0 &= 2\,\langle (\mathcal{E}u)' - (\mathcal{E}\hat{u})' + \mathcal{M}u - \mathcal{M}\hat{u} - f + \hat{f}, u - \hat{u}\rangle_{L^2(S;Y)} \\ &\geq \|(\mathcal{K}u)(t_1) - (\mathcal{K}\hat{u})(t_1)\|_H^2 - \|w - \hat{w}\|_H^2 \\ &\quad + M\,\|u - \hat{u}\|_{L^2(S;Y)}^2 - \tfrac{1}{M}\,\|f - \hat{f}\|_{L^2(S;Y^*)}^2, \end{aligned}$$

which means we have

$$M\,\|u - \hat{u}\|_{L^2(S;Y)}^2 \leq \|w - \hat{w}\|_H^2 + \tfrac{1}{M}\,\|f - \hat{f}\|_{L^2(S;Y^*)}^2.$$

Note that in the case $f = \hat{f}$, $w = \hat{w}$ this yields $u = \hat{u}$. Hence, we have shown the unique solvability of problem (2.3).

Moreover, using the Lipschitz continuity of $\mathcal{M}$ we obtain

$$\begin{aligned} \|(\mathcal{E}u)' - (\mathcal{E}\hat{u})'\|_{L^2(S;Y^*)}^2 &\leq 2\,\|f - \hat{f}\|_{L^2(S;Y^*)}^2 + 2\,\|\mathcal{M}u - \mathcal{M}\hat{u}\|_{L^2(S;Y^*)}^2 \\ &\leq 2\,\|f - \hat{f}\|_{L^2(S;Y^*)}^2 + 2L^2\,\|u - \hat{u}\|_{L^2(S;Y)}^2. \end{aligned}$$

From the last estimates it follows that the assignment $(f, w) \mapsto u$ is Lipschitz continuous from $L^2(S;Y^*) \times H$ into $W_E(S;Y)$. □

Let $\alpha \in \mathbb{R}$ be given. In the following we show that the result of the preceeding theorem remains true for the class of more general problems

$$(\mathcal{E}u)' + \mathcal{M}u - \alpha\mathcal{E}u = f, \quad (\mathcal{K}u)(0) = w,$$

if we additionally assume that the operator $\mathcal{M} : L^2(S;Y) \to L^2(S;Y^*)$ has the Volterra property. For the proof we need some preparation:

Lemma 2.3. *Let $e_\alpha : [t_0, t_1] \to \mathbb{R}$ be the exponential function given by*

$$e_\alpha(s) = \exp(\alpha(t_0 - s)) \quad \textit{for } \alpha \in \mathbb{R},\ s \in [t_0, t_1],$$

and $\mathcal{M} : L^2(S;Y) \to L^2(S;Y^*)$ be a strongly monotone, Lipschitz-continuous Volterra operator. For $\alpha \geq 0$ the map $\mathcal{M}_\alpha : L^2(S;Y) \to L^2(S;Y^*)$, defined as

$$\mathcal{M}_\alpha u = e_\alpha\,\mathcal{M}(e_{-\alpha}u) \quad \textit{for } u \in L^2(S;Y),$$

is a strongly monotone and Lipschitz-continuous Volterra operator, too.

Proof. 1. Let $\alpha \geq 0$. The operator $\mathcal{M}_\alpha : L^2(S;Y) \to L^2(S;Y^*)$ is correctly defined, because for all $u \in L^2(S;Y)$ and $f \in L^2(S;Y^*)$ we have $e_\alpha u$, $e_{-\alpha} u \in L^2(S;Y)$ and $e_\alpha f$, $e_{-\alpha} f \in L^2(S;Y^*)$.

2. Let u_α, $v_\alpha \in L^2(S;Y)$ be fixed, and set $u = e_{-\alpha} u_\alpha \in L^2(S;Y)$, $v = e_{-\alpha} v_\alpha \in L^2(S;Y)$. If $u_\alpha|(t_0, s) = v_\alpha|(0, s)$ holds true for all $s \in S$, then we obtain $u|(t_0, s) = v|(t_0, s)$, and the Volterra property of $\mathcal{M}$ yields $(\mathcal{M}u)|(t_0, s) = (\mathcal{M}v)|(t_0, s)$, which leads to $(\mathcal{M}_\alpha u_\alpha)|(t_0, s) = (\mathcal{M}_\alpha v_\alpha)|(t_0, s)$.

If $L > 0$ is a Lipschitz constant of $\mathcal{M}$, for $L_\alpha = L e_{-\alpha}(t_1)$ we get

$$\|\mathcal{M}_\alpha u_\alpha - \mathcal{M}_\alpha v_\alpha\|_{L^2(S;Y^*)} \leq L\, \|u - v\|_{L^2(S;Y)} \leq L_\alpha\, \|u_\alpha - v_\alpha\|_{L^2(S;Y)},$$

which means $L_\alpha > 0$ is some Lipschitz constant for $\mathcal{M}_\alpha$.

3. Note that for all functions $h \in L^1(S)$ Fubini's theorem yields

$$\begin{aligned} &-2\alpha \int_{t_0}^{t_1} e_{2\alpha}(s) \int_{t_0}^{s} h(\tau)\, d\tau\, ds = \int_{t_0}^{t_1} e'_{2\alpha}(s) \int_{t_0}^{s} h(\tau)\, d\tau\, ds \\ &\quad = \int_{t_0}^{t_1} h(s) \int_{s}^{t_1} e'_{2\alpha}(\tau)\, d\tau\, ds = e_{2\alpha}(t_1) \int_{t_0}^{t_1} h(s)\, ds - \int_{t_0}^{t_1} e_{2\alpha}(s) h(s)\, ds. \end{aligned}$$

Applying this identity to the function $h \in L^1(S)$ defined by

$$h(s) = \langle (\mathcal{M}u)(s) - (\mathcal{M}v)(s), u(s) - v(s) \rangle_Y \quad \text{for } s \in S,$$

we obtain

$$\begin{aligned} &\int_S e_{2\alpha}(s) \langle (\mathcal{M}u)(s) - (\mathcal{M}v)(s), u(s) - v(s) \rangle_Y\, ds \\ &\qquad = e_{2\alpha}(t_1) \int_S \langle (\mathcal{M}u)(s) - (\mathcal{M}v)(s), u(s) - v(s) \rangle_Y\, ds \\ &\qquad\quad + 2\alpha \int_S e_{2\alpha}(s) \int_{t_0}^{s} \langle (\mathcal{M}u)(\tau) - (\mathcal{M}v)(\tau), u(\tau) - v(\tau) \rangle_Y\, d\tau\, ds. \end{aligned}$$

Due to the monotonicity and the Volterra property of $\mathcal{M}$, the second summand is nonnegative. If $M > 0$ is a monotonicity constant of $\mathcal{M}$, for $M_\alpha = M e_{2\alpha}(t_1)$ this yields

$$\langle \mathcal{M}_\alpha u_\alpha - \mathcal{M}_\alpha v_\alpha, u_\alpha - v_\alpha \rangle_{L^2(S;Y)} \geq M_\alpha\, \|u - v\|^2_{L^2(S;Y)} \geq M_\alpha\, \|u_\alpha - v_\alpha\|^2_{L^2(S;Y)};$$

in other words, $M_\alpha > 0$ is some monotonicity constant for $\mathcal{M}_\alpha$. □

Theorem 2.4 (Unique solvability)**.** *Assume that* $\mathcal{M} : L^2(S;Y) \to L^2(S;Y^*)$ *is a strongly monotone and Lipschitz-continuous operator Volterra operator such that* $\operatorname{dom}(\mathcal{M}) = L^2(S;Y)$*. Under the general assumptions, for every* $\alpha \in \mathbb{R}$*,* $f \in L^2(S;Y^*)$*, and* $w \in H$*, the initial-value problem*

$$(\mathcal{E}u)' + \mathcal{M}u - \alpha \mathcal{E}u = f, \quad (\mathcal{K}u)(t_0) = w, \tag{2.4}$$

has a uniquely determined solution $u \in W_E(S;Y)$*. Moreover, the assignment* $(f,w) \mapsto u$ *is Lipschitz continuous from* $L^2(S;Y^*) \times H$ *into* $W_E(S;Y)$*.*

Proof. 1. In the case $\alpha \le 0$ the result follows immediately from Theorem 2.2, because the positive semidefiniteness of $E \in \mathcal{L}(Y;Y^*)$ yields that $\mathcal{M} - \alpha\mathcal{E} : L^2(S;Y) \to L^2(S;Y^*)$ is a strongly monotone and Lipschitz-continuous Volterra operator.

2. Let $\alpha \ge 0$, $f \in L^2(S;Y^*)$, and $w \in H$ be given. Setting $f_\alpha = e_\alpha f \in L^2(S;Y^*)$ and applying Theorem 2.2 and Lemma 2.3 we get the uniquely determined solution $u_\alpha \in W_E(S;Y)$ of the auxiliary problem

$$(\mathcal{E}u_\alpha)' + \mathcal{M}_\alpha u_\alpha = f_\alpha, \quad (\mathcal{K}u_\alpha)(t_0) = w. \tag{2.5}$$

Consequently, the function $u = e_{-\alpha}u_\alpha \in W_E(S;Y)$ solves problem (2.4).

Conversely, if $u \in W_E(S;Y)$ is a solution of problem (2.4), then $u_\alpha = e_\alpha u \in W_E(S;Y)$ solves the auxiliary problem (2.5). Hence, the solution $u \in W_E(S;Y)$ of problem (2.4) is uniquely determined, too.

3. Let w, $\hat{w} \in H$ and f, $\hat{f} \in L^2(S;Y^*)$ be given data. Due to Step 2 of the proof we find unique solutions u, $\hat{u} \in W_E(S;Y)$ of the problems

$$\begin{aligned} (\mathcal{E}u)' + \mathcal{M}u - \alpha\mathcal{E}u &= f, \quad (\mathcal{K}u)(t_0) = w, \\ (\mathcal{E}\hat{u})' + \mathcal{M}\hat{u} - \alpha\mathcal{E}\hat{u} &= \hat{f}, \quad (\mathcal{K}\hat{u})(t_0) = \hat{w}. \end{aligned}$$

Defining as before $f_\alpha = e_\alpha f \in L^2(S;Y^*)$ and $\hat{f}_\alpha = e_\alpha \hat{f} \in L^2(S;Y^*)$, the functions $u_\alpha = e_\alpha u \in W_E(S;Y)$ and $\hat{u}_\alpha = e_\alpha \hat{u} \in W_E(S;Y)$ solve

$$\begin{aligned} (\mathcal{E}u_\alpha)' + \mathcal{M}_\alpha u_\alpha &= f_\alpha, \quad (\mathcal{K}u_\alpha)(t_0) = w, \\ (\mathcal{E}\hat{u}_\alpha)' + \mathcal{M}_\alpha \hat{u}_\alpha &= \hat{f}_\alpha, \quad (\mathcal{K}\hat{u}_\alpha)(t_0) = \hat{w}. \end{aligned}$$

As in the proof of Theorem 2.2 we obtain

$$\begin{aligned} M_\alpha \|u_\alpha - \hat{u}_\alpha\|^2_{L^2(S;Y)} &\le \|w - \hat{w}\|^2_H + \tfrac{1}{M_\alpha} \|f_\alpha - \hat{f}_\alpha\|^2_{L^2(S;Y^*)}, \\ \|(\mathcal{E}u_\alpha)' - (\mathcal{E}\hat{u}_\alpha)'\|^2_{L^2(S;Y^*)} &\le 2\|f_\alpha - \hat{f}_\alpha\|^2_{L^2(S;Y^*)} + 2L_\alpha^2 \|u_\alpha - \hat{u}_\alpha\|^2_{L^2(S;Y)}, \end{aligned}$$

where $M_\alpha > 0$, $L_\alpha > 0$ are monotonicity and Lipschitz constants of $\mathcal{M}_\alpha$, respectively. To get the desired estimates for $u - \hat{u} \in W_E(S;Y)$ in terms of $f - \hat{f} \in L^2(S;Y^*)$ and $w - \hat{w} \in H$, we start with

$$\begin{aligned} \|u - \hat{u}\|_{L^2(S;Y)} &\le e_{-\alpha}(t_1)\|u_\alpha - \hat{u}_\alpha\|_{L^2(S;Y)}, \\ \|f_\alpha - \hat{f}_\alpha\|_{L^2(S;Y^*)} &\le \|f - \hat{f}\|_{L^2(S;Y^*)}. \end{aligned}$$

Due to $(\mathcal{E}u)' = e_{-\alpha}(\mathcal{E}u_\alpha)' + \alpha e_{-\alpha}\mathcal{E}u_\alpha$ we see that

$$\|(\mathcal{E}u)' - (\mathcal{E}\hat{u})'\|^2_{L^2(S;Y^*)} \le 2e_{-2\alpha}(t_1)\|(\mathcal{E}u_\alpha)' - (\mathcal{E}\hat{u}_\alpha)'\|^2_{L^2(S;Y^*)}$$

$$+ 2\alpha^2 e_{-2\alpha}(t_1)\|E\|^2_{\mathcal{L}(Y;Y^*)}\|u_\alpha - \hat{u}_\alpha\|^2_{L^2(S;Y)},$$

Summing up, we arrive at the Lipschitz continuity of the assignment $(f,w) \mapsto u$ from $L^2(S;Y^*) \times H$ into $W_E(S;Y)$. □

3. Morrey and Campanato spaces

We collect classical results concerning Morrey and Campanato spaces with regard to the parabolic metric. Based on these, later on we introduce new classes of Sobolev–Morrey spaces adequate for the treatment of the regularity problem formulated in the introduction.

Let us introduce some notation. Throughout this section we assume S to be a bounded open interval in $\mathbb{R}$. For $r > 0$ we define the set of subintervals $\mathcal{S}_r = \{S \cap (t - r^2, t) : t \in S\}$. The symbol $|\ |$ is used for both the absolute value and the maximum norm in $\mathbb{R}^n$, whereas $\|\ \|$ denotes the Euclidean norm in $\mathbb{R}^n$. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we write $\hat{x} = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. We introduce the notation $Q_r(x) = \{\xi \in \mathbb{R}^n : |\xi - x| < r\}$ and $Q_r^-(x) = \{\xi \in Q_r(x) : \xi_n - x_n < 0\}$ for the open cube and the open half-cube with center $x \in \mathbb{R}^n$ and radius $r > 0$, respectively. In the case $x = 0$ we write Q_r and Q_r^- for short. If, additionally, $r = 1$, then we use Q and Q^-.

For subsets G of $\mathbb{R}^n$ we write G°, $\overline{G}$ and ∂G for the topological interior, the closure, and the boundary of G, respectively. For $r > 0$ and subsets $G \subset \mathbb{R}^n$ we use the corresponding calligraphic letter to denote by $\mathcal{G}_r$ the set $\mathcal{G}_r = \{G \cap Q_r(x) : x \in G\}$ of intersections. To introduce the function spaces we are interested in we need the following definition:

Definition 3.1 (Integral mean value)**.** Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space and $w : F \to \mathbb{R}$ be an integrable function given on the measurable set $F \in \mathfrak{A}$ of finite positive measure. We define the integral mean value of w over F by

$$\fint_F w\, d\mu = \frac{1}{\mu(F)} \int_F w\, d\mu.$$

Remark 3.1 (Minimal property)**.** If $w : F \to \mathbb{R}$ is square-integrable on the set $F \in \mathfrak{A}$ of finite positive measure with respect to $(\Omega, \mathfrak{A}, \mu)$, then we have

$$\min_{c \in \mathbb{R}} \int_F |w - c|^2\, d\mu = \int_F \left| w - \fint_F w\, d\mu \right|^2 d\mu.$$

The case of open sets. We define Morrey and Campanato spaces for bounded open sets $U \subset \mathbb{R}^n$; see Campanato [2] and Da Prato [3]:

Definition 3.2 (Morrey spaces)**.** 1. For $\omega \in [0, n+2]$ we introduce the Morrey space $L_2^\omega(S; L^2(U))$ as the set of all $u \in L^2(S; L^2(U))$ such that

$$[u]^2_{L_2^\omega(S;L^2(U))} = \sup_{\substack{(I,V)\in \mathcal{S}_r \times \mathcal{U}_r \\ r>0}} r^{-\omega} \int_I \int_V |u(s)|^2 \, d\lambda^n \, ds$$

remains finite. The norm of $u \in L_2^\omega(S; L^2(U))$ is defined by

$$\|u\|^2_{L_2^\omega(S;L^2(U))} = \|u\|^2_{L^2(S;L^2(U))} + [u]^2_{L_2^\omega(S;L^2(U))}.$$

2. For $\sigma \in [0, n+4]$ we denote by $\mathfrak{L}_2^\sigma(S; L^2(U))$ the Campanato space of all $u \in L^2(S; L^2(U))$ such that

$$[u]^2_{\mathfrak{L}_2^\sigma(S;L^2(U))} = \sup_{\substack{(I,V)\in \mathcal{S}_r \times \mathcal{U}_r \\ r>0}} r^{-\sigma} \int_I \int_V \left| u(s) - \fint_I \fint_V u(\tau) \, d\lambda^n \, d\tau \right|^2 d\lambda^n \, ds$$

has a finite value, and we define the norm of $u \in \mathfrak{L}_2^\sigma(S; L^2(U))$ by

$$\|u\|^2_{\mathfrak{L}_2^\sigma(S;L^2(U))} = \|u\|^2_{L^2(S;L^2(U))} + [u]^2_{\mathfrak{L}_2^\sigma(S;L^2(U))}.$$

For $\omega \le 0$ we set $L_2^\omega(S; L^2(U)) = \mathfrak{L}_2^\omega(S; L^2(U)) = L^2(S; L^2(U))$.

3. Let $H_0^1(U) \subset X \subset H^1(U^\circ)$ be some closed subspace equipped with the usual scalar product of $H^1(U^\circ)$. For $\omega \in [0, n+2]$ we introduce the Sobolev–Morrey space

$$L_2^\omega(S; X) = \big\{ u \in L^2(S; X) : u \in L_2^\omega(S; L^2(U)),\ \|\nabla u\| \in L_2^\omega(S; L^2(U)) \big\},$$

and we define the norm of $u \in L_2^\omega(S; X)$ by

$$\|u\|^2_{L_2^\omega(S;X)} = \|u\|^2_{L_2^\omega(S;L^2(U))} + \| \|\nabla u\| \|^2_{L_2^\omega(S;L^2(U))}.$$

For $\omega \le 0$ we set $L_2^\omega(S; X) = L^2(S; X)$.

Remark 3.2. Note that the spaces $L_2^\omega(S; L^2(U))$ and $\mathfrak{L}_2^\sigma(S; L^2(U))$ are usually denoted by $L^{2,\omega}(S \times U)$ and $\mathfrak{L}^{2,\sigma}(S \times U)$, respectively. Apart from these, later on we introduce further Morrey-type function spaces. Hence, we have decided to use a different but integrated naming scheme. Let us collect some well-known properties:

1. The function spaces introduced above are Banach spaces.

2. If we take the suprema over $0 < r \le r_0$ only, then the corresponding r_0-depending norms are equivalent to the original norms, respectively.

3. For $\omega \in [0, n+2]$ the set $L^\infty(S \times U)$ is a space of multipliers for $L_2^\omega(S; L^2(U))$, and $C^{0,1}(\overline{S} \times \overline{U})$ is a space of multipliers for $L_2^\omega(S; H^1(U))$.

Definition 3.3 (Restriction)**.** Let $I \subset \mathbb{R}$ be an open subinterval of S and $V \subset \mathbb{R}^n$ be an open subset of U. We define $R_V v \in L^2(V)$ by $(R_V v)(x) = v(x)$ for all $v \in L^2(U)$ and $x \in V$. We carry over this definition to $\mathcal{R}_{I,V} u \in L^2(I; L^2(V))$ by $(\mathcal{R}_{I,V} u)(s) = R_V u(s)$ for all $u \in L^2(S; L^2(U))$ and $s \in I$.

