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HEAT KERNEL AND RESOLVENT PROPERTIES FOR SECOND ORDER ELLIPTIC DIFFERENTIAL OPERATORS WITH GENERAL BOUNDARY CONDITIONS ON L^p

Dedicated to Professor Dr. Herbert Gajewski on the occasion of his sixtieth birthday

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Abstract. Under general (including mixed) boundary condition, nonsmooth coeffcients and weak assumptions on the spatial domain of arbitrary space dimension, resolvent estimates for second order elliptic operators in divergence form are proved. The semigroups generated by them are analytic, map into Hölder spaces, are positivity improving, and their heat kernels are Hölder continuous in both arguments. We regard perturbations of the elliptic operator by nonnegative potentials, by first order differential operators and multiplicative perturbations. The results prove that the solutions of the corresponding linear and semilinear parabolic equations are Hölder continuous in space and time.
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1 Introduction

During the last years considerable progress has been made in the investigation of elliptic differential operators in connection with nonsmooth situations. This concerns results covering Lipschitz domains, cf. Jerison, Kenig [19], as well as possibly jumping coefficients. In particular, in case of Dirichlet boundary conditions explicit ranges of p's are known, where $-\Delta$ provides an isomorphism from $W_0^{1,p}$ and the dual of $W_0^{1,p'}$. Little effort has been made, however, to tackle mixed boundary conditions, although they play an important role in applied problems, cf. Amann [2] or Gajewski, Gröger [17] and the references cited there.

The present work is motivated by the study of rection-diffusion equations of the type

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(D(u)\operatorname{grad} u\big) = f(u, \operatorname{grad} u), \tag{1.1}$$

where u is a concentration, $D(u)$ is a diffusion coefficient, $J = -D(u)\operatorname{grad} u$ is the current, and f represents external sources and reactions, cf. also AMANN [2] and GRÖGER, GAJEWSKI [17]. In nonsmooth situations equations of this type have usually been regarded in negatively indexed Sobolev spaces, cf. [17] and the references cited there. The serious disadvantage of this approach is that one does *not* know in the end that for any time point the divergence of the current is a function from L^p; one only obtains that it is a distribution.

However, it would be highly satisfactory that the normal flow over any part of the Dirichlet boundary is well defined by Gauss' theorem, because the continuity of the normal component plays an essential role in connecting and embedding of potential flow systems (1.1), not least in electronic device simulation, cf. GAJEWSKI [11].

In order to deal with equation (1.1) in a function space we investigate elliptic partial differential operators in divergence form on L^p. Inspecting existing theories which can be possibly applied, cf. AMANN [1, 3], LUNARDI [23], PAZY [25] and references cited there, one recognizes that a cornerstone are always resolvent estimates uniform on the left complex half plane which imply the generation property of an analytic semigroup in the appropriate space. The generator property of an analytic semigroup on L^p for operators $\operatorname{div} a \operatorname{grad}$ with general boundary conditions has already been proved by ARENDT, TER ELST [4, Sect. 4]. However, the approach in [4] rests on a Nash–Moser type iteration by FABES, STROOCK [10], and explicit resolvent estimates are not available. Moreover, the underlying concept of minimally smooth boundary in the sense of STEIN [28], while admitting unbounded sets, excludes some possibly useful bounded domains, cf. Remark 2.3. We take a different approach to the problem: By means of recently obtained C^α regularity results of GRIEPENTROG, RECKE [14], operating in the conceptual framework of regular sets in the sense of GRÖGER [16], and an old estimation technique taken from PAZY [25, Ch. 7.3, Th. 3.6], we are able to give explicit reolvent estimates in terms of the coefficient function. Moreover, we prove that a finite power of the resolvent maps L^2 into C^α. A fortiori the semigroup operators map L^2 continuously into C^α, are nuclear and the corresponding heat kernel is not only essentially bounded but Hölder continuous in both arguments. This provides the persistence of spectral properties of the elliptic operator on the scale of L^p spaces, cf. DAVIES [6]. Moreover, the semigroup is positivity improving.

The reader will notice that one of the main results, Theorem 5.2, is not only formulated for operators $\operatorname{div} a \operatorname{grad}$ but for operators $U \operatorname{div} a \operatorname{grad}$, where U is a measurable, positive L^∞ function, bounded from below be a stricly positive constant, cf. also OUHABAZ [24]. This is motivated as follows: In material heterostructures the concentration u in (1.1) may be given by a function relative to another, $u = \tilde{u}/U$, where U is a fixed function representing material properties, cf. e.g. GRÖGER, GAJEWSKI [17]. Multiplying (1.1) by the reference density U leads to an operator $U \operatorname{div} a \operatorname{grad}$. There are other settings dealing with reference functions U from L^∞, cf. GRIEPENTROG [13, Ch. 2]; however, in L^p spaces they canonically act as multipliers.

Using, that the operator $U \operatorname{div} a \operatorname{grad}$ generates an analytic semigroup on L^p and the continuous embedding of the elliptic operators domain into a C^α space, we prove that the solutions of corresponding linear and semilinear parabolic equations are Hölder continuous in space and time.

2 Notations, definitions, prerequisities

In the sequel Ω will always be a bounded domain in $\mathbb{R}^d$ and Γ a part of its boundary, which may be empty. If p is from $[1, \infty[$, then $L^p = L^p(\Omega)$ is the space of complex, Lebesgue measurable, p-integrable functions on Ω, and $W^{1,p} = W^{1,p}(\Omega)$ is the usual Sobolev space on Ω. The L^p-$L^{p'}$ duality shall be given by the extended L^2 duality

$$\langle \psi_1, \psi_2 \rangle = \int_\Omega \psi_1(x)\, \overline{\psi_2(x)}\, dx. \tag{2.1}$$

$L^\infty = L^\infty(\Omega)$ is the space of Lebesgue measurable, essentially bounded functions on Ω, and $C^\alpha = C^\alpha(\overline{\Omega})$ the space of up to the boundary α-Hölder continuous functions on Ω.

We assume that $\Omega \cup \Gamma$ is a regular set in the following sense:

Definition 2.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $\Gamma \subset \partial\Omega$ be a part of its boundary. $\Omega \cap \Gamma$ is a *regular set* if for every point $\tilde{x} \in \partial\Omega$ there exist two open sets $\mathcal{U}$, $\mathcal{V} \subset \mathbb{R}^d$ and a bi-Lischitz transformation L from $\mathcal{U}$ onto $\mathcal{V}$ such that, $\tilde{x} \in \mathcal{U}$, and $L\big(\mathcal{U} \cap (\Omega \cup \Gamma)\big)$ coincides with one of the three model sets

$$\begin{aligned} E_1 &= \big\{x \in \mathbb{R}^d : |x| < 1,\, x_d < 0\big\}, \\ E_2 &= \big\{x \in \mathbb{R}^d : |x| < 1,\, x_d \le 0\big\}, \\ E_3 &= \big\{x \in E_2 : x_d < 0 \text{ or } x_1 > 0\big\}. \end{aligned} \tag{2.2}$$

Definition 2.2. We define $W_0^{1,p}$ as the closure in $W^{1,p}$ of the set

$$C_0^\infty(\Omega \cup \Gamma) \stackrel{\text{def}}{=} \big\{u|\Omega : u \in C_0^\infty(\mathbb{R}^d),\, \operatorname{supp}(u) \cap \big(\overline{\Omega} \setminus (\Omega \cup \Gamma)\big) = \varnothing\big\}, \tag{2.3}$$

and $W^{-1,p}$ as the dual space to $W_0^{1,p'}$, where $1/p + 1/p' = 1$.

Remark 2.3. The above concept coincides exactly with Gröger's definition of regular sets, cf. [16], which is well-adjusted to mixed boundary value problems. N.B. from the definition of the regular set follows that Γ is relatively open in $\partial\Omega$. We can identify Γ with the Neumann and $\partial\Omega \setminus \Gamma$ with the Dirichlet part of the boundary $\partial\Omega$. Please note, that every bounded open set $\Omega \subset \mathbb{R}^d$ with minimally smooth boundary in the sense of Stein, cf. [28, Ch. VI, § 3.3], i.e. a Lipschitz boundary, is regular, but the converse statement is not true. Indeed, Grisvard's lightning set is regular but does not have a minimally smooth bondary, cf. Grisvard [15, Ch. 1.2.1.4]. Nevertheless, it is easy to prove the $W^{1,p}$ extension domain property of Ω in $\mathbb{R}^d$, by means of the localization, transformation and reflection principles, cf. e.g. Griepentrog [13, Ch. 1.1]. Thus one obtains the usual embedding theorems $W^{1,p} \hookrightarrow L^q$. Furthermore, an adequate concept of surface measure σ on the boundary can be established by passing the boundary measure from the three model sets (2.2) via the bi-Lipschitz transformation L to the boundary of Ω. In particular, the embedding $W^{1,2} \hookrightarrow L^2(\partial\Omega, \sigma)$ is compact, cf. e.g. Griepentrog, Recke [14], and there is

$$\int_{\partial\Omega} |\psi|^2 \, d\sigma \leq M \, \|\psi\|_{L^2} \sqrt{\int_{\Omega} \left(\| \operatorname{grad} \psi \|_{\mathbb{C}^d}^2 + |\psi|^2 \right) dx} \quad \text{for } \psi \in W^{1,2}, \tag{2.4}$$

with a positive constant M, cf. [13, Ch. 1.1].

Throughout this paper $\mathcal{B}(X; Y)$ denotes the space of bounded linear operators from X to Y, X and Y being Banach spaces.

Definition 2.4. Let $a = \{a_{k,l}\}_{k,l} : \Omega \to \mathcal{B}(\mathbb{C}^d; \mathbb{C}^d)$, be a measurable mapping into the set of *real, symmetric* $d \times d$-matrices, with

$$\|a(x)\|_{\mathcal{B}(\mathbb{C}^d;\mathbb{C}^d)} \leq a^\bullet \quad \text{and} \quad \sum_{k,l=1}^{d} a_{k,l}(x) \, \xi_k \, \xi_l \geq a_\bullet \sum_{k=1}^{d} \xi_k^2 \tag{2.5}$$

for almost every $x \in \Omega$, all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and two strictly positive constants $a_\bullet$ and $a^\bullet$. Further, let β be a nonnegative function from $L^\infty(\Gamma, d\sigma)$. Then, $\mathfrak{t}_\Omega$ and $\mathfrak{t}_\Gamma$ are the following sesquilinear forms on $W_0^{1,2} \times W_0^{1,2}$:

$$\mathfrak{t}_\Omega[\psi_1, \psi_2] \stackrel{\text{def}}{=} \int_\Omega \langle a \operatorname{grad} \psi_1, \operatorname{grad} \psi_2 \rangle_{\mathbb{C}^d} \, dx, \tag{2.6a}$$

$$\mathfrak{t}_\Gamma[\psi_1, \psi_2] \stackrel{\text{def}}{=} \int_\Gamma \beta \, \psi_1 \, \overline{\psi_2} \, d\sigma. \tag{2.6b}$$

The form $\mathfrak{t}$ is defined as the sum of the forms $\mathfrak{t}_\Omega$ and $\mathfrak{t}_\Gamma$.

