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Local existence, uniqueness and smooth dependence for nonsmooth quasilinear parabolic problems

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Abstract. We prove local existence, uniqueness, Hölder regularity in space and time, and smooth dependence in Hölder spaces for a general class of quasilinear parabolic initial boundary value problems with nonsmooth data. As a result the gap between low smoothness of the data, which is typical for many applications, and high smoothness of the solutions, which is necessary for the applicability of differential calculus to abstract formulations of the initial boundary value problems, has been closed. The theory works for any space dimension, and the nonlinearities in the equations as well as in the boundary conditions are allowed to be nonlocal and to have any growth. The main tools are new maximal regularity results (Griepentrog in Adv Differ Equ 12:781–840, 1031–1078, 2007) in Sobolev–Morrey spaces for linear parabolic initial boundary value problems with nonsmooth data, linearization techniques and the Implicit Function Theorem.

1. Introduction

This paper concerns initial boundary value problems for quasilinear second order parabolic equations in divergence form with nonsmooth data and for weakly coupled systems of such equations. Here *nonsmooth data* means that the domain can be nonsmooth (but has to be a set with Lipschitz boundary), that the coefficients of the equations and the boundary conditions may be discontinuous with respect to the space and time variables (but have to be smooth with respect to the unknown function u), and that the boundary conditions can change type (mixed boundary conditions, where the Dirichlet and the Neumann boundary parts can touch). The coefficients in the equations as well as in the boundary conditions may be local or nonlocal functions of u, they can have any growth with respect to u, and the space dimension can be arbitrary. Typical applications are transport processes of charged particles in semiconductor heterostructures, phase separation processes of nonlocally interacting particles, chemotactic aggregation in heterogeneous environments as well as optimal control by means of quasilinear elliptic and parabolic PDEs with nonsmooth data.

Main results: The main results are Theorem 4.1 about regularity and smooth dependence, Theorem 4.2 about local in time existence, Theorem 4.3 about uniqueness and Theorem 4.5 about Hölder continuity of the first time derivative of the solution.

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Here regularity and smooth dependence means that the solutions are Hölder continuous in space and time and depend smoothly on the data in parabolic Hölder space norms over the space–time cylinder. So the door is open to apply the powerful theorems of differential calculus (principle of linearized stability, analytic bifurcation theory, existence and persistence of smooth invariant manifolds, enabling a reduction of the study of the long-time dynamics to finite dimensions) to those initial boundary value problems. In particular, Theorem 4.1 shows how to apply the classical Newton iteration procedure with quadratic convergence rate in parabolic Hölder space norms.

Remark that here in the introduction we formulate the results in the language of Hölder spaces, which is satisfactory for most of the applications. But it turns out that the proofs cannot be done by working in Hölder spaces or in Sobolev spaces because the linearized diffential operators do not have the maximal regularity property between such spaces in the case of general nonsmooth data and arbitrary space dimension.

Main techniques: We work in parabolic Sobolev–Morrey–Campanato spaces. Those spaces for functions are known (but much less used than Sobolev spaces) since 40 years, but for functionals almost unknown and not used. Concerning the delicate questions about embedding theorems, traces on Lipschitz hypersurfaces and behavior under Lipschitz transformations and pointwise multiplication, which appear necessarily in the analysis, there existed only a few results, and those mainly under unrealistic high smoothness assumptions on the data. In [19] a general theory was developed for parabolic Sobolev–Morrey–Campanato spaces on domains with Lipschitz boundary and Lipschitz hypersurfaces as well for functions as for functionals. In [20] it was shown that a general class of linear second order parabolic differential operators has the maximal regularity property between such spaces. Now, in the present paper we show that these maximal regularity properties together with linearization techniques and the Implicit Function Theorem give local existence, uniqueness, regularity and smooth dependence for a general class of quasilinear parabolic initial boundary value problems with nonsmooth data.

Remark that the authors together with GRÖGER realized the same program for quasilinear elliptic boundary value problems with nonsmooth data: applications of differential calculus to nonlinear problems [23] via maximal regularity in Sobolev–Morrey–Campanato spaces for linear problems [18] after investigation of the needed properties of the spaces [17,22].

Related work: Let us close this introduction by some remarks concerning the so far existing literature about quasilinear parabolic initial boundary value problems.

As far as we know up to now there did not exist any results about smoothness (at least continuous differentiability) of the data-to-solution-map for quasilinear parabolic initial boundary value problems with nonsmooth data.

What concerns local existence, uniqueness and continuous dependence for quasilinear parabolic initial boundary value problems, we learned a lot from the work of AMANN [2–6]. There the main tool is maximal L^p regularity of the corresponding linear operators. The smoothness assumptions on the data are slightly, but, from the point of view of applications, essentially stronger than ours: the leading order coefficients of the elliptic differential operator have to be continuous in space and time, and the Dirichlet and Neumann boundary parts in the mixed boundary conditions are not allowed to touch. On the other hand, AMANN's assumptions on the possibly nonlocal coefficient functions are weaker than ours: he includes time delay, we do not. Remark that [3, Theorem 4.1] gives Gateaux differentialbility of the data-to-solution-map on Fréchet spaces of coefficient functions, which is a first step to smoothness.

What concerns nonsmoothness of the data, the assumptions in [10,24,25] for local existence and uniqueness for quasilinear parabolic initial boundary value problems are as weak as ours. In particular, some domains, which are not Lipschitz domains in the commonly used sense (like two crossing three-dimensional cuboids) are allowed as well as nonlinear Robin or Neumann boundary conditions. Further, in [10,24,25] as well as in our paper the concept of GRÖGER's regular sets, see [21], is used, which enables to handle mixed boundary value problems with touching Dirichlet and Neumann boundary parts. In [10,24,25] the assumptions concerning the space dimension (they suppose $n \leq 3$) and the allowed discontinuities in the leading order coefficients are slightly more restrictive than ours (we suppose only L^∞ in space and time), but general enough for most applications.

The idea, to use maximal regularity properties together with linearization techniques and the Implicit Function Theorem in order to get solution regularity is known in the case of problems with sufficiently smooth data, see, for instance [8,11,12]. Also, in the case of problems with sufficiently smooth data there exist proofs of local existence results for nonlinear parabolic problems using *classical* (see [27]) as well as *hard* (see [26]) Implicit Function Theorems.

What concerns strongly coupled systems with nonsmooth data, it is known that Hölder regularity of the solutions cannot be expected in the case $n \geq 3$, in general. Similarly, it turns out that one cannot expect smooth dependence in the case of nonsmooth data, in general, if the equations contain terms which are not affine with respect to the spatial gradient of the solution. Therefore we consider only equations and boundary conditions, which are affine with respect to the spatial gradient. For equations, which are nonlinear with respect to the spatial gradient, even the question of uniqueness is much more difficult, see, for instance [1,14–16].

Organization of the paper: In Sect. 2 we introduce some notation and results about parabolic Sobolev–Morrey–Campanato spaces, mainly summarizing results from [19]. Further, we introduce the nonlinear operators which are needed for the rigorous statement of the abstract quasilinear parabolic problem (2.7).

Section 3 is devoted to the formulation and the proof of a maximal regularity result for abstract linear parabolic problems with nonsmooth data in Sobolev–Morrey–Campanato spaces. There we use results from [20] and generalize them (from scalar equations to weakly coupled systems and from local to nonlocal operators).

In Sect. 4, we formulate and prove our main results concerning abstract quasilinear parabolic problems. There we use the maximal regularity result from Sect. 3.

Finally, in Sect. 5 we present classes of nonlinear differential operators, defined in the domain or on the Neumann boundary part, which lead to abstract quasilinear parabolic problems of the type considered in Sect. 4.

2. Notation and setting

Let us introduce some notation. Throughout this text we assume $S = (t_0, t_1)$ to be a bounded open interval in $\mathbb{R}$. For $r > 0$ we define the set of subintervals

$$\mathcal{S}_r = \Big\{ S \cap (t - r^2, t) : t \in S \Big\}.$$

The symbol $|\ \ |$ is used for both the absolute value and the maximum norm in $\mathbb{R}^n$. We denote by

$$Q_r(x) = \big\{ \xi \in \mathbb{R}^n : |\xi - x| < r \big\}$$

the open cube with center $x \in \mathbb{R}^n$ and radius $r > 0$. For subsets Y of $\mathbb{R}^n$ we write Y°, $\overline{Y}$ and ∂Y for the topological interior, the closure, and the boundary of Y, respectively. For $r > 0$ and subsets $Y \subset \mathbb{R}^n$ we use the corresponding calligraphic letter to introduce the set

$$\mathcal{Y}_r = \{ Y \cap Q_r(y) : y \in Y \}$$

of intersections. Let λ^n be the n-dimensional Lebesgue measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^n$.