As an obvious consequence of the above definitions and the minimal property of the integral mean value we get the following result:

Lemma 3.1 (Restriction)**.** 1. *For $\omega \in [0, n+2]$ the assignment $u \to \mathcal{R}_{I,V} u$ is a bounded linear map from $L_2^\omega(S; L^2(U))$ into $L_2^\omega(I; L^2(V))$ as well as from $L_2^\omega(S; H^1(U))$ into $L_2^\omega(I; H^1(V))$.*

2. *For $\sigma \in [0, n+4]$ the linear assignment $u \to \mathcal{R}_{I,V} u$ maps $\mathfrak{L}_2^\sigma(S; L^2(U))$ continuously into $\mathfrak{L}_2^\sigma(I; L^2(V))$.*

Remark 3.3 (Zero extension)**.** 1. Let $V \subset \mathbb{R}^n$ be an open subset of U and $\omega \in [0, n+2]$. Then, by Definition 3.2 the zero extension is a bounded linear map from $L_2^\omega(S; L^2(V))$ into $L_2^\omega(S; L^2(U))$.

2. Let $V \subset \mathbb{R}^n$ be an open set with $U \cap V \neq \varnothing$, take $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\chi) \subset V$, and fix $\delta > 0$ such that $Q_\delta(x) \subset V$ for all $x \in \operatorname{supp}(\chi)$. Hence, if $Q_r(x) \cap \operatorname{supp}(\chi) \neq \varnothing$ for some $0 < r \leq \frac{\delta}{2}$ and $x \in U$, then $Q_r(x) \subset V$.

Assume that $v \in L^2(S; L^2(U \cap V))$ satisfies $\chi v \in \mathfrak{L}_2^\sigma(S; L^2(U \cap V))$ for some $\sigma \in [0, n+4]$. For the zero extension $u \in L^2(S; L^2(U))$ of χv we get

$$\begin{aligned}\int_S \int_{U \cap Q_r(x)} &\left| u(s) - \fint_S \fint_{U \cap Q_r(x)} u(\tau)\, d\lambda^n\, d\tau \right|^2 d\lambda^n\, ds \\ &= \int_S \int_{U \cap V \cap Q_r(x)} \left| \chi v(s) - \fint_S \fint_{U \cap V \cap Q_r(x)} \chi v(\tau)\, d\lambda^n\, d\tau \right|^2 d\lambda^n\, ds\end{aligned}$$

provided that $0 < r \leq \frac{\delta}{2}$ and $x \in U$. Consequently, using Remark 3.2 we obtain $u \in \mathfrak{L}_2^\sigma(S; L^2(U))$, and we find some $c = c(\chi, \delta, V, U) > 0$ such that

$$\|u\|_{\mathfrak{L}_2^\sigma(S; L^2(U))} \leq c\, \|\chi v\|_{\mathfrak{L}_2^\sigma(S; L^2(U \cap V))}.$$

Remark 3.4. 1. Using Hölder's inequality we obtain the continuous embedding of the usual Lebesgue space $L^q(S; L^p(U))$ into $L_2^\omega(S; L^2(U))$ for p, $q \geq 2$ satisfying $\omega = n(1 - 2/p) + 2(1 - 2/q) \in [0, n+2]$.

2. For $\omega \in [0, n]$, the Banach space of all functions $u \in L^2(S; L^2(U))$ such that the norm of u, defined by

$$\|u\|_\omega^2 = \int_S \int_U |u(s)|^2\, d\lambda^n\, ds + \operatorname*{ess\,sup}_{s \in S} \sup_{\substack{V \in \mathfrak{U}_r \\ r > 0}} r^{-\omega} \int_V |u(s)|^2\, d\lambda^n,$$

remains finite, is continuously embedded into $L_2^{\omega+2}(S; L^2(U))$.

Definition 3.4 (Lipschitz transformation)**.** 1. A bijective map T between two subsets of $\mathbb{R}^n$ such that T and T^{-1} are Lipschitz continuous is called Lipschitz transformation.

2. Let T be a Lipschitz transformation from an open set $U \subset \mathbb{R}^n$ onto $U^* \subset \mathbb{R}^n$. We define $T_* u = u \circ T \in L^2(U)$ for $u \in L^2(U^*)$ and carry over this definition to $\mathcal{T}_* u \in L^2(S; L^2(U))$ by $(\mathcal{T}_* u)(s) = T_* u(s)$ for $u \in L^2(S; L^2(U^*))$ and $s \in S$.

Lemma 3.2 (Transformation)**.** 1. *For $\omega \in [0, n+2]$ the assignment $u \mapsto \mathcal{T}_* u$ is a linear isomorphism from $L_2^\omega(S; L^2(U^*))$ onto $L_2^\omega(S; L^2(U))$ as well as between $L_2^\omega(S; H^1(U^*))$ and $L_2^\omega(S; H^1(U))$.*

2. *For $\sigma \in [0, n+4]$ the assignment $u \mapsto \mathcal{T}_* u$ is a linear isomorphism from $\mathfrak{L}_2^\sigma(S; L^2(U^*))$ onto $\mathfrak{L}_2^\sigma(S; L^2(U))$.*

Proof. Let $L \geq 1$ be a Lipschitz constant of T and set $\delta = Lr$. For all $r > 0$, $t \in S$, $x \in U$ we consider $S_r = S \cap (t - r^2, t)$, $S_\delta = S \cap (t - \delta^2, t)$, $U_r = U \cap Q_r(x)$, $U_\delta^* = U^* \cap Q_\delta(T(x))$.

1. Due to the change-of-variable formula $\mathcal{T}_*$ is a bounded linear operator from $L^2(S; L^2(U^*))$ into $L^2(S; L^2(U))$. For all $r > 0$, $t \in S$, $x \in U$, and $u \in L^2(S; L^2(U^*))$ the inclusion $T[U_r] \subset U_\delta^*$ leads to

$$\int_{S_r} \int_{U_r} |T_* u(s)|^2 \, d\lambda^n \, ds \leq L^n \int_{S_\delta} \int_{U_\delta^*} |u(s)|^2 \, d\lambda^n \, ds,$$

which yields some constant $c_1 = c_1(n, L) > 0$ such that

$$\|\mathcal{T}_* u\|^2_{L_2^\omega(S; L^2(U))} \leq c_1 \|u\|^2_{L_2^\omega(S; L^2(U^*))} \quad \text{for all } u \in L_2^\omega(S; L^2(U^*)).$$

2. Applying both the chain rule and the change-of-variable formula we obtain that $\mathcal{T}_*$ maps $L^2(S; H^1(U^*))$ continuously into $L^2(S; H^1(U))$: For all $r > 0$, $t \in S$, $x \in U$, and $u \in L^2(S; H^1(U^*))$ we have

$$\begin{aligned} \int_{S_r} \int_{U_r} |\nabla T_* u(s)|^2 \, d\lambda^n \, ds &\leq \int_{S_r} \int_{U_r} \|DT\|^2 \|T_* \nabla u(s)\|^2 \, d\lambda^n \, ds \\ &\leq L^{n+2} \int_{S_\delta} \int_{U_\delta^*} \|\nabla u(s)\|^2 \, d\lambda^n \, ds. \end{aligned}$$

In view of Step 1 we find some constant $c_2 = c_2(n, L) > 0$ such that

$$\|\mathcal{T}_* u\|^2_{L_2^\omega(S; H^1(U))} \leq c_2 \|u\|^2_{L_2^\omega(S; H^1(U^*))} \quad \text{for all } u \in L_2^\omega(S; H^1(U^*)).$$

3. Using the change-of-variable formula for all $r > 0$, $t \in S$, $x \in U$, and $u \in L^2(S; L^2(U^*))$ we get

$$\int_{S_r}\int_{U_r}\left|T_* u(s) - \fint_{S_\delta}\fint_{U_\delta^*} u(\tau)\,d\lambda^n\,d\tau\right|^2 d\lambda^n\,ds$$
$$\leq L^n \int_{S_\delta}\int_{U_\delta^*}\left|u(s) - \fint_{S_\delta}\fint_{U_\delta^*} u(\tau)\,d\lambda^n\,d\tau\right|^2 d\lambda^n\,ds.$$

Applying the minimal property of the integral mean value, we find some constant $c_3 = c_3(n, L) > 0$ such that

$$\|\mathcal{T}_* u\|^2_{\mathfrak{L}_2^\sigma(S;L^2(U))} \leq c_3 \|u\|^2_{\mathfrak{L}_2^\sigma(S;L^2(U^*))} \quad \text{for all } u \in \mathfrak{L}_2^\sigma(S; L^2(U^*)).$$

Analogously, we prove the statements for the inverse transformation. □

Definition 3.5 (Reflection)**.** Let the map $N : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Nx = (\hat{x}, -x_n)$ for $x = (\hat{x}, x_n) \in \mathbb{R}^n$.

1. We introduce reflection $R^+u \in L^2(Q)$ and antireflection $R^-u \in L^2(Q)$ of $u \in L^2(Q^-)$ by

$$(R^+u)(x) = \begin{cases} u(x) & \text{if } x \in Q^-, \\ u(Nx) & \text{otherwise,} \end{cases} \quad (R^-u)(x) = \begin{cases} u(x) & \text{if } x \in Q^-, \\ -u(Nx) & \text{otherwise,} \end{cases}$$

and define $\mathcal{R}^+u$, $\mathcal{R}^-u \in L^2(S; L^2(Q))$ for $u \in L^2(S; L^2(Q^-))$ by

$$(\mathcal{R}^+u)(s) = R^+u(s), \quad (\mathcal{R}^-u)(s) = R^-u(s) \quad \text{for } s \in S.$$

2. For vector-valued functions $g \in L^2(Q^-; \mathbb{R}^n)$ we define reflection $R^+g \in L^2(Q; \mathbb{R}^n)$ and antireflection $R^-g \in L^2(Q; \mathbb{R}^n)$ by

$$(R^+g)(x) = \begin{cases} g(x) & \text{if } x \in Q^-, \\ Ng(Nx) & \text{otherwise,} \end{cases} \quad (R^-g)(x) = \begin{cases} g(x) & \text{if } x \in Q^-, \\ -Ng(Nx) & \text{otherwise.} \end{cases}$$

We carry over the definitions to $g \in L^2(S; L^2(Q^-; \mathbb{R}^n))$ by

$$(\mathcal{R}^+g)(s) = R^+g(s), \quad (\mathcal{R}^-g)(s) = R^-g(s) \quad \text{for } s \in S.$$

3. Let $\mathbb{S}^n$ be the set of real symmetric $(n \times n)$-matrices. For matrix-valued functions $A \in L^\infty(Q^-; \mathbb{S}^n)$ we define the reflection $R^+A \in L^\infty(Q; \mathbb{S}^n)$ by

$$(R^+A)(x) = \begin{cases} A(x) & \text{if } x \in Q^-, \\ NA(Nx)N & \text{otherwise.} \end{cases}$$

For $A \in L^\infty(S \times Q^-; \mathbb{S}^n)$ we set $(\mathcal{R}^+A)(s, y) = R^+A(s, y)$ for $(s, y) \in S \times Q^-$.

Lemma 3.3 (Reflection)**.** *For $\sigma \in [0, n+4]$ and $\omega \in [0, n+2]$ the map $\mathcal{R}^+ : \mathfrak{L}_2^\sigma(S; L^2(Q^-)) \to \mathfrak{L}_2^\sigma(S; L^2(Q))$ as well as $\mathcal{R}^-$, $\mathcal{R}^+ : L_2^\omega(S; L^2(Q^-)) \to L_2^\omega(S; L^2(Q))$ are bounded linear operators, and we have*

$$\|\mathcal{R}^+u\|_{\mathfrak{L}_2^\sigma(S;L^2(Q))} \leq \sqrt{2}\,\|u\|_{\mathfrak{L}_2^\sigma(S;L^2(Q^-))} \quad \textit{for all } u \in \mathfrak{L}_2^\sigma(S; L^2(Q^-)),$$

$$\|\mathcal{R}^- u\|_{L_2^\omega(S;L^2(Q))} \le \sqrt{2}\,\|u\|_{L_2^\omega(S;L^2(Q^-))} \quad \textit{for all } u \in L_2^\omega(S;L^2(Q^-)).$$

Proof. Let $P : Q \to Q$ be defined by $Px = (\hat{x}, -|x_n|)$ for $x = (\hat{x}, x_n) \in Q$.

1. Obviously, the map $\mathcal{R}^+ : L^2(S;L^2(Q^-)) \to L^2(S;L^2(Q))$ is continuous. By construction, for all $r > 0$, $x \in Q$, $I \in \mathcal{S}_r$, and $u \in L^2(I;L^2(Q^-))$ we get

$$\int_I \int_{Q\cap Q_r(x)} \left| R^+ u(s) - \fint_I \fint_{Q^-\cap Q_r(Px)} u(\tau)\, d\lambda^n\, d\tau \right|^2 d\lambda^n\, ds$$
$$\le 2 \int_I \int_{Q^-\cap Q_r(Px)} \left| u(s) - \fint_I \fint_{Q^-\cap Q_r(Px)} u(\tau)\, d\lambda^n\, d\tau \right|^2 d\lambda^n\, ds.$$

Hence, the minimal property of the integral mean value yields

$$\|\mathcal{R}^+ u\|^2_{\mathfrak{L}_2^\sigma(S;L^2(Q))} \le 2\,\|u\|^2_{\mathfrak{L}_2^\sigma(S;L^2(Q^-))} \quad \text{for all } u \in \mathfrak{L}_2^\sigma(S;L^2(Q^-)).$$

2. The map $\mathcal{R}^- : L^2(S;L^2(Q^-)) \to L^2(S;L^2(Q))$ is continuous. Due to the definition for all $r > 0$, $x \in Q$, $I \in \mathcal{S}_r$, and $u \in L^2(S;L^2(Q^-))$ we obtain

$$\int_I \int_{Q\cap Q_r(x)} |R^- u(s)|^2\, d\lambda^n\, ds \le 2 \int_I \int_{Q^-\cap Q_r(Px)} |u(s)|^2\, d\lambda^n\, ds.$$

This leads to the estimates

$$\|\mathcal{R}^- u\|^2_{L_2^\omega(S;L^2(Q))} \le 2\,\|u\|^2_{L_2^\omega(S;L^2(Q^-))} \quad \text{for all } u \in L_2^\omega(S;L^2(Q^-)),$$
$$\|\mathcal{R}^+ u\|^2_{L_2^\omega(S;L^2(Q))} \le 2\,\|u\|^2_{L_2^\omega(S;L^2(Q^-))} \quad \text{for all } u \in L_2^\omega(S;L^2(Q^-)),$$

where the second one follows analogously. □

For the following classical results concerning Morrey and Campanato spaces we suppose some regularity property of the boundary ∂U; see again Campanato [2] and Da Prato [3]:

Theorem 3.4 (Equivalence)**.** *Let $U \subset \mathbb{R}^n$ be an open set without outward cusps; that means there exist constants $r_0 > 0$ and $c_0 > 0$ such that*

$$\lambda^n(V) \ge c_0 r^n \quad \textit{for all } 0 < r \le r_0,\ V \in \mathcal{U}_r.$$

Then the following holds true:

1. *For $\omega \in [0, n+2)$ the Morrey space $L_2^\omega(S;L^2(U))$ is isomorphic to the Campanato space $\mathfrak{L}_2^\omega(S;L^2(U))$.*

2. *For $\sigma \in (n+2, n+4]$, $\alpha = (\sigma - n - 2)/2$ the Campanato space $\mathfrak{L}_2^\sigma(S;L^2(U))$ is isomorphic to the space $C(\overline{S};C^{0,\alpha}(\overline{U})) \cap C^{0,\alpha/2}(\overline{S};C(\overline{U}))$ of Hölder-continuous functions.*

The case of hypersurfaces. Analogously, we define functions spaces on Lipschitz hypersurfaces in $\mathbb{R}^n$. To do so, for $x \in \mathbb{R}^n$ and $r > 0$ we introduce the $(n-1)$-dimensional equatorial plate

$$\Sigma_r(x) = \big\{\xi \in \mathbb{R}^n : |\xi - x| < r,\, \xi_n = x_n\big\}$$

of the cube $Q_r(x)$. In the case $x = 0$ we write Σ_r for short. If, additionally, $r = 1$, we use the notation Σ.

Definition 3.6 (Lipschitz hypersurface)**.** A subset M of $\mathbb{R}^n$ is called Lipschitz hypersurface in $\mathbb{R}^n$ if for each point $x \in M$ there exist an open neighborhood U of x and a Lipschitz transformation T from U onto Q such that $T[U \cap M] = \Sigma$ and $T(x) = 0$.

Let M be a compact Lipschitz hypersurface in $\mathbb{R}^n$. By λ_M we denote the $(n-1)$-dimensional Lebesgue measure on the σ-algebra $\mathfrak{L}_M$ of Lebesgue-measurable subsets of M; see Evans, Gariepy [5], Simon [29]. In Griepentrog [9] we have carried over both the definition and classical properties of Morrey and Campanato spaces to the case of relatively open subsets F of M; see also Geisler [7]:

Definition 3.7 (Morrey spaces)**.** 1. For $\omega \in [0, n+1]$ we introduce the Morrey space $L_2^\omega(S; L^2(F))$ as the set of all $u \in L^2(S; L^2(F))$ such that

$$[u]^2_{L_2^\omega(S;L^2(F))} = \sup_{\substack{(I,\Gamma)\in\mathcal{S}_r\times\mathcal{F}_r \\ r>0}} r^{-\omega} \int_I \int_\Gamma |u(s)|^2 \, d\lambda_M \, ds$$

remains finite, and we define the norm of $u \in L_2^\omega(S; L^2(F))$ by

$$\|u\|^2_{L_2^\omega(S;L^2(F))} = \|u\|^2_{L^2(S;L^2(F))} + [u]^2_{L_2^\omega(S;L^2(F))}.$$

2. For $\sigma \in [0, n+3]$ we denote by $\mathfrak{L}_2^\sigma(S; L^2(F))$ the Campanato space of all $u \in L^2(S; L^2(F))$ such that

$$[u]^2_{\mathfrak{L}_2^\sigma(S;L^2(F))} = \sup_{\substack{(I,\Gamma)\in\mathcal{S}_r\times\mathcal{F}_r \\ r>0}} r^{-\sigma} \int_I \int_\Gamma \left| u(s) - \fint_I \fint_\Gamma u(\tau) \, d\lambda_M \right|^2 d\lambda_M \, ds$$

has a finite value; we define the norm of $u \in \mathfrak{L}_2^\sigma(S; L^2(F))$ by

$$\|u\|^2_{\mathfrak{L}_2^\sigma(S;L^2(F))} = \|u\|^2_{L^2(S;L^2(F))} + [u]^2_{\mathfrak{L}_2^\sigma(S;L^2(F))}.$$

For $\omega \le 0$ we set $L_2^\omega(S; L^2(F)) = \mathfrak{L}_2^\omega(S; L^2(F)) = L^2(S; L^2(F))$.

Remark 3.5. We collect some facts concerning the above function spaces:

1. The spaces introduced in Definition 3.7 are Banach spaces.

2. If we take the suprema over $0 < r \le r_0$ only, then the associated r_0-depending norms are equivalent to the original norms, respectively.

3. For $\omega \in [0, n+1]$ the set $L^\infty(S \times F)$ is a space of multipliers for $L_2^\omega(S; L^2(F))$.

Remark 3.6 (Zero extension)**.** 1. Let $\Gamma \subset M$ be a relatively open subset of F and $\omega \in [0, n+1]$. By Definition 3.7 the zero extension is a bounded linear map from $L_2^\omega(S; L^2(\Gamma))$ into $L_2^\omega(S; L^2(F))$.

2. Let $V \subset \mathbb{R}^n$ be an open set with $F \cap V \neq \varnothing$, consider $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\chi) \subset V$, and choose $\delta > 0$ such that $Q_\delta(x) \subset V$ for all $x \in \operatorname{supp}(\chi)$. If $Q_r(x) \cap \operatorname{supp}(\chi) \neq \varnothing$ for some $0 < r \le \frac{\delta}{2}$ and $x \in F$, then $Q_r(x) \subset V$.