We intend to define an operator on L^2 which corresponds to the form $\mathfrak{t}$ by the representation theorem of forms. Before doing this, we collect some properties of the forms.

Lemma 2.5. *Let* $\mathfrak{t}_\Omega$*,* $\mathfrak{t}_\Gamma$*, and* $\mathfrak{t}$ *be according to Definition 2.4.*

(i) *The forms* $\mathfrak{t}_\Omega$*,* $\mathfrak{t}_\Gamma$*, and* $\mathfrak{t}$ *are well-defined and symmetric on* $W_0^{1,2}$.

(ii) *The quadratic forms associated to* $\mathfrak{t}_\Omega$*,* $\mathfrak{t}_\Gamma$*, and* $\mathfrak{t}$ *are nonnegative.*

(iii) *The forms* $\mathfrak{t}_\Omega$ *and* $\mathfrak{t}$ *are densely defined on* L^2 *and closed.*

The proof is standard; the closedness of $\mathfrak{t}$ on $W_0^{1,2}$ results from the closedness of $\mathfrak{t}_\Omega$ and the relative boundedness of $\mathfrak{t}_\Gamma$ with respect to $\mathfrak{t}_\Omega$, which easily follows from (2.4).

Definition 2.6. A_2 is the selfadjoint, nonnegative operator on L^2 which corresponds to the form $\mathfrak{t}$ from Definition 2.4 by the first representation theorem of forms, cf. Kato [20, Ch. VI, § 2, Th. 2.1 and Th. 2.6]. For $p > 2$, A_p is the restriction of A_2 to L^p.

From the properties of the forms one can conclude

Theorem 2.7. *The resolvent of A_2 is compact. The semigroup generated by $-A_2$ is contractive. If $\sigma(\partial\Omega \setminus \Gamma) > 0$ or $\int_\Gamma \beta \, d\sigma > 0$, then the operator A_2 has a strictly positive lower bound and the semigroup generated by $-A_2$ is even strictly contractive.*

The next theorems concern positivity properties of the semigroup generated by A_2+W, where W is nonnegative potential. We give the precise statements, because they have not been proved as yet for the underlying concept of regular sets, cf. Definition 2.1 and Remark 2.3, but we will not work the proofs out in detail, since the ideas are standard.

Theorem 2.8. *Let W be a nonnegative L^∞ function. Then the operators $\mathrm{e}^{-t(A_2+W)}$, $t > 0$ and $(A_2 + W + \rho)^{-1}$, $\rho > 0$ are positivity preserving.*

Proof. According to the first Beurling–Deny criterion, cf. Davies [6, Th. 1.3.2], the statements on $\mathrm{e}^{-t(A_2+W)}$ and $(A_2 + W + \rho)^{-1}$ imply each other. One has to show two things, cf. Liskevich, Semenov [22, Proposition 1.6], namely

(i) Each operator $\mathrm{e}^{-t(A_2+W)}$, $t > 0$ maps real valued functions from L^2 onto real valued functions.

(ii) Phillips's condition

$$\big\langle (A_2 + W)\psi, \psi^+ \big\rangle \geq 0 \quad \text{for all } \psi \in \operatorname{dom}(A_2) \cap L^2(\Omega; \mathbb{R}), \tag{2.7}$$

ψ^+ being the positive part of ψ, holds.

To see that (i) holds true, we first note that $A_2 + W$ commutes with the complex conjugation, so does its resolvent. From the formula

$$\mathrm{e}^{-t(A_2+W)} = \operatorname*{s-lim}_{n\to\infty} \left(1 + \frac{t}{n}(A_2 + W)\right)^{-n},$$

cf. Kato [20, Ch. IX, § 1.2], follows that $\mathrm{e}^{-t(A_2+W)}$ also commutes with the complex conjugation.

In order to show (2.7) one first proves that for real valued functions ψ from $\operatorname{dom}(A_2) \subset W_0^{1,2}$ the positive part ψ^+ belongs to $W_0^{1,2}$, the form domain of $A_2 + W$. This is obtained by the usual formula for the partial derivatives of ψ^+

$$\frac{\partial \psi^+}{\partial x_k} = \begin{cases} \frac{\partial \psi}{\partial x_k} & \text{a.e. on } \{\psi > 0\}, \\ 0 & \text{a.e. on } \{\psi \leq 0\}, \end{cases} \tag{2.8}$$

cf. Evans, Gariepy [9, Ch. 4.2.2], and a standard mollifier argument to assure that ψ^+ has the correct boundary behaviour, i.e. belongs to the $W^{1,2}$ closure of $C_0^\infty(\Omega \cup \Gamma)$. Knowing $\psi^+ \in W_0^{1,2}$, one calculates by means of (2.8)

$$\begin{aligned} \big\langle (A_2 + W)\psi, \psi^+ \big\rangle &= \int_\Omega \big\langle a \operatorname{grad} \psi, \operatorname{grad} \psi^+ \big\rangle_{\mathbb{R}^d} \, dx + \int_\Omega W \psi \, \psi^+ \, dx + \int_\Gamma \beta \, \psi \, \psi^+ \, d\sigma \\ &= \int_\Omega \big\langle a \operatorname{grad} \psi^+, \operatorname{grad} \psi^+ \big\rangle_{\mathbb{R}^d} \, dx + \int_\Omega W \, |\psi^+|^2 \, dx + \int_\Gamma \beta \, |\psi^+|^2 \, d\sigma \end{aligned}$$

which indeed is a nonnegative expression. $\square$

Theorem 2.9. *If W is a nonnegative L^∞ function, then $(A_2 + W + \rho)^{-1}$ is positivity improving, hence ergodic, at least for all $\rho > -\operatorname{ess\,inf}_\Omega W$, and $\mathrm{e}^{-t(A_2+W)}$ is positivity improving, hence ergodic, for all $t > 0$.*

According to Reed, Simon [26, Vol. IV, Th. XIII.44], Theorem 2.7 and Theorem 2.8 the proof follows from

Lemma 2.10. *If W is a nonnegative L^∞ function, then the eigenspace belonging to the smallest eigenvalue λ_1 of $A_2 + W$ contains a function ψ which is strictly positive on Ω. Any other function from this eigenspace is a scalar multiple of ψ.*

A proof of this runs along the same lines as that of Gilbarg, Trudinger [12, Th. 8.38]. The cornerstone is Harnack's inequality, which also holds for the underlying concept of regular sets in the sense of Gröger, cf. Remark 2.3.

The following regularity result for elliptic boundary value problems is an essential ingredient in our subsequent proofs.

Proposition 2.11 (Cf. Griepentrog, Recke [14])**.** *Let $\Omega \cup \Gamma$ be a regular set in the sense of Definition 2.1, W be a nonnegative L^∞ function, A_2 be according to Definition 2.6, and $0 < a_\bullet \le a^\bullet < \infty$ be the constants from (2.5). For every number p with $p \ge 2$ and $p > d/2$ there is a constant $\alpha = \alpha(p, a_\bullet, a^\bullet, \Omega, \Gamma) \in \,]0, 1[$ such that for every $f \in L^p$ the solution $u \in W_0^{1,2}$ of the elliptic boundary value problem $(A_2 + W)\,u = f$ is Hölder continuous up to the boundary, and*

$$\left\|(A_2 + W + 1)^{-1}\right\|_{\mathcal{B}(L^p;C^\alpha)} < \infty. \tag{2.9}$$

Remark 2.12. This result corresponds to one being known since long for the Dirichlet problem, cf. Gilbarg, Trudinger [12, Ch. 8.10].

From Proposition 2.11 one easily deduces the following

Theorem 2.13. *Let α be the Hölder exponent from Proposition 2.11. There is a positive number j such that for any nonnegative L^∞ function W the mapping*

$$(A_2 + W + 1)^{-j} : L^2 \longrightarrow C^\alpha \hookrightarrow L^\infty \tag{2.10}$$

is well-defined and continuous. If $d \in \{2, 3\}$, then $j = 1$ suffices. If $d \in \{4, 5\}$, then $j = 3/2$ works. Furthermore, each semigroup operator $\mathrm{e}^{-t(A_2+W)}$, $t > 0$, maps L^2 continuously into $C^\alpha \hookrightarrow L^\infty$.

Proof. If $d \le 3$, then (2.10) holds with $j = 1$, according to Proposition 2.11. If $d \in \{4, 5\}$, then $\frac{2d}{d-2} > \frac{d}{2}$ and Proposition 2.11 yields

$$L^2 \xrightarrow{(A_2+W+1)^{-1/2}} \operatorname{dom}(\mathfrak{t}) = W_0^{1,2} \hookrightarrow L^{\frac{2d}{d-2}} \xrightarrow{(A_2+W+1)^{-1}} C^\alpha.$$

If $d > 5$, then by Definition 2.6 and Proposition 2.11 one has

$$(A_2 + W + 1)^{-1} : L^2 \longrightarrow \operatorname{dom}(A_2) \hookrightarrow \operatorname{dom}(\mathfrak{t}) = W_0^{1,2} \hookrightarrow L^{\frac{2d}{d-2}}$$

as well as

$$(A_2 + W + 1)^{-1} : L^{\frac{d+1}{2}} \longrightarrow C^\alpha \hookrightarrow L^\infty.$$

Hence, by the Riesz–Thorin interpolation theorem the mapping

$$(A_2 + W + 1)^{-1} : L^{p_\theta} \longrightarrow L^{q_\theta} \hookrightarrow L^{\frac{d}{d-2} p_\theta}$$

is continuous provided that

$$\frac{1}{p_\theta} = \frac{1-\theta}{2} + \frac{2\theta}{d+1} \quad \text{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{2} \frac{d-2}{d} \quad \text{for } \theta \in [0,1[.$$

Consequently, the mapping

$$(A_2 + W + 1)^{-1} : L^{\frac{2d}{d-2}\left(\frac{d}{d-2}\right)^k} \longrightarrow L^{\frac{2d}{d-2}\left(\frac{d}{d-2}\right)^{k+1}}$$

is continuous for all nonnegative integers k such that

$$p_{\theta(k)} = \frac{2d}{d-2}\left(\frac{d}{d-2}\right)^k \leq \frac{d+1}{2}.$$

Thus with a finite resolvent power $(A_2 + W + 1)^{1-j}$ one ends up in $L^{\frac{d+1}{2}}$. Now, applying once more $(A_2 + W + 1)^{-1}$ one arrives at C^α, due to Proposition 2.11.