2.1. Parabolic Morrey–Campanato spaces

Let $X \subset \mathbb{R}^n$ be some bounded open set. The following definition goes back to DA PRATO [9] and CAMPANATO [7]: for $\omega \in [0, n+2]$ the Morrey space $L_2^\omega(S; L^2(X))$ consists of all $u \in L^2(S; L^2(X))$ such that

$$[u]^2_{L_2^\omega(S;L^2(X))} = \sup_{r>0} \sup_{(I,Y)\in \mathcal{S}_r \times \mathcal{X}_r} r^{-\omega} \int_I \int_Y |u(s)|^2 \, \mathrm{d}\lambda^n \, \mathrm{d}s$$

remains finite. The norm of $u \in L_2^\omega(S; L^2(X))$ is defined by

$$\|u\|^2_{L_2^\omega(S;L^2(X))} = \|u\|^2_{L^2(S;L^2(X))} + [u]^2_{L_2^\omega(S;L^2(X))}.$$

Let $H_0^1(X) \subset H \subset H^1(X)$ be some closed subspace equipped with the usual scalar product of $H^1(X)$. For $\omega \in [0, n+2]$ we introduce the Sobolev–Morrey space

$$L_2^\omega(S; H) = \Big\{ u \in L^2(S; H) : u \in L_2^\omega(S; L^2(X)),\ |\nabla u| \in L_2^\omega(S; L^2(X)) \Big\},$$

and we define the norm of $u \in L_2^\omega(S; H)$ by

$$\|u\|^2_{L_2^\omega(S;H)} = \|u\|^2_{L_2^\omega(S;L^2(X))} + \||\nabla u|\|^2_{L_2^\omega(S;L^2(X))}.$$

Note that the spaces $L_2^{\omega}(S; L^2(X))$ are usually denoted by $L^{2,\omega}(S \times X)$. Apart from these, later on we use further Morrey-type function spaces. Hence, we have decided to use a different but integrated naming scheme. The set $L^{\infty}(S \times X)$ of bounded measurable functions is a space of multipliers for $L_2^{\omega}(S; L^2(X))$.

Analogously, we consider function spaces on Lipschitz hypersurfaces in $\mathbb{R}^n$. Here, a subset M of $\mathbb{R}^n$ is called *Lipschitz hypersurface* in $\mathbb{R}^n$ if for each point $x \in M$ there exist a neighborhood U of x and a Lipschitz transformation Φ from U onto the cube $Q_1(0)$ such that $\Phi[U \cap M] = \{y \in \mathbb{R}^n : |y| < 1,\ y_n = 0\}$ and $\Phi(x) = 0$.

Let M be a compact Lipschitz hypersurface in $\mathbb{R}^n$ and λ_M the $(n-1)$-dimensional Lebesgue measure on the σ-algebra $\mathfrak{L}_M$ of Lebesgue measurable subsets of M, see [13]. For $\varkappa \in [0, n+1]$ and relatively open subsets K of M we define the Morrey space $L_2^{\varkappa}(S; L^2(K))$ as the set of all $u \in L^2(S; L^2(K))$ such that

$$[u]^2_{L_2^{\varkappa}(S;L^2(K))} = \sup_{r>0} \sup_{(I,\Gamma)\in\mathcal{S}_r\times\mathcal{K}_r} r^{-\varkappa} \int_I \int_\Gamma |u(s)|^2 \,\mathrm{d}\lambda_M \,\mathrm{d}s$$

remains finite, and we introduce the norm of $u \in L_2^{\varkappa}(S; L^2(K))$ by

$$\|u\|^2_{L_2^{\varkappa}(S;L^2(K))} = \|u\|^2_{L^2(S;L^2(K))} + [u]^2_{L_2^{\varkappa}(S;L^2(K))}.$$

The set $L^{\infty}(S \times K)$ is a space of multipliers for $L_2^{\varkappa}(S; L^2(K))$.

2.2. Regular sets

For our investigations on global regularity we use a notion of regular sets $G \subset \mathbb{R}^n$, which is equivalent to the version introduced by Gröger, see [21]. Being the natural generalization of sets with Lipschitz boundary it allows the proper functional analytic description of elliptic and parabolic problems with mixed boundary conditions in nonsmooth domains. For $x \in \mathbb{R}^n$ and $r > 0$ we introduce the halfcubes

$$Q_r^-(x) = \left\{\xi \in \mathbb{R}^n : |\xi - x| < r, \xi_n - x_n < 0\right\},$$
$$Q_r^+(x) = \left\{\xi \in \mathbb{R}^n : |\xi - x| < r, \xi_n - x_n \le 0\right\},$$
$$Q_r^{\pm}(x) = \left\{\xi \in Q_r^+(x) : \xi_1 - x_1 > 0 \text{ or } \xi_n - x_n < 0\right\}.$$

A bounded set $G \subset \mathbb{R}^n$ is called *regular* if for each $x \in \partial G$ we find some neighborhood U of x in $\mathbb{R}^n$ and a Lipschitz transformation Φ from U onto $Q_1(0)$ such that $\Phi[U \cap G] \in \left\{Q_1^-(0), Q_1^+(0), Q_1^{\pm}(0)\right\}$ and $\Phi(x) = 0$.

From now on throughout the paper we fix a regular set $G \subset \mathbb{R}^n$ and we use its representation as a disjoint union

$$G = X \cup \Gamma \quad \text{with} \quad X = G^{\circ} \quad \text{and} \quad \Gamma = G \setminus X. \tag{2.1}$$

2.3. Parabolic Sobolev–Morrey–Campanato spaces of functionals

We define function spaces associated with relatively open subsets Y of the regular set $G \subset \mathbb{R}^n$. By $H_0^1(Y)$ we denote the closure of

$$C_0^{\infty}(Y) = \left\{u|Y^{\circ} : u \in C_0^{\infty}(\mathbb{R}^n),\ \operatorname{supp}(u) \cap (\overline{Y} \setminus Y) = \varnothing\right\}$$

in the space $H^1(Y^\circ)$, and we write $H^{-1}(Y)$ for the dual space of $H_0^1(Y)$. In particular, $H_0^1(G)$ is the subspace of the classical Sobolev space $H^1(X)$ of all functions which vanish on $\partial G \setminus \overline{\Gamma}$ in the sense of traces. That is why $\partial G \setminus \overline{\Gamma}$ and Γ usually are called Dirichlet and Neumann boundary parts of G. Especially, if Γ is empty, that means $G = X$, then the just introduced notation coincides with the classical notation $H_0^1(G)$ of all $H^1(X)$-functions which vanish on ∂G in the sense of traces, and if $\Gamma = \partial G$, which means $G = \overline{X}$, then $H_0^1(G) = H^1(X)$.

Let $I \subset \mathbb{R}$ be an open subinterval of S. If $Z_G : H_0^1(Y) \to H_0^1(G)$ is the zero extension map, then we define $\mathcal{Z}_{S,G} : L^2(I; H_0^1(Y)) \to L^2(S; H_0^1(G))$ by

$$(\mathcal{Z}_{S,G}u)(s) = \begin{cases} Z_G u(s) & \text{if } s \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } u \in L^2(I; H_0^1(Y)).$$

Note that $\mathcal{Z}_{S,G}$ is a linear isometry from $L^2(I; H_0^1(Y))$ into $L^2(S; H_0^1(G))$.

In the same spirit as the well-established Morrey spaces of functions, in [19] we have constructed a new scale of Sobolev–Morrey spaces of functionals as subspaces of $L^2(S; H^{-1}(G))$.

To localize a functional $f \in L^2(S; H^{-1}(G))$ we define the assignment $f \mapsto \mathcal{L}_{S,G} f$ from $L^2(S; H^{-1}(G))$ into $L^2(I; H^{-1}(Y))$ as the adjoint operator to the zero extension map $\mathcal{Z}_{S,G} : L^2(I; H_0^1(Y)) \to L^2(S; H_0^1(G))$:

$$\langle \mathcal{L}_{I,Y} f, w \rangle_{L^2(I; H_0^1(Y))} = \langle f, \mathcal{Z}_{S,G} w \rangle_{L^2(S; H_0^1(G))} \quad \text{for } w \in L^2(I; H_0^1(Y)).$$

Here and in what follows we denote, as usual, by $\langle \, , \rangle$ dual pairings. Using the isometric property of $\mathcal{Z}_{S,G}$, we get

$$\|\mathcal{L}_{I,Y} f\|_{L^2(I; H^{-1}(Y))} \le \|f\|_{L^2(S; H^{-1}(G))} \quad \text{for all } f \in L^2(S; H^{-1}(G)).$$

For $\omega \in [0, n+2]$ we define the Sobolev–Morrey space $L_2^\omega(S; H^{-1}(G))$ as the set of all elements $f \in L^2(S; H^{-1}(G))$ for which

$$[f]^2_{L_2^\omega(S; H^{-1}(G))} = \sup_{r>0} \sup_{(I,Y) \in \mathcal{S}_r \times \mathcal{G}_r} r^{-\omega} \int_I \|(\mathcal{L}_{I,Y} f)(s)\|^2_{H^{-1}(Y)} \, \mathrm{d}s$$

has a finite value. We introduce the norm of $f \in L_2^\omega(S; H^{-1}(G))$ by

$$\|f\|^2_{L_2^\omega(S; H^{-1}(G))} = \|f\|^2_{L^2(S; H^{-1}(G))} + [f]^2_{L_2^\omega(S; H^{-1}(G))}.$$

The assignment $(g, g_0, g_\Gamma) \mapsto \Psi(g, g_0, g_\Gamma)$ defined by

$$\begin{aligned} \langle \Psi(g, g_0, g_\Gamma), \varphi \rangle_{L^2(S; H_0^1(G))} &= \int_S \int_X g(s) \cdot \nabla \varphi(s) \, \mathrm{d}\lambda^n \, \mathrm{d}s \\ &\quad + \int_S \int_X g_0(s) \, \varphi(s) \, \mathrm{d}\lambda^n \, \mathrm{d}s \\ &\quad + \int_S \int_\Gamma g_\Gamma(s) \, \varphi(s) \, \mathrm{d}\lambda_\Gamma \, \mathrm{d}s \end{aligned} \tag{2.2}$$

for $\varphi \in L^2(S; H_0^1(G))$, generates a linear continuous operator

$$\Psi : L_2^{\omega}(S; L^2(X; \mathbb{R}^n)) \times L_2^{\omega-2}(S; L^2(X)) \times L_2^{\omega-1}(S; L^2(\Gamma)) \to L_2^{\omega}(S; H^{-1}(G)),$$
$$(g, g_0, g_\Gamma) \mapsto \Psi(g, g_0, g_\Gamma), \tag{2.3}$$

and its norm depends on n and G, only, see [19, Theorem 5.6].