Suppose, that $v \in L^2(S; L^2(F \cap V))$ satisfies $\chi v \in \mathfrak{L}_2^\sigma(S; L^2(F \cap V))$ for some $\sigma \in [0, n+3]$. For the zero extension $u \in L^2(S; L^2(F))$ of χv we obtain

$$\int_S \int_{F \cap Q_r(x)} \left| u(s) - \fint_S \fint_{F \cap Q_r(x)} u(\tau)\, d\lambda_M\, d\tau \right|^2 d\lambda_M\, ds$$
$$= \int_S \int_{F \cap V \cap Q_r(x)} \left| \chi v(s) - \fint_S \fint_{F \cap V \cap Q_r(x)} \chi v(\tau)\, d\lambda_M\, d\tau \right|^2 d\lambda_M\, ds$$

whenever $0 < r \le \frac{\delta}{2}$ and $x \in F$. Thus, Remark 3.5 yields $u \in \mathfrak{L}_2^\sigma(S; L^2(F))$, and we find some $c = c(\chi, \delta, V, F) > 0$ such that

$$\|u\|_{\mathfrak{L}_2^\sigma(S; L^2(F))} \le c\, \|\chi v\|_{\mathfrak{L}_2^\sigma(S; L^2(F \cap V))}.$$

Remark 3.7. 1. Applying Hölder's inequality we get the continuous embedding of the usual Lebesgue space $L^q(S; L^p(F))$ into $L_2^\omega(S; L^2(F))$ for p, $q \ge 2$ satisfying $\omega = (n-1)(1 - 2/p) + 2(1 - 2/q) \in [0, n+1]$.

2. For $\omega \in [0, n-1]$, the Banach space of all functions $u \in L^2(S; L^2(F))$ such that the norm of u, defined by

$$\|u\|_\omega^2 = \int_S \int_F |u(s)|^2\, d\lambda_M\, ds + \operatorname*{ess\,sup}_{s \in S} \sup_{\substack{\Gamma \in \mathcal{F}_r \\ r > 0}} r^{-\omega} \int_\Gamma |u(s)|^2\, d\lambda_M,$$

remains finite, is continuously embedded into $L_2^{\omega+2}(S; L^2(F))$.

Definition 3.8 (Lipschitz transformation)**.** Let M and M^* be two compact Lipschitz hypersurfaces in $\mathbb{R}^n$, and T be a Lipschitz transformation from the relatively open subset F of M onto a subset F^* of M^*. We define $T_* u = u \circ T \in L^2(F)$ for $u \in L^2(F^*)$ and carry over this definition to $\mathcal{T}_* u \in L^2(S; L^2(F))$ by $(\mathcal{T}_* u)(s) = T_* u(s)$ for $u \in L^2(S; L^2(F^*))$ and $s \in S$.

Lemma 3.5 (Transformation)**.** 1. *For $\omega \in [0, n+1]$ the assignment $u \mapsto \mathcal{T}_* u$ is a linear isomorphism from $L_2^\omega(S; L^2(F^*))$ onto $L_2^\omega(S; L^2(F))$.*

2. *For* $\sigma \in [0, n+3]$ *the assignment* $u \mapsto \mathcal{T}_* u$ *is a linear isomorphism from* $\mathfrak{L}_2^\sigma(S; L^2(F^*))$ *onto* $\mathfrak{L}_2^\sigma(S; L^2(F))$.

Proof. Let $L \geq 1$ be a Lipschitz constant of T and set $\delta = Lr$. For $r > 0$, $t \in S$, $x \in F$ we define

$$S_r = S \cap (t - r^2, t), \quad S_\delta = S \cap (t - \delta^2, t),$$
$$F_r = F \cap Q_r(x), \quad F_\delta^* = F^* \cap Q_\delta(T(x)).$$

1. In view of the change-of-variable formula $\mathcal{T}_*$ is a bounded linear map from $L^2(S; L^2(F^*))$ into $L^2(S; L^2(F))$. For all $r > 0$, $t \in S$, and $x \in F$ the relation $T[F_r] \subset F_\delta^*$ yields

$$\int_{S_r} \int_{F_r} |T_* u(s)|^2 \, d\lambda_M \, ds \leq c_1 \int_{S_\delta} \int_{F_\delta^*} |u(s)|^2 \, d\lambda_{M^*} \, ds$$

for all $u \in L^2(S; L^2(F^*))$ and some constant $c_1 = c_1(n, L, M, M^*) > 0$. Hence, $\mathcal{T}_*$ maps $L_2^\omega(S; L^2(F^*))$ continuously into $L_2^\omega(S; L^2(F))$.

2. Similarly, for all $r > 0$, $t \in S$, $x \in F$, and $u \in L^2(S; L^2(F^*))$ we get

$$\int_{S_r} \int_{F_r} \left| T_* u(s) - \fint_{S_\delta} \fint_{F_\delta^*} u(\tau) \, d\lambda_{M^*} \, d\tau \right|^2 d\lambda_M \, ds$$
$$\leq c_2 \int_{S_\delta} \int_{F_\delta^*} \left| u(s) - \fint_{S_\delta} \fint_{F_\delta^*} u(\tau) \, d\lambda_{M^*} \, d\tau \right|^2 d\lambda_{M^*} \, ds,$$

where $c_2 = c_2(n, L, M, M^*) > 0$ is a suitable constant. Applying the minimal property of the integral mean value, we obtain the continuity of the map $\mathcal{T}_*$ from $\mathfrak{L}_2^\sigma(S; L^2(F^*))$ into $\mathfrak{L}_2^\sigma(S; L^2(F))$. Analogously, we prove the statements for the inverse transformation. $\square$

In order to get properties of Morrey and Campanato spaces analogous to Theorem 3.4 we suppose the relatively open subset F of M to have no outward cusps; that means we find constants $r_0 > 0$ and $c_0 > 0$ such that

$$\lambda_M(\Gamma) \geq c_0 r^{n-1} \quad \text{for all } 0 < r \leq r_0, \, \Gamma \in \mathcal{F}_r.$$

Theorem 3.6 (Equivalence)**.** *For relatively open subsets F of M without outward cusps the following holds true:*

1. *For* $\omega \in [0, n+1)$ *the Morrey space* $L_2^\omega(S; L^2(F))$ *is isomorphic to the Campanato space* $\mathfrak{L}_2^\omega(S; L^2(F))$.

2. *For* $\sigma \in (n+1, n+3]$, $\alpha = (\sigma - n - 1)/2$ *the Campanato space* $\mathfrak{L}_2^\sigma(S; L^2(F))$ *is isomorphic to the space* $C(\overline{S}; C^{0,\alpha}(\overline{F})) \cap C^{0,\alpha/2}(\overline{S}; C(\overline{F}))$ *of Hölder-continuous functions.*

Sets with Lipschitz boundary. Instead of using graphs of Lipschitz-continuous functions, we prefer a more general definition of sets with Lipschitz boundary; see Giusti [8], Grisvard [14], Gröger [16], and Wloka [32]:

Definition 3.9 (Set with Lipschitz boundary)**.** A bounded subset Ω of $\mathbb{R}^n$ is called set with Lipschitz boundary if for each $x \in \partial\Omega$ there exist an open neighborhood U of x and a Lipschitz transformation T from U onto Q such that $T[U \cap \Omega] = Q^-$ and $T(x) = 0$.

Remark 3.8. Every set with Lipschitz boundary is an open subset of $\mathbb{R}^n$ without outward cusps. Moreover, let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\Upsilon = \mathbb{R}^n \setminus \overline{\Omega}$ be its exterior. Then Ω is a set with Lipschitz boundary if and only if $\partial\Omega$ is a compact Lipschitz hypersurface in $\mathbb{R}^n$ with $\partial\Omega = \partial\Upsilon$.

Remark 3.9. Following Giusti [8] every set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary is an extension domain; that means there exists a linear extension operator which maps $H^1(\Omega)$ continuously into $H^1(\mathbb{R}^n)$. Because $C_0^\infty(\mathbb{R}^n)$ is a dense subset of $H^1(\mathbb{R}^n)$, the set of restrictions $\{u|\Omega : u \in C_0^\infty(\mathbb{R}^n)\}$ is dense in $H^1(\Omega)$, too. Together with the properties of the Lebesgue measure $\lambda_{\partial\Omega}$ this ensures the complete continuity of the trace operator $K_{\partial\Omega}$ from $H^1(\Omega)$ in $L^2(\partial\Omega)$. Due to Mazya [24] we find some constant $c_\Omega > 0$ such that the following multiplicative inequality holds true

$$\|K_{\partial\Omega} v\|^2_{L^2(\partial\Omega)} \le c_\Omega \|v\|_{H^1(\Omega)} \|v\|_{L^2(\Omega)} \quad \text{for all } v \in H^1(\Omega). \tag{3.1}$$

Definition 3.10 (Trace map)**.** Let $\Omega \subset \mathbb{R}^n$ be a set with Lipschitz boundary and F be relatively open in $\partial\Omega$. For the trace map we introduce the notation $K_F \in \mathcal{L}(H^1(\Omega); L^2(F))$, and we define the bounded linear map $\mathcal{K}_{S,F} : L^2(S; H^1(\Omega)) \to L^2(S; L^2(F))$ by $(\mathcal{K}_{S,F} u)(s) = K_F u(s)$ for $u \in L^2(S; H^1(\Omega))$ and $s \in S$.

Remark 3.10. If T is some Lipschitz transformation from an open neighborhood of $\overline{\Omega}$ into $\mathbb{R}^n$, then $\Omega^* = T[\Omega]$ is a set with Lipschitz boundary. Let F be relatively open in $\partial\Omega$ and set $F^* = T[F]$. Following Griepentrog, Rehberg [9, 13], for $T^F = T|F$ we have

$$T_*^F K_{F^*} v = K_F T_* v \quad \text{for all } v \in H^1(\Omega^*).$$

Lemma 3.7 (Transformation)**.** *We have the identity*

$$\mathcal{T}_*^F \mathcal{K}_{S,F^*} u = \mathcal{K}_{S,F} \mathcal{T}_* u \quad \textit{for all } u \in L^2(S; H^1(\Omega^*)).$$

4. Regular sets and associated function spaces

For our investigations on global regularity we use the terminology of regular sets $G \subset \mathbb{R}^n$ introduced by Gröger. Being the natural generalization of sets with Lipschitz boundary it allows the proper functional analytic description of elliptic and parabolic problems with mixed boundary conditions in nonsmooth domains; see Gröger, Rehberg [16, 17, 18] and Griepentrog, Recke [9, 10, 12].

Topological concept. Regular sets $G \subset \mathbb{R}^n$ are to be understood as the union of some set with Lipschitz boundary and some relatively open Neumann part of this boundary. Note that the Dirichlet part of the Lipschitz boundary is defined as the relative exterior of the Neumann part. This concept enables us to reduce the global regularity theory for general regular sets to the case of three elementary half-cubes representing the standard boundary conditions under consideration; see Figure 1.

For $x \in \mathbb{R}^n$ and $r > 0$ we introduce the half-cubes

$$Q_r^-(x) = \bigl\{\xi \in \mathbb{R}^n : |\xi - x| < r,\, \xi_n - x_n < 0\bigr\},$$
$$Q_r^+(x) = \bigl\{\xi \in \mathbb{R}^n : |\xi - x| < r,\, \xi_n - x_n \le 0\bigr\},$$
$$Q_r^\pm(x) = \bigl\{\xi \in Q_r^+(x) : \xi_1 - x_1 > 0 \text{ or } \xi_n - x_n < 0\bigr\}.$$

In the case $x = 0$ we write Q_r^-, Q_r^+, $Q_r^\pm$, respectively, for short. If, additionally, $r = 1$, then we use the notation Q^-, Q^+, $Q^\pm$.

Definition 4.1 (Regular set)**.** A bounded set $G \subset \mathbb{R}^n$ is called regular if for each $x \in \partial G$ we find some open neighborhood U of x in $\mathbb{R}^n$ and a Lipschitz transformation T from U onto Q such that $T[U \cap G] \in \bigl\{Q^-, Q^+, Q^\pm\bigr\}$ and $T(x) = 0$.

We collect some frequently used properties of regular sets; see Griepentrog, Recke [9, 10, 12]:

Lemma 4.1 (Topological properties)**.** 1. *Every set with Lipschitz boundary is a regular set. Conversely, the interior of a regular set is a set with Lipschitz boundary. The closure of a regular set is regular, too.*

2. *For regular sets $G \subset \mathbb{R}^n$ both the Neumann boundary part $\partial_+G = G \cap \partial G$ and the Dirichlet boundary part $\partial_-G = \partial G \setminus \overline{\partial_+G}$ are relatively open subsets of ∂G without outward cusps.*

3. *If $G \subset \mathbb{R}^n$ is a regular set and T is a Lipschitz transformation from an open neighborhood of $\overline{G}$ into $\mathbb{R}^n$, then $T[G]$ is regular, too.*

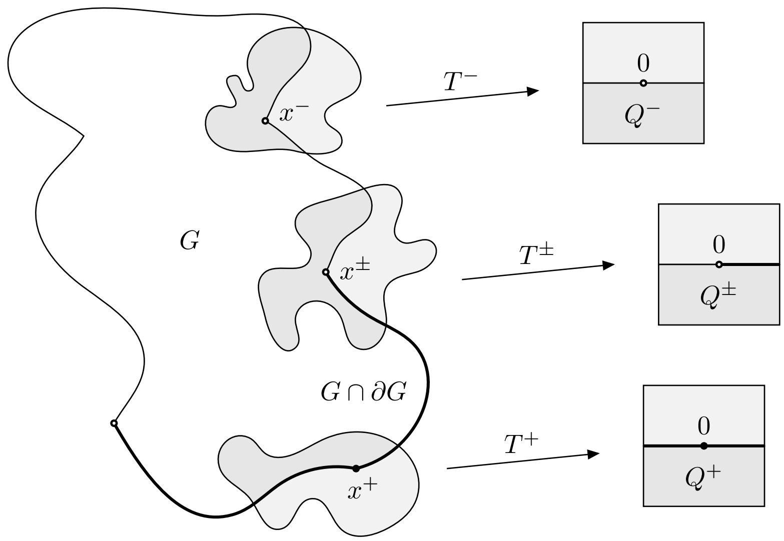

FIGURE 1. Regular set $G \subset \mathbb{R}^n$ with Neumann boundary part $\partial_+ G = G \cap \partial G$ (bold line): Transformation of different boundary regions near the points x^-, $x^\pm$, $x^+ \in \partial G$ to corresponding half-cubes Q^-, $Q^\pm$, Q^+ representing the cases of Dirichlet, Zaremba (or mixed), and Neumann boundary conditions.

Lemma 4.2 (Atlas)**.** *For every regular set $G \subset \mathbb{R}^n$ we find an atlas of charts $(T_1, U_1), \dots, (T_m, U_m)$ with the following properties:*

1. *$U_1, \dots, U_m$ are open neighborhoods of points $x_1, \dots, x_m \in \overline{G}$ in $\mathbb{R}^n$.*
2. *$T_1, \dots, T_m$ are Lipschitz transformations from $U_1, \dots, U_m$ into $\mathbb{R}^n$.*
3. *Introducing the index sets*

$$J_0 = \big\{ i \in \{1, \dots, m\} : x_i \in G^\circ \big\}, \quad J_1 = \big\{ i \in \{1, \dots, m\} : x_i \in \partial G \big\},$$

we have the inclusions

$$\partial G \subset \bigcup_{i \in J_1} U_i, \quad \bigcup_{i \in J_0} \overline{U_i} \subset G^\circ, \quad \overline{G} \subset \bigcup_{i=1}^{m} U_i. \tag{4.1}$$

4. *For all $i \in \{1, \dots, m\}$ the above transformations satisfy*

$$T_i(x_i) = 0, \quad T_i[U_i] = Q, \quad T_i[U_i \cap G] \in \big\{ Q, Q^-, Q^+, Q^\pm \big\}. \tag{4.2}$$

5. *The subfamily $\big\{ (T_i, U_i) : i \in J_1 \big\}$ is an atlas of ∂G.*

Function spaces and invariance principles. We define function spaces associated with relatively open subsets $U \subset \mathbb{R}^n$ of regular sets $G \subset \mathbb{R}^n$. Let $V \subset \mathbb{R}^n$ be relatively open in U, and $I \subset \mathbb{R}$ be an open subinterval of S.

Definition 4.2 (Sobolev space)**.** By $H_0^1(U)$ we denote the closure of

$$C_0^\infty(U) = \left\{u|U^\circ : u \in C_0^\infty(\mathbb{R}^n),\ \mathrm{supp}(u) \cap (\overline{U} \setminus U) = \varnothing\right\}$$

in the space $H^1(U^\circ)$. We write $H^{-1}(U)$ for the dual space of $H_0^1(U)$.

In the following we collect extension, transformation, and reflection principles for Sobolev spaces:

Definition 4.3 (Zero extension)**.** For the zero extension map we introduce the notation $Z_U : H_0^1(V) \to H_0^1(U)$, and we define the operator $\mathcal{Z}_{S,U} : L^2(I; H_0^1(V)) \to L^2(S; H_0^1(U))$ by

$$(\mathcal{Z}_{S,U}u)(s) = \begin{cases} Z_U u(s) & \text{if } s \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } u \in L^2(I; H_0^1(V)).$$

Because the zero extension map Z_U is a linear isometry from $H_0^1(V)$ into $H_0^1(U)$ (see Griepentrog, Rehberg [9, 13]), we get

Lemma 4.3 (Zero extension)**.** *$\mathcal{Z}_{S,U}$ is a linear isometry from $L^2(I; H_0^1(V))$ into $L^2(S; H_0^1(U))$.*

Let T be a Lipschitz transformation from an open neighborhood of $\overline{G}$ into $\mathbb{R}^n$, and set $U^* = T[U]$, $V^* = T[V]$. Then T_* is a linear isomorphism from $H_0^1(U^*)$ onto $H_0^1(U)$, where

$$T_* Z_{U^*} u = Z_U T_* u \quad \text{for all } u \in H_0^1(V^*);$$

see Griepentrog, Rehberg [9, 13]. Hence, Lemma 3.2 leads to

Lemma 4.4 (Transformation)**.** *For $\omega \in [0, n+2]$ the operator $\mathcal{T}_*$ is a linear isomorphism between $L_2^\omega(S; H_0^1(U^*))$ and $L_2^\omega(S; H_0^1(U))$. We have*

$$\mathcal{T}_* \mathcal{Z}_{S,U^*} u = \mathcal{Z}_{S,U} \mathcal{T}_* u \quad \textit{for all } u \in L^2(I; H_0^1(V^*)).$$

Following Giusti [8] and Griepentrog, Rehberg [9, 13], the reflection R^+ is a bounded linear operator from $H_0^1(Q^+)$ into $H_0^1(Q)$ as well as from $H^1(Q^-)$ into $H^1(Q)$. The antireflection R^- maps $H_0^1(Q^-)$ continuously into $H_0^1(Q)$. Due to Definition 3.5 we have

$$\nabla R^+ u = R^+ \nabla u \quad \text{for all } u \in H^1(Q^-),$$
$$\nabla R^- u = R^- \nabla u \quad \text{for all } u \in H_0^1(Q^-).$$

In view of Lemma 3.3 this yields

Lemma 4.5 (Reflection)**.** *For $\omega \in [0, n+2]$ the map $\mathcal{R}^+$ is a bounded linear map from $L_2^\omega(S; H_0^1(Q^+))$ into $L_2^\omega(S; H_0^1(Q))$ as well as from $L_2^\omega(S; H^1(Q^-))$ into $L_2^\omega(S; H^1(Q))$. In addition to that, $\mathcal{R}^-$ is a bounded linear operator from $L_2^\omega(S; H_0^1(Q^-))$ into $L_2^\omega(S; H_0^1(Q))$, and we have*

$$\|\mathcal{R}^+ u\|_{L_2^\omega(S;H^1(Q))} \leq \sqrt{2}\,\|u\|_{L_2^\omega(S;H^1(Q^-))} \quad \textit{for all } u \in L_2^\omega(S; H^1(Q^-)),$$
$$\|\mathcal{R}^- u\|_{L_2^\omega(S;H^1(Q))} \leq \sqrt{2}\,\|u\|_{L_2^\omega(S;H^1(Q^-))} \quad \textit{for all } u \in L_2^\omega(S; H_0^1(Q^-)).$$

Definition 4.4 (Even and odd part)**.** Let the maps $N : \mathbb{R}^n \to \mathbb{R}^n$ and $P : \mathbb{R}^n \to \mathbb{R}^n$ defined by $Nx = (\hat{x}, -x_n)$, $Px = (\hat{x}, -|x_n|)$ for $x = (\hat{x}, x_n) \in \mathbb{R}^n$, and consider the symmetric union

$$Q_r^2(x) = Q_r(x) \cup Q_r(Nx) \quad \text{for } x \in \mathbb{R}^n,\ r > 0.$$

Then, for $x \in Q$, $r > 0$, and $u \in L^2(Q_r^2(x) \cap Q)$ we define the even part $O_r^+(x)u \in L^2(Q_r(Px) \cap Q^-)$ and the odd part $O_r^-(x)u \in L^2(Q_r(Px) \cap Q^-)$ of $2u$ by

$$(O_r^+(x)u)(\xi) = u(\xi) + u(N\xi) \quad \text{for } \xi \in Q_r(Px) \cap Q^-,$$
$$(O_r^-(x)u)(\xi) = u(\xi) - u(N\xi) \quad \text{for } \xi \in Q_r(Px) \cap Q^-.$$

In the case $x = 0$, $r = 1$ we simply write O^+ and O^-.