As for the second assertion there is

$$\left\| \mathrm{e}^{-t(A_2+W)} \right\|_{\mathcal{B}(L^2;C^\alpha)} \leq \left\| (A_2 + W + 1)^{-j} \right\|_{\mathcal{B}(L^2;C^\alpha)} \left\| (A_2 + W + 1)^j \, \mathrm{e}^{-t(A_2+W)} \right\|_{\mathcal{B}(L^2;L^2)}.$$

The first factor on the right hand side is finite according to the first assertion, the second one is finite due to the spectral theorem. □

Remark 2.14. The first assertion of Theorem 2.13 can be equivalently formulated: For each of the spaces $X = C^\beta$, $0 \leq \beta \leq \alpha$, α according to Proposition 2.11, and $X = L^p$, $1 \leq p \leq \infty$, there is a constant γ_X such that

$$\|\psi\|_X \leq \gamma_X \left\| (A_2 + W + 1)^j \psi \right\|_{L^2} \quad \text{for all } \psi \in \operatorname{dom}\big((A_2 + W + 1)^j\big). \tag{2.11}$$

Theorem 2.15. *Let W be a nonnegative L^∞ function and let j be the number from Theorem 2.13. The operator $(A_2 + W + 1)^{-j} : L^2 \to L^2$ is Hilbert–Schmidt, and all semigroup operators $\mathrm{e}^{-t(A_2+W)} : L^2 \to L^2$, $t > 0$, are nuclear.*

Proof. For each of the operators $(A_2 + W + 1)^{-j}$ and $\mathrm{e}^{-t(A_2+W)}$, $t > 0$, there is a factorization over L^∞, cf. Theorem 2.13. Hence, by the Pietsch factorization theorem, cf. Diestel, Jarchow, Tonge [8, 2.13 item iv], these operators are Hilbert–Schmidt. By splitting up $\mathrm{e}^{-t(A_2+W)} = \big(\mathrm{e}^{-t(A_2+W)/2}\big)^2$ one now obtains that each semigroup operator $\mathrm{e}^{-t(A_2+W)}$, $t > 0$, is nuclear. □

3 Hölder continuity of the heat kernel

Theorem 3.1. *Let W be a nonnegative L^∞ function and let α be the Hölder exponent from Proposition 2.11. Each semigroup operator* $\mathrm{e}^{-t(A_2+W)} : L^2 \to L^2$, $t > 0$, *is an integral operator and the corresponding kernels are from the space* $C^\alpha(\overline{\Omega} \times \overline{\Omega}; \mathbb{R})$.

Proof. According to Theorem 2.15 each semigroup operator $\mathrm{e}^{-t(A_2+W)}$, $t > 0$, is nuclear and, consequently, an integral operator. Let $\{\lambda_r\}_{r=1}^\infty$ be the sequence of eigenvalues of A_2+W, counting multiplicity, and $\{\psi_r\}_{r=1}^\infty$ a corresponding complete, orthonormal system of real eigenfunctions. Such a system can always be found because $A_2 + W$ commutes with the complex conjugation on L^2. We prove that the series

$$\sum_{r=1}^{\infty} \mathrm{e}^{-t\lambda_r} \, \psi_r \otimes \psi_r \tag{3.1}$$

converges absolutely in $C^\alpha(\overline{\Omega} \times \overline{\Omega}; \mathbb{R})$. This implies that the series represents the integral kernel of $\mathrm{e}^{-t(A_2+W)}$, because for any eigenfunction ψ_r one obtains the correct image under $\mathrm{e}^{-t(A_2+W)}$. First, it follows from Theorem 2.13 that all eigenfunctions ψ_r belong to C^α because they belong to $\cap_{l=1}^\infty \operatorname{dom}\big((A_2+W)^l\big)$, thus in particular to $\operatorname{dom}\big((A_2+W)^j\big)$. Further, it is easy to check the inequality

$$\|\psi \otimes \varphi\|_{C^\alpha(\overline{\Omega}\times\overline{\Omega};\mathbb{R})} \le 2\, \|\psi\|_{C^\alpha} \|\varphi\|_{C^\alpha} \quad \text{for all } \psi,\, \varphi \in C^\alpha. \tag{3.2}$$

Now one can estimate the terms of the sum (3.1) by means of (3.2) and (2.11) as follows:

$$\|\psi_r \otimes \psi_r\|_{C^\alpha(\overline{\Omega}\times\overline{\Omega};\mathbb{R})} \le 2\gamma_{C^\alpha}^2 \big\|(A_2+W+1)^j \psi_r\big\|_{L^2}^2 \le 2\gamma_{C^\alpha}^2 (\lambda_r+1)^{2j}. \tag{3.3}$$

N.B. the ψ_r are L^2-normalized. According to Theorem 2.15 $(A_2+W+1)^{-j}$ is a Hilbert–Schmidt operator. Hence, the series $\sum_{r=1}^\infty (\lambda_r+1)^{-2j}$ is convergent and due to (3.3) and the exponential decay of the factor $\mathrm{e}^{-t(\lambda_r+1)}$ the series (3.1) converges absolutely in $C^\alpha(\overline{\Omega} \times \overline{\Omega}; \mathbb{R})$. □

4 The operators A_p

In this section we will regard more closely the operators A_p from Definition 2.6 and operators $A_p + W$, where W is a multiplication operator, induced by a nonnegative, not necessarily bounded function. Functions W of this type frequently occur as potentials of Schrödinger operators, cf. e.g. Reed, Simon [26, Vol. IV, Ch. XIII].

4.a Basic properties of the operators A_p

Theorem 4.1. *Suppose* $p \in \,]2,\infty[$. *For any* $\rho > 0$ *the operator* $(A_p+\rho)^{-1}$ *exists and is compact, hence,* A_p *is closed.*

Proof. Let first p be greater than $d/2$. According to Proposition 2.11 $(A_p+\rho)^{-1}$ is a continuous mapping from L^p into a Hölder space, hence it is compact as a mapping from

L^p into itself. By Theorem 2.7, also $(A_2+\rho)^{-1}$ is compact. Thus, using a well-known interpolation theorem for L^p spaces, cf. Davies [6, Th. 1.6.1], one obtains that $(A_p+\rho)^{-1}$ exists for $p \in]2, d/2]$ and is compact.

$(A_p+\rho)^{-1}$ is continuous for any $\rho > 0$, and, consequently, closed. Thus, $A_p+\rho$ and A_p are also closed operators. □

For any $p \in [2, \infty[$ let $J_p : L^p \longrightarrow L^{p'}$, $1/p + 1/p' = 1$, denote the *duality mapping*

$$J_p : \psi \longmapsto \frac{1}{\|\psi\|_{L^p}^{p-2}} \, |\psi|^{p-2} \, \psi \tag{4.1}$$

from L^p into $L^{p'}$. N.B. the duality was defined by (2.1) as the extended L^2 duality, i.e. antilinear in the second argument. The duality mapping (4.1) has the following properties:

Lemma 4.2. *Suppose $p \geq \max\left\{4, \frac{d+1}{2}\right\}$. If $\psi \in \operatorname{dom}(A_p)$, then $J_p\psi \in W_0^{1,2}$ and the generalized derivatives of $J_p\psi$ may be calculated as*

$$\frac{\partial}{\partial x_l} \, J_p\psi = \frac{1}{\|\psi\|_{L^p}^{p-2}} \left(|\psi|^{p-2} \, \frac{\partial \psi}{\partial x_l} + \frac{p-2}{2} \, |\psi|^{p-4} \, \psi \left(\overline{\psi} \, \frac{\partial \psi}{\partial x_l} + \psi \, \frac{\partial \overline{\psi}}{\partial x_l} \right) \right). \tag{4.2}$$

Proof. As $p \geq \max\left\{4, \frac{d+1}{2}\right\}$ we have $\psi \in C(\overline{\Omega})$ due to Proposition 2.11. Moreover, one has

$$\operatorname{dom}(A_p) \subset \operatorname{dom}(A_2) \hookrightarrow \operatorname{dom}(\mathfrak{t}) = W_0^{1,2}. \tag{4.3}$$

Hence, due to the product rule it is sufficient to prove that $|\psi|^{p-2}$ is from $W_0^{1,2}$ and that its partial derivatives may be calculated as

$$\frac{\partial}{\partial x_l} \, |\psi|^{p-2} = \frac{p-2}{2} \, |\psi|^{p-4} \left(\overline{\psi} \, \frac{\partial \psi}{\partial x_l} + \psi \, \frac{\partial \overline{\psi}}{\partial x_l} \right). \tag{4.4}$$

For $p = 4$, what is permitted in the cases $d \leq 7$, the statement follows immediately by the product rule. Let now p be greater than 4 and not smaller than $\frac{d+1}{2}$. With $\varphi = |\psi|^2$ the left hand side of (4.4) can be written as $\frac{\partial}{\partial x_l}\left(\varphi^{p/2-1}\right)$. The function $\varphi \in W_0^{1,2} \cap C(\overline{\Omega})$ is positive; we denote its supremum by M, and define the function $g : \mathbb{R} \longrightarrow [0, \infty[$ by

$$g(x) = \begin{cases} 0 & \text{if } x \in]-\infty, 0[, \\ x^{p/2-1} & \text{if } x \in [0, M+1], \\ (M+1)^{p/2-1} & \text{if } x \in]M+1, \infty[. \end{cases} \tag{4.5}$$

Because the function φ takes its values only in the interval $[0, M]$, we have

$$|\psi|^{p-2} = \varphi^{p/2-1} = g(\varphi).$$

By construction, g is a continuous and piecewise continuously differentiable function with $g' \in L^\infty(\mathbb{R})$; thus the weak partial derivatives of $g(\varphi)$ are

$$\frac{\partial}{\partial x_l} \, g(\varphi) = g'(\varphi) \, \frac{\partial \varphi}{\partial x_l} \, ,$$

cf. Gilbarg, Trudinger [12, Th. 7.8]. □

Lemma 4.2 implies the following result on the numerical range of the operators A_p.