2.4. Parabolic Sobolev–Morrey–Campanato spaces of functions

Based upon the preceeding definitions, in [19] we have constructed new function classes suitable for the regularity theory of second order parabolic boundary value problems with nonsmooth data, see [20]. Here, we present a version being adequate for systems of equations with $m \in \mathbb{N}$ unknowns. In particular, for the modeling of instationary drift-diffusion problems we are interested in dealing with nonsmooth capacity-like coefficients $a^1, \ldots, a^m \in L^\infty(X)$, which are supposed to be δ-definite with respect to X and $\delta \in (0, 1]$, that means, we assume that

$$\delta \le \operatorname*{ess\,inf}_{x\in X} a^\alpha(x), \quad \operatorname*{ess\,sup}_{x\in X} a^\alpha(x) \le \frac{1}{\delta} \quad \text{for all } \alpha \in \{1, \ldots, m\}. \tag{2.4}$$

We consider the linear continuous operators $E^1, \ldots, E^m$ from $H^1(X)$ into $H^{-1}(G)$ being defined by

$$\langle E^\alpha w, \varphi\rangle_{H_0^1(G)} = \int_X a^\alpha w\varphi \,\mathrm{d}\lambda^n \quad \text{for } \varphi \in H_0^1(G), \tag{2.5}$$

the corresponding linear continuous operator $E = (E^1, \ldots, E^m)$ from $H^1(X; \mathbb{R}^m)$ into $H^{-1}(G; \mathbb{R}^m)$ and, associated with S and E, the linear continuous map $\mathcal{E} = (\mathcal{E}^1, \ldots, \mathcal{E}^m)$ from $L^2(S; H^1(X; \mathbb{R}^m))$ into $L^2(S; H^{-1}(G; \mathbb{R}^m))$ being defined as

$$\langle \mathcal{E}u, \varphi\rangle_{L^2(S; H_0^1(G;\mathbb{R}^m))} = \sum_{\alpha=1}^{m} \int_S \langle E^\alpha u^\alpha(s), \varphi^\alpha(s)\rangle_{H_0^1(G)} \,\mathrm{d}s \tag{2.6}$$

for $\varphi \in L^2(S; H_0^1(G; \mathbb{R}^m))$. For $\omega \in [0, n+2]$ and $\alpha \in \{1, \ldots, m\}$ we introduce the Sobolev–Morrey space $W_{E^\alpha}^{\omega}(S; H^1(X))$ as the set of all functions $v \in L_2^{\omega}(S; H^1(X))$, such that the weak time derivative $(\mathcal{E}^\alpha v)'$ of $\mathcal{E}^\alpha v \in L^2(S; H^{-1}(G))$ exists and belongs to $L_2^{\omega}(S; H^{-1}(G))$. We define the norm of $v \in W_{E^\alpha}^{\omega}(S; H^1(X))$ by

$$\|v\|^2_{W_{E^\alpha}^{\omega}(S;H^1(X))} = \|v\|^2_{L_2^{\omega}(S;H^1(X))} + \|(\mathcal{E}^\alpha v)'\|^2_{L_2^{\omega}(S;H^{-1}(G))},$$

and consider the following closed subspaces:

$$W_{E^\alpha}^{\omega}(S; H_0^1(G)) = \Big\{v \in W_{E^\alpha}^{\omega}(S; H^1(X)) : v \in L_2^{\omega}(S; H_0^1(G))\Big\},$$
$$W_{0E^\alpha}^{\omega}(S; H_0^1(G)) = \Big\{v \in W_{E^\alpha}^{\omega}(S; H_0^1(G)) : v(t_0) = 0\Big\}.$$

In the definition of $W^{\omega}_{0E^\alpha}(S;H^1_0(G))$ it is used that the spaces $W^{\omega}_{E^\alpha}(S;H^1_0(G))$ are continuously embedded into $C(\overline{S};L^2(X))$. Finally, as a natural generalization, we set

$$\begin{aligned} W^{\omega}_{E}(S;H^1(X;\mathbb{R}^m)) &= W^{\omega}_{E^1}(S;H^1(X)) \times \cdots \times W^{\omega}_{E^m}(S;H^1(X)),\\ W^{\omega}_{E}(S;H^1_0(G;\mathbb{R}^m)) &= W^{\omega}_{E^1}(S;H^1_0(G)) \times \cdots \times W^{\omega}_{E^m}(S;H^1_0(G)),\\ W^{\omega}_{0E}(S;H^1_0(G;\mathbb{R}^m)) &= W^{\omega}_{0E^1}(S;H^1_0(G)) \times \cdots \times W^{\omega}_{0E^m}(S;H^1_0(G)) \end{aligned}$$

and equip these spaces with the maximum norm of the components, respectively. For $\omega = 0$ we drop superscripted indices.

Note that for $\omega \in (n, n+2]$ and $\beta = (\omega - n)/4$ the space $W^{\omega}_{E^\alpha}(S;H^1(X))$ is continuously embedded into the space $C^{0,\beta}(\overline{S};C(\overline{X})) \cap C(\overline{S};C^{0,2\beta}(\overline{X}))$ of functions, which are Hölder continuous is space and time, see [19, Theorems 3.4, 6.8]. For every parameter $0 < \beta < (\omega - n)/4$ this embedding is completely continuous.

From now on throughout the paper the coefficients a^α with (2.4) as well as the corresponding operators E^α an E are given and fixed.

2.5. Formulation of the problem

Let U be an open subset in $C(\overline{S};C(\overline{X};\mathbb{R}^m))$, Λ a Banach space, and V an open subset in Λ. We look for solutions

$$(u,\lambda) = (u^1,\ldots,u^m,\lambda) \in \Big(U \cap W_E(S;H^1(X;\mathbb{R}^m))\Big) \times V$$

of weakly coupled systems of quasilinear operator equations

$$(\mathcal{E}^\alpha u^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(u,\lambda),u^\alpha) = \mathcal{F}^\alpha(u,\lambda), \quad \alpha \in \{1,\ldots,m\}, \tag{2.7}$$

where $\lambda \in V$ plays the role of a control parameter, which includes, for instance, inhomogeneities of the problem. The linear continuous operator

$$\mathcal{E} = (\mathcal{E}^1,\ldots,\mathcal{E}^m) : L^2(S;H^1(X;\mathbb{R}^m)) \to L^2(S;H^{-1}(G;\mathbb{R}^m))$$

was already defined in (2.6) via δ-definite leading order coefficients $a^1,\ldots,a^m \in L^\infty(X)$, see (2.4) and (2.5). The bilinear continuous map

$$\mathcal{B} : L^\infty(S\times X;\mathbb{R}^{n\times n}) \times L^2(S;H^1(X)) \to L^2(S;H^{-1}(G))$$

is given by

$$\langle \mathcal{B}(A,v),\varphi\rangle_{L^2(S;H^1_0(G))} = \int_S\int_X A\nabla v(s)\cdot\nabla\varphi(s)\,\mathrm{d}\lambda^n\,\mathrm{d}s \tag{2.8}$$

for $\varphi \in L^2(S;H^1_0(G))$. Concerning the nonlinearities we suppose that

$$\mathcal{A}^\alpha \in C^1\left(U\times V;L^\infty(S\times X;\mathbb{R}^{n\times n})\right), \tag{2.9}$$

$$\mathcal{F}^\alpha \in C^1\Big(U\times V;L^{\omega_0}_2(S;H^{-1}(G))\Big), \tag{2.10}$$

where $\omega_0 \in (n, n+2]$ is a common Morrey exponent for all $\alpha \in \{1,\ldots,m\}$.