We carry over the definition to $u \in L^2(I; L^2(Q_r^2(x) \cap Q))$ by setting

$$(\mathcal{O}_r^+(x)u)(s) = O_r^+(x)u(s) \quad \text{for } s \in I,$$
$$(\mathcal{O}_r^-(x)u)(s) = O_r^-(x)u(s) \quad \text{for } s \in I,$$

and we use the notation $\mathcal{O}^+$ and $\mathcal{O}^-$ in the case $x = 0$, $r = 1$.

Following Griepentrog, Rehberg [9, 13], for all $x \in Q$ and $r > 0$ the maps $O_r^+(x) : H_0^1(Q_r^2(x) \cap Q) \to H_0^1(Q_r(Px) \cap Q^+)$ and $O_r^-(x) : H_0^1(Q_r^2(x) \cap Q) \to H_0^1(Q_r(Px) \cap Q^-)$ are bounded linear operators, and

$$O^+ Z_Q u = Z_{Q^+} O_r^+(x)u \quad \text{for all } u \in H_0^1(Q_r^2(x) \cap Q),$$
$$O^- Z_Q u = Z_{Q^-} O_r^-(x)u \quad \text{for all } u \in H_0^1(Q_r^2(x) \cap Q).$$

Consequently, this yields

Lemma 4.6 (Even and odd part)**.** *Let $x \in Q$ and $r > 0$ be given. Then both the operators $\mathcal{O}_r^+(x) : L^2(I; H_0^1(Q_r^2(x) \cap Q)) \to L^2(I; H_0^1(Q_r(Px) \cap Q^+))$ and $\mathcal{O}_r^-(x) : L^2(I; H_0^1(Q_r^2(x) \cap Q)) \to L^2(I; H_0^1(Q_r(Px) \cap Q^-))$ are bounded linear operators, and we have*

$$\mathcal{O}^+ \mathcal{Z}_{S,Q} u = \mathcal{Z}_{S,Q^+} \mathcal{O}_r^+(x)u \quad \textit{for all } u \in L^2(I; H_0^1(Q_r^2(x) \cap Q)),$$
$$\mathcal{O}^- \mathcal{Z}_{S,Q} u = \mathcal{Z}_{S,Q^-} \mathcal{O}_r^-(x)u \quad \textit{for all } u \in L^2(I; H_0^1(Q_r^2(x) \cap Q)).$$

5. Sobolev–Morrey spaces of functionals

Again, we assume that $U \subset \mathbb{R}^n$ is a relatively open subset of the regular set $G \subset \mathbb{R}^n$. Moreover, let $V \subset \mathbb{R}^n$ be relatively open in U, $I \subset \mathbb{R}$ be an open subinterval of S, and $\omega \in [0, n+2]$.

Function spaces and invariance principles. In the same spirit as the well-established Morrey spaces of functions, we construct a new scale of Sobolev–Morrey spaces of functionals as subspaces of $L^2(S; H^{-1}(U))$. We generalize an idea of Rakotoson [25, 26] to the purpose of evolution equations; see Griepentrog [9]:

Definition 5.1 (Localization)**.** 1. We define the localization $f \mapsto L_V f$ from $H^{-1}(U)$ into $H^{-1}(V)$ as the adjoint operator to the zero extension map $Z_U : H_0^1(V) \to H_0^1(U)$; that means

$$\langle L_V f, w \rangle_{H_0^1(V)} = \langle f, Z_U w \rangle_{H_0^1(U)} \quad \text{for } w \in H_0^1(V).$$

2. To localize a functional $f \in L^2(S; H^{-1}(U))$ we define the assignment $f \mapsto \mathcal{L}_{I,V} f$ from $L^2(S; H^{-1}(U))$ into $L^2(I; H^{-1}(V))$ as the adjoint operator to the zero extension map $\mathcal{Z}_{S,U} : L^2(I; H_0^1(V)) \to L^2(S; H_0^1(U))$:

$$\langle \mathcal{L}_{I,V} f, w \rangle_{L^2(I; H_0^1(V))} = \langle f, \mathcal{Z}_{S,U} w \rangle_{L^2(S; H_0^1(U))} \quad \text{for } w \in L^2(I; H_0^1(V)).$$

Remark 5.1. Using the properties of $\mathcal{Z}_{S,U}$ (see Lemma 4.3), we get

$$\|\mathcal{L}_{I,V} f\|_{L^2(I; H^{-1}(V))} \le \|f\|_{L^2(S; H^{-1}(U))} \quad \text{for all } f \in L^2(S; H^{-1}(U)).$$

Definition 5.2 (Sobolev–Morrey space)**.** We define the Sobolev–Morrey space $L_2^\omega(S; H^{-1}(U))$ as the set of all elements $f \in L^2(S; H^{-1}(U))$ for which

$$[f]^2_{L_2^\omega(S; H^{-1}(U))} = \sup_{\substack{(I,V) \in \mathcal{S}_r \times \mathcal{U}_r \\ r>0}} r^{-\omega} \int_I \|L_V f(s)\|^2_{H^{-1}(V)} \, ds$$

has a finite value. We introduce the norm of $f \in L_2^\omega(S; H^{-1}(U))$ by

$$\|f\|^2_{L_2^\omega(S; H^{-1}(U))} = \|f\|^2_{L^2(S; H^{-1}(U))} + [f]^2_{L_2^\omega(S; H^{-1}(U))}.$$

For $\omega \le 0$ we set $L_2^\omega(S; H^{-1}(U)) = L^2(S; H^{-1}(U))$.

Remark 5.2. Note that for fixed $r_0 > 0$ we get an equivalent norm on $L_2^\omega(S; H^{-1}(U))$, if we take the supremum over $0 < r \le r_0$, only.

Lemma 5.1. *The spaces $L_2^\omega(S; H^{-1}(U))$ are Banach spaces.*

Proof. To prove of the completeness of the space $L_2^\omega(S;H^{-1}(U))$ let (f_ℓ) be a Cauchy sequence in $L_2^\omega(S;H^{-1}(U))$. Due to the continuous embedding of $L_2^\omega(S;H^{-1}(U))$ in $L^2(S;H^{-1}(U))$ the sequence (f_ℓ) converges in $L^2(S;H^{-1}(U))$ to some $f \in L^2(S;H^{-1}(U))$. We fix $\delta > 0$ and choose $\ell_0(\delta) \in \mathbb{N}$ such that

$$\|f_{\ell+k} - f_\ell\|_{L_2^\omega(S;H^{-1}(U))} \le \delta \quad \text{for all } \ell,\, k \in \mathbb{N} \text{ with } \ell \ge \ell_0(\delta).$$

For all $r > 0$, $I \in \mathcal{S}_r$, and $V \in \mathcal{U}_r$ we get

$$r^{-\omega}\,\|\mathcal{L}_{I,V}(f - f_\ell)\|^2_{L^2(I;H^{-1}(U))} \le 2r^{-\omega}\,\|\mathcal{L}_{I,V}(f - f_{\ell+k})\|^2_{L^2(I;H^{-1}(V))} + 2\delta^2.$$

Passing to the limit $k \to \infty$ and taking the supremum over all $r > 0$, $I \in \mathcal{S}_r$, and $V \in \mathcal{U}_r$, we arrive at

$$\|f - f_\ell\|^2_{L_2^\omega(S;H^{-1}(U))} \le 2\delta^2 \quad \text{for all } \ell \in \mathbb{N} \text{ with } \ell \ge \ell_0(\delta);$$

in other words, (f_ℓ) converges to f in $L_2^\omega(S;H^{-1}(U))$. □

We show that the above Sobolev–Morrey spaces are invariant with respect to localization, Lipschitz transformations, and reflection.

Definition 5.3 (Multiplication)**.** The product $\chi f \in L^2(S;H^{-1}(U))$ of the function $\chi \in C_0^\infty(\mathbb{R}^n)$ and the functional $f \in L^2(S;H^{-1}(U))$ is defined by

$$\langle \chi f, w\rangle_{L^2(S;H_0^1(U))} = \langle f, \chi w\rangle_{L^2(S;H_0^1(U))} \quad \text{for } w \in L^2(S;H_0^1(U)).$$

Remark 5.3. Obviously, we find some constant $c = c(\chi) > 0$ such that

$$\|\chi f\|_{L^2(S;H^{-1}(U))} \le c\,\|f\|_{L^2(S;H^{-1}(U))} \quad \text{for all } f \in L^2(S;H^{-1}(U)).$$

Lemma 5.2 (Multiplication)**.** *For all $\chi \in C_0^\infty(\mathbb{R}^n)$ the assignment $f \mapsto \chi f$ is a bounded linear map from $L_2^\omega(S;H^{-1}(U))$ into itself.*

Proof. Let $f \in L_2^\omega(S;H^{-1}(U))$ be given. For all $r > 0$, $I \in \mathcal{S}_r$, $V \in \mathcal{U}_r$, and $w \in L^2(I;H_0^1(V))$ we obtain

$$\begin{aligned}\langle \mathcal{L}_{I,V}(\chi f), w\rangle_{L^2(I;H_0^1(V))} &= \langle \chi f, \mathcal{Z}_{S,U} w\rangle_{L^2(S;H_0^1(U))}\\ &= \langle f, \mathcal{Z}_{S,U}(\chi w)\rangle_{L^2(S;H_0^1(U))} = \langle \mathcal{L}_{I,V} f, \chi w\rangle_{L^2(I;H_0^1(V))},\end{aligned}$$

and $[\chi f]_{L_2^\omega(S;H^{-1}(U))} \le c\,[f]_{L_2^\omega(S;H^{-1}(U))}$, which finishes the proof. □

Lemma 5.3 (Localization)**.** *The restriction $f \mapsto \mathcal{L}_{I,V} f$ defines a bounded linear operator from $L_2^\omega(S;H^{-1}(U))$ into $L_2^\omega(I;H^{-1}(V))$.*

Proof. Let $f \in L_2^\omega(S; H^{-1}(U))$, and $t \in I$, $x \in V$, $r > 0$ be given. Setting $I_r = I \cap (t - r^2, t)$, $S_r = S \cap (t - r^2, t)$, $U_r = U \cap Q_r(x)$, and $V_r = V \cap Q_r(x)$, for all $w \in L^2(I_r; H_0^1(V_r))$ we get

$$\begin{aligned}\langle \mathcal{L}_{I_r, V_r} f, w \rangle_{L^2(I_r; H_0^1(V_r))} &= \langle f, \mathcal{Z}_{S,U} w \rangle_{L^2(S; H_0^1(U))} \\ &= \langle \mathcal{L}_{S_r, U_r} f, \mathcal{Z}_{S_r, U_r} w \rangle_{L^2(S_r; H_0^1(U_r))}.\end{aligned}$$

Therefore, we obtain $[\mathcal{L}_{I,V} f]_{L_2^\omega(I; H^{-1}(V))} \leq [f]_{L_2^\omega(S; H^{-1}(U))}$, which proves the assertion. $\square$

Definition 5.4 (Lipschitz transformation)**.** Let T be a Lipschitz transformation from an open neighborhood of $\overline{G}$ into $\mathbb{R}^n$ and set $U^* = T[U]$. We define the assignment $f \mapsto \mathcal{T}^* f$ from $L^2(S; H^{-1}(U))$ into $L^2(S; H^{-1}(U^*))$ as the adjoint operator of $\mathcal{T}_* : L^2(S; H_0^1(U^*)) \to L^2(S; H_0^1(U))$; that means

$$\langle \mathcal{T}^* f, w \rangle_{L^2(S; H_0^1(U^*))} = \langle f, \mathcal{T}_* w \rangle_{L^2(S; H_0^1(U))} \quad \text{for } w \in L^2(S; H_0^1(U^*)).$$

Remark 5.4. Using the transformation invariance of Sobolev spaces (see Lemma 4.4), we find some constant $c = c(T) > 0$ such that

$$\|\mathcal{T}^* f\|_{L^2(S; H^{-1}(U^*))} \leq c \, \|f\|_{L^2(S; H^{-1}(U))} \quad \text{for all } f \in L^2(S; H^{-1}(U)).$$

Lemma 5.4 (Transformation)**.** *The assignment $f \mapsto \mathcal{T}^* f$ is a bounded linear map from $L_2^\omega(S; H^{-1}(U))$ into $L_2^\omega(S; H^{-1}(U^*))$.*

Proof. Let $L \geq 1$ be a Lipschitz constant of T and set $\delta = Lr$. For $r > 0$, $t \in S$, and $y \in U^*$ we introduce the sets $S_r = S \cap (t - r^2, t)$, $S_\delta = S \cap (t - \delta^2, t)$, $U_r^* = U^* \cap Q_r(y)$, $U_\delta = U \cap Q_\delta(T^{-1}(y))$.

Let $f \in L_2^\omega(S; H^{-1}(U))$. In view of the properties of the extension operators with respect to Lipschitz transformations (see Lemma 4.4), and the inclusion $T^{-1}[U_r^*] \subset U_\delta$, for all $w \in L^2(S_r; H_0^1(U_r^*))$ we get

$$\begin{aligned}\langle \mathcal{L}_{S_r, U_r^*} \mathcal{T}^* f, w \rangle_{L^2(S_r; H_0^1(U_r^*))} &= \langle \mathcal{T}^* f, \mathcal{Z}_{S, U^*} w \rangle_{L^2(S; H_0^1(U^*))} \\ &= \langle f, \mathcal{T}_* \mathcal{Z}_{S, U^*} w \rangle_{L^2(S; H_0^1(U))} = \langle f, \mathcal{Z}_{S,U} \mathcal{Z}_{S_\delta, U_\delta} \mathcal{T}_* w \rangle_{L^2(S; H_0^1(U))} \\ &= \langle \mathcal{L}_{S_\delta, U_\delta} f, \mathcal{Z}_{S_\delta, U_\delta} \mathcal{T}_* w \rangle_{L^2(S_\delta; H_0^1(U_\delta))},\end{aligned}$$

and $[\mathcal{T}^* f]_{L_2^\omega(S; H^{-1}(U^*))} \leq c \, [f]_{L_2^\omega(S; H^{-1}(U))}$ for all $f \in L_2^\omega(S; H^{-1}(U))$ and some constant $c = c(T) > 0$, which proves the result. $\square$

Definition 5.5 (Reflection)**.** We introduce the reflection $f \mapsto \mathcal{R}^+ f$ from $L^2(S; H^{-1}(Q^+))$ to $L^2(S; H^{-1}(Q))$ as the adjoint operator of the map $\mathcal{O}^+ : L^2(S; H_0^1(Q)) \to L^2(S; H_0^1(Q^+))$; that means

$$\langle \mathcal{R}^+ f, w \rangle_{L^2(S; H_0^1(Q))} = \langle f, \mathcal{O}^+ w \rangle_{L^2(S; H_0^1(Q^+))} \quad \text{for } w \in L^2(S; H_0^1(Q)).$$

The antireflection $f \mapsto \mathcal{R}^- f$ from $L^2(S; H^{-1}(Q^-))$ to $L^2(S; H^{-1}(Q))$ is defined as the adjoint operator of $\mathcal{O}^- : L^2(S; H_0^1(Q)) \to L^2(S; H_0^1(Q^-))$,

$$\langle \mathcal{R}^- f, w\rangle_{L^2(S;H_0^1(Q))} = \langle f, \mathcal{O}^- w\rangle_{L^2(S;H_0^1(Q^-))} \quad \text{for } w \in L^2(S; H_0^1(Q)).$$

Remark 5.5. By the properties of $\mathcal{O}^+$ and $\mathcal{O}^-$ (see Lemma 4.6), we get

$$\|\mathcal{R}^+ f\|_{L^2(S;H^{-1}(Q))} \le \sqrt{2}\, \|f\|_{L^2(S;H^{-1}(Q^+))} \quad \text{for all } f \in L^2(S; H^{-1}(Q^+)),$$
$$\|\mathcal{R}^- f\|_{L^2(S;H^{-1}(Q))} \le \sqrt{2}\, \|f\|_{L^2(S;H^{-1}(Q^-))} \quad \text{for all } f \in L^2(S; H^{-1}(Q^-)).$$

Lemma 5.5 (Reflection)**.** *The assignment $f \mapsto \mathcal{R}^+ f$ is a continuous map from $L_2^\omega(S; H^{-1}(Q^+))$ into $L_2^\omega(S; H^{-1}(Q))$. Analogously, the assignment $f \mapsto \mathcal{R}^- f$ maps $L_2^\omega(S; H^{-1}(Q^-))$ continuously into $L_2^\omega(S; H^{-1}(Q))$.*

Proof. We consider $r > 0$, $I \in \mathcal{S}_r$, $x \in Q$ and introduce the sets

$$C = Q \cap Q_r(x), \quad C_2 = C \cup N[C] = Q_r^2(x) \cap Q, \quad C^+ = Q^+ \cap Q_r(Px).$$

Let $f \in L_2^\omega(S; H^{-1}(Q^+))$. Due to the properties of $\mathcal{O}_r^+(x)$ (see Lemma 4.6), and the zero extension operators, for all $w \in L^2(I; H_0^1(C))$ we get

$$\begin{aligned}
&\langle \mathcal{L}_{I,C} \mathcal{R}^+ f, w\rangle_{L^2(I;H_0^1(C))} = \langle \mathcal{R}^+ f, \mathcal{Z}_{S,Q} w\rangle_{L^2(S;H_0^1(Q))} \\
&\quad = \langle f, \mathcal{O}^+ \mathcal{Z}_{S,Q} \mathcal{Z}_{I,C_2} w\rangle_{L^2(S;H_0^1(Q^+))} = \langle f, \mathcal{Z}_{S,Q^+} \mathcal{O}_r^+(x) \mathcal{Z}_{I,C_2} w\rangle_{L^2(S;H_0^1(Q^+))} \\
&\quad = \langle \mathcal{L}_{I,C^+} f, \mathcal{O}_r^+(x) \mathcal{Z}_{I,C_2} w\rangle_{L^2(I;H_0^1(C^+))},
\end{aligned}$$

and, therefore, the estimates

$$[\mathcal{R}^+ f]_{L_2^\omega(S;H^{-1}(Q))} \le \sqrt{2}\, [f]_{L_2^\omega(S;H^{-1}(Q^+))} \quad \text{for all } f \in L_2^\omega(S; H^{-1}(Q^+)),$$
$$[\mathcal{R}^- f]_{L_2^\omega(S;H^{-1}(Q))} \le \sqrt{2}\, [f]_{L_2^\omega(S;H^{-1}(Q^-))} \quad \text{for all } f \in L_2^\omega(S; H^{-1}(Q^-)),$$

where the second one follows analogously to the first one. □

Examples of functionals. Next, we consider examples of functionals from $L_2^\omega(S; H^{-1}(G))$, which cover a broad class of applications; see also Remarks 3.4 and 3.7.