Theorem 4.3. *Suppose* $p \geq \max\left\{4, \frac{d+1}{2}\right\}$. *If* $\psi \in \operatorname{dom}(A_p)$, *then*

$$\left|\Im\langle A_p\psi, J_p\psi\rangle\right| \leq \frac{a^\bullet}{a_\bullet}\,\frac{p-2}{2\sqrt{p-1}}\,\Re\langle A_p\psi, J_p\psi\rangle. \tag{4.6}$$

In particular, $-A_p$ *is dissipative, and* $-A_p$ *is the infinitesimal generator of a strongly continuous semigroup of contractions.*

Proof. By Proposition 2.11 and our assumption on p, $\operatorname{dom}(A_p)$ is contained in L^∞. Hence, the L^p-$L^{p'}$ duality $\langle A_p\psi, J_p\psi\rangle$ is equal to the scalar product between $A_p\psi$ and $J_p\psi$ in L^2. Further, $\psi \in \operatorname{dom}(A_p)$ implies by (4.3) and Lemma 4.2 that ψ and $J_p\psi$ belong to the space $\operatorname{dom}(\mathfrak{t}) = W_0^{1,2}$, which yields $\langle A_p\psi, J_p\psi\rangle = \mathfrak{t}[\psi, J_p\psi]$. Hence, due to (4.2) there is

$$\begin{aligned}\|\psi\|_{L^p}^{p-2}\,\langle A_p\psi, J_p\psi\rangle = \int_\Omega \sum_{k,l=1}^d a_{k,l}\,\frac{\partial\psi}{\partial x_k}\left(|\psi|^{p-2}\,\frac{\partial\overline{\psi}}{\partial x_l} + \frac{p-2}{2}\,|\psi|^{p-4}\,\overline{\psi}\left(\psi\,\frac{\partial\overline{\psi}}{\partial x_l} + \overline{\psi}\,\frac{\partial\psi}{\partial x_l}\right)\right)dx \\ + \int_\Gamma \beta\,|\psi|^p\,d\sigma.\end{aligned} \tag{4.7}$$

If one neglects the (nonnegative) term $\int_\Gamma \beta\,|\psi|^p\,d\sigma$, then the real part of the right hand side of (4.7) decreases. We split

$$|\psi|^{\frac{p-4}{2}}\,\overline{\psi}\,\frac{\partial\psi}{\partial x_k} \stackrel{\text{def}}{=} \varphi_k + i\phi_k$$

into real and imaginary parts and write down what remains on the right hand side of (4.7), thereby observing that the coefficient matrix $a = \{a_{k,l}\}_{k,l}$ is *real symmetric:*

$$\begin{aligned}\int_\Omega \sum_{k,l=1}^d a_{k,l}\,(\varphi_k + i\phi_k)\left(\overline{(\varphi_l + i\phi_l)} + \frac{p-2}{2}\left(\overline{(\varphi_l + i\phi_l)} + (\varphi_l + i\phi_l)\right)\right)dx \\ = \int_\Omega \sum_{k,l=1}^d a_{k,l}\left((p-1)\,\varphi_k\,\varphi_l + \phi_k\,\phi_l + i(p-2)\,\varphi_k\,\phi_l\right)dx.\end{aligned} \tag{4.8}$$

By means of (2.5), the real part of (4.8) may be estimated from below by

$$a_\bullet\left((p-1)\sum_{k=1}^d \int_\Omega \varphi_k^2\,dx + \sum_{k=1}^d \int_\Omega \phi_k^2\,dx\right) \geq 0, \tag{4.9}$$

while the absolute value of the imaginary part of (4.8) can be estimated due to (2.5) and (4.9) as follows:

$$\begin{aligned}(p-2)\left|\int_\Omega \sum_{k,l=1}^{d} a_{k,l}\,\varphi_k\,\phi_l\,dx\right| &\le a^\bullet (p-2)\sqrt{\int_\Omega \sum_{k=1}^{d} \varphi_k^2\,dx}\,\sqrt{\int_\Omega \sum_{k=1}^{d} \phi_k^2\,dx}\\ &\le a^\bullet\,\frac{p-2}{2}\left(\sqrt{p-1}\int_\Omega \sum_{k=1}^{d} \varphi_k^2\,dx + \frac{1}{\sqrt{p-1}}\int_\Omega \sum_{k=1}^{d} \phi_k^2\,dx\right)\\ &\le a^\bullet\,\frac{p-2}{2\sqrt{p-1}}\left((p-1)\int_\Omega \sum_{k=1}^{d} \varphi_k^2\,dx + \int_\Omega \sum_{k=1}^{d} \phi_k^2\,dx\right)\end{aligned}$$

what proves the assertion (4.6). Now (4.6) implies immediately the dissipativity of $-A_p$, cf. Pazy [25, Ch. 1.4, Def. 4.1], and this together with Theorem 4.1 ensures by the Lumer–Phillips theorem [25, Ch. 1.4, Th. 4.3], that $-A_p$ is the infinitesimal generator of a strongly continuous semigroup of contractions. □

Remark 4.4. The proof of Theorem 4.3 follows exactly the proof of Pazy [25, Ch. 7.3, Th. 3.6] for the case of smooth domains, smooth coefficients and homogeneous Dirichlet boundary conditions. Indeed, the crucial part of the proof in our setting is to show that the duality mapping J_p maps the domain of the operator A_p into the form domain of $\mathfrak{t}$.

Theorem 4.3 permits an essential conclusion:

Theorem 4.5. *If p is any number from $[2,\infty[$, then $\mathrm{dom}(A_p)$ is dense in L^p.*

Proof. At first let p be not smaller than $\max\left\{4, \frac{d+1}{2}\right\}$. According to a well-known theorem, cf. Pazy [25, Ch. 1.4, Th. 4.6], it suffices to show that $A_p + 1$ has the whole space L^p as its range. Indeed, for any $\rho > 0$ the operator $A_p + \rho$ is surjective, because, due to the compactness of the resolvent, cf. Theorem 4.1, in the opposite case ρ would be an eigenvalue of $-A_p$. But this is impossible because $-A_p$ is dissipative.

Thus, the assertion is proved for $p \ge p_0 = \left\{4, \frac{d+1}{2}\right\}$. Let now p be from $[2, p_0[$. There is $\mathrm{dom}(A_{p_0}) \subset \mathrm{dom}(A_p)$ for all $p \in [2, p_0[$. Hence, as $\mathrm{dom}(A_{p_0})$ is dense in L^{p_0} and L^{p_0} is dense in L^p, $\mathrm{dom}(A_{p_0})$ must be dense in L^p. □

Theorem 4.5 justifies the following definition, supplementing Definition 2.6.

Definition 4.6. For $p < 2$, A_p is the adjoint of $A_{p'}$, where $1/p + 1/p' = 1$.

With the help of classical duality results, cf. Kato [20, Ch. III, § 5, Th. 5.29 and Th. 5.30], one easily reproduces the statements on A_p for the case $p \in \,]1,2[$.

Theorem 4.7. *Suppose $p \in \,]1,2[$. A_p is closed and densely defined. The restriction of A_p to L^2 is equal to A_2. For any $\rho > 0$ the operator $(A_p + \rho)^{-1}$ exists and is compact.*

Remark 4.8. According to Theorem 4.1 and Theorem 4.7, the operator $(A_p + 1)^{-1}$ is compact. Hence, $\mathrm{dom}(A_p)$ equipped with the graph norm $\|\psi\|_{\mathrm{dom}(A_p)} = \|(A_p + 1)\psi\|_{L^p}$ embeds compactly into L^p for all $p \in \,]1,\infty[$.

Now we may conclude the dissipativity of all the operators $-A_p$, more precisely:

Theorem 4.9. *If $p \in \,]1,\infty[$ and $\rho > 0$, then*

$$\left\|(A_p+\rho)^{-1}\right\|_{\mathcal{B}(L^p;L^p)} \le \frac{1}{\rho}\,, \tag{4.10}$$

hence, $-A_p$ is dissipative.

Proof. In view of $(A_p+\rho)^{-1} = \big((A_{p'}+\rho)^{-1}\big)^*$, where $1/p + 1/p' = 1$, it suffices to prove (4.10) for $p \in [2,\infty[$. For the cases $p = 2$ and $p \ge \max\left\{4, \frac{d+1}{2}\right\}$ the inequality follows from the dissipativity of $-A_p$ and the surjectivity of $A_p + \rho$ and a well-known theorem, cf. Pazy [25, Th. 1.4.2]. For $p \in \left]2, \max\left\{4, \frac{d+1}{2}\right\}\right[$, interpolation leads to (4.10), and the dissipativity of $-A_p$ follows again from [25, Th. 1.4.2]. □

4.b $\quad A_p$: Perturbations by nonnegative potentials W

For $A_p + W$ to generate a strongly continuous semigroup of contractions and to allow resolvent estimates it is sufficient to know that the multiplication operator induced by the function W is relatively compact with respect to the operator A_p. First we prove a general lemma about relatively bounded perturbations of A_p+1, which also will be of use when we later regard perturbations by first order operators.

Lemma 4.10. *Suppose $p \in \,]1,\infty[$ and let $T : \mathrm{dom}(A_p) \longrightarrow L^p$ be relatively bounded with respect to $A_p + 1$, i.e. there are numbers $a \ge 0$ and $b \ge 0$, such that*

$$\|T\psi\|_{L^p} \le a\,\|\psi\|_{L^p} + b\,\|(A_p+1)\psi\|_{L^p} \quad \textit{for all } \psi \in \mathrm{dom}(A_p). \tag{4.11}$$

If $\rho > 1$, then

$$\|T\psi\|_{L^p} \le a\,\|\psi\|_{L^p} + 2b\,\|(A_p+\rho)\psi\|_{L^p} \quad \textit{for all } \psi \in \mathrm{dom}(A_p). \tag{4.12}$$

Moreover, if $b < 1/2$, then the operators A_p and $A_p + T$ have the same domain $\mathrm{dom}(A_p)$, are closed and the resolvent of $A_p + T$ is compact.

Proof. First, (4.12) results by means of (4.10) and (4.11) from the inequality

$$\begin{aligned}
\|(A_p+1)\,\psi\|_{L^p} &\le \left\|\big((A_p+\rho)-(\rho-1)\big)(A_p+\rho)^{-1}\right\|_{\mathcal{B}(L^p;L^p)}\|(A_p+\rho)\psi\|_{L^p} \\
&\le \big(1+(\rho-1)\left\|(A_p+\rho)^{-1}\right\|_{\mathcal{B}(L^p;L^p)}\big)\,\|(A_p+\rho)\psi\|_{L^p} \\
&\le 2\,\|(A_p+\rho)\psi\|_{L^p}.
\end{aligned}$$

A_p is closed due to Theorem 4.1 and Theorem 4.7. If $b < 1$, then the operators A_p and A_p+T have the same domain $\mathrm{dom}(A_p)$ and are closed, cf. Kato [20, Ch. IV, § 1, Th. 1.1]. According to Theorem 4.1 and Theorem 4.7 the operator $A_p + \rho$ is compactly invertible for all $\rho > 0$. If $b < 1/2$, then from (4.10) follows

$$a\left\|(A_p+\rho)^{-1}\right\|_{\mathcal{B}(L^p;L^p)} + 2b \le \frac{a}{\rho} + 2b < 1$$

for sufficently great ρ. Hence, due to (4.12) by a general perturbation theorem, cf. [20, Ch. IV, § 1, Th. 1.16], the resolvent of $A_p + T$ is compact. □

Theorem 4.11. *Suppose $p \in\,]1,\infty[$, and let W be a nonnegative, measurable function on Ω with lower bound $W_\bullet$. If the multiplication operator, induced by W on L^p is relatively compact with respect to A_p, then* $\operatorname{dom}(A_p + W)$ *equals* $\operatorname{dom}(A_p)$*, the operator $A_p + W$ is closed, its resolvent is compact, and*

$$\left\|(A_p + W + \rho)^{-1}\right\|_{\mathcal{B}(L^p;L^p)} \leq \frac{1}{\rho + W_\bullet} \quad \textit{for all } \rho > -W_\bullet. \tag{4.13}$$

Moreover, the operator $-(A_p + W)$ generates a strongly continuous semigroup of contractions on L^p. If $\sigma(\partial\Omega \setminus \Gamma) > 0$, or $\int_\Gamma \beta \, d\sigma > 0$, or $W_\bullet > 0$, then this semigroup is even strictly contractive.