Moreover, we suppose that the operators $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$ are Volterra operators with respect to u, which means that for all $t \in S = (t_0, t_1)$ and $(u, \lambda), (w, \lambda) \in U \times V$ it holds

$$\mathcal{A}^\alpha(u, \lambda)|((t_0, t) \times X) = \mathcal{A}^\alpha(w, \lambda)|((t_0, t) \times X),$$
$$\mathcal{F}^\alpha(u, \lambda)|(t_0, t) = \mathcal{F}^\alpha(w, \lambda)|(t_0, t),$$

whenever $u|(t_0, t) = w|(t_0, t)$. Here, as usual, $u|(t_0, t)$ denotes the restriction of the function u on the subinterval (t_0, t) of S, $\mathcal{A}^\alpha(u, \lambda)|((t_0, t) \times X)$ is the restriction of the function $\mathcal{A}^\alpha(u, \lambda)$ on $(t_0, t) \times X$, and $\mathcal{F}^\alpha(u, \lambda)|(t_0, t)$ denotes the restriction of the functional $\mathcal{F}^\alpha(u, \lambda)$ on (t_0, t), which is defined by

$$\langle \mathcal{F}^\alpha(u, \lambda)|(t_0, t), w\rangle_{L^2((t_0,t);H_0^1(G))} = \langle \mathcal{F}^\alpha(u, \lambda), \mathcal{Z}_t w\rangle_{L^2(S;H_0^1(G))}$$

for $w \in L^2((t_0, t); H_0^1(G))$, where $\mathcal{Z}_t : L^2((t_0, t); H_0^1(G)) \to L^2(S; H_0^1(G))$ is the zero extension operator. Note, that for all $(u, \lambda) \in U \times V$ the linear continuous operators

$$\frac{\partial \mathcal{A}^\alpha}{\partial u}(u, \lambda) : C(\overline{S}; C(\overline{X}; \mathbb{R}^m)) \to L^\infty(S \times X; \mathbb{R}^{n\times n})), \tag{2.11}$$
$$\frac{\partial \mathcal{F}^\alpha}{\partial u}(u, \lambda) : C(\overline{S}; C(\overline{X}; \mathbb{R}^m)) \to L_2^{\omega_0}(S; H^{-1}(G)), \tag{2.12}$$

have the Volterra property, too.

2.6. Homogenization of initial and Dirichlet boundary conditions

Let W be an open subset in $W_E^{\omega_0}(S; H^1(X; \mathbb{R}^m))$ containing regular inhomogeneities we are interested in. Here $\omega_0 \in (n, n+2]$ is the Morrey exponent from assumption (2.10). Further, let U_h be a neighborhood of zero in $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ such that the inclusion $\{u_h + w : u_h \in U_h,\ w \in W\} \subset U$ holds true. We define nonlinear Volterra operators

$$\mathcal{A}_h^\alpha \in C^1\left(U_h \times W \times V; L^\infty(S \times X; \mathbb{R}^{n\times n})\right),$$
$$\mathcal{F}_h^\alpha \in C^1\left(U_h \times W \times V; L_2^{\omega_0}(S; H^{-1}(G))\right)$$

by setting

$$\mathcal{A}_h^\alpha(u_h, w, \lambda) = \mathcal{A}^\alpha(u_h + w, \lambda),$$
$$\mathcal{F}_h^\alpha(u_h, w, \lambda) = \mathcal{F}^\alpha(u_h + w, \lambda) - \mathcal{B}(\mathcal{A}^\alpha(u_h + w, \lambda), w^\alpha) - (\mathcal{E}^\alpha w^\alpha)',$$

for $(u_h, w, \lambda) \in U_h \times W \times V$ and $\alpha \in \{1, \ldots, m\}$. These have the same mapping properties as $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$, respectively, where the old control parameter $\lambda \in V$ has to be replaced by the new control parameter $(w, \lambda) \in W \times V$. Moreover, if

$$(u_h, w, \lambda) \in \left(U_h \cap W_{0E}(S; H_0^1(G; \mathbb{R}^m))\right) \times W \times V$$

is a solution to the system

$$(\mathcal{E}^\alpha u_h^\alpha)' + \mathcal{B}(\mathcal{A}_h^\alpha(u_h, w, \lambda), u_h^\alpha) = \mathcal{F}_h^\alpha(u_h, w, \lambda), \quad \alpha \in \{1, \dots, m\},$$

then the pair $(u, \lambda) = (u_h + w, \lambda) \in \big(U \cap W_E(S; H^1(X; \mathbb{R}^m))\big) \times V$ solves problem (2.7). Hence, for given inhomogeneities $w \in W$, we can restrict ourselves to look for homogeneous solutions $(u, \lambda) \in \big(U \cap W_{0E}(S; H_0^1(G; \mathbb{R}^m))\big) \times V$ of problem (2.7).

2.7. Linearization

In addition to the nonlinear operator equation (2.7) we also consider solutions $v \in W_{0E}(S; H_0^1(G; \mathbb{R}^m))$ of its linearization

$$\begin{aligned}&(\mathcal{E}^\alpha v^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(u_0, \lambda_0), v^\alpha)\\ &\quad = \frac{\partial \mathcal{F}^\alpha}{\partial u}(u_0, \lambda_0)\, v - \mathcal{B}\left(\frac{\partial \mathcal{A}^\alpha}{\partial u}(u_0, \lambda_0)\, v, u_0^\alpha\right), \quad \alpha \in \{1, \dots, m\}, \end{aligned} \tag{2.13}$$

at $(u_0, \lambda_0) \in \big(U \cap W_{0E}(S; H_0^1(G; \mathbb{R}^m))\big) \times V$. Further, we investigate the linear operator equations, determining the sequence of Newton iterations $u_{k+1} \in U \cap W_{0E}(S; H_0^1(G; \mathbb{R}^m))$ for given u_k by

$$\begin{aligned}&\left(\mathcal{E}^\alpha u_{k+1}^\alpha\right)' + \mathcal{B}\left(\mathcal{A}^\alpha(u_k, \lambda_0), u_{k+1}^\alpha\right) + \mathcal{B}\left(\frac{\partial \mathcal{A}^\alpha}{\partial u}(u_k, \lambda_0)(u_{k+1} - u_k), u_k^\alpha\right)\\ &\quad = \mathcal{F}^\alpha(u_k, \lambda_0) + \frac{\partial \mathcal{F}^\alpha}{\partial u}(u_k, \lambda_0)(u_{k+1} - u_k), \quad \alpha \in \{1, \dots, m\}. \end{aligned} \tag{2.14}$$

3. Maximal Sobolev–Morrey regularity for abstract linear parabolic problems

Recall that $S = (t_0, t_1)$ is the fixed time interval, $G \subset \mathbb{R}^n$ is a fixed regular set, and that we use the notation (2.1).

The following maximal regularity result for linear parabolic boundary value problems in Sobolev–Morrey spaces will serve as the main tool of our considerations. It generalizes the results [20, Theorems 6.8 and 7.5] to weakly coupled systems of linear parabolic equations including nonlocal operators.

Analogously to the notion of δ-definiteness with respect to X and $\delta \in (0, 1]$ of scalar coefficient functions $a \in L^\infty(X)$, see (2.4), a matrix valued coefficient function $A \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ is called δ-definite with respect to S, X, and $\delta \in (0, 1]$ if

$$\delta\, \xi \cdot \xi \le \operatorname*{ess\,inf}_{s\in S,\, x\in X} A(s, x)\xi \cdot \xi, \qquad \operatorname*{ess\,sup}_{s\in S,\, x\in X} A(s, x)\xi \cdot \xi \le \frac{1}{\delta}\, \xi \cdot \xi, \tag{3.1}$$

holds true for all $\xi \in \mathbb{R}^n$.

THEOREM 3.1. *Assume that*

$$\mathcal{N}^\alpha : C(\overline{S}; C(\overline{X}; \mathbb{R}^m)) \to L_2^{\omega_0}(S; H^{-1}(G)), \quad \alpha \in \{1, \dots, m\}, \tag{3.2}$$

are linear continuous Volterra operators having the Morrey exponent $\omega_0 \in (n, n+2]$ *in common. Suppose that for all* $\alpha \in \{1, \dots, m\}$ *the leading coefficients* $a^\alpha \in L^\infty(X)$ *and* $A^\alpha \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ *are* δ*-definite with respect to* S, X, *and* $\delta \in (0, 1]$.

Then there exists $\bar{\omega} \in (n, \omega_0]$, *depending on* δ *and* G *only, such that for every* $\omega \in (n, \bar{\omega})$ *the assignment*

$$v \mapsto \Big((\mathcal{E}^1 v^1)' + \mathcal{B}(A^1, v^1) + \mathcal{N}^1 v, \dots, (\mathcal{E}^m v^m)' + \mathcal{B}(A^m, v^m) + \mathcal{N}^m v\Big) \tag{3.3}$$

is a linear isomorphism from $W_{0E}^{\omega}(S; H_0^1(G; \mathbb{R}^m))$ *onto* $L_2^{\omega}(S; H^{-1}(G; \mathbb{R}^m))$.