Theorem 5.6. *Let F be a relatively open subset of the boundary ∂G. Then the map $(g, g_0, g_F) \mapsto f$ defined by*

$$\begin{aligned}
\langle f, w\rangle_{L^2(S;H_0^1(G))} &= \int_S \int_G \big(g(s) \cdot \nabla w(s) + g_0(s) w(s)\big)\, d\lambda^n\, ds \\
&\quad + \int_S \int_F g_F(s)\, K_F w(s)\, d\lambda_{\partial G}\, ds
\end{aligned}$$

for $w \in L^2(S; H_0^1(G))$*, is a bounded linear operator from*

$$L_2^{\omega}(S; L^2(G^\circ; \mathbb{R}^n)) \times L_2^{\omega-2}(S; L^2(G^\circ)) \times L_2^{\omega-1}(S; L^2(F))$$

into $L_2^{\omega}(S; H^{-1}(G))$*, and its norm depends on* n*,* F*, and* G*, only.*

Proof. 1. Let $\{(T_1, U_1), \dots, (T_m, U_m)\}$ be an atlas of G which satisfies (4.1) and (4.2); see Lemma 4.2. Furthermore, let $L \geq 1$ be a common Lipschitz constant for all the transformations. In view of the Lebesgue property of the covering we choose some $\delta > 0$ such that for all $x \in G$ the open cube $Q_\delta(x)$ is contained in one of the neighborhoods $U_1, \dots, U_m$. We decompose $\{1, \dots, m\}$ into the sets

$$J_0 = \big\{i \in \{1, \dots, m\} : \overline{U_i} \subset G^\circ\big\}, \quad J_1 = \big\{i \in \{1, \dots, m\} : U_i \cap \partial G \neq \varnothing\big\}.$$

We consider three different types of functionals:

2. Let $f \in L^2(S; H^{-1}(G))$ be defined by

$$\langle f, w \rangle_{L^2(S; H_0^1(G))} = \int_S \int_G g(s) \cdot \nabla w(s) \, d\lambda^n \, ds \quad \text{for } w \in L^2(S; H_0^1(G)).$$

Then, for all $r > 0$, $I \in \mathcal{S}_r$, $V \in \mathcal{G}_r$, and $w \in L^2(I; H_0^1(V))$ we have

$$\big|\langle \mathcal{L}_{I,V} f, w \rangle_{L^2(I; H_0^1(V))}\big|^2 \leq \int_I \int_V \|g(s)\|^2 \, d\lambda^n \, ds \int_I \int_V \|\nabla w(s)\|^2 \, d\lambda^n \, ds.$$

Hence, we get

$$\|\mathcal{L}_{I,V} f\|^2_{L^2(I; H^{-1}(V))} \leq \int_I \int_V \|g(s)\|^2 \, d\lambda^n \, ds. \tag{5.1}$$

3. Next we consider the functional $f \in L^2(S; H^{-1}(G))$ defined as

$$\langle f, w \rangle_{L^2(S; H_0^1(G))} = \int_S \int_G g_0(s) w(s) \, d\lambda^n \, ds \quad \text{for } w \in L^2(S; H_0^1(G)).$$

For all $r > 0$, $I \in \mathcal{S}_r$, $V \in \mathcal{G}_r$, and $w \in L^2(I; H_0^1(V))$ we obtain

$$\big|\langle \mathcal{L}_{I,V} f, w \rangle_{L^2(I; H_0^1(V))}\big|^2 \leq \int_I \int_V |g_0(s)|^2 \, d\lambda^n \, ds \int_I \int_V |w(s)|^2 \, d\lambda^n \, ds.$$

3.1. Assume that $Q_\delta(x) \subset U_i$ holds true for some index $i \in J_0$. Note that for all $0 < r \leq \delta$ we have $Q_r(x) \subset G$. For all $I \in \mathcal{S}_r$ and $w \in L^2(I; H_0^1(Q_r(x)))$ the Sobolev–Friedrichs inequality yields

$$\int_I \int_{Q_r(x)} |w(s)|^2 \, d\lambda^n \, ds \leq 4r^2 \int_I \int_{Q_r(x)} \|\nabla w(s)\|^2 \, d\lambda^n \, ds.$$

3.2. Otherwise we have $Q_\delta(x) \subset U_i$ for some index $i \in J_1$. Setting

$$y = T_i(x) \in Q^+, \quad U = T_i^{-1}[Q_{Lr}(y)], \quad V = G \cap Q_r(x),$$

for all $0 < r \le \delta/L^2$ we get the inclusions

$$Q_r(x) \subset U \subset Q_\delta(x), \quad T_i[V] \subset Q^+ \cap Q_{Lr}(y), \quad V \subset G \cap U.$$

For all $0 < r \le \delta/L^2$, $I \in \mathcal{S}_r$, and $w \in L^2(I; H_0^1(V))$ we set

$$w_i = [\mathcal{T}_i^{-1}]_* \mathcal{Z}_{I,G\cap U} w \in L^2(I; H_0^1(Q_{Lr}(y) \cap Q^+)).$$

Applying the Sobolev–Friedrichs inequality we get the estimate

$$\begin{aligned} &\int_I \int_V |w(s)|^2 \, d\lambda^n \, ds \le c_1 \int_I \int_{Q_{Lr}(y)\cap Q^-} |w_i(s)|^2 \, d\lambda^n \, ds \\ &\le 4c_1 (Lr)^2 \int_I \int_{Q_{Lr}(y)\cap Q^-} \|\nabla w_i(s)\|^2 \, d\lambda^n \, ds \le c_2 r^2 \int_I \int_V \|\nabla w(s)\|^2 \, d\lambda^n \, ds, \end{aligned}$$

where the constants c_1, $c_2 > 0$ depend on n and L, only.

3.3. Summing up we find a constant $c_3 = c_3(n, L) > 0$, such that for all $0 < r \le \delta/L^2$, $I \in \mathcal{S}_r$, $V \in \mathcal{G}_r$, and $w \in L^2(I; H_0^1(V))$ we have

$$\left|\langle \mathcal{L}_{I,V} f, w\rangle_{L^2(I;H_0^1(V))}\right|^2 \le c_3 r^2 \int_I \int_V |g_0(s)|^2 \, d\lambda^n \, ds \int_I \int_V \|\nabla w(s)\|^2 \, d\lambda^n \, ds,$$

and, hence,

$$\|\mathcal{L}_{I,V} f\|^2_{L^2(I;H^{-1}(V))} \le c_3 r^2 \int_I \int_V |g_0(s)|^2 \, d\lambda^n \, ds. \tag{5.2}$$

4. We extend $g_F \in L_2^{\omega-1}(S; L^2(F))$ by zero to a function which belongs to the space $L_2^{\omega-1}(S; L^2(\partial G))$. Hence, it suffices to consider the case $F = \partial G$, only. Since the trace map K_F from $H_0^1(G)$ into $L^2(F)$ is continuous, we define a functional $f \in L^2(S; H^{-1}(G))$ by

$$\langle f, w\rangle_{L^2(S;H_0^1(G))} = \int_S \int_F g_F(s) \, K_F w(s) \, d\lambda_F \, ds \quad \text{for } w \in L^2(S; H_0^1(G)).$$

Let $r > 0$, $x \in G$, $I \in \mathcal{S}_r$, and set $V = G \cap Q_r(x)$, $\Gamma = F \cap Q_r(x)$. For all $w \in L^2(I; H_0^1(V))$ we obtain

$$\left|\langle \mathcal{L}_{I,V} f, w\rangle_{L^2(I;H_0^1(V))}\right|^2 \le \int_I \int_\Gamma |g_F(s)|^2 \, d\lambda_F \, ds \int_I \int_\Gamma |K_F w(s)|^2 \, d\lambda_F \, ds.$$

To estimate the second integral of the right-hand side it is sufficient to consider $0 < r \le \delta/L^2$ and $x \in G$ that satisfy $\Gamma = F \cap Q_r(x) \ne \varnothing$. In that case we find an index $i \in J_1$ with $Q_\delta(x) \subset U_i$. Moreover, setting again $y = T_i(x) \in Q^+$, $U = T_i^{-1}[Q_{Lr}(y)]$, we get the inclusions

$$Q_r(x) \subset U \subset Q_\delta(x), \quad T_i[V] \subset Q^+ \cap Q_{Lr}(y), \quad V \subset G \cap U.$$

For all $0 < r \le \delta/L^2$ and $x \in G$ with $\Gamma = F \cap Q_r(x) \neq \varnothing$, every $I \in \mathcal{S}_r$ and $w \in L^2(I; H_0^1(V))$ we set $w_i = [\mathcal{T}_i^{-1}]_* \mathcal{Z}_{I,G\cap U} w \in L^2(I; H_0^1(Q_{Lr}(y) \cap Q^+))$. Using the Sobolev–Friedrichs trace inequality we obtain the estimate

$$\begin{aligned}\int_I \int_\Gamma |K_\Gamma w(s)|^2 \, d\lambda_F \, ds &\le c_4 \int_I \int_{\Sigma \cap Q_{Lr}(y)} |K_{\Sigma\cap Q_{Lr}(y)} w_i(s)|^2 \, d\lambda_\Sigma \, ds \\ &\le 2c_4 Lr \int_I \int_{Q^-\cap Q_{Lr}(y)} \|\nabla w_i(s)\|^2 \, d\lambda^n \, ds \le c_5 r \int_I \int_V \|\nabla w(s)\|^2 \, d\lambda^n \, ds,\end{aligned}$$

where the constants c_4, $c_5 > 0$ depend on n and L, only. Thus for all $0 < r \le \delta/L^2$, $x \in G$, $I \in \mathcal{S}_r$, and $w \in L^2(I; H_0^1(V))$ we have proved

$$\left|\langle \mathcal{L}_{I,V} f, w \rangle_{L^2(I;H_0^1(V))}\right|^2 \le c_5 r \int_I \int_\Gamma |g_F(s)|^2 \, d\lambda_F \, ds \int_I \int_V \|\nabla w(s)\|^2 \, d\lambda^n \, ds,$$

where $V = G \cap Q_r(x)$ and $\Gamma = F \cap Q_r(x)$, which yields

$$\|\mathcal{L}_{I,V} f\|^2_{L^2(I;H^{-1}(V))} \le c_5 r \int_I \int_\Gamma |g_F(s)|^2 \, d\lambda_F \, ds. \tag{5.3}$$

5. Combining (5.1), (5.2), and (5.3) we end up with the desired result. $\square$

6. Sobolev–Morrey spaces for evolution equations

Suppose that $U \subset \mathbb{R}^n$ is a relatively open subset of the regular set $G \subset \mathbb{R}^n$. Furthermore, let $V \subset \mathbb{R}^n$ be relatively open in U, $I \subset \mathbb{R}$ be an open subinterval of $S = (t_0, t_1)$, and $\omega \in [0, n+2]$.

For our following considerations let $\varepsilon \in (0, 1]$ be constant. We define the duality map $J_H : H \to H^*$ on the Hilbert space $H = L^2(U^\circ)$ as usual by $\langle J_H v, w \rangle_H = (v|w)_H$ for v, $w \in H$, but note that we equip H with the weighted scalar product

$$(v|w)_H = \int_U a v w \, d\lambda^n \quad \text{for } v,\, w \in H,$$

where $a \in L^\infty(U^\circ)$ is a nonsmooth capacity coefficient, which is supposed to be ε-definite with respect to U°; that means we assume that

$$\varepsilon \le \operatorname*{ess\,inf}_{y\in U^\circ} a(y), \qquad \operatorname*{ess\,sup}_{y \in U^\circ} a(y) \le \frac{1}{\varepsilon}$$

holds true. Corresponding to Section 1 we set $Y = H_0^1(U)$, and we consider some closed subspace $H_0^1(U) \subset X \subset H^1(U^\circ)$ of $H^1(U^\circ)$ equipped with the usual scalar product. We choose for K the completely continuous embedding of X into H. Then, the set $K[Y]$ is dense in H. The map $E = (K|Y)^* J_H K : X \to Y^*$, associated with the coefficient a being ε-definite with respect to U°, is a bounded linear operator. Its restriction $E|Y$ onto Y is symmetric and positively semidefinite:

$$\langle Ev, w\rangle_Y = \langle J_H Kv, Kw\rangle_H = (v|w)_H \quad \text{for all } v,\, w \in Y.$$

We define the bounded linear operator $\mathcal{E} : L^2(S;X) \to L^2(S;Y^*)$ associated with S and E via $(\mathcal{E}u)(s) = Eu(s)$ for $u \in L^2(S;X)$ and $s \in S$.

Based on Definitions 1.1, 3.2, and 5.2, we construct our class of Sobolev–Morrey spaces suitable for the regularity theory of second-order parabolic boundary-value problems with nonsmooth data; see Griepentrog [9, 11]:

Definition 6.1 (Sobolev–Morrey space)**.** We define the Sobolev–Morrey space $W_E^\omega(S;X)$ as the set of all elements $u \in L_2^\omega(S;X)$ for which the weak time derivative $(\mathcal{E}u)'$ of $\mathcal{E}u \in L^2(S;Y^*)$ exists and belongs to $L_2^\omega(S;Y^*)$:

$$W_E^\omega(S;X) = \big\{u \in L_2^\omega(S;X) : (\mathcal{E}u)' \in L_2^\omega(S;Y^*)\big\}.$$

We introduce the norm of an element $u \in W_E^\omega(S;X)$ by

$$\|u\|^2_{W_E^\omega(S;X)} = \|u\|^2_{L_2^\omega(S;X)} + \|(\mathcal{E}u)'\|^2_{L_2^\omega(S;Y^*)}.$$

For $\omega \le 0$ we set $W_E^\omega(S;X) = W_E(S;X)$.

Lemma 6.1. *The function space $W_E^\omega(S;X)$ is a Banach space.*

Proof. 1. Note that the functional is correctly defined for $u \in W_E^\omega(S;X)$, and that it has the properties of a norm.

2. To prove the completeness of $W_E^\omega(S;X)$, let (u_k) be a Cauchy sequence in $W_E^\omega(S;X)$. Then (u_k) and $((\mathcal{E}u_k)')$ are Cauchy sequences in the Banach spaces $L_2^\omega(S;X)$ and $L_2^\omega(S;Y^*)$, respectively. Hence, for $k \to \infty$ both sequences converge in $L_2^\omega(S;X)$ and $L_2^\omega(S;Y^*)$ to some functions $u \in L_2^\omega(S;X)$ and $f \in L_2^\omega(S;Y^*)$, respectively. The continuity of the operator $\mathcal{E} : L^2(S;X) \to L^2(S;Y^*)$ and the embeddings from $L_2^\omega(S;X)$ into $L^2(S;X)$ and from $L_2^\omega(S;Y^*)$ into $L^2(S;Y^*)$ yields the convergence of $(\mathcal{E}u_k)$ to $\mathcal{E}u$ in $L^2(S;Y^*)$ and of $((\mathcal{E}u_k)')$ to f in $L^2(S;Y^*)$. Passing to the limit $k \to \infty$ in

$$\int_S \langle (\mathcal{E}u_k)'(s), w\rangle_Y\, \vartheta(s)\, ds = -\int_S \langle (\mathcal{E}u_k)(s), w\rangle_Y\, \vartheta'(s)\, ds,$$

for all $\vartheta \in C_0^\infty(S)$ and $w \in Y$ we get

$$\int_S \langle f(s), w\rangle_Y\, \vartheta(s)\, ds = -\int_S \langle (\mathcal{E}u)(s), w\rangle_Y\, \vartheta'(s)\, ds,$$

which proves $(\mathcal{E}u)' = f \in L_2^\omega(S;Y^*)$ and $u \in W_E^\omega(S;X)$. $\square$

Invariance principles. We consider the behaviour of functions from the above Sobolev–Morrey spaces with respect to localization, Lipschitz transformations, and reflections.

Lemma 6.2 (Multiplication). *Let $X = H^1(U^\circ)$ and $Y = H^1_0(U)$. Then, for every $\chi \in C_0^\infty(\mathbb{R}^n)$ the assignment $u \mapsto \chi u$ is a bounded linear map from $W_E^\omega(S;X)$ into itself as well as from $W^\omega_{E|Y}(S;Y)$ into itself.*

Proof. Because of Remark 3.2 and Lemma 5.2, for all $u \in W_E^\omega(S;X)$ we have $\chi u \in L_2^\omega(S;X)$, $\chi(\mathcal{E}u)' \in L_2^\omega(S;Y^*)$, and we find some constant $c = c(\chi) > 0$ such that

$$\|\chi u\|_{L_2^\omega(S;X)} \le c\,\|u\|_{L_2^\omega(S;X)}, \quad \|\chi(\mathcal{E}u)'\|_{L_2^\omega(S;Y^*)} \le c\,\|(\mathcal{E}u)'\|_{L_2^\omega(S;Y^*)}.$$

Because of the identities $\mathcal{E}(\chi u) = \chi\mathcal{E}u$ and $(\chi\mathcal{E}u)' = \chi(\mathcal{E}u)'$ this yields $\chi u \in W_E^\omega(S;X)$ with corresponding norm estimates. The multiplier property in $W^\omega_{E|Y}(S;Y)$ follows analogously. □

Lemma 6.3 (Localization). *Let $E \in \mathcal{L}(H^1(U^\circ); H^{-1}(U))$ be associated with the coefficient a and consider the map $E_V \in \mathcal{L}(H^1(V^\circ); H^{-1}(V))$ associated with the restricted coefficient $a_V = R_V a$. Then the assignment $u \mapsto \mathcal{R}_{I,V}u$ defines a bounded linear map from $W_E^\omega(S;H^1(U^\circ))$ into $W^\omega_{E_V}(I;H^1(V^\circ))$.*

Proof. Due to Lemmas 3.1 and 5.3 we see that for all $u \in W_E^\omega(S;H^1(U^\circ))$

$$\mathcal{R}_{I,V}u \in L_2^\omega(I;H^1(V^\circ)), \quad \mathcal{L}_{I,V}(\mathcal{E}u)' \in L_2^\omega(I;H^{-1}(V)),$$

and we find some constant $c > 0$ such that

$$\|\mathcal{R}_{I,V}u\|_{L_2^\omega(I;H^1(V^\circ))} \le c\,\|u\|_{L_2^\omega(S;H^1(U^\circ))},$$
$$\|\mathcal{L}_{I,V}(\mathcal{E}u)'\|_{L_2^\omega(I;H^{-1}(V))} \le c\,\|(\mathcal{E}u)'\|_{L_2^\omega(S;H^{-1}(U))}.$$

Let $\mathcal{E}_V : L^2(I;H^1(V^\circ)) \to L^2(I;H^{-1}(V))$ be associated with I and E_V. Since $\mathcal{E}_V\mathcal{R}_{I,V}u = \mathcal{L}_{I,V}\mathcal{E}u$ and $(\mathcal{L}_{I,V}\mathcal{E}u)' = \mathcal{L}_{I,V}(\mathcal{E}u)'$ holds true this yields $\mathcal{R}_{I,V}u \in W^\omega_{E_V}(I;H^1(V^\circ))$. The result follows from the above estimates. □

Lemma 6.4 (Transformation). *Suppose that T is a Lipschitz transformation from an open neighborhood of $\overline{G}$ into $\mathbb{R}^n$ with a Lipschitz constant $L \ge 1$. Set $U_* = T[U]$, and consider*

$$X = H^1(U^\circ),\ Y = H^1_0(U), \quad X_* = H^1(U_*^\circ),\ Y_* = H^1_0(U_*).$$

Let $\varepsilon_* = \varepsilon/L^n$ and $E_* \in \mathcal{L}(X_*;Y_*^*)$ be associated with the transformed coefficient $a_* = |JT^{-1}| \cdot T_*^{-1}a$, which is ε_*-definite with respect to U_*°. Then the assignment $u \mapsto \mathcal{T}_*^{-1}u$ defines a bounded linear map from $W_E^\omega(S;X)$ into $W^\omega_{E_*}(S;X_*)$ as well as from $W^\omega_{E|Y}(S;Y)$ into $W^\omega_{E_*|Y_*}(S;Y_*)$.