Proof. The multiplication operator induced by W maps $\operatorname{dom}(A_p)$ compactly into L^p. N.B. $\operatorname{dom}(A_p)$ equipped with the graph norm compactly embeds into L^p, cf. Remark 4.8. Further, the multiplication operator induced on L^p by the function $\frac{1}{1+W}$ is continuous and injective, because W is nonnegative, i.e.

$$\operatorname{dom}(A_p) \xrightarrow[\text{compact}]{W} L^p \xrightarrow[\text{continuous, injective}]{\frac{1}{1+W}} L^p,$$

and by Ehrling's lemma, cf. e.g. Wloka [32, Ch. I, § 7, Th. 7.3], for every $b > 0$ there is a number $a > 0$, such that

$$\|W\psi\|_{L^p} \leq a \left\| \frac{W\psi}{1+W} \right\|_{L^p} + b\,\|\psi\|_{\operatorname{dom}(A_p)} \leq a\,\|\psi\|_{L^p} + b\,\|(A_p+1)\psi\|_{L^p}.$$

Thus, the multiplication operator induced by W on L^p is relatively bounded by $A_p + 1$ with bound zero, and the assertions about $A_p + W$ follow from Lemma 4.10.

The multiplication operator, induced by $W - W_\bullet$ on L^p, is dissipative. According to Theorem 4.3 the operator $-A_p$ is the infinitesimal generator of a strongly continuous semigroup of contractions, and with the up to now obtained properties of $W - W_\bullet$ a perturbation theorem for such generators, cf. Pazy [25, Ch. 3.3, Cor. 3.3], applies. Thus, $-(A_p + W - W_\bullet)$ is the infinitesimal generator of a strongly continuous semigroup of contractions, and in particular dissipative. Now the criterion [25, Ch. 1.4, Th. 4.2] for dissipativity provides

$$\|(A_p + W - W_\bullet + \rho)\psi\|_{L^p} \geq \rho\,\|\psi\|_{L^p} \quad \text{for all } \rho > 0,\ \psi \in \operatorname{dom}(A_p + W - W_\bullet), \tag{4.14}$$

i.e. the operator $A_p + W - W_\bullet + \rho$ is injective. Due to the compactness of the resolvent, $A_p + W - W_\bullet + \rho$ is also surjective. Consequently, (4.14) implies (4.13).

If $\sigma(\partial\Omega \setminus \Gamma) > 0$ or $\int_\Gamma \beta \, d\sigma > 0$, then the semigroup generated by $-(A_2 + W - W_\bullet)$ on L^2 is strictly contractive, cf. Theorem 2.7. Because the semigroups generated by $-(A_p + W - W_\bullet)$ on L^p are at least contractive, it follows by interpolation that the semigroups must be strictly contractive for all $p \in\,]1,\infty[$. If $W_\bullet > 0$, then the strict contractivity follows from [25, Ch. 1.3, Cor. 3.8]. □

4.c A_p: Semigroups on L^∞ and L^1

Next we will regard the semigroup $\{e^{-t(A_2+W)}\}_{t>0}$ with a nonnegative, bounded potential W on the spaces L^1 and L^∞.

Theorem 4.12. *Let W be a nonnegative L^∞ function. Then the semigroup $e^{-t(A_2+W)}$, $t > 0$, induces semigroups of contractions on L^∞ and L^1. The latter semigroup is strongly continuous, while the first is not.*

Proof. From Theorem 2.13 we know that $e^{-t(A_2+W)} \in \mathcal{B}(L^\infty; L^\infty)$, and $\{e^{-t(A_2+W)}\}_{t>0}$ forms a semigroup on L^∞. It remains to show that $e^{-t(A_2+W)}$ is contractive on L^∞. Indeed, due to the contractivity of $e^{-t(A_2+W)}$ on L^p for any $p \in [2, \infty[$, there is

$$\left\|e^{-t(A_2+W)}\psi\right\|_{L^\infty} \xleftarrow{\infty \leftarrow p} \left\|e^{-t(A_2+W)}\psi\right\|_{L^p} \leq \|\psi\|_{L^p} \xrightarrow{p \to \infty} \|\psi\|_{L^\infty} \quad \text{for all } \psi \in L^\infty.$$

N.B. if $\psi \in L^\infty$, then $\|\psi\|_{L^\infty} = \lim_{p\to\infty} \|\psi\|_{L^p}$. The statement for L^1 follows by a duality argument and the strong continuity in L^2. The semigroup is not strongly continuous on L^∞ because in the opposite case its generator would have to be densely defined in L^∞ according to the Hille–Yosida theorem, cf. Pazy [25, Ch. 1.3, Th. 3.1]. But, due to Proposition 2.11, $\operatorname{dom}(A_\infty + W)$ is contained in a Hölder space C^α, never being dense in L^∞. □

Remark 4.13. Fitting together the results of the Theorems 2.7, 2.8, 2.13, and 4.12 one obtains that $A_2 + W$ is generating a symmetric Markov semigroup which is ultracontractive, cf. Davies [6, Ch. 2]. In particular this implies that it is hypercontractive, cf. Reed, Simon [26, Vol. II, Ch. X.9]. The integral kernels $\mathcal{K}_t$ belonging to the operators $e^{-t(A_2+W)}$, cf. Theorem 3.1, are nonnegative. This follows immediately from the symmetric Markov property and the ultracontractivity, cf. [6, Lemma 2.1.2]. Furthermore, the following statement on the spectral properties may be deduced, cf. [6, Ch. 1.6].

Theorem 4.14. *Suppose $p \in]1, \infty[$, and let W be a nonnegative L^∞ function. The spectrum of $A_p + W$ coincides with the spectrum of $A_2 + W$ and the geometric multiplicities are the same. For every eigenvalue λ of $A_p + W$ the algebraic multiplicity equals the geometric multiplicity, or, in other words, there are no nontrivial Jordan chains. Moreover, the eigenspaces coincide for all p.*

The next result is again a consequence of Theorem 2.13.

Theorem 4.15. *Let V be a real valued L^∞ function. The set of eigenvectors of the operator $A_p + V$ is total in L^p for every $p \in]1, \infty[$.*

Proof. The statement holds true for $p = 2$, due to the selfadjointness of $A_2 + V$. As the sets of eigenvectors for A_p and A_2 are identical, this set is also total in L^p for $p \in]1, 2[$, because in that case L^2 is dense in L^p. Let now p be from $]2, \infty[$. It is easy to see that it suffices to prove the statement for potentials $V = W + 1$, where $W \in L^\infty$ is nonnegative, as one can shift the operator by a scalar without changing the set of eigenvevtors. Let j be the number from Theorem 2.13. Because $-(A_p + W + 1)$ generates a strongly continuous semigroup of contractions on L^p, cf. Theorem 4.11, $\operatorname{dom}\big((A_p+W+1)^j\big)$ is dense in L^p, cf. Pazy [25, Ch. 1.2, Th. 2.7]. Hence, it suffices to show that any element from the space

$$\operatorname{dom}\big((A_p+W+1)^j\big) \subset L^p \hookrightarrow L^2$$

may be approximated by linear combinations of eigenvectors of A_p+W+1. Let for this purpose γ_{L^p} be the embedding constant of $\mathrm{Id} : \operatorname{dom}\big((A_2+W+1)^j\big) \to L^p$, which is finite, cf. Remark 2.14. Further, let $\psi \in \operatorname{dom}\big((A_p+W+1)^j\big)$ and $\varepsilon > 0$ be given. Because the system of eigenvectors of A_2+W+1 is total in L^2, there is a finite sequence $\{\lambda_r\}_r$ of eigenvectors of A_2+W+1, and a finite sequence $\{\mu_r\}_r$ of complex numbers such that

$$\begin{aligned}\varepsilon > \gamma_{L^p}\Big\|\sum_r \mu_r\psi_r - (A_2+W+1)^j\psi\Big\|_{L^2} &= \gamma_{L^p}\Big\|(A_2+W+1)^j\Big(\sum_r \frac{\mu_r}{\lambda_r^j}\,\psi_r - \psi\Big)\Big\|_{L^2}\\ &\ge \Big\|\sum_r \frac{\mu_r}{\lambda_r^j}\,\psi_r - \psi\Big\|_{L^p},\end{aligned}$$

i.e. the eigenvectors of A_p+W+1 form a total set in $\operatorname{dom}\big((A_p+W+1)^j\big)$. □

5 The operators UA_p

Now we turn to the investigation of operators $U \operatorname{div} a \operatorname{grad}$, where U is in all what follows a positive L^∞ function, bounded from below by a strictly positive constant. We will prove that these operators generate analytic semigroups on L^p and that this property is stable with respect to perturbations by first order differential operators, at least for certain p.

5.a Resolvent estimates

This section is devoted to resolvent estimates for operators UA_p+W, where A_p is according to Definition 2.6 and Definition 4.6. The L^∞ function W is always supposed to be nonnegative; by $W_\bullet$ we denote the essential infimum of W on Ω. In the sequel $\mathbb{H}$ will always be the closed complex right half plane. We abbreviate

$$\tau_p = \frac{a_\bullet}{a^\bullet}\,\frac{2\sqrt{p-1}}{p-2} \quad \text{if } p \in \big[\max\big\{4, \tfrac{d+1}{2}\big\}, \infty\big[, \tag{5.1}$$

cf. Theorem 4.3.

Theorem 5.1. *For any $p \in\,]1,\infty[$ there exist a constant M_p such that*

$$\big\|\big(A_p + WU^{-1} + \rho U^{-1}\big)^{-1}\big\|_{\mathcal{B}(L^p;L^p)} \le \frac{M_p\,\|U\|_{L^\infty}}{|\rho| + W_\bullet} \quad \textit{for all } \rho \in \mathbb{H}\setminus\{0\}.$$

The constant M_p can be specified as follows:

(i) *If $p=2$, then $M_2 = \sqrt{2}$.*

(ii) *If* $p \in \big[\max\big\{4, \frac{d+1}{2}\big\}, \infty\big[$, *then* $M_p = \dfrac{\sqrt{1+\tau_p^2}}{\min\{1, \tau_p\}}$.