Proof. 1. In view of the maximal regularity result in [20, Theorem 6.8], there exists some exponent $\bar{\omega} = \bar{\omega}(\delta, G) \in (n, \omega_0]$ such the linear parabolic operator

$$v \mapsto \Big((\mathcal{E}^1 v^1)' + \mathcal{B}(A^1, v^1), \dots, (\mathcal{E}^m v^m)' + \mathcal{B}(A^m, v^m)\Big)$$

is an isomorphism from $W_{0E}^{\omega}(S; H_0^1(G; \mathbb{R}^m))$ onto $L_2^{\omega}(S; H^{-1}(G; \mathbb{R}^m))$ for every $\omega \in (n, \bar{\omega})$. Due to the completely continuous embedding of $W_{0E}^{\omega}(S; H_0^1(G; \mathbb{R}^m))$ into $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$, see [19, Theorem 6.9], $\mathcal{N}^\alpha$ maps $W_{0E}^{\omega}(S; H_0^1(G; \mathbb{R}^m))$ completely continuous into $L_2^{\omega}(S; H^{-1}(G))$ for $\alpha \in \{1, \dots, m\}$. Hence, the map defined in (3.3) is a Fredholm operator of index zero from $W_{0E}^{\omega}(S; H_0^1(G; \mathbb{R}^m))$ into $L_2^{\omega}(S; H^{-1}(G; \mathbb{R}^m))$. For the assertion of the theorem, it suffices to prove the injectivity of this map.

2. Suppose that $v \in W_{0E}^{\omega}(S; H_0^1(G; \mathbb{R}^m))$ is a solution to the system of homogeneous initial boundary value problems

$$(\mathcal{E}^\alpha v^\alpha)' + \mathcal{B}(A^\alpha, v^\alpha) + \mathcal{N}^\alpha v = 0, \quad \alpha \in \{1, \dots, m\}. \tag{3.4}$$

For fixed $t \in S$ we consider the subinterval (t_0, t) of S and the restrictions $v|(t_0, t) \in W_{0E}^{\omega}((t_0, t); H_0^1(G; \mathbb{R}^m))$ and $(\mathcal{N}^\alpha v)|(t_0, t) \in L_2^{\omega}((t_0, t); H^{-1}(G))$. Due to [20, Remark 6.2] we get estimates

$$\|v|(t_0, t))\|_{W_{0E}^{\omega}((t_0,t); H_0^1(G;\mathbb{R}^m))} \le c_1 \sup_{1\le\alpha\le m} \|(\mathcal{N}^\alpha v)|(t_0, t)\|_{L_2^{\omega}((t_0,t); H^{-1}(G))}, \tag{3.5}$$

where the constant $c_1 > 0$ may depend on S but not on t. To estimate the right hand side of (3.5) we arbitrarily choose $t^* \in S$ with $t^* > t$ and some cut-off function $\vartheta \in C^\infty(\mathbb{R})$ with

$$0 \le \vartheta \le 1, \quad \vartheta(s) = 1 \quad \text{for all } s \le t, \quad \vartheta(s) = 0 \quad \text{for all } t \ge t^*.$$

The Volterra property of the maps $\mathcal{N}^\alpha : C(\overline{S}; C(\overline{X}; \mathbb{R}^m)) \to L_2^{\omega}(S; H^{-1}(G))$ and the definition of the norm in $L_2^{\omega}(S; H^{-1}(G))$ for all $\alpha \in \{1, \dots, m\}$ yield that

$$\begin{aligned}\|(\mathcal{N}^\alpha v)|(t_0, t)\|_{L_2^{\omega}((t_0,t); H^{-1}(G))} &= \|(\mathcal{N}^\alpha(\vartheta v))|(t_0, t)\|_{L_2^{\omega}((t_0,t); H^{-1}(G))}\\ &\le \|\mathcal{N}^\alpha(\vartheta v)\|_{L_2^{\omega}(S; H^{-1}(G))}\\ &\le c_2 \|\vartheta v\|_{C(\overline{S}; C(\overline{X}; \mathbb{R}^m))},\end{aligned} \tag{3.6}$$

where

$$c_2 = \sup\left\{\|\mathcal{N}^\alpha w\|_{L_2^\omega(S;H^{-1}(G))} : \|w\|_{C(\overline{S};C(\overline{X};\mathbb{R}^m))} \le 1,\ \alpha \in \{1,\dots,m\}\right\}$$

is the maximum of the operator norms of $\mathcal{N}^1,\dots,\mathcal{N}^m$. In view of the continuity of the embedding from $W_{0E}^\omega(S;H_0^1(G;\mathbb{R}^m))$ into the Hölder space $C^{0,\beta}(\overline{S};C(\overline{X};\mathbb{R}^m))$ for $\beta=(\omega-n)/4$, for all $s\in[t,t^*]$ we get

$$\begin{aligned}\|v(s)\|_{C(\overline{X};\mathbb{R}^m)} &\le \|v(s)-v(t)\|_{C(\overline{X};\mathbb{R}^m)} + \|v(t)\|_{C(\overline{X};\mathbb{R}^m)}\\ &\le (t^*-t)^\beta\|v\|_{C^{0,\beta}(\overline{S};C(\overline{X};\mathbb{R}^m))} + \|v|[t_0,t]\|_{C([t_0,t];C(\overline{X};\mathbb{R}^m))},\end{aligned}$$

and, hence,

$$\|\vartheta v\|_{C(\overline{S};C(\overline{X};\mathbb{R}^m))} \le (t^*-t)^\beta\|v\|_{C^{0,\beta}(\overline{S};C(\overline{X};\mathbb{R}^m))} + \|v|[t_0,t]\|_{C([t_0,t];C(\overline{X};\mathbb{R}^m))}.$$

Together with (3.6), for all $\alpha\in\{1,\dots,m\}$ this leads to

$$\begin{aligned}\|(\mathcal{N}^\alpha v)|(t_0,t)\|_{L_2^\omega((t_0,t);H^{-1}(G))} \le c_2(t^*-t)^\beta\|v\|_{C^{0,\beta}(\overline{S};C(\overline{X};\mathbb{R}^m))}\\ +c_2\|v|[t_0,t]\|_{C([t_0,t];C(\overline{X};\mathbb{R}^m))}.\end{aligned}$$

Since $t^*\in S$, $t^*>t$ was arbitrarily fixed at the beginning, we arrive at

$$\sup_{1\le\alpha\le m}\|(\mathcal{N}^\alpha v)|(t_0,t)\|_{L_2^\omega((t_0,t);H^{-1}(G))} \le c_2\|v|[t_0,t]\|_{C([t_0,t];C(\overline{X};\mathbb{R}^m))}. \tag{3.7}$$

To estimate the left hand side of (3.5) we consider the interval $(t+t_0-t_1,t)$, which contains (t_0,t) and has the same length than S, and we define the zero extension $v_0\in W_{0E}^\omega((t+t_0-t_1,t);H_0^1(G;\mathbb{R}^m))$ of $v|(t_0,t)$ to $(t+t_0-t_1,t)$ by

$$v_0(s)=\begin{cases}v(s) & \text{if } s\in[t_0,t),\\ 0 & \text{if } s\in(t+t_0-t_1,t_0].\end{cases}$$

Using the continuity of the embedding from $W_{0E}^\omega((t+t_0-t_1,t_0);H_0^1(G;\mathbb{R}^m))$ into the Hölder space $C^{0,\beta}([t+t_0-t_1,t_0];C(\overline{X};\mathbb{R}^m))$ for $\beta=(\omega-n)/4$, and the definition of the norms in Morrey and Hölder spaces, the above construction yields

$$\begin{aligned}\|v|[t_0,t]\|_{C^{0,\beta}([t_0,t];C(\overline{X};\mathbb{R}^m))} &\le \|v_0\|_{C^{0,\beta}([t+t_0-t_1,t];C(\overline{X};\mathbb{R}^m))}\\ &\le c_3\|v_0\|_{W_{0E}^\omega((t+t_0-t_1,t);H_0^1(G;\mathbb{R}^m))}\\ &\le c_3\|v|(t_0,t)\|_{W_{0E}^\omega((t_0,t);H_0^1(G;\mathbb{R}^m))},\end{aligned}$$

where the constant $c_3>0$ may depend on S but not on t. Together with (3.5) and (3.7) this leads to the key estimate

$$\|v|[t_0,t]\|_{C^{0,\beta}([t_0,t];C(\overline{X};\mathbb{R}^m))} \le c_4\|v|[t_0,t]\|_{C([t_0,t];C(\overline{X};\mathbb{R}^m))} \quad \text{for all } t\in S, \tag{3.8}$$

where the constant $c_4=c_1c_2c_3\ge 0$ does not depend on $t\in S$.

3. Set

$$s_k = t_0 + \frac{k}{\ell}(t_1 - t_0) \quad \text{for } k \in \{0, 1, \dots, \ell\},$$

where $\ell \in \mathbb{N}$, $\ell > 1$ is large enough to satisfy the condition

$$2c_4(t_1 - t_0)^\beta < \ell^\beta. \tag{3.9}$$

We prove that for every $k \in \{1, \dots, \ell\}$ it follows that $v(s) = 0$ for all $t_0 \le s \le s_k$, whenever $v(s) = 0$ holds true for every $t_0 \le s \le s_{k-1}$: indeed, applying (3.8) to the case $t = s_k$, condition (3.9) ensures that for all $s \in [s_{k-1}, s_k]$ we obtain

$$\begin{aligned}\|v(s) - v(s_{k-1})\|_{C(\overline{X};\mathbb{R}^m)} &\le (s - s_{k-1})^\beta \|v|[s_{k-1}, s_k]\|_{C^{0,\beta}([s_{k-1},s_k];C(\overline{X};\mathbb{R}^m))}\\ &\le (s - s_{k-1})^\beta \|v|[t_0, s_k]\|_{C^{0,\beta}([t_0,s_k];C(\overline{X};\mathbb{R}^m))}\\ &\le \frac{1}{2}\|v|[t_0, s_k]\|_{C([t_0,s_k];C(\overline{X};\mathbb{R}^m))}.\end{aligned}$$

Since $v(s_{k-1}) = 0$ this leads to $v(s) = 0$ for every $s \in [t_0, s_k]$.