Proof. Using Lemma 3.2 and 5.4 for all $u \in W^\omega_E(S;X)$ we get $\mathcal{T}_*^{-1}u \in L^\omega_2(S;X_*)$, $\mathcal{T}^*(\mathcal{E}u)' \in L^\omega_2(S;Y^*_*)$, and we find some constant $c = c(T) > 0$ such that

$$\|\mathcal{T}_*^{-1}u\|_{L^\omega_2(S;X_*)} \le c\,\|u\|_{L^\omega_2(S;X)}, \quad \|\mathcal{T}^*(\mathcal{E}u)'\|_{L^\omega_2(S;Y^*_*)} \le c\,\|(\mathcal{E}u)'\|_{L^\omega_2(S;Y^*)}.$$

Let $\mathcal{E}_* : L^2(S;X_*) \to L^2(S;Y^*_*)$ be associated with S and E_*. Due to $\mathcal{E}_*\mathcal{T}_*^{-1}u = \mathcal{T}^*\mathcal{E}u$ and $(\mathcal{T}^*\mathcal{E}u)' = \mathcal{T}^*(\mathcal{E}u)'$ this leads to the desired result $\mathcal{T}_*^{-1}u \in W^\omega_{E_*}(S;X_*)$ together with the norm estimates. The proof of the continuity of the map $u \mapsto \mathcal{T}_*^{-1}u$ from $W^\omega_{E|Y}(S;Y)$ into $W^\omega_{E_*|Y_*}(S;Y_*)$ is exactly the same. □

Lemma 6.5 (Reflection)**.** *Let both the maps $E^+ \in \mathcal{L}(H^1_0(Q^+);H^{-1}(Q^+))$ and $E^- \in \mathcal{L}(H^1_0(Q^-);H^{-1}(Q^-))$ be associated with the coefficient a^-, which is assumed to be ε-definite with respect to Q^-, and consider the operator $E \in \mathcal{L}(H^1_0(Q);H^{-1}(Q))$ associated with the reflected coefficient $a = R^+a^-$ being ε-definite with respect to Q. Then the assignment $u \mapsto \mathcal{R}^+u$ maps $W^\omega_{E^+}(S;H^1_0(Q^+))$ continuously into $W^\omega_E(S;H^1_0(Q))$ and the assignment $u \mapsto \mathcal{R}^-u$ is a bounded linear operator from $W^\omega_{E^-}(S;H^1_0(Q^-))$ into $W^\omega_E(S;H^1_0(Q))$.*

Proof. Let the maps $\mathcal{E}^+$, $\mathcal{E}^-$, $\mathcal{E}$ be associated with S, E^+, E^-, E, respectively. Applying Lemma 4.5 and 5.5, for all $u \in W^\omega_{E^+}(S;H^1_0(Q^+))$ we obtain

$$\mathcal{R}^+u \in L^\omega_2(S;H^1_0(Q)), \quad \mathcal{R}^+(\mathcal{E}^+u)' \in L^\omega_2(S;H^{-1}(Q))$$

together with the estimates

$$\begin{aligned}\|\mathcal{R}^+u\|_{L^\omega_2(S;H^1_0(Q))} &\le \sqrt{2}\,\|u\|_{L^\omega_2(S;H^1_0(Q^+))},\\ \|\mathcal{R}^+(\mathcal{E}^+u)'\|_{L^\omega_2(S;H^{-1}(Q))} &\le \sqrt{2}\,\|(\mathcal{E}u)'\|_{L^\omega_2(S;H^{-1}(Q^+))}.\end{aligned}$$

Using the identities $\mathcal{E}\mathcal{R}^+u = \mathcal{R}^+\mathcal{E}^+u$ and $(\mathcal{R}^+\mathcal{E}^+u)' = \mathcal{R}^+(\mathcal{E}^+u)'$ this yields $\mathcal{R}^+u \in W^\omega_E(S;H^1_0(Q))$, which leads to the desired result. Analogously, for all $u \in W^\omega_{E^-}(S;H^1_0(Q^-))$ we get $\mathcal{E}\mathcal{R}^-u = \mathcal{R}^-\mathcal{E}^-u$ and $(\mathcal{R}^-\mathcal{E}^-u)' = \mathcal{R}^-(\mathcal{E}^-u)'$ and, hence, $\mathcal{R}^-u \in W^\omega_E(S;H^1_0(Q))$ together with the corresponding norm estimates. □

Embedding theorems. We prove the (complete) continuity of the embedding of Sobolev–Morrey spaces in Campanato spaces. Here, the key estimate is some variant of the Poincaré inequality; see Theorem A.3.

Lemma 6.6. *Let $0 < \delta < 1$ and $E \in \mathcal{L}(H^1(Q);H^{-1}(Q))$ be associated with the coefficient a, which is assumed to be ε-definite with respect to Q. Then* *we find a constant* $c = c(\varepsilon, \omega, n, \delta) > 0$ *such that for all* $u \in W_E^\omega(S; H^1(Q))$

$$[u]^2_{\mathfrak{L}_2^{\omega+2}(S;L^2(Q_\delta))} \le c \left(\| \|\nabla u\| \|^2_{L_2^\omega(S;L^2(Q))} + \|(\mathcal{E}u)'\|^2_{L_2^\omega(S;H^{-1}(Q))} \right).$$

Proof. Let $0 < r < 1 - \delta$, $I \in \mathcal{S}_r$, and $x \in Q_\delta$. Applying the minimal property of integral mean values and Theorem A.3 we find some constant $c_1 = c_1(\varepsilon, n) > 0$ such that

$$\begin{aligned}\int_I \int_{Q_r(x)\cap Q_\delta} & \left| u(s) - \fint_I \fint_{Q_r(x)\cap Q_\delta} u(\tau)\, d\lambda^n\, d\tau \right|^2 d\lambda^n\, ds \\ & \le \int_I \int_{Q_r(x)} \left| u(s) - \fint_I \fint_{Q_r(x)} u(\tau)\, d\lambda^n\, d\tau \right|^2 d\lambda^n\, ds \\ & \le c_1 r^2 \int_I \left(\int_{Q_r(x)} \|\nabla u(s)\|^2\, d\lambda^n + \|L_{Q_r(x)}(\mathcal{E}u)'(s)\|^2_{H^{-1}(Q_r(x))} \right) ds.\end{aligned}$$

Hence, there exists a constant $c_2 = c_2(\varepsilon, \omega, n, \delta) > 0$ such that for all functions $u \in W_E^\omega(S; H^1(Q))$ the estimate

$$[u]^2_{\mathfrak{L}_2^{\omega+2}(S;L^2(Q_\delta))} \le c_2 \left(\| \|\nabla u\| \|^2_{L_2^\omega(S;L^2(Q))} + \|(\mathcal{E}u)'\|^2_{L_2^\omega(S;H^{-1}(Q))} \right)$$

holds true. □

Lemma 6.7. *Let* $0 < \delta < 1$ *and* $E \in \mathcal{L}(H^1(Q^-); H^{-1}(Q^-))$ *be associated with the coefficient* a*, which is assumed to be* ε*-definite with respect to* Q^-*. Then we find a constant* $c = c(\varepsilon, \omega, n, \delta) > 0$ *such that for all* $u \in W_E^\omega(S; H^1(Q^-))$

$$[u]^2_{\mathfrak{L}_2^{\omega+2}(S;L^2(Q_\delta^-))} \le c \left(\| \|\nabla u\| \|^2_{L_2^\omega(S;L^2(Q^-))} + \|(\mathcal{E}u)'\|^2_{L_2^\omega(S;H^{-1}(Q^-))} \right).$$

Proof. Let $0 < r < \min\{\delta, 1 - \delta\}$. Furthermore, we consider $I \in \mathcal{S}_r$ and $x \in Q_\delta^-$. Note that $Q_r(x) \cap Q_\delta^-$ is included in the cube $Q_r(y) \subset Q^-$ if we set $y = (\hat{y}, y_n)$, $\hat{y} = \hat{x}$ and $y_n = \min\{x_n, -r\}$. Using the minimal property of integral mean values and Theorem A.3, we obtain

$$\begin{aligned}\int_I \int_{Q_r(x)\cap Q_\delta^-} & \left| u(s) - \fint_I \fint_{Q_r(x)\cap Q_\delta^-} u(\tau)\, d\lambda^n\, d\tau \right|^2 d\lambda^n\, ds \\ & \le \int_I \int_{Q_r(y)} \left| u(s) - \fint_I \fint_{Q_r(y)} u(\tau)\, d\lambda^n\, d\tau \right|^2 d\lambda^n\, ds \\ & \le c_1 r^2 \int_I \left(\int_{Q_r(y)} \|\nabla u(s)\|^2\, d\lambda^n + \|L_{Q_r(y)}(\mathcal{E}u)'(s)\|^2_{H^{-1}(Q_r(y))} \right) ds,\end{aligned}$$

where $c_1 = c_1(\varepsilon, n) > 0$ is some constant. Therefore, we find a further constant $c_2 = c_2(\varepsilon, \omega, n, \delta) > 0$ such that for all $u \in W_E^\omega(S; H^1(Q^-))$

$$[u]^2_{\mathfrak{L}_2^{\omega+2}(S;L^2(Q_\delta^-))} \le c_2 \Big(\|\|\nabla u\|\|^2_{L_2^\omega(S;L^2(Q^-))} + \|(\mathcal{E}u)'\|^2_{L_2^\omega(S;H^{-1}(Q^-))} \Big)$$

holds true. □

Theorem 6.8 (Continuous embedding)**.** *If $E \in \mathcal{L}(H^1(G^\circ); H^{-1}(G))$ is associated with a coefficient a being ε-definite with respect to G°, then the space $W_E^\omega(S; H^1(G^\circ))$ is continuously embedded in $\mathfrak{L}_2^{\omega+2}(S; L^2(G^\circ))$.*

Proof. 1. Let $\Omega = G^\circ$ and $\{(T_1, U_1), \ldots, (T_m, U_m)\}$ be an atlas of G; see Lemma 4.2. Furthermore, let $L \ge 1$ be a common Lipschitz constant for all the transformations, and $\{\chi_1, \ldots, \chi_m\} \subset C_0^\infty(\mathbb{R}^n)$ be a smooth partition of unity subordinate to the above covering. In view of the Lebesgue property of the covering we choose radii $0 < \delta' < \delta < 1$ such that the sets $V_i' = T_i^{-1}[Q_{\delta'}]$ and $V_i = T_i^{-1}[Q_\delta]$ satisfy the condition

$$\operatorname{supp}(\chi_i) \subset V_i' \subset V_i \quad \text{for all } i \in \{1, \ldots, m\}.$$

Hence, $\{V_1', \ldots, V_m'\}$ and $\{V_1, \ldots, V_m\}$ are still open coverings of $\overline{G}$. We consider the decomposition of the set $\{1, \ldots, m\}$ into the index sets

$$J_0 = \big\{ i \in \{1, \ldots, m\} : \overline{U_i} \subset \Omega \big\}, \quad J_1 = \big\{ i \in \{1, \ldots, m\} : U_i \cap \partial\Omega \ne \varnothing \big\}.$$

2. Let $u \in W_E^\omega(S; H^1(\Omega))$. For every $i \in \{1, \ldots, m\}$ we define the map $E^i \in \mathcal{L}(H^1(V_i \cap \Omega); H^{-1}(V_i \cap \Omega))$ associated with the restricted coefficient $a^i = R_{V_i \cap \Omega} a$ being ε-definite with respect to $V_i \cap \Omega$. Since $V_i \cap \Omega$ is an open subset of G for every $i \in \{1, \ldots, m\}$, the localization principle (see Lemmas 6.2 and 6.3), leads to $u_i = \mathcal{R}_{S, V_i \cap \Omega}(\chi_i u) \in W_{E^i}^\omega(S; H^1(V_i \cap \Omega))$ and the estimate

$$\|u_i\|_{W_{E^i}^\omega(S;H^1(V_i \cap \Omega))} \le c_1 \|u\|_{W_E^\omega(S;H^1(\Omega))},$$

where $c_1 > 0$ is some constant depending on ω and G, only.

We introduce inverse transformations and transformed coefficients by

$$T^i = T_i^{-1} : Q \to U_i = U_i \cap \Omega, \quad a_*^i = |JT^i| \cdot T_*^i a^i \quad \text{for } i \in J_0,$$
$$T^i = T_i^{-1}|Q^- : Q^- \to U_i \cap \Omega, \quad a_*^i = |JT^i| \cdot T_*^i a^i \quad \text{for } i \in J_1.$$

Due to the properties of the Jacobi determinant JT^i, the coefficient a_*^i is ε_*-definite with respect to Q_δ for $i \in J_0$ and to Q_δ^- for $i \in J_1$, where $\varepsilon_* = \varepsilon / L^n$. We consider the operators $E_*^i \in \mathcal{L}(H^1(Q_\delta); H^{-1}(Q_\delta))$ associated with a_*^i for

$i \in J_0$ and $E_*^i \in \mathcal{L}(H^1(Q_\delta^-); H^{-1}(Q_\delta^-))$ associated with a_*^i for $i \in J_1$. From the transformation invariance (see Lemma 6.4), it follows that

$$\mathcal{T}_*^i u_i \in W_{E_*^i}^\omega(S; H^1(Q_\delta)) \quad \text{for all } i \in J_0,$$
$$\mathcal{T}_*^i u_i \in W_{E_*^i}^\omega(S; H^1(Q_\delta^-)) \quad \text{for all } i \in J_1.$$

Moreover, we find some constant $c_2 = c_2(n, \varepsilon, \omega, \delta, L, G) > 0$ such that

$$\|\mathcal{T}_*^i u_i\|_{W_{E_*^i}^\omega(S;H^1(Q_\delta))} \leq c_2 \|u\|_{W_E^\omega(S;H^1(\Omega))} \quad \text{for all } i \in J_0,$$
$$\|\mathcal{T}_*^i u_i\|_{W_{E_*^i}^\omega(S;H^1(Q_\delta^-))} \leq c_2 \|u\|_{W_E^\omega(S;H^1(\Omega))} \quad \text{for all } i \in J_1.$$

Setting $v_i = \mathcal{R}_{S,V_i'\cap\Omega}(\chi_i u)$ and applying Lemmas 6.6 and 6.7, we get

$$\mathcal{T}_*^i v_i \in \mathfrak{L}_2^{\omega+2}(S; L^2(Q_{\delta'})) \quad \text{for all } i \in J_0,$$
$$\mathcal{T}_*^i v_i \in \mathfrak{L}_2^{\omega+2}(S; L^2(Q_{\delta'}^-)) \quad \text{for all } i \in J_1.$$

Additionally, we find some constant $c_3 > 0$ depending on the quantities n, ε, ω, δ', δ, L, and G such that

$$\|\mathcal{T}_*^i v_i\|_{\mathfrak{L}_2^{\omega+2}(S;L^2(Q_{\delta'}))} \leq c_3 \|u\|_{W_E^\omega(S;H^1(\Omega))} \quad \text{for all } i \in J_0,$$
$$\|\mathcal{T}_*^i v_i\|_{\mathfrak{L}_2^{\omega+2}(S;L^2(Q_{\delta'}^-))} \leq c_3 \|u\|_{W_E^\omega(S;H^1(\Omega))} \quad \text{for all } i \in J_1.$$

Using Lemma 3.2, the transformation invariance of Campanato spaces leads to $v_i \in \mathfrak{L}_2^{\omega+2}(S; L^2(V_i' \cap \Omega))$ together with the estimates

$$\|v_i\|_{\mathfrak{L}_2^{\omega+2}(S;L^2(V_i'\cap\Omega))} \leq c_4 \|u\|_{W_E^\omega(S;H^1(\Omega))} \quad \text{for all } i \in \{1, \ldots, m\},$$

where $c_4 = c_4(n, \varepsilon, \omega, \delta, \delta', L, G) > 0$ is some constant. Since $\chi_i \in C_0^\infty(\mathbb{R}^n)$ satisfies $\operatorname{supp}(\chi_i) \subset V_i'$ for every $i \in \{1, \ldots, m\}$, we extend v_i by zero to see that $\chi_i u \in \mathfrak{L}_2^{\omega+2}(S; L^2(\Omega))$ and

$$\|\chi_i u\|_{\mathfrak{L}_2^{\omega+2}(S;L^2(\Omega))} \leq c_5 \|u\|_{W_E^\omega(S;H^1(\Omega))} \quad \text{for all } i \in \{1, \ldots, m\},$$

where the constant $c_5 > 0$ depends n, ε, ω, δ, δ', L, and G; see Remark 3.3. Summing up, for all $u \in W_E^\omega(S; H^1(\Omega))$ we obtain the estimate

$$\|u\|_{\mathfrak{L}_2^{\omega+2}(S;L^2(\Omega))} \leq \sum_{i=1}^m \|\chi_i u\|_{\mathfrak{L}_2^{\omega+2}(S;L^2(\Omega))} \leq c_5 m \|u\|_{W_E^\omega(S;H^1(\Omega))},$$

which finishes the proof. □

Theorem 6.9 (Completely continuous embedding)**.** *Let $\sigma \in [0, \omega + 2)$ be given, and let $E \in \mathcal{L}(H^1(G^\circ); H^{-1}(G))$ be associated with the coefficient a being ε-definite with respect to G°. Then the embedding of $W_E^\omega(S; H^1(G^\circ))$ in $\mathfrak{L}_2^\sigma(S; L^2(G^\circ))$ is completely continuous.*

Proof. Let (u_k) be a bounded sequence in $W_E^\omega(S;H^1(G^\circ))$ and $\delta > 0$. Due to Theorem 1.10 the embedding of $W_E(S;H^1(G^\circ))$ in $L^2(S;L^2(G^\circ))$ is completely continuous. Together with the continuity of the embedding of $W_E^\omega(S;H^1(G^\circ))$ in $W_E(S;H^1(G^\circ))$ we find an increasing subsequence $(k_\ell) \subset \mathbb{N}$ and some $\ell_0 = \ell_0(\delta) > 0$ such that

$$\|u_{k_i} - u_{k_\ell}\|^2_{L^2(S;L^2(G^\circ))} \le \delta \quad \text{for all } i,\, \ell \ge \ell_0.$$

We introduce the notation $\theta = \sigma/(\omega+2)$ and use the minimal property of the integral mean value to get

$$\begin{aligned}[u_{k_i} - u_{k_\ell}]^2_{\mathfrak{L}_2^\sigma(S;L^2(G^\circ))} &\le \|u_{k_i} - u_{k_\ell}\|^{2-2\theta}_{L^2(S;L^2(G^\circ))}\,[u_{k_i} - u_{k_\ell}]^{2\theta}_{\mathfrak{L}_2^{\omega+2}(S;L^2(G^\circ))}\\ &\le \delta^{1-\theta}\,[u_{k_i} - u_{k_\ell}]^{2\theta}_{\mathfrak{L}_2^{\omega+2}(S;L^2(G^\circ))}.\end{aligned}$$

In view of the boundedness of (u_k) in $W_E^\omega(S;H^1(G^\circ))$ and the arbitrary choice of $\delta > 0$ at the beginning of the proof, the continuity of the embedding operator from $W_E^\omega(S;H^1(G^\circ))$ in $\mathfrak{L}_2^{\omega+2}(S;L^2(G^\circ))$ (see Theorem 6.8) yields the convergence of the sequence (u_{k_ℓ}) in $\mathfrak{L}_2^\sigma(S;L^2(G^\circ))$. □

Trace theorems. We show the (complete) continuity of the trace map from Sobolev–Morrey spaces in Campanato spaces. Again, these results are based on some variant of the Poincaré inequality; see Theorem A.4.