(iii) *If* $p \in \big]2, \max\big\{4, \frac{d+1}{2}\big\}\big[$, *then* $M_p = \big(\sqrt{2}\big)^{1-\theta_p}\left(\dfrac{\sqrt{1+\tau^2}}{\min\{1, \tau\}}\right)^{\theta_p}$.

(iv) *If* $p \in]1, 2[$, *then* $M_p = M_{p'}$ *for* $1/p + 1/p' = 1$.

Here, the constants $\tau > 0$ *and* $\theta_p \in [0, 1]$ *are defined by*

$$\tau = \tau_{\max\{4,(d+1)/2\}} \quad \text{and} \quad \frac{1}{p} = \frac{\theta_p}{\max\big\{4, \frac{d+1}{2}\big\}} + \frac{1-\theta_p}{2}.$$

Proof. We regard firstly the selfadjoint case $p = 2$. Let γ be the lower form bound for $\mathfrak{t}$, which is nonnegative, cf. Lemma 2.5. It follows

$$\begin{aligned}\big\|\big(A_2 + WU^{-1} + \rho\, U^{-1}\big)\psi\big\|_{L^2}\|\psi\|_{L^2} &\geq \left|\mathfrak{t}[\psi, \psi] + \int_\Omega (\rho + W)U^{-1}|\psi|^2\, dx\right| \\ &\geq \left|\gamma + \frac{\rho + W_\bullet}{\|U\|_{L^\infty}}\right| \|\psi\|_{L^2}^2 \geq \frac{|\rho + W_\bullet|}{\|U\|_{L^\infty}}\|\psi\|_{L^2}^2 \geq \frac{|\rho| + W_\bullet}{\sqrt{2}\,\|U\|_{L^\infty}}\|\psi\|_{L^2}^2, \end{aligned} \tag{5.2}$$

i.e. the operator $A_2 + WU^{-1} + \rho\, U^{-1}$ is injective. If one can show additionally the surjectivity of $A_2 + WU^{-1} + \rho\, U^{-1}$, then (5.2) implies the assertion. Indeed, $(W + \rho)U^{-1}$ is a bounded linear operator on L^2, hence it is A_2-bounded with bound equal to zero. Thus, Lemma 4.10 applies and provides that $A_2 + WU^{-1} + \rho\, U^{-1}$ has a compact resolvent, hence, it is surjective as it is injective.

Now let $p \in \big[\max\big\{4, \frac{d+1}{2}\big\}, \infty\big[$, $1/p + 1/p' = 1$, and τ_p according to (5.1). We define

$$\hat\rho = \begin{cases} 1 + \tau_p i & \text{if } \rho \in \mathbb{R} \text{ and } \rho > 0, \\ 1 - \tau_p \operatorname{sign}(\Im\rho) i & \text{if } \rho \in \mathbb{H} \setminus \mathbb{R}. \end{cases} \tag{5.3}$$

If $\psi \in \operatorname{dom}(A_p)$ and $\rho \in \mathbb{H} \setminus \{0\}$, then due to (4.1)

$$\begin{aligned}\sqrt{1+\tau_p^2}\,\big\|\big(A_p + (W+\rho)U^{-1}\big)\psi\big\|_{L^p}\|\psi\|_{L^p} &= |\hat\rho|\,\big\|\big(A_p + (W+\rho)U^{-1}\big)\psi\big\|_{L^p}\,\|J_p\psi\|_{L^{p'}} \\ &\geq \big|\hat\rho\,\big\langle\big(A_p + (W+\rho)U^{-1}\big)\psi, J_p\psi\big\rangle\big| \\ &\geq \Re\big(\hat\rho\,\big\langle\big(A_p + (W+\rho)U^{-1}\big)\psi, J_p\psi\big\rangle\big) \\ &= \Re\big(\hat\rho\,\Re\langle A_p\psi, J_p\psi\rangle + i\hat\rho\,\Im\langle A_p\psi, J_p\psi\rangle\big) \\ &\quad + \Re(\hat\rho\rho)\big\langle U^{-1}\psi, J_p\psi\big\rangle + \Re(\hat\rho)\big\langle WU^{-1}\psi, J_p\psi\big\rangle. \end{aligned}$$

Using (5.3) the summands on the right hand side can be estimated by

$$\Re\big(\hat\rho\,\Re\langle A_p\psi, J_p\psi\rangle + i\hat\rho\,\Im\langle A_p\psi, J_p\psi\rangle\big) \geq \Re\langle A_p, J_p\psi\rangle - \tau_p\,\big|\Im\langle A_p\psi, J_p\psi\rangle\big|$$

and

$$\Re(\hat\rho\rho)\big\langle U^{-1}\psi, J_p\psi\big\rangle + \Re(\hat\rho)\big\langle WU^{-1}\psi, J_p\psi\big\rangle \geq \frac{\Re\rho + \tau_p\,|\Im\rho| + W_\bullet}{\|U\|_{L^\infty}}\|\psi\|_{L^p}^2.$$

Together with (4.6) this yields

$$\sqrt{1+\tau_p^2}\,\big\|\big(A_p+(W+\rho)U^{-1}\big)\psi\big\|_{L^p}\|\psi\|_{L^p} \geq \min\{1,\tau_p\}\,\frac{|\rho|+W_\bullet}{\|U\|_{L^\infty}}\,\|\psi\|_{L^p}^2.$$

Hence, for all $\psi \in \mathrm{dom}(A_p)$ and all $\rho \in \mathbb{H}\setminus\{0\}$ we obtain

$$\|\psi\|_{L^p} \leq \frac{\|U\|_{L^\infty}\sqrt{1+\tau_p^2}}{(|\rho|+W_\bullet)\min\{1,\tau_p\}}\,\big\|\big(A_p+(W+\rho)U^{-1}\big)\psi\big\|_{L^p}. \tag{5.4}$$

Now the assertion follows in the same way as in the case $p=2$.

If $p \in \left]2, \max\left\{4, \frac{d+1}{2}\right\}\right[$, then interpolation between the previous cases $p=2$ and $p = \max\left\{4, \frac{d+1}{2}\right\}$ with the Riesz–Thorin theorem provides the stated result.

If $p \in \left]1,2\right[$, then one obtains the assertion by duality: Definition 4.6 and Kato [20, Ch. III, § 5, Th. 5.30] provide

$$\big(A_p+(W+\rho)U^{-1}\big)^{-1} = \Big(\big(A_{p'}+(W+\overline{\rho})U^{-1}\big)^{-1}\Big)^*$$

for $1/p+1/p'=1$, and the already proved cases imply the assertion. □

Theorem 5.2. *For any $p \in \left]1,\infty\right[$ the operator UA_p+W is closed, has the same domain as A_p, and is the infinitesimal generator of an analytic semigroup on L^p. More precisely, for any $\rho \in \mathbb{H}\setminus\{0\}$ one has*

$$\big\|\big(UA_p+W+\rho\big)^{-1}\big\|_{\mathcal{B}(L^p;L^p)} \leq M_p\,\big\|U\big\|_{L^\infty}\big\|U^{-1}\big\|_{L^\infty}\,\frac{1}{|\rho|+W_\bullet}\,, \tag{5.5}$$

where the constants M_p are those from Theorem 5.1.

Proof. The multiplication operator induced by U provides a linear homeomorphism on L^p. Thus the operators UA_p+W and $A_p+U^{-1}W$ have the same domain and are closed simultaneously, i.e. UA_p+W is closed (cf. Theorem 4.1 and Theorem 4.7) and has the domain $\mathrm{dom}(A_p)$.

Let ρ be in $\mathbb{H}\setminus\{0\}$. According to Theorem 5.1 the operator $A_p+(W+\rho)U^{-1}$ is continuously invertible. Hence, $UA_p+W+\rho$ is continuously invertible and

$$\big\|\big(UA_p+W+\rho\big)^{-1}\big\|_{\mathcal{B}(L^p;L^p)} \leq \big\|\big(A_p+(W+\rho)U^{-1}\big)^{-1}\big\|_{\mathcal{B}(L^p;L^p)}\,\big\|U^{-1}\big\|_{L^\infty}$$

holds. Thus, the asserted inequality (5.5) follows immediately from Theorem 5.1. Estimate (5.5) implies that UA_p+W is the infinitesimal generator of an analytic semigroup on L^p, cf. e.g. Pazy [25, Ch. 2.5]. □

Remark 5.3. If $\sigma(\partial\Omega\setminus\Gamma)>0$, or $\int_\Gamma \beta\,d\sigma>0$, or $W_\bullet>0$, then $\rho=0$ can be admitted in (5.5).

5.b Perturbations by first order differential operators

Our next aim is to investigate the influence of perturbations upon an operator UA_p by first order differential operators. First we quote a regularity result for elliptic boundary value problems which we will need in the sequel.

Proposition 5.4 (Cf. GRÖGER, REHBERG [18]). *Let $\Omega \cup \Gamma$ be a regular set in the sense of Definition 2.1, A_2 be according to Definition 2.6, and $0 < a_\bullet \leq a^\bullet < \infty$ be the constants from* (2.5). *There is a real constant $\varepsilon = \varepsilon(\Omega, \Gamma, a_\bullet, a^\bullet) > 0$, such that $(A_2 + 1)^{-1}$ continuously extends to a topological isomorphism between $W^{-1,p}$ and $W_0^{1,p}$ for all $p \in [2, 2+\varepsilon[$. Denoting the inverse of this isomorphism by B_p, one has the following resolvent estimate:*

$$\left\|(B_p + \rho)^{-1}\right\|_{\mathcal{B}(W^{-1,p};W^{-1,p})} \leq \frac{N_p}{|\rho| + 1} \quad \textit{for all } \rho \in \mathbb{H}, \tag{5.6}$$

where the constant N_p depends on Ω, Γ, $a_\bullet$, and $a^\bullet$, and $\mathbb{H}$ is the closed complex right half plane.

Remark 5.5. It should be noticed that in view of the example of SHAMIR [27, p. 151] one cannot expect in general that ε becomes much greater than zero, even for a smooth domain and constant coefficients.

Theorem 5.6. *Let $b_1, b_2, \ldots, b_d$, and c be essentially bounded, complex valued functions on Ω. We define the first order differential operator $T_p : W_0^{1,p} \longrightarrow L^p$ by*

$$T_p : \psi \longmapsto \sum_{k=1}^{d} b_k \frac{\partial \psi}{\partial x_k} + c\psi. \tag{5.7}$$

Let ε be the constant from Proposition 5.4. If $p \in \left]\frac{2d}{d+2}, 2+\varepsilon\right[$, then

(i) $\operatorname{dom}(UA_p)$ *compactly embeds into* $\operatorname{dom}(T_p) = W_0^{1,p}$.