Because for $k = 1$ the initial condition $v(t_0) = 0$ is satisfied, we inductively apply the last argument to prove that v vanishes on the whole interval $S = (t_0, t_1)$. Hence, $v = 0$ is the unique solution of the homogeneous problem (3.4) in the space $W^\omega_{0E}(S; H^1_0(G; \mathbb{R}^m))$. Due to Step 1 the proof is finished. □

4. Abstract quasilinear parabolic problems

In this section, again, $S = (t_0, t_1)$ is the fixed time interval, $G \subset \mathbb{R}^n$ is a fixed regular set, and we use the notation (2.1). Moreover, we suppose the Volterra operators $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$ with (2.9) and (2.10) to be given and fixed. Recall that the operators $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$ are defined on $U \times V$, where U is an open subset of $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ and V is an open subset in a Banach space Λ.

4.1. Regularity and smooth dependence

We prove regularity and smooth dependence of solutions to the nonlinear problem (2.7) nearby a known solution $(u_0, \lambda_0) \in \big(U \cap W_{0E}(S; H^1_0(G; \mathbb{R}^m))\big) \times V$.

THEOREM 4.1. *Let $(u_0, \lambda_0) \in \big(U \cap W_{0E}(S; H^1_0(G; \mathbb{R}^m))\big) \times V$ be a solution to (2.7) and suppose that there exists some constant $\delta \in (0, 1]$ such that $a^\alpha \in L^\infty(X)$ and $\mathcal{A}^\alpha(u_0, \lambda_0) \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ are δ-definite with respect to S and X for all $\alpha \in \{1, \dots, m\}$.*

Then we find some $\omega \in (n, \omega_0]$, depending on δ and G only, and a neighborhood U_0 of u_0 in $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ with $U_0 \subset U$ such that the following holds true:

1. *There exists a neighborhood V_0 of λ_0 in Λ with $V_0 \subset V$ and a solution map $\Phi \in C^1\big(V_0; W^\omega_{0E}(S; H^1_0(G; \mathbb{R}^m))\big)$ such that $(u, \lambda) \in U_0 \times V_0$ is a solution to (2.7)*

if and only if $u = \Phi(\lambda)$. *In particular, for each solution* $(u, \lambda) \in U_0 \times V_0$ *to* (2.7) *we get* $u \in W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$.

2. *If for all* $\alpha \in \{1, \dots, m\}$ *the maps*

$$u \in U \mapsto \frac{\partial \mathcal{A}^\alpha}{\partial u}(u, \lambda_0) \in \mathscr{L}\left(C(\overline{S}; C(\overline{X}; \mathbb{R}^m)); L^\infty(S \times X; \mathbb{R}^{n\times n})\right),$$
$$u \in U \mapsto \frac{\partial \mathcal{F}^\alpha}{\partial u}(u, \lambda_0) \in \mathscr{L}\left(C(\overline{S}; C(\overline{X}; \mathbb{R}^m)); L^{\omega_0}_2(S; H^{-1}(G))\right),$$

are locally Lipschitz continuous, then for each $u_1 \in U_0 \cap W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$ *the Eqs.* (2.14) *define a sequence of Newton iterations* $u_k \in U_0$ *for* $k \in \mathbb{N}$, $k \geq 2$, *which converges to* u_0 *in* $W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$ *for* $k \to \infty$.

Proof. 1. Let us prove the first assertion. Since the leading coefficients $\mathcal{A}^\alpha(u_0, \lambda_0)$ are supposed to be δ-definite with respect to S and X for all $\alpha \in \{1, \dots, m\}$, the coefficients $\mathcal{A}^\alpha(u, \lambda)$ are $\delta/2$-definite with respect to S and X for all u, which are close to u_0 in $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$, all λ, which are close to λ_0 in Λ, and all $\alpha \in \{1, \dots, m\}$. Hence, Theorem 3.1 yields that there exists an exponent $\omega \in (n, \omega_0]$, depending on δ and G only, and neighborhoods U_0 of u_0 in $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ with $U_0 \subset U$ and V_0 of λ_0 in Λ with $V_0 \subset V$ such that for all solutions $(u, \lambda) \in U_0 \times V_0$ to (2.7) we get $u \in W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$, in particular, we obtain $u_0 \in W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$. Hence, close to the solution (u_0, λ_0) it is equivalent to look for solutions $(u, \lambda) \in \left(U_0 \cap W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))\right) \times V_0$ of problem (2.7). To do so, we will apply the classical Implicit Function Theorem.

Because of $\omega > n$ the space $W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$ is continuously embedded into $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$. Therefore, the set $U_0 \cap W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$ is open in $W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$. Since

$$\mathcal{B} : L^\infty(S \times X; \mathbb{R}^{n\times n}) \times L^{\omega}_2(S; H^1_0(G)) \to L^{\omega}_2(S; H^{-1}(G))$$

is a bilinear continuous map, see (2.2) and (2.3), for every $\alpha \in \{1, \dots, m\}$ the operator

$$(u, \lambda) \mapsto \mathcal{P}^\alpha(u, \lambda) = (\mathcal{E}^\alpha u^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(u, \lambda), u^\alpha) - \mathcal{F}^\alpha(u, \lambda)$$

is a C^1-map from $\left(U_0 \cap W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))\right) \times V_0$ into $L^{\omega}_2(S; H^{-1}(G))$. Its partial derivative with respect to u at the solution (u_0, λ_0) is the linear continuous map from $W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$ into $L^{\omega}_2(S; H^{-1}(G))$ given by

$$\frac{\partial \mathcal{P}^\alpha}{\partial u}(u_0, \lambda_0)\, v = (\mathcal{E}^\alpha v^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(u_0, \lambda_0), v^\alpha) + \mathcal{B}\left(\frac{\partial \mathcal{A}^\alpha}{\partial u}(u_0, \lambda_0)\, v, u_0^\alpha\right) - \frac{\partial \mathcal{F}^\alpha}{\partial u}(u_0, \lambda_0)\, v$$

for $v \in W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$; it corresponds to the linearization (2.13). Applying Theorem 3.1, the derivative

$$v \mapsto \left(\frac{\partial \mathcal{P}^1}{\partial u}(u_0, \lambda_0), \dots, \frac{\partial \mathcal{P}^m}{\partial u}(u_0, \lambda_0)\right), \tag{4.1}$$

is a linear isomorphism from $W^{\omega}_{0E}(S; H^1_0(G;\mathbb{R}^m))$ onto $L^{\omega}_2(S; H^{-1}(G;\mathbb{R}^m))$, because the map $\mathcal{N}^\alpha$ defined by

$$\mathcal{N}^\alpha v = \mathcal{B}\left(\frac{\partial \mathcal{A}^\alpha}{\partial u}(u_0,\lambda_0)\, v, u_0^\alpha\right) - \frac{\partial \mathcal{F}^\alpha}{\partial u}(u_0,\lambda_0)\, v \quad \text{for } v \in C(\overline{S}; C(\overline{X};\mathbb{R}^m)),$$

is a linear continuous Volterra operator from $C(\overline{S}; C(\overline{X};\mathbb{R}^m))$ into $L^{\omega}_2(S; H^{-1}(G))$ for every $\alpha \in \{1,\ldots,m\}$, see (2.2), (2.3), (2.8), (2.11) and (2.12). Hence, the Implicit Function Theorem, see [28, Theorem 4.B], works for the first assertion.

2. Finally, we prove the second assertion of the theorem. Remembering (2.14), the sequence of Newton iterations is defined by

$$\begin{aligned}
&\left(\mathcal{E}^\alpha u^\alpha_{k+1}\right)' + \mathcal{B}\left(\mathcal{A}^\alpha(u_k,\lambda_0), u^\alpha_{k+1}\right) + \mathcal{B}\left(\frac{\partial \mathcal{A}^\alpha}{\partial u}(u_k,\lambda_0)\, u_{k+1}, u_k^\alpha\right)\\
&\quad - \frac{\partial \mathcal{F}^\alpha}{\partial u}(u_k,\lambda_0)\, u_{k+1}\\
&= \mathcal{B}\left(\frac{\partial \mathcal{A}^\alpha}{\partial u}(u_k,\lambda_0)\, u_k, u_k^\alpha\right) + \mathcal{F}^\alpha(u_k,\lambda_0) - \frac{\partial \mathcal{F}^\alpha}{\partial u}(u_k,\lambda_0)\, u_k, \quad \alpha \in \{1,\ldots,m\}.
\end{aligned} \tag{4.2}$$