Lemma 6.10. *Let $0 < \delta < 1$ and $E \in \mathcal{L}(H^1(Q^-);H^{-1}(Q^-))$ be associated with the coefficient a, which is supposed to be ε-definite with respect to Q^-. We find some constant $c = c(\varepsilon,\omega,n,\delta) > 0$ such that for all $u \in W_E^\omega(S;H^1(Q^-))$ we have*

$$[\mathcal{K}_{S,\Sigma_\delta} u]^2_{\mathfrak{L}_2^{\omega+1}(S;L^2(\Sigma_\delta))} \le c\left(\| \|\nabla u\| \|^2_{L_2^\omega(S;L^2(Q^-))} + \|(\mathcal{E}u)'\|^2_{L_2^\omega(S;H^{-1}(Q^-))}\right).$$

Proof. Let $0 < r < 1-\delta$, $I \in \mathcal{S}_r$, and $x \in \Sigma_\delta$. Using the minimal property of integral mean values and Theorem A.4 we find a constant $c_1 = c_1(\varepsilon,n) > 0$ such that

$$\begin{aligned}&\int_I \int_{Q_r(x)\cap\Sigma_\delta} \left| K_\Sigma u(s) - \fint_I \fint_{Q_r(x)\cap\Sigma_\delta} K_\Sigma u(\tau)\, d\lambda_\Sigma\, d\tau \right|^2 d\lambda_\Sigma\, ds\\ &\qquad \le \int_I \int_{\Sigma_r(x)} \left| K_\Sigma u(s) - \fint_I \fint_{\Sigma_r(x)} K_\Sigma u(\tau)\, d\lambda_\Sigma\, d\tau \right|^2 d\lambda_\Sigma\, ds\\ &\quad \le c_1 r \int_I \left(\int_{Q_r^-(x)} \|\nabla u(s)\|^2\, d\lambda^n\, ds + \|L_{Q_r^-(x)}(\mathcal{E}u)'(s)\|^2_{H^{-1}(Q_r^-(x))} \right) ds.\end{aligned}$$

Hence, there exists a constant $c_2 = c_2(\varepsilon, \omega, n, \delta) > 0$ such that for all functions $u \in W_E^\omega(S; H^1(Q^-))$ the estimate

$$[\mathcal{K}_{S,\Sigma_\delta} u]^2_{\mathfrak{L}_2^{\omega+1}(S;L^2(\Sigma_\delta))} \le c_2 \left(\| \| \nabla u \| \|^2_{L_2^\omega(S;L^2(Q^-))} + \| (\mathcal{E}u)' \|^2_{L_2^\omega(S;H^{-1}(Q^-))} \right)$$

holds true. □

Theorem 6.11 (Continuous trace map)**.** *Let $E \in \mathcal{L}(H^1(G^\circ); H^{-1}(G))$ be associated with the coefficient a being ε-definite with respect to G°. Then the trace map $\mathcal{K}_{S,\partial G}$ is a bounded linear operator from $W_E^\omega(S; H^1(G^\circ))$ in $\mathfrak{L}_2^{\omega+1}(S; L^2(\partial G))$.*

Proof. 1. Let $\Omega = G^\circ$ and $\{(T_1, U_1), \dots, (T_m, U_m)\}$ be an atlas of $\Gamma = \partial G$; see Lemma 4.2. Moreover, let $L \ge 1$ be a common Lipschitz constant for all the transformations, and $\{\chi_1, \dots, \chi_m\} \subset C_0^\infty(\mathbb{R}^n)$ be a smooth partition of unity subordinate to the above covering. Because of the Lebesgue property of the covering we find radii $0 < \delta' < \delta < 1$ such that $V_i' = T_i^{-1}[Q_{\delta'}]$ and $V_i = T_i^{-1}[Q_\delta]$ satisfy the relations

$$\operatorname{supp}(\chi_i) \subset V_i' \subset V_i \quad \text{for all } i \in \{1, \dots, m\}.$$

Therefore, $\{V_1', \dots, V_m'\}$ and $\{V_1, \dots, V_m\}$ are open coverings of Γ, too.

2. Let $u \in W_E^\omega(S; H^1(\Omega))$. For every $i \in \{1, \dots, m\}$ we introduce the operator $E^i \in \mathcal{L}(H^1(V_i \cap \Omega); H^{-1}(V_i \cap \Omega))$ associated with the restricted coefficient $a^i = R_{V_i \cap \Omega} a$ which is ε-definite with respect to $V_i \cap \Omega$. Because $V_i \cap \Omega$ is an open subset of G for every $i \in \{1, \dots, m\}$, the localization principle (see Lemmas 6.2 and 6.3) yields

$$u_i = \mathcal{R}_{S, V_i \cap \Omega}(\chi_i u) \in W_{E^i}^\omega(S; H^1(V_i \cap \Omega))$$

and the estimate

$$\|u_i\|_{W_{E^i}^\omega(S;H^1(V_i\cap\Omega))} \le c_1 \, \|u\|_{W_E^\omega(S;H^1(\Omega))},$$

where $c_1 > 0$ is some constant depending on ω and G, only.

We consider inverse transformations and transformed coefficients given by

$$T^i = T_i^{-1} | Q^- : Q^- \to U_i \cap \Omega, \quad a_*^i = |JT^i| \cdot T_*^i a^i \quad \text{for } i \in \{1, \dots, m\}.$$

In view of the properties of the Jacobi determinant JT^i, the coefficient a_*^i is ε_*-definite with respect to Q_δ^-, where $\varepsilon_* = \varepsilon / L^n$. We define the operator $E_*^i \in \mathcal{L}(H^1(Q_\delta^-); H^{-1}(Q_\delta^-))$ associated with a_*^i for $i \in \{1, \dots, m\}$. The transformation invariance yields

$$\mathcal{T}_*^i u_i \in W_{E_*^i}^\omega(S; H^1(Q_\delta^-)) \quad \text{for } i \in \{1, \dots, m\}$$

(see Lemma 6.4), and we find some constant $c_2 = c_2(\varepsilon, n, \omega, \delta, L, G) > 0$ with

$$\|\mathcal{T}_*^i u_i\|_{W^\omega_{E^i_*}(S;H^1(Q_\delta^-))} \leq c_2 \, \|u\|_{W^\omega_E(S;H^1(\Omega))} \quad \text{for all } i \in \{1, \dots, m\}.$$

Setting $v_i = \mathcal{K}_{S,V_i'\cap\Gamma} u_i$ and using Lemmas 3.7 and 6.10, we obtain

$$\mathcal{T}_*^i v_i = \mathcal{K}_{S,\Sigma_{\delta'}} \mathcal{T}_*^i u_i \in \mathfrak{L}_2^{\omega+1}(S; L^2(\Sigma_{\delta'})) \quad \text{for all } i \in \{1, \dots, m\},$$

and we find some $c_3 > 0$ depending on ε, n, ω, δ, δ', L, and G such that

$$\|\mathcal{T}_*^i v_i\|_{\mathfrak{L}_2^{\omega+1}(S;L^2(\Sigma_{\delta'}))} \leq c_3 \, \|u\|_{W^\omega_E(S;H^1(\Omega))} \quad \text{for all } i \in \{1, \dots, m\}.$$

Due to Lemma 3.5, the transformation invariance of Campanato spaces yields $v_i \in \mathfrak{L}_2^{\omega+1}(S; L^2(V_i' \cap \Gamma))$ and the norm estimate

$$\|v_i\|_{\mathfrak{L}_2^{\omega+1}(S;L^2(V_i'\cap\Gamma))} \leq c_4 \, \|u\|_{W^\omega_E(S;H^1(\Omega))} \quad \text{for every } i \in \{1, \dots, m\},$$

where $c_4 = c_4(\varepsilon, n, \omega, \delta, \delta', L, G) > 0$ is a constant. Because $\chi_i \in C_0^\infty(\mathbb{R}^n)$ satisfies $\operatorname{supp}(\chi_i) \subset V_i'$ for every $i \in \{1, \dots, m\}$, we extend v_i by zero to see that $\mathcal{K}_{S,\Gamma}(\chi_i u) \in \mathfrak{L}_2^{\omega+1}(S; L^2(\Gamma))$ and

$$\|\mathcal{K}_{S,\Gamma}(\chi_i u)\|_{\mathfrak{L}_2^{\omega+1}(S;L^2(\Gamma))} \leq c_5 \, \|u\|_{W^\omega_E(S;H^1(\Omega))} \quad \text{for all } i \in \{1, \dots, m\},$$

where $c_5 > 0$ depends on ε, n, ω, δ, δ', L, and G, only; see Remark 3.6. Finally, for all $u \in W^\omega_E(S; H^1(\Omega))$ we end up with the estimate

$$\begin{aligned} \|\mathcal{K}_{S,\Gamma} u\|_{\mathfrak{L}_2^{\omega+1}(S;L^2(\Gamma))} &\leq \sum_{i=1}^m \|\mathcal{K}_{S,\Gamma}(\chi_i u)\|_{\mathfrak{L}_2^{\omega+1}(S;L^2(\Gamma))} \\ &\leq c_5 m \, \|u\|_{W^\omega_E(S;H^1(\Omega))}, \end{aligned}$$

and, therefore, $\mathcal{K}_{S,\Gamma} u \in \mathfrak{L}_2^{\omega+1}(S; L^2(\Gamma))$. □

Theorem 6.12 (Completely continuous trace map)**.** *Let $\sigma \in [0, \omega + 1)$ be a given parameter and $E \in \mathcal{L}(H^1(G^\circ); H^{-1}(G))$ be associated with the coefficient a being ε-definite with respect to G°. Then the trace operator $\mathcal{K}_{S,\partial G}$ maps $W^\omega_E(S; H^1(G^\circ))$ completely continuous into $\mathfrak{L}_2^\sigma(S; L^2(\partial G))$.*

Proof. 1. Let $\Gamma = \partial G$. Due to the multiplicative inequality (3.1) we find some constant $c > 0$ such that for all $v \in H^1(G^\circ)$ and $\delta > 0$ Young's inequality yields

$$\|K_\Gamma v\|^2_{L^2(\Gamma)} \leq c \, \|v\|_{H^1(G^\circ)} \|v\|_{L^2(G^\circ)} \leq \delta \, \|v\|^2_{H^1(G^\circ)} + \frac{c^2}{4\delta} \, \|v\|^2_{L^2(G^\circ)}.$$

In view of the complete continuity of both the embedding of $H^1(G^\circ)$ in $L^2(G^\circ)$ and the trace map $K_\Gamma : H^1(G^\circ) \to L^2(\Gamma)$ we apply Corollary 1.11 to get the complete continuity of the trace operator $\mathcal{K}_{S,\Gamma}$ from $W_E(S;H^1(G^\circ))$ in $L^2(S;L^2(\Gamma))$.

2. Let (u_k) be a bounded sequence in $W_E^\omega(S;H^1(G^\circ))$ and $\delta>0$ a fixed constant. Because of the boundedness of the embedding operator from $W_E^\omega(S;H^1(G^\circ))$ in $W_E(S;H^1(G^\circ))$ and the complete continuity of the trace operator $\mathcal{K}_{S,\Gamma}$ from $W_E(S;H^1(G^\circ))$ in $L^2(S;L^2(\Gamma))$, we find an increasing subsequence $(k_\ell)\subset\mathbb{N}$ and some $\ell_0=\ell_0(\delta)>0$ such that

$$\|\mathcal{K}_{S,\Gamma}u_{k_i}-\mathcal{K}_{S,\Gamma}u_{k_\ell}\|^2_{L^2(S;L^2(\Gamma))}\le\delta\quad\text{for all } i,\,\ell\ge\ell_0.$$

We introduce the notation $\theta=\sigma/(\omega+1)$, and use the minimal property of the integral mean value to get

$$\begin{aligned}&[\mathcal{K}_{S,\Gamma}u_{k_i}-\mathcal{K}_{S,\Gamma}u_{k_\ell}]^2_{\mathfrak{L}_2^\sigma(S;L^2(\Gamma))}\\&\quad\le\|\mathcal{K}_{S,\Gamma}u_{k_i}-\mathcal{K}_{S,\Gamma}u_{k_\ell}\|^{2-2\theta}_{L^2(S;L^2(\Gamma))}[\mathcal{K}_{S,\Gamma}u_{k_i}-\mathcal{K}_{S,\Gamma}u_{k_\ell}]^{2\theta}_{\mathfrak{L}_2^{\omega+1}(S;L^2(\Gamma))}\\&\qquad\qquad\le\delta^{1-\theta}[\mathcal{K}_{S,\Gamma}u_{k_i}-\mathcal{K}_{S,\Gamma}u_{k_\ell}]^{2\theta}_{\mathfrak{L}_2^{\omega+1}(S;L^2(\Gamma))}.\end{aligned}$$

Because of the boundedness of (u_k) in $W_E^\omega(S;H^1(G^\circ))$ and the arbitrary choice of $\delta>0$ at the beginning, the continuity of the trace map $\mathcal{K}_{S,\Gamma}$ from $W_E^\omega(S;H^1(G^\circ))$ into $\mathfrak{L}_2^{\omega+1}(S;L^2(\Gamma))$ (see Theorem 6.11) yields the convergence of the sequence $(\mathcal{K}_{S,\Gamma}u_{k_\ell})$ in $\mathfrak{L}_2^\sigma(S;L^2(\Gamma))$. □

Appendix A. Some variants of Poincaré inequalities

Here, we collect some variants of Poincaré inequalities. As a starting point we cite the following generalized version; see Ziemer [34]:

Theorem A.1 (Poincaré inequality)**.** *Let $X\subset H$ be two Banach spaces equipped with the norms $\|\ \|_X$ and $\|\ \|_H$, respectively, where*

$$\|w\|_X^2=\|w\|_H^2+|w|_X^2\quad\textit{for all } w\in X$$

holds true for some seminorm $|\ |_X$ on X. We assume that the embedding of $(X,\|\ \|_X)$ in $(H,\|\ \|_H)$ is completely continuous and consider the subspace $X_0=\{w\in X:|w|_X=0\}$ of X. Then, we find some constant $c>0$ such that for every projector $L\in\mathcal{L}(X;X_0)$ from X onto X_0 we have

$$\|w-Lw\|_H\le c\,\|L\|_{\mathcal{L}(X;X_0)}|w|_X\quad\textit{for all } w\in X.$$

We apply this theorem to get some weighted variant of the Poincaré inequality for cubes $Q_r(x)$ with $x\in\mathbb{R}^n$ and $r>0$.

Lemma A.2 (Poincaré inequality). *For all $\alpha, \varepsilon \in (0,1]$ we find some constant $c = c(\alpha, \varepsilon, n) > 0$ such that for all $r > 0$, $x \in \mathbb{R}^n$, and $w \in H^1(Q_r(x))$ we have*

$$\int_{Q_r(x)} \left| w - \fint_{Q_r(x)} w \, d\lambda_a^n \right|^2 d\lambda^n \leq cr^2 \int_{Q_r(x)} \|\nabla w\|^2 \, d\lambda^n, \tag{A.1}$$

whenever the weight function $a \in L^\infty(Q_r(x))$ satisfies

$$0 \leq a(y) \leq \frac{1}{\varepsilon}, \quad a(\tilde{y}) \geq \varepsilon \quad \text{for } \lambda^n\text{-almost all } y \in Q_r(x) \text{ and } \tilde{y} \in F,$$

where $F \subset Q_r(x)$ is some Lebesgue-measurable set with $\lambda^n(F) \geq \alpha r^n$, and the weighted Lebesgue measure λ_a^n is defined as

$$\lambda_a^n(\Omega) = \int_\Omega a \, d\lambda^n \quad \text{for Lebesgue-measurable subsets } \Omega \subset Q_r(x).$$

Proof. Let $T : Q \to Q_r(x)$ be the homothecy defined by $T(y) = x + ry$ for $y \in Q$. We consider the projector $L \in \mathcal{L}(H^1(Q); \mathbb{R})$ given by

$$Lv = \frac{\int_Q v \, T_* a \, d\lambda^n}{\int_Q T_* a \, d\lambda^n} \quad \text{for } v \in H^1(Q).$$

Note that there exists some constant $c_1 > 0$ which depends on n, α, and ε, only, such that $\|L\|_{\mathcal{L}(H^1(Q);\mathbb{R})} \leq c_1$ holds true, whenever the assumptions on $a \in L^\infty(Q_r(x))$ are satisfied. Using the complete continuity of the embedding of $H^1(Q)$ into $L^2(Q)$ and Theorem A.1, for all $w \in H^1(Q_r(x))$ we obtain

$$\int_Q |T_* w - L T_* w|^2 \, d\lambda^n \leq c_2 \int_Q \|\nabla T_* w\|^2 \, d\lambda^n,$$

where $c_2 = c_2(\alpha, \varepsilon, n) > 0$ is some constant. Together with the identity

$$L T_* w = \frac{\int_Q T_*(aw) \, d\lambda^n}{\int_Q T_* a \, d\lambda^n} = \frac{\int_{Q_r(x)} aw \, d\lambda^n}{\int_{Q_r(x)} a \, d\lambda^n} = \fint_{Q_r(x)} w \, d\lambda_a^n,$$

for all $w \in H^1(Q_r(x))$ we get the desired estimate

$$\int_{Q_r(x)} \left| w - \fint_{Q_r(x)} w \, d\lambda_a^n \right|^2 d\lambda^n \leq c_3 r^2 \int_{Q_r(x)} \|\nabla w\|^2 \, d\lambda^n,$$

where $c_3 = c_3(\alpha, \varepsilon, n) > 0$ is some constant. $\square$

Finally, we present versions of the Poincaré inequality which are of main importance for both the regularity theory of solutions to parabolic boundary-value problems and the theory of Sobolev–Morrey spaces suitable for the treatment of these problems; see Section 6 and Griepentrog [9, 11]. The proof generalizes ideas of Struwe [30].

Theorem A.3 (Poincaré inequality)**.** *There exists some constant $c > 0$ depending on n and ε, only, such that for all $0 < r \le 1$, $x \in \mathbb{R}^n$, and $u \in W_E(I; H^1(Q_r(x)))$ the inequality*

$$\int_I \int_{Q_r(x)} \left| u(s) - \fint_I \fint_{Q_r(x)} u(\tau)\, d\lambda^n\, d\tau \right|^2 d\lambda^n\, ds \\ \le cr^2 \left(\int_I \int_{Q_r(x)} \|\nabla u(s)\|^2\, d\lambda^n\, ds + \int_I \|(\mathcal{E}u)'(s)\|^2_{H^{-1}(Q_r(x))}\, ds \right),$$

holds true, whenever $I = (\theta, t) \subset \mathbb{R}$ is an interval with $0 < t - \theta \le r^2$. Here, the operator $E \in \mathcal{L}(H^1(Q_r(x)); H^{-1}(Q_r(x)))$ is associated with the coefficient a being ε-definite with respect to $Q_r(x)$.