(ii) *T_p is relatively bounded with respect to UA_p and the bound is equal to zero.*

(iii) *$UA_p + T_p$ has the same domain as A_p and is closed.*

(iv) *$UA_p + T_p$ generates an analytic semigroup on L^p.*

(v) *The resolvent of $UA_p + T_p$ is compact.*

Proof. According to Theorem 5.2 there is $\operatorname{dom}(A_p) = \operatorname{dom}(UA_p)$. Let $\mathcal{M} \subset L^p$ be a set such that $(A_p + 1)\mathcal{M}$ is bounded in L^p. Thus, $(A_p + 1)\mathcal{M}$ is a precompact set in $W^{-1,p}$. If $p \in [2, 2+\varepsilon[$, then Proposition 5.4 implies that $\mathcal{M}$ is precompact in $W_0^{1,p}$. If $p \in \left]\frac{2d}{d+2}, 2\right[$, then the compactness of the embedding $L^p \hookrightarrow W^{-1,2}$ provides the precompactness of $(A_p + 1)\mathcal{M}$ in $W^{-1,2}$. Knowing this, the Lax–Milgram theorem implies the precompactness of $\mathcal{M}$ in $W_0^{1,2}$ and, by embdding, also in $W_0^{1,p}$. Thus, taking onto account

$$\|T_p \psi\|_{L^p} \leq \max\left\{\|c\|_{L^\infty}, \|b_1\|_{L^\infty}, \ldots, \|b_d\|_{L^\infty}\right\} \|\psi\|_{W^{1,p}} \quad \text{for all } \psi \in W_0^{1,p},$$

the claim (i) is proved. Assertion (ii) is implied by Ehrling's lemma, cf. WLOKA [32, Ch. I, § 7, Th. 7.3].

The claims (iii) and (iv) follow from (ii), Theorem 5.2 and abstract perturbation theorems, cf. KATO [20, Ch. IV, § 1, Th. 1.1 and Ch. IX, § 2, Th. 2.4] and PAZY [25, Ch. 3.2, Th. 2.1]. In view of $UA_p + T_p + \rho = U\bigl(A_p + U^{-1}T_p + U^{-1}\rho\bigr)$ it sufficed to prove the assertion (v) for $U \equiv 1$. In this case it follows from (i), (ii) and Lemma 4.10. □

Remark 5.7. Theorem 5.6 is primarily relevant in the low dimensional cases $d \in \{2, 3, 4\}$ where the permitted interval for p intersects the p-interval where $\mathrm{dom}(A_p)$ continuously embeds into a space C^α, cf. Proposition 2.11. Further, Theorem 5.6 is in correspondence to the results of ARENDT, TER ELST [4], which also require restrictions on the first order differential operators.

6 Induced analytic semigroups on fractional Sobolev and Besov spaces

The operator $-A_2$ induces analytic semigroups on L^p space, cf. Theorem 4.3 and on certain $W^{-1,q}$ spaces, cf. Proposition 5.4 and GRÖGER, REHBERG [18]. By interpolation it induces analytic semigroups also on fractional Sobolev spaces and on Besov spaces.

Theorem 6.1. *Let ε be the constant from Proposition 5.4. If*

$$q \in [2, 2+\varepsilon[, \quad p \in]1, \infty[, \quad \theta \in]0, 1[, \quad s \in [1, \infty[, \tag{6.1}$$

then the operator $-A_2$, cf. Definition 2.6, induces an analytic semigroup on any of the interpolation spaces

$$\bigl(W^{-1,q}, L^p\bigr)_{\theta,s} \quad \text{and} \quad \bigl[W^{-1,q}, L^p\bigr]_\theta\,. \tag{6.2}$$

Proof. Let A_p be the operators on L^p from Definition 2.6 and Definition 4.6, let B_q be the operators on $W^{-1,q}$ from Proposition 5.4, and let $\mathbb{H}$ be the closed complex right half plane. The resolvent estimates for the infinitesimal generators in the interpolation spaces, which imply the analyticity of the semigroups, cf. PAZY [25, Ch. 2.5], result by interpolation due to the following facts:

(i) $W^{-1,q}$ and L^p form an interpolation couple, because both embed continuously into $W^{-1,r}$, $r = \min\{p, q\}$.

(ii) For any $\rho \in \mathbb{H}$ the operators $(A_p + 1 + \rho)^{-1}$ and $(B_q + \rho)^{-1}$ coincide on $L^{\max\{2,p\}}$. This set is dense in L^p and $W^{-1,q}$; thus $(A_p+1+\rho)^{-1}$ and $(B_q+\rho)^{-1}$ may be viewed as the same operator.

(iii) Real and complex interpolation are exact interpolation functors of type θ, cf. e.g. TRIEBEL [29, Ch. 1.2.2].

It remains to show that the domain of the operator on the corresponding interpolation space is dense in this space: one knows, cf. [29, Ch. 1.6.2 and 1.9.3], that $W^{-1,q} \cap L^p$ is dense in $\big(W^{-1,q}, L^p\big)_{\theta,s}$ and in $\big[W^{-1,q}, L^p\big]_\theta$. (Because this is not necessarily true for the real interpolation index (θ,∞), cf. [29, Rem. 1.6.2] one has to exclude this index in the assertion.) Further, the norm

$$\max\big\{\|\psi\|_{L^p}, \|\psi\|_{W^{-1,q}}\big\} \tag{6.3}$$

on $W^{-1,q} \cap L^p$ is stronger than the induced norm from any of the interpolation spaces, cf. [29, Ch. 1.3.3 and 1.9.3].

Let now $p_0 \geq \max\{2, p\}$ be chosen, such that L^{p_0} continuously embeds into L^p and into $W^{-1,q}$. Then one has $\mathrm{dom}(A_{p_0}) \subset \mathrm{dom}(A_p|W^{-1,q} \cap L^p)$ and the images under the embedding mappings $L^{p_0} \hookrightarrow W^{-1,q}$ and $L^{p_0} \hookrightarrow L^p$ are dense in both spaces. By Theorem 4.5 $\mathrm{dom}(A_p)$ is dense in L^{p_0} and, consequently, dense in $W^{-1,q} \cap L^p$ in the norm (6.3). □

Remark 6.2. According to duality theorems from interpolation theory, cf. Triebel [29, Ch. 1.11], there is

$$\big(W^{-1,q}, L^p\big)_{\theta,s} = \big(\big(W_0^{1,q'}, L^{p'}\big)_{\theta,s'}\big)^* \quad \text{and} \quad \big[W^{-1,q}, L^p\big]_\theta = \big(\big[W_0^{1,q'}, L^{p'}\big]_\theta\big)^*$$

for $s \in\,]1,\infty[$, where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, and $1/s + 1/s' = 1$.

7 Applications to parabolic equations

The property of $-UA_p$ to be an infinitesimal generator of an analytic semigroup on L^p paves the way for treating the corresponding parabolic equations on L^p, cf. Amann [3], Lunardi [23], Pazy [25]. With respect to the results obtained by Arendt, ter Elst [4] the basic finding in our context is the Hölder continuity of solutions to the parabolic equation in space and time, which ultimatily rests on Proposition 2.11.

Lemma 7.1. *Suppose $p > d/2$ and $p \geq 2$ and let $\alpha \in\,]0,1]$ be the Hölder exponent from Proposition 2.11. There are numbers $\Theta \in\,]0,1[$ and $0 < \tilde{\alpha} < \alpha$ such that*

$$\mathrm{dom}\left((UA_p + 1)^\Theta\right) \hookrightarrow C^{\tilde{\alpha}}. \tag{7.1}$$

Any function $u \in C^1(S; L^p) \cap C\big(S; \mathrm{dom}(UA_p)\big)$, where $S = [T_0, T]$ is an interval of the real axis, is Hölder continuous in space and time, more precisely $u \in C^{1-\Theta}(S; C^{\tilde{\alpha}}) \hookrightarrow C^\beta(S \times \overline{\Omega})$.

Proof. Let $\Theta \in\,]0,1[$. If $\theta < \Theta$, then the embeddings

$$\mathrm{dom}\left((UA_p + 1)^\Theta\right) \hookrightarrow \big(\mathrm{dom}(UA_p + 1), L^p\big)_{1-\Theta,\infty} \hookrightarrow \big(\mathrm{dom}(UA_p + 1), L^p\big)_{1-\theta,1}$$

are continuous, cf. Triebel [29, Ch. 1.15.2 and 1.3.3]. The chain of continuous embeddings may be continued by applying Proposition 2.11 and [29, Ch. 1.10.3]

$$\big(\mathrm{dom}(UA_p + 1), L^p\big)_{1-\theta,1} \hookrightarrow \big(\mathrm{dom}(A_p), L^p\big)_{1-\theta,1} \hookrightarrow \big(C^\alpha, L^p\big)_{1-\theta,1} \hookrightarrow \big[C^\alpha, L^p\big]_{1-\theta}.$$

Thus, it remains to show that

$$\big[C^\alpha, L^p\big]_{1-\theta} = \big[L^p, C^\alpha\big]_\theta \hookrightarrow C^{\tilde{\alpha}} \quad \text{for some } \tilde{\alpha} > 0. \tag{7.2}$$

By means of localization, transformation and reflection principles, cf. Griepentrog [13, Ch. 1.1], one can construct a simultaneous extension operator for

$$C^\alpha \longrightarrow C^\alpha(\mathbb{R}^d) \quad \text{and} \quad L^p \longrightarrow L^p(\mathbb{R}^d),$$

cf. Troianiello [31, Ch. 1.2.2, Th. 1.2]. Hence, it is sufficient to prove

$$\bigl[L^p(\mathbb{R}^d), C^\alpha(\mathbb{R}^d)\bigr]_\theta \hookrightarrow C^{\tilde\alpha}(\mathbb{R}^d),$$

cf. e.g. [29, Ch. 1.2.4] instead of (7.2). According to Triebel [30], Stein [28, Ch. VI.2.2] the space $C^\alpha(\mathbb{R}^d)$ is identical with the Besov space $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$. Further, the interpolation space with L^p is identical with a Lizorkin–Triebel space and continuously embeds into a Hölder space. More precisely,

$$\bigl[L^p(\mathbb{R}^d), B^\alpha_{\infty,\infty}(\mathbb{R}^d)\bigr]_\theta \overset{\text{def}}{=} F^{\theta\alpha}_{p/(1-\theta),2/(1-\theta)}(\mathbb{R}^d) \hookrightarrow C^{\tilde\alpha}(\mathbb{R}^d) \quad \text{if } \tilde\alpha \overset{\text{def}}{=} \theta\alpha - (1-\theta)\,\frac{d}{p} > 0,$$

cf. [30] and [29, Ch. 2.8.1], respectively. By choosing θ and Θ sufficiently close to 1 on always finds a strictly positive $\tilde\alpha$.