Starting the iteration with $k = 1$ and $u_1 \in U \cap W^{\omega}_{0E}(S; H^1_0(G;\mathbb{R}^m))$, the right hand side of (4.2) belongs to $L^{\omega}_2(S; H^{-1}(G))$. Since we have

$$\frac{\partial \mathcal{P}^\alpha}{\partial u}(u,\lambda_0)\, v = (\mathcal{E}^\alpha v^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(u,\lambda_0), v^\alpha) + \mathcal{B}\left(\frac{\partial \mathcal{A}^\alpha}{\partial u}(u,\lambda_0)\, v, u^\alpha\right) - \frac{\partial \mathcal{F}^\alpha}{\partial u}(u,\lambda_0)\, v$$

for all $v \in W^{\omega}_{0E}(S; H^1_0(G;\mathbb{R}^m))$ and $\alpha \in \{1,\ldots,m\}$, the derivative

$$v \mapsto \left(\frac{\partial \mathcal{P}^1}{\partial u}(u,\lambda_0), \ldots, \frac{\partial \mathcal{P}^m}{\partial u}(u,\lambda_0)\right), \tag{4.3}$$

is close to the isomorphism defined by (4.1) with respect to the operator norm in the space $\mathscr{L}\left(W^{\omega}_{0E}(S; H^1_0(G;\mathbb{R}^m)); L^{\omega}_2(S; H^{-1}(G;\mathbb{R}^m))\right)$ and, therefore, an isomorphism from $W^{\omega}_{0E}(S; H^1_0(G;\mathbb{R}^m))$ onto $L^{\omega}_2(S; H^{-1}(G;\mathbb{R}^m))$, too, whenever u is sufficiently close to u_0 in $W^{\omega}_{0E}(S; H^1_0(G;\mathbb{R}^m))$. Hence, if u_1 is sufficiently close to u_0 in $W^{\omega}_{0E}(S; H^1_0(G;\mathbb{R}^m))$, then the new iteration u_2 is uniquely defined by (4.2), belongs to $W^{\omega}_{0E}(S; H^1_0(G;\mathbb{R}^m))$ and is close to u_0 in $W^{\omega}_{0E}(S; H^1_0(G;\mathbb{R}^m))$. Now, the classical Newton iteration procedure, see [28, Proposition 5.1], works for problem (2.7), since the norm of the map (4.3) in $\mathscr{L}\left(W^{\omega}_{0E}(S; H^1_0(G;\mathbb{R}^m)); L^{\omega}_2(S; H^{-1}(G;\mathbb{R}^m))\right)$ depends even Lipschitz continuously on u in a neighborhood of u_0 in the space $W^{\omega}_{0E}(S; H^1_0(G;\mathbb{R}^m))$. □

4.2. Local existence

One cannot expect that solutions $(u,\lambda) \in \left(U \cap W_{0E}(S; H^1_0(G;\mathbb{R}^m))\right) \times V$ to problem (2.7) exist on arbitrarily long time intervals $S = (t_0, t_1)$ without imposing further structural or growth conditions on the nonlinear operators $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$. Setting

$$U_\tau = \{u|S_\tau : u \in U\},$$

our next assertion deals with the fact, that in the case $(0, \lambda) \in U \times V$ we can always find a solution $(u_\tau, \lambda) \in \big(U_\tau \cap W_{0E}(S_\tau; H_0^1(G; \mathbb{R}^m))\big) \times V$ to the problem (2.7) restricted to the subinterval

$$S_\tau = (t_0, t_0 + \tau)$$

of S, whenever we choose $\tau \in (0, t_1 - t_0]$ small enough. This restricted problem is of the type

$$\left(\mathcal{E}_\tau^\alpha u_\tau^\alpha\right)' + \mathcal{B}_\tau\left(\mathcal{A}_\tau^\alpha(u_\tau, \lambda), u_\tau^\alpha\right) = \mathcal{F}_\tau^\alpha(u_\tau, \lambda), \quad \alpha \in \{1, \dots, m\}, \tag{4.4}$$

where, for indices $\alpha \in \{1, \dots, m\}$ and leading order coefficients $a^\alpha \in L^\infty(X)$ and $A_\tau^\alpha \in L^\infty(S_\tau \times X; \mathbb{R}^{n\times n})$, the linear continuous operator

$$\mathcal{E}_\tau = \left(\mathcal{E}_\tau^1, \dots, \mathcal{E}_\tau^m\right) : L^2(S_\tau; H^1(X; \mathbb{R}^m)) \to L^2(S_\tau; H^{-1}(G; \mathbb{R}^m))$$

associated with S_τ and E as well as the bilinear continuous map

$$\mathcal{B}_\tau : L^\infty(S_\tau \times X; \mathbb{R}^{n\times n}) \times L^2(S_\tau; H^1(X)) \to L^2(S_\tau; H^{-1}(G))$$

are defined analogously to (2.6) and (2.8). Furthermore, using the Volterra property of $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$ with respect to u, both the nonlinear operators

$$\mathcal{A}_\tau^\alpha \in C^1\left(U_\tau \times V; L^\infty(S_\tau \times X; \mathbb{R}^{n\times n})\right),$$
$$\mathcal{F}_\tau^\alpha \in C^1\left(U_\tau \times V; L_2^{\omega_0}(S_\tau; H^{-1}(G))\right),$$

are uniquely defined by the identities

$$\mathcal{A}_\tau^\alpha(u|S_\tau, \lambda) = \mathcal{A}^\alpha(u, \lambda)|(S_\tau \times X), \quad \mathcal{F}_\tau^\alpha(u|S_\tau, \lambda) = \mathcal{F}^\alpha(u, \lambda)|S_\tau,$$

for $(u, \lambda) \in U \times V$ and $\alpha \in \{1, \dots, m\}$.

THEOREM 4.2. *Assume that* $(0, \lambda_0)$ *belongs to* $U \times V$*, and let* $\delta \in (0, 1]$ *be a constant such that* $a^\alpha \in L^\infty(X)$ *and* $\mathcal{A}^\alpha(0, \lambda_0) \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ *are* δ*-definite with respect to* S *and* X *for all* $\alpha \in \{1, \dots, m\}$.

Then we find a parameter $\omega \in (n, \omega_0]$*, depending on* δ *and* G *only, some* $\tau \in (0, t_1 - t_0]$ *and a neighborhood* V_0 *in* Λ *of* λ_0 *with* $V_0 \subset V$ *such that for all* $\lambda \in V_0$ *there exists a solution* $u_\tau \in U_\tau \cap W_{0E}^\omega(S_\tau; H_0^1(G; \mathbb{R}^m))$ *to* (4.4).

Proof. 1. Due to Theorem 4.1 it suffices to find some $\omega \in (n, \omega_0]$, depending on δ and G only, and some $\tau \in (0, t_1 - t_0]$ such that there exists a solution $u_\tau \in U_\tau \cap W_{0E}^\omega(S_\tau; H_0^1(G; \mathbb{R}^m))$ to (4.4) with $\lambda = \lambda_0$. Hence, from now on $\lambda = \lambda_0$ is fixed.

2. Because $\mathcal{F}^\alpha(0,\lambda_0) \in L_2^{\omega_0}(S; H^{-1}(G))$ holds true for every $\alpha \in \{1,\dots,m\}$, Theorem 3.1 implies that there exist some $\bar\omega = \bar\omega(\delta, G) \in (n, \omega_0]$ such that the solution $v \in W_{0E}(S; H_0^1(G;\mathbb{R}^m))$ of the linear auxiliary problem

$$(\mathcal{E}^\alpha v^\alpha)' + \mathcal{B}(\mathcal{A}^\alpha(0,\lambda_0), v^\alpha) = \mathcal{F}^\alpha(0,\lambda_0), \quad \alpha \in \{1,\dots,m\}, \tag{4.5}$$

belongs to $W_{0E}^{\bar\omega}(S; H_0^1(G;\mathbb{R}^m))$.

3. We choose two neighborhoods U_0 and W_0 of zero in $C(\overline{S}; C(\overline{X};\mathbb{R}^m))$ such that the inclusion $\{u + w : (u,w) \in U_0 \times W_0\} \subset U$ holds true. Now, we look for solutions $(u,w) \in \big(U_0 \cap W_{0E}(S; H_0^1(G;\mathbb{R}^m))\big) \times W_0$ of the nonlinear auxiliary problem

$$(\mathcal{E}^\alpha u^\alpha)' + \mathcal{B}(\mathcal{A}_0^\alpha(u,w), u^\alpha) = \mathcal{F}_0^\alpha(u,w), \quad \alpha \in \{1,\dots,m\}, \tag{4.6}$$

where the nonlinear Volterra operators $\mathcal{A}_0^\alpha \in C^1\left(U_0 \times W_0; L^\infty(S \times X; \mathbb{R}^{n\times n})\right)$ and $\mathcal{F}_0^\alpha \in C^1\left(U_0 \times W_0; L_2^{\bar\omega}(S; H^{-1}(G))\right)$ are defined by

$$\mathcal{A}_0^\alpha(u,w) = \mathcal{A}^\alpha(u+w,\lambda_0), \tag{4.7}$$

$$\begin{aligned}\mathcal{F}_0^\alpha(u,w) = {} & \big(\mathcal{F}^\alpha(u+w,\lambda_0) - \mathcal{F}^\alpha(0,\lambda_0)\big) \\ & - \mathcal{B}\left(\mathcal{A}^\alpha(u+w,\lambda_0) - \mathcal{A}^\alpha(0,\lambda_0), v^\alpha\right),\end{aligned} \tag{4.8}$$

for $(u,w) \in U_0 \times W_0$ and $\alpha \in \{1,\dots,m\}$. Since $\mathcal{A}_0^\alpha(0,0) = \mathcal{A}^\alpha(0,\lambda_0)$ and $\mathcal{F}_0^\alpha(0,0) = 0$ hold true, the pair $(u,w) = (0,0) \in U_0 \times W_0$ is a solution of (4.6).

In view of Theorem 4.1 we find a parameter $\omega \in (n, \bar\omega]$, two neighborhoods U_1 and W_1 of zero in $C(\overline{S}; C(\overline{X};\mathbb{R}^m))$ satisfying $U_1 \times W_1 \subset U_0 \times W_0$ and a solution map $\Phi \in C^1\left(W_1; W_{0E}^\omega(S; H_0^1(G;\mathbb{R}^m))\right)$ such that $(u,w) \in U_1 \times W_1$ is a solution of (4.6) if and only if $u = \Phi(w)$. In particular, for each solution $(u,w) \in U_1 \times W_1$ of (4.6) we get $u \in W_{0E}^\omega(S; H_0^1(G;\mathbb{R}^m))$.

4. Next, we make use of the fact that the solution $v \in W_{0E}^{\bar\omega}(S; H_0^1(G;\mathbb{R}^m))$ of (4.5) is small in the norm of $C(\overline{S_t}; C(\overline{X};\mathbb{R}^m))$ on subintervals $S_t = (t_0, t_0 + t)$ of S, whenever $t \in (0, t_1 - t_0]$ is small enough: indeed, due to the continuous embedding of $W_E^{\bar\omega}(S; H_0^1(G;\mathbb{R}^m))$ into $C^{0,\beta}(\overline{S}; C(\overline{X};\mathbb{R}^m))$ for $\beta = (\bar\omega - n)/4$, see [19, Theorems 3.4, 6.8], for all $t \in (0, t_1 - t_0]$ and $s \in [t_0, t_0 + t]$ we get

$$\|v(s) - v(t_0)\|_{C(\overline{X};\mathbb{R}^m)} \le (s - t_0)^\beta \|v\|_{C^{0,\beta}(\overline{S}; C(\overline{X};\mathbb{R}^m))} \le c_1 t^\beta \|v\|_{W_E^{\bar\omega}(S; H_0^1(G;\mathbb{R}^m))},$$

where the constant $c_1 > 0$ does not depend on t. Because $v(t_0) = 0$ holds true, we can find some $\tau, t \in (0, t_1 - t_0]$ with $\tau < t$ and a cut-off function $\vartheta \in C^\infty(\mathbb{R})$ with

$$0 \le \vartheta \le 1, \quad \vartheta(s) = 1 \quad \text{for all } s \le t_0 + \tau, \quad \vartheta(s) = 0 \quad \text{for all } s \ge t_0 + t,$$

such that $\vartheta v \in C(\overline{S}; C(\overline{X};\mathbb{R}^m))$ belongs to W_1. Hence, choosing $w = \vartheta v$ in (4.6) we get that $u = \Phi(w) = \Phi(\vartheta v) \in W_{0E}^\omega(S; H_0^1(G;\mathbb{R}^m))$ solves (4.6) with $w = \vartheta v$. Because of (4.5), by (4.7) and (4.8) we arrive at

$$(\mathcal{E}^\alpha(u^\alpha + v^\alpha))' + \mathcal{B}(\mathcal{A}^\alpha(u+w,\lambda_0), u^\alpha + v^\alpha) = \mathcal{F}^\alpha(u+w,\lambda_0), \quad \alpha \in \{1,\dots,m\}.$$

Note, that $u_\tau = (u+w)|S_\tau = (u+v)|S_\tau$ belongs to $W^\omega_{0E}(S_\tau; H^1_0(G;\mathbb{R}^m))$. Hence, restricting the functionals on both sides of the last identity to the subinterval S_τ, the Volterra property of the maps $\mathcal{E}$, $\mathcal{A}^\alpha$, $\mathcal{B}$, $\mathcal{F}^\alpha$ and the definition of their restrictions $\mathcal{E}_\tau$, $\mathcal{A}^\alpha_\tau$, $\mathcal{B}_\tau$, $\mathcal{F}^\alpha_\tau$ to S_τ yield that (u_τ, λ_0) is a solution of problem (4.4). □

4.3. Uniqueness

This section is to prove the following

THEOREM 4.3. *Let $\lambda \in V$ be fixed, let $u, v \in U \cap W_{0E}(S; H^1_0(G;\mathbb{R}^m))$ be solutions to (2.7), and suppose that there exists a constant $\delta \in (0,1]$ such that $a^\alpha \in L^\infty(X)$ and $\mathcal{A}^\alpha(u,\lambda)$, $\mathcal{A}^\alpha(v,\lambda) \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ are δ-definite with respect to S and X for all $\alpha \in \{1,\dots,m\}$.*

Then we have $u = v$.

Proof. 1. Because of the continuous embedding of the space $W_E(S; H^1_0(G;\mathbb{R}^m))$ into $C(\overline{S}; L^2(X;\mathbb{R}^m))$ and since $u(t_0) = v(t_0) = 0$ holds true we can define $t^* \in \overline{S} = [t_0, t_1]$ by

$$t^* = \sup\left\{t \in \overline{S} : u(s) = v(s) \text{ for all } t_0 \le s \le t\right\}.$$

We have to show that $t^* = t_1$.

Suppose, to the contrary, that $t^* < t_1$. Consider $\tau \in (t^*, t_1)$ and the corresponding subinterval $S_\tau = (t_0, \tau)$. Obviously, the restrictions $u_\tau = u|S_\tau$ and $v_\tau = v|S_\tau$ are solutions to the restricted problem (4.4). We are going to show that, if τ is sufficiently close to t^*, we have $u_\tau = v_\tau$ and, hence, a contradiction to the definition of t^*.

2. Since U is open in $C(\overline{S}; C(\overline{X};\mathbb{R}^m))$ there exists $\varepsilon > 0$ such that

$$v + w \in U \text{ for all } w \in C(\overline{S}; C(\overline{X};\mathbb{R}^m)) \text{ with } \|w\|_{C(\overline{S};C(\overline{X};\mathbb{R}^m))} \le \varepsilon.$$

Because of Theorem 4.1 we find $\omega \in (n, \omega_0]$ such that $u, v \in W^\omega_{0E}(S; H^1_0(G;\mathbb{R}^m))$. Using the continuous embedding from $W^\omega_E(S; H^1_0(G;\mathbb{R}^m))$ into $C^{0,\beta}(\overline{S}; C(\overline{X};\mathbb{R}^m))$ for $\beta = (\omega - n)/4$, see [19, Theorems 3.4, 6.8], and the fact that $u(s) = v(s)$ holds true for all $t_0 \le s \le t^*$, for all $s \in [t^*, t_1]$ we get

$$\begin{aligned}\|u(s) - v(s)\|_{C(\overline{X};\mathbb{R}^m)} &\le (s-t^*)^\beta \|u - v\|_{C^{0,\beta}(\overline{S};C(\overline{X};\mathbb{R}^m))}\\ &\le c_1 (s-t^*)^\beta \|u - v\|_{W^\omega_E(S;H^1_0(G;\mathbb{R}^m))},\end{aligned}$$

where the constant $c_1 > 0$ does not depend on t^* or τ. Hence, we can choose τ, $t \in (t^*, t_1)$ with $\tau < t$ and a cut-off function $\vartheta \in C^\infty(\mathbb{R})$ satisfying

$$0 \le \vartheta \le 1, \quad \vartheta(s) = 1 \quad \text{for all } s \le \tau, \quad \vartheta(s) = 0 \quad \text{for all } s \ge t,$$

such that for $\vartheta(u - v) \in C(\overline{S}; C(\overline{X};\mathbb{R}^m))$ we have $\|\vartheta(u-v)\|_{C(\overline{S};C(\overline{X};\mathbb{R}^m))} \le \varepsilon$. This implies $v + \sigma\vartheta(u - v) \in U$ and, consequently,

$$v_\tau + \sigma(u_\tau - v_\tau) \in U_\tau = \{w|S_\tau : w \in U\} \text{ for all } \sigma \in [0,1].$$

Therefore we can use the Mean Value Theorem to write the right hand sides of the system

$$
\begin{aligned}
&\left(\mathcal{E}_\tau^\alpha \left(u_\tau^\alpha - v_\tau^\alpha\right)\right)' + \mathcal{B}_\tau \left(\mathcal{A}_\tau^\alpha(u_\tau, \lambda), u_\tau^\alpha - v_\tau^\alpha\right) \\
&\quad = \left(\mathcal{F}_\tau^\alpha(u_\tau, \lambda) - \mathcal{F}_\tau^\alpha(v_\tau, \lambda)\right) - \mathcal{B}_\tau \left(\mathcal{A}_\tau^\alpha(u_\tau, \lambda) - \mathcal{A}_\tau^\alpha(v_\tau, \lambda), v_\tau^\alpha\right), \\
&\qquad \alpha \in \{1, \ldots, m\},
\end{aligned}
$$

in the form

$$
\begin{aligned}
&\int_0^1 \frac{\partial \mathcal{F}_\tau^\alpha}{\partial u}(\sigma u_\tau + (1-\sigma)