Proof. 1. Let s_1, $s_2 \in I$ with $s_1 < s_2$ be given. We choose some cut-off function $\zeta \in C_0^\infty(Q_r(x))$ which satisfies

$$0 \le \zeta(y) \le 1, \ \ \|\nabla \zeta(y)\| \le 4/r \quad \text{for } y \in Q_r(x), \quad \zeta(y) = 1 \quad \text{for } y \in Q_{r/2}(x).$$

In addition to that, for $k \in \mathbb{N}$ we consider cut-off functions $\vartheta_k \in H_0^1(\mathbb{R})$ with $\operatorname{supp}(\vartheta_k) = [s_1, s_2] \subset I$ defined by

$$\vartheta_k(s) = \begin{cases} 0 & \text{if } s \le s_1, \\ \frac{4k(s-s_1)}{s_2-s_1} & \text{if } s_1 \le s \le s_{1k}, \\ 1 & \text{if } s_{1k} \le s \le s_{2k}, \\ \frac{4k(s_2-s)}{s_2-s_1} & \text{if } s_{2k} \le s \le s_2, \\ 0 & \text{if } s_2 \le s, \end{cases}$$

where we have used the notation

$$s_{1k} = s_1 + \frac{s_2 - s_1}{4k}, \quad s_{2k} = s_2 - \frac{s_2 - s_1}{4k} \quad \text{for } k \in \mathbb{N}.$$

2. For a shorter notation we use the abbreviations $X = H^1(Q_r(x))$ and $Y = H_0^1(Q_r(x))$. We introduce the weighted Lebesgue measure μ by

$$\mu(\Omega) = \int_\Omega \zeta^2 a\, d\lambda^n \quad \text{for Lebesgue-measurable sets } \Omega \subset Q_r(x).$$

To prove the desired result, we start with the case $u \in C_0^\infty(\mathbb{R}; X)$ and estimate the following difference of weighted mean values

$$\bar{u}(s_1, s_2) = \fint_{Q_r(x)} u(s_2)\, d\mu - \fint_{Q_r(x)} u(s_1)\, d\mu.$$

To do so, for $k \in \mathbb{N}$ we introduce functions

$$w_k = \bar{u}(s_1, s_2)\, \zeta \vartheta_k \in H^1(\mathbb{R}; Y).$$

In view of $\mathrm{supp}(\vartheta_k) \subset I$ and $\zeta u \in C_0^\infty(\mathbb{R}; Y)$, for every $k \in \mathbb{N}$ we integrate by parts to see that

$$\int_I \langle (\mathcal{E}(\zeta u))'(s), w_k(s) \rangle_Y \, ds + \int_I \langle (\mathcal{E} w_k)'(s), \zeta u(s) \rangle_Y \, ds = 0. \tag{A.2}$$

Using the properties of the cut-off functions and applying Young's inequality to the first integral, we find some constant $c_1 = c_1(\varepsilon, n) > 0$ such that for all $k \in \mathbb{N}$

$$\begin{aligned} \int_I \langle (\mathcal{E}(\zeta u))'(s), w_k(s) \rangle_Y \, ds &= \int_{s_1}^{s_2} \langle (\mathcal{E} u)'(s), \bar{u}(s_1, s_2)\, \zeta^2 \vartheta_k(s) \rangle_Y \, ds \\ &\le c_1 \int_{s_1}^{s_2} \|(\mathcal{E} u)'(s)\|_{Y^*}^2 \, ds + \frac{\varepsilon r^n}{2} \, |\bar{u}(s_1, s_2)|^2. \end{aligned}$$

For all $k \in \mathbb{N}$ the second integral in (A.2) equals

$$\begin{aligned} \int_I \langle (\mathcal{E} w_k)'(s), \zeta u(s) \rangle_Y \, ds &= \int_I \int_{Q_r(x)} a w_k'(s)\, \zeta u(s) \, d\lambda^n \, ds \\ &= \int_I \int_{Q_r(x)} \bar{u}(s_1, s_2)\, \vartheta_k'(s) u(s) \, d\mu \, ds. \end{aligned}$$

Due to the properties of the cut-off functions this yields

$$\begin{aligned} \lim_{k \to \infty} \int_I \langle (\mathcal{E} w_k)'(s), \zeta u(s) \rangle_Y \, ds &= \int_{Q_r(x)} \bar{u}(s_1, s_2) \big(u(s_1) - u(s_2) \big) \, d\mu \\ &= - \int_{Q_r(x)} |\bar{u}(s_1, s_2)|^2 \, \zeta^2 a \, d\lambda^n \le -\varepsilon r^n |\bar{u}(s_1, s_2)|^2. \end{aligned}$$

Hence, passing to the limit $k \to \infty$ in (A.2) we obtain

$$|\bar{u}(s_1, s_2)|^2 \le \frac{2c_1}{\varepsilon r^n} \int_{s_1}^{s_2} \|(\mathcal{E} u)'(s)\|_{Y^*}^2 \, ds \tag{A.3}$$

for all s_1, $s_2 \in I$ with $s_1 \le s_2$.

3. Next, we use the minimal property of integral mean values to get

$$\begin{aligned} &\int_I \int_{Q_r(x)} \left| u(s) - \fint_I \fint_{Q_r(x)} u(\tau) \, d\lambda^n \, d\tau \right|^2 d\lambda^n \, ds \\ &\qquad \le 2 \int_I \int_{Q_r(x)} \left| u(s) - \fint_{Q_r(x)} u(s) \, d\mu \right|^2 d\lambda^n \, ds \end{aligned}$$

$$+ 2 \int_I \int_{Q_r(x)} \left| \fint_{Q_r(x)} u(s)\, d\mu - \fint_I \fint_{Q_r(x)} u(\tau)\, d\mu\, d\tau \right|^2 d\lambda^n\, ds. \quad \text{(A.4)}$$

To estimate the first integral on the right-hand side of (A.4) we apply a weighted version of the Poincaré inequality (see Lemma A.2): We make use of the fact, that there is some constant $c_2 = c_2(\varepsilon, n) > 0$ such that for all $s \in I$ we have

$$\int_{Q_r(x)} \left| u(s) - \fint_{Q_r(x)} u(s)\, d\mu \right|^2 d\lambda^n \le c_2 r^2 \int_{Q_r(x)} \|\nabla u(s)\|^2\, d\lambda^n. \quad \text{(A.5)}$$

For the second integral on the right-hand side of (A.4) we use Hölder's inequality and Fubini's Theorem to obtain

$$\begin{aligned} &\int_I \int_{Q_r(x)} \left| \fint_{Q_r(x)} u(s_1)\, d\mu - \fint_I \fint_{Q_r(x)} u(s_2)\, d\mu\, ds_2 \right|^2 d\lambda^n\, ds_1 \\ &\qquad \le \int_I \fint_I \int_{Q_r(x)} |\bar{u}(s_1, s_2)|^2\, d\lambda^n\, ds_2\, ds_1 \\ &\qquad = 2 \fint_I \int_{s_1}^t \int_{Q_r(x)} |\bar{u}(s_1, s_2)|^2\, d\lambda^n\, ds_2\, ds_1. \quad \text{(A.6)} \end{aligned}$$

In view of (A.3) we find some constant $c_3 = c_3(\varepsilon, n) > 0$ such that

$$\begin{aligned} \fint_I \int_{s_1}^t \int_{Q_r(x)} |\bar{u}(s_1, s_2)|^2\, d\lambda^n\, ds_2\, ds_1 &\le c_3 \fint_I \int_{s_1}^t \int_{s_1}^{s_2} \|(\mathcal{E}u)'(s)\|_{Y^*}^2\, ds\, ds_2\, ds_1 \\ &\le c_3 r^2 \int_I \|(\mathcal{E}u)'(s)\|_{Y^*}^2\, ds. \quad \text{(A.7)} \end{aligned}$$

Applying (A.4), (A.5), (A.6), and (A.7), for all $u \in C_0^\infty(\mathbb{R}; X)$ we end up with

$$\begin{aligned} &\int_I \int_{Q_r(x)} \left| u(s) - \fint_I \fint_{Q_r(x)} u(\tau)\, d\lambda^n\, d\tau \right|^2 d\lambda^n\, ds \\ &\qquad \le 2c_2 r^2 \int_I \int_{Q_r(x)} \|\nabla u(s)\|^2\, d\lambda^n\, ds + 4c_3 r^2 \int_I \|(\mathcal{E}u)'(s)\|_{Y^*}^2\, ds. \end{aligned}$$

Due to the density of the set $\{u|I : u \in C_0^\infty(\mathbb{R}; X)\}$ in $W_E(I; X)$ (see Theorem 1.6), this estimate holds true for all $u \in W_E(I; X)$, too. $\square$

Remark A.1. A simple rescaling argument shows, that the result of Theorem A.3 remains true if we replace the cube $Q_r(x)$ by the half-cube $Q_r^-(x)$.

Theorem A.4 (Poincaré inequality). *There exists some constant $c > 0$ depending on n and ε, only, such that for all $0 < r \leq 1$, $x \in \mathbb{R}^n$, and $u \in W_E(I; H^1(Q_r^-(x)))$ the inequality*

$$\int_I \int_{\Sigma_r(x)} \left| K_{\Sigma_r(x)} u(s) - \fint_I \fint_{\Sigma_r(x)} K_{\Sigma_r(x)} u(\tau)\, d\lambda_{\Sigma_r(x)}\, d\tau \right|^2 d\lambda_{\Sigma_r(x)}\, ds$$
$$\leq cr \biggl(\int_I \int_{Q_r^-(x)} \|\nabla u(s)\|^2\, d\lambda^n\, ds + \int_I \|(\mathcal{E}u)'(s)\|^2_{H^{-1}(Q_r^-(x))}\, ds \biggr)$$

holds true, whenever $I = (\theta, t) \subset \mathbb{R}$ is an interval with $0 < t - \theta \leq r^2$. Here, the map $E \in \mathcal{L}(H^1(Q_r^-(x)); H^{-1}(Q_r^-(x)))$ is associated with the coefficient a, which is supposed to be ε-definite with respect to $Q_r^-(x)$.

Proof. Let $T : Q^- \to Q_r^-(x)$ be defined by $T(y) = x + ry$ for $y \in Q^-$. Using the continuity of the trace operator $K_\Sigma : H^1(Q^-) \to L^2(\Sigma)$ we find some constant $c = c(n) > 0$ such that for all $v \in H^1(Q_r(x))$ we have

$$\int_\Sigma |K_\Sigma T_* v|^2\, d\lambda_\Sigma \leq c \int_{Q^-} \bigl(|T_* v|^2 + \|\nabla T_* v\|^2 \bigr)\, d\lambda^n.$$

Hence, for all $v \in H^1(Q_r(x))$ we get

$$\int_{\Sigma_r(x)} |K_{\Sigma_r(x)} v|^2\, d\lambda_{\Sigma_r(x)} \leq cr \int_{Q_r^-(x)} \left(\frac{|v|^2}{r^2} + \|\nabla v\|^2 \right) d\lambda^n.$$

As a consequence, for all $u \in W_E(I; H^1(Q_r^-(x)))$ we obtain

$$\int_I \int_{\Sigma_r(x)} \left| K_{\Sigma_r(x)} u(s) - \fint_I \fint_{Q_r^-(x)} u(\tau)\, d\lambda^n\, d\tau \right|^2 d\lambda_{\Sigma_r(x)}\, ds$$
$$\leq cr \int_I \int_{Q_r^-(x)} \left(\frac{1}{r^2} \left| u(s) - \fint_I \fint_{Q_r^-(x)} u(\tau)\, d\lambda^n\, d\tau \right|^2 + \|\nabla u(s)\|^2 \right) d\lambda^n\, ds.$$

In view of the minimal property of integral mean values and Remark A.1, this yields the desired result. $\square$

Appendix B. A special chain rule

Based on the notation of Section 6, as a further technical instrument, we provide the following combination of integration by parts and chain rule:

Lemma B.1 (Chain rule). *Let $S = (t_0, t_1) \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ be some bounded open set, $\zeta \in C_0^\infty(\Omega)$, $\vartheta \in C^\infty(\mathbb{R})$, and let $\iota \in C^2(\mathbb{R})$ satisfy $\iota'' \in BC(\mathbb{R})$. Then for all $v \in W_E(S; H^1(\Omega)) \cap C(\overline{S}; L^2(\Omega))$ we have*

$$\int_S \langle (\mathcal{E}v)'(s), \zeta^2 \iota'(v(s))\rangle_{H_0^1(\Omega)}\, \vartheta(s)\, ds + \int_S \int_\Omega \iota(v(s))\, \vartheta'(s)\, \zeta^2 a\, d\lambda^n\, ds$$
$$= \vartheta(t_1) \int_\Omega \iota(v(t_1))\, \zeta^2 a\, d\lambda^n - \vartheta(t_0) \int_\Omega \iota(v(t_0))\, \zeta^2 a\, d\lambda^n. \quad \text{(B.1)}$$

Here, the operator $E \in \mathcal{L}(H^1(\Omega); H^{-1}(\Omega))$ is associated with the coefficient a being ε-definite with respect to Ω.

Proof. In our proof we set $Y = H_0^1(\Omega)$, $X = H^1(\Omega)$, and $H = L^2(\Omega)$; see Section 6. Let $v \in W_E(S; X) \cap C(\overline{S}; H)$ be given.

1. Because of $\iota \in C^2(\mathbb{R})$, $\iota'' \in BC(\mathbb{R})$ there exists some constant $c_1 = c_1(\iota) > 0$ such that we have $|\iota''(z)| \le c_1$, and, consequently,

$$|\iota'(z)| \le |\iota'(0)| + |\iota'(z) - \iota'(0)| \le |\iota'(0)| + c_1|z|$$

for all $z \in \mathbb{R}$. This yields $\iota' \circ u \in L^2(S; X)$, and, therefore, $\zeta^2 \iota' \circ u \in L^2(S; V)$, which shows that the first integral on the left-hand side of formula (B.1) is correctly defined. In addition to that, for all $z \in \mathbb{R}$ we obtain

$$|\iota(z)| \le |\iota(0)| + |\iota'(0)||z| + \tfrac{1}{2} c_1 |z|^2 \le |\iota(0)| + \tfrac{1}{2}|\iota'(0)|^2 + \tfrac{1}{2}(c_1 + 1)|z|^2.$$

This ensures that all the other terms in (B.1) are correctly defined.

Due to Theorem 1.6 we find a sequence $(v_k) \subset C_0^\infty(\mathbb{R}; X)$ such that $(v_k|S)$ converges to v in $W_E(S; X)$ for $k \to \infty$. Note that the identity

$$\langle (\mathcal{E}v_k)'(s), \zeta^2 \iota'(v_k(s))\rangle_Y = \langle E v_k'(s), \zeta^2 \iota'(v_k(s))\rangle_Y$$
$$= \int_\Omega v_k'(s)\, \iota'(v_k(s))\, \zeta^2 a\, d\lambda^n = \int_\Omega (\iota \circ v_k)'(s)\, \zeta^2 a\, d\lambda^n$$

holds true for all $s \in \mathbb{R}$ and $k \in \mathbb{N}$ due to the classical chain rule. Using integration by parts, for every $k \in \mathbb{N}$ we obtain

$$\int_S \langle (\mathcal{E}v_k)'(s), \zeta^2 \iota'(v_k(s))\rangle_Y\, \vartheta(s)\, ds + \int_S \int_\Omega \iota(v_k(s))\, \vartheta'(s)\, \zeta^2 a\, d\lambda^n\, ds$$
$$= \int_S \int_\Omega (\iota \circ v_k)'(s)\, \vartheta(s)\, \zeta^2 a\, d\lambda^n\, ds + \int_S \int_\Omega \iota(v_k(s))\, \vartheta'(s)\, \zeta^2 a\, d\lambda^n\, ds$$
$$= \vartheta(t_1) \int_\Omega \iota(v_k(t_1))\, \zeta^2 a\, d\lambda^n - \vartheta(t_0) \int_\Omega \iota(v_k(t_0))\, \zeta^2 a\, d\lambda^n. \quad \text{(B.2)}$$

Our plan is to prove formula (B.1) by passing to the limit $k \to \infty$ in (B.2):

2. We start with the first integral term on the left-hand side of (B.2). Due to the convergence of $(v_k|S)$ to v in $W_E(S; X)$ it suffices to show that $(\zeta^2 \iota' \circ v_k|S)$ converges weakly to $\zeta^2 \iota' \circ v$ in $L^2(S; Y)$.

Having in mind that $|\iota'(z) - \iota'(\hat{z})| \le c_1|z - \hat{z}|$ holds true for all z, $\hat{z} \in \mathbb{R}$, we obtain the convergence result

$$\lim_{k\to\infty} \int_S \int_\Omega |\iota'(v_k(s)) - \iota'(v(s))|^2 \, d\lambda^n \, ds = 0.$$

Since $|\iota''(z)| \le c_1$ holds for all $z \in \mathbb{R}$, the sequence $(\iota' \circ v_k|S)$ is bounded in $L^2(S;X)$. Therefore, $(\zeta^2\iota' \circ v_k|S)$ is a bounded sequence in $L^2(S;Y)$.

Let $(k_\ell) \subset \mathbb{N}$ be an increasing sequence such that $(\zeta^2\iota' \circ v_{k_\ell}|S)$ converges weakly in $L^2(S;Y)$ to some weak limit $w \in L^2(S;Y)$. Due to the definition of weak differentiablity and the convergence of $(\zeta^2\iota' \circ v_{k_\ell}|S)$ to $\zeta^2\iota' \circ u$ in $L^2(S;H)$, the limit process $\ell \to \infty$ yields that the weak spatial derivatives of w coincide with the weak spatial derivatives of $\zeta^2\iota' \circ v$. This shows that every subsequence of $(\zeta^2\iota' \circ v_k|S)$, which converges weakly in $L^2(S;Y)$, has the weak limit $\zeta^2\iota' \circ v$. Consequently, the whole sequence $(\zeta^2\iota' \circ v_k|S)$ converges weakly in $L^2(S;Y)$ to $\zeta^2\iota' \circ v$.

Since $((\mathcal{E}v_k)'|S)$ converges to $(\mathcal{E}v)'$ in $L^2(S;Y^*)$, it follows that

$$\lim_{k\to\infty} \int_S \langle(\mathcal{E}v_k)'(s), \zeta^2\iota'(v_k(s))\rangle_Y \, \vartheta(s) \, ds = \int_S \langle(\mathcal{E}v)'(s), \zeta^2\iota'(v(s))\rangle_Y \, \vartheta(s) \, ds.$$

3. Similar to Step 1, for all $\delta > 0$ and z, $\hat{z} \in \mathbb{R}$ we get

$$\begin{aligned} |\iota(z) - \iota(\hat{z})| &\le |\iota'(\hat{z})||z - \hat{z}| + \tfrac{1}{2}c_1|z - \hat{z}|^2 \\ &\le \delta\big(|\iota'(0)|^2 + c_1^2|\hat{z}|^2\big) + \tfrac{1}{2}\big(c_1 + \tfrac{1}{\delta}\big)|z - \hat{z}|^2. \end{aligned} \tag{B.3}$$

Therefore, for all $\delta > 0$ and $k \in \mathbb{N}$ we obtain

$$\begin{aligned} &\int_S \int_\Omega |\iota(v_k(s)) - \iota(v(s))| \, |\vartheta'(s)| \, \zeta^2 a \, d\lambda^n \, ds \\ &\qquad \le \delta \int_S \int_\Omega \big(|\iota'(0)|^2 + c_1^2|v(s)|^2\big) \, |\vartheta'(s)| \, \zeta^2 a \, d\lambda^n \, ds \\ &\qquad\qquad + \frac{1}{2}\left(c_1 + \frac{1}{\delta}\right) \int_S \int_\Omega |v_k(s) - v(s)|^2 \, |\vartheta'(s)| \, \zeta^2 a \, d\lambda^n \, ds. \end{aligned}$$

Passing to the limit $k \to \infty$, the convergence of $(v_k|S)$ to v in $W_E(S;X)$ and the arbitrary choice of $\delta > 0$ leads to the convergence

$$\lim_{k\to\infty} \int_S \int_\Omega \iota(v_k(s)) \, \vartheta'(s) \, \zeta^2 a \, d\lambda^n \, ds = \int_S \int_\Omega \iota(v(s)) \, \vartheta'(s) \, \zeta^2 a \, d\lambda^n \, ds$$

of the second integral term on the left-hand side of (B.2).

4. Again using (B.3), for all $\delta > 0$, $s \in \overline{S}$, and $k \in \mathbb{N}$ we get the estimate

$$\int_\Omega |\iota(v_k(s)) - \iota(v(s))| \,\zeta^2 a\, d\lambda^n \le \delta \int_\Omega \left(|\iota'(0)|^2 + c_1^2 |v(s)|^2\right) \zeta^2 a\, d\lambda^n + \frac{1}{2}\left(c_1 + \frac{1}{\delta}\right) \int_\Omega |\zeta v_k(s) - \zeta v(s)|^2\, a\, d\lambda^n.$$

Since the map $C \in \mathcal{L}(X;Y)$, defined by $Cw = \zeta w$ for $w \in X$, satisfies

$$\langle J_H K\hat{w}, KCw\rangle_H = (\hat{w}|\zeta w)_H = (w|\zeta\hat{w})_H = \langle J_H K w, KC\hat{w}\rangle_H$$

that means, the condition $\langle E\hat{w}, Cw\rangle_Y = \langle Ew, C\hat{w}\rangle_Y$ for all w, $\hat{w} \in X$, the sequence $(\zeta v_k|S)$ converges to ζv in $W_{E|Y}(S;Y)$ for $k \to \infty$; see Lemma 1.2. The continuity of the embedding from $W_{E|Y}(S;Y)$ into $BC(\overline{S};H)$ (see Theorem 1.7) and the arbitrary choice of $\delta > 0$ yields

$$\lim_{k\to\infty} \int_\Omega \iota(v_k(s))\,\zeta^2 a\, d\lambda^n = \int_\Omega \iota(v(s))\,\zeta^2 a\, d\lambda^n \quad \text{for all } s \in \overline{S},$$

and, therefore, the convergence of the integral terms on the right-hand side of (B.2). □

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