Let now u be from $C^1(S; L^p) \cap C\bigl(S; \mathrm{dom}(UA_p)\bigr)$ and let s, t be different numbers from the interval S. We have by the first statement of this lemma

$$\frac{\|u(s)-u(t)\|_{C^{\tilde\alpha}}}{|s-t|^{1-\Theta}} \le \|\mathrm{Id}\|_{\mathcal{B}(\mathrm{dom}((UA_p+1)^\Theta);C^{\tilde\alpha})}\,\frac{\bigl\|(UA_p+1)^\Theta(u(s)-u(t))\bigr\|_{L^p}}{|s-t|^{1-\Theta}}\,.$$

There is a constant γ such that this inequality may be prolonged, cf. Pazy [25, Ch. 2, Th. 6.10], as follows

$$\frac{\|u(s)-u(t)\|_{C^{\tilde\alpha}}}{|s-t|^{1-\Theta}} \le \gamma\,\bigl\|(UA_p+1)(u(s)-u(t))\bigr\|^\Theta_{L^p}\left(\frac{\|u(s)-u(t)\|_{L^p}}{|s-t|}\right)^{1-\Theta}.$$

Due to the supposition on u, the expression on the right hand side is uniformly bounded for all s, $t \in S$, $s \neq t$. □

7.a Linear parabolic equations

Let us consider an initial-boundary value problem

$$\frac{\partial u}{\partial t} - U \operatorname{div} a \operatorname{grad} u = f, \quad u(0) = u_0, \quad \text{and boundary conditions.}$$

If we regard this equation in L^p, then $-U \operatorname{div} a \operatorname{grad}$ gets the precise meaning of the operator UA_p, cf. Section 5, and the fully elaborated existence, uniqueness and regularity theory for parabolic equations related to the infinitesimal generator of an analytic semigroup applies, cf. Amann [3], Lunardi [23], Pazy [25]. We formulate the new and essential facts for operators with mixed boundary conditions.

Theorem 7.2. *Suppose $p > d/2$, $p \geq 2$, $T > 0$ and let u_0 be from L^p. If f is a Hölder continuous mapping from $[0, T]$ into L^p, then for any $T_0 \in]0, T[$ the solution of*

$$\frac{\partial u}{\partial t} + UA_p u = f, \quad u(0) = u_0 \tag{7.3}$$

is Hölder continuous on the set $[T_0, T] \times \overline{\Omega}$. If in addition the initial value u_0 is from $\mathrm{dom}(UA_p) = \mathrm{dom}(A_p)$, *then the solution is Hölder continuous on $[0, T] \times \overline{\Omega}$.*

Proof. The proof results from Lemma 7.1 and classical regularity results, cf. PAZY [25, Ch. 4.3]. □

Remark 7.3. In the case $u_0 = 0$ the Hölder continuity of solutions of (7.3) on $[0, T] \times \overline{\Omega}$ has been obtained in GRIEPENTROG [13, Ch. 2.3] under weaker assumptions on the right hand side f in a completely different way.

If $U \equiv 1$, then the suppositions on the right hand side f may be considerably relaxed.

Theorem 7.4. *Let W be a nonnnegative L^∞ function, and let $S = [0, T]$, $T > 0$ be an interval. Then for any $q \in]1, \infty[$ the operator $A_p + W$ satisfies q-regularity, i.e. the operator $\frac{\partial}{\partial t} + A_p + W$ provides a topological isomorphism between*

$$L^q\big(S; \mathrm{dom}(A_p)\big) \cap \big\{v \in W^{1,q}(S; L^p) : v(0) = 0\big\} \quad \text{and} \quad L^q(S; L^p).$$

If $p > d/2$, $p \geq 2$, $u_0 \in \mathrm{dom}(A_p)$, and $f \in L^\infty(S; L^p)$, then the solution u of the initial value problem

$$\frac{\partial u}{\partial t} + (A_p + W)u = f, \quad u(0) = u_0 \tag{7.4}$$

is Hölder continuous on $S \times \overline{\Omega}$.

Proof. The first statement follows from the positivity of the operator $A_2 + W$, Theorem 4.12 and a result of LAMBERTON, cf. [21].

Let α be the Hölder exponent from Proposition 2.11. By the trace method in interpolation theory, cf. ASHYRALYEV, SOBOLEVSKII [5, Ch. 1.3] or TRIEBEL [29, Ch. 1.8.2], follows for any $q \in]1, \infty[$ the existence of a continuous embedding

$$L^q\big(S; \mathrm{dom}(A_p)\big) \cap W^{1,q}(S; L^p) \hookrightarrow C\big(S; \big(\mathrm{dom}(A_p), L^p\big)_{1/q,q}\big) \hookrightarrow C\big(S; \big(C^\alpha, L^p\big)_{1/q,q}\big).$$

We choose q great enough such that $\big(C^\alpha, L^p\big)_{1/q,q}$ continuously embeds into a space C^β with some $\beta > 0$, and η small enough such that $\big(C^\beta, L^p\big)_{\eta,r}$ still embeds into a space C^γ for some $\gamma > 0$; both is possible by Lemma 7.1. Defining $\Theta = \eta + (1 - \eta)/q$, we have by the reiteration theorem for real interpolation

$$\big(C^\alpha, L^p\big)_{\Theta,r} = \big(\big(C^\alpha, L^p\big)_{1/q,q}, L^p\big)_{\eta,r}$$

and by the suppositions on q and η the continuity of the embedding

$$\big(\big(C^\alpha, L^p\big)_{1/q,q}, L^p\big)_{\eta,r} \hookrightarrow \big(C^\beta, L^p\big)_{\eta,r} \hookrightarrow C^\gamma,$$

cf. [29, Ch. 1.10.3]. Using the corresponding interpolation inequality, one can estimate with some $\delta > 0$ for any $s, t \in S$, $s \neq t$:

$$\begin{aligned}
\frac{\|u(s)-u(t)\|_{(C^\alpha,L^p)_{\Theta,r}}}{|s-t|^{\Theta-1/q}} &\le \delta\, \frac{\|u(s)-u(t)\|_{L^p}^{\eta}}{|s-t|^{\Theta-1/q}}\, \|u(s)-u(t)\|_{(C^\alpha,L^p)_{1/q,q}}^{1-\eta} \\
&\le \delta\, \frac{\left\|\int_s^t u'(\tau)\,d\tau\right\|_{L^p}^{\eta}}{|s-t|^{\Theta-1/q}} \left(2\sup_{\tau\in S}\|u(\tau)\|_{(C^\alpha,L^p)_{1/q,q}}\right)^{1-\eta} \\
&\le \delta\, \frac{\left(\int_s^t \|u'(\tau)\|_{L^p}^q\,d\tau\right)^{\eta/q} |s-t|^{\eta/q'}}{|s-t|^{\Theta-1/q}} \left(2\sup_{\tau\in S}\|u(\tau)\|_{(C^\alpha,L^p)_{1/q,q}}\right)^{1-\eta}
\end{aligned}$$

By the definition of Θ we have $n/q' = \Theta - 1/q$, what proves the boundedness of the right hand side, independently from $s, t \in S$, $s \neq t$. $\square$

7.b Semilinear parabolic equations

Theorem 7.5. *Let $\mathcal{F} : [0,T] \times \mathbb{C} \longrightarrow \mathbb{C}$ be a function which is Hölder continuous in the first argument and locally Lipschitz continuous in the second. (For $t \in [0,T]$ we identify the function $\mathcal{F}(t,\cdot)$ with the induced Nemytskii operator on L^∞.) We assume the existence of a uniform Hölder exponent for every bounded set of $z \in \mathbb{C}$, and that there are local Lipschitz constants uniform over $[0,T]$. Suppose $p > d/2$ and $p \ge 2$ and let $\Theta \in \,]0,1[$ be such that* $\operatorname{dom}\big((A_p+1)^\Theta\big) \hookrightarrow C^{\tilde\alpha}$ *for some $\tilde\alpha > 0$. (Such numbers Θ and $\tilde\alpha$ exist according to Lemma 7.1.) Then the equation*

$$\frac{\partial u}{\partial t} + UA_p u = \mathcal{F}(t,u), \quad u(0) = u_0 \in \operatorname{dom}\big((A_p+1)^\Theta\big) \tag{7.5}$$

has a unique local solution

$$u \in C\big([0,T_1[;L^p\big) \cap C^1\big(]0,T_1[;L^p\big) \cap C\big(]0,T_1[;\operatorname{dom}(A_p)\big),$$

which, by Lemma 7.1, is Hölder continuous on any set $[T_0,T_2]\times\overline{\Omega}$, when $0 < T_0 < T_2 < T_1$.

Proof. The local existence, uniqueness and asserted regularity follow from standard results, cf. Pazy [25, Ch. 6.3, Th. 3.1 and Ch. 4.3, Th. 3.5], provided one can prove that

$$[0,T] \times \operatorname{dom}\big((A_p+1)^\Theta\big) \ni (t,\psi) \longmapsto \mathcal{F}(t,\psi) \in L^\infty \hookrightarrow L^p$$

is Hölder continuous in the first variable and Lipschitzian in the second. But this follows immediately from our supposition $\operatorname{dom}\big((A_p+1)^\Theta\big) \hookrightarrow C^{\tilde\alpha}$ and the suppositions on $\mathcal{F}$. $\square$

Remark 7.6. Much more could be said about fine properties of solutions to (7.3), (7.4) and (7.5) in dependence of the initial values u_0 and $\mathcal{F}(0,u_0)$, respectively, for particulars we refer to Lunardi [23]. We do not expatiate this here because in our highly nonsmooth constellation it is impossible in general to determine $\operatorname{dom}(A_p)$ or $\operatorname{dom}\big((A_p+1)^\Theta\big)$ explicitly, or to say how regular $\mathcal{F}(0,u_0)$ is.

As mentioned in the introduction, we are interested primarily in reaction-diffusion equations, especially in semiconductor equations. This requires a solution theory for coupled evolution equations, where, among others, the following two problems, cf. Pazy [25, Ch. 5.6], arise:

Problem 7.7. Under what conditions on two L^∞ functions a and $\tilde{a}$ with stricly positive lower bounds the domains of the corresponding operators A_p and $\tilde{A}_p$ coincide? Do, at least the domains of fractional powers of A_p and $\tilde{A}_p$ coincide?

Problem 7.8. Let $t \longmapsto a_t$ be a function from $[0,T]$ into L^∞ and let $A_{p,t}$ be the operator corresponding to a_t, according to Definition 2.6. What can be said about Hölder continuity, in an appropriate sense, of the function $t \longmapsto A_{p,t}$?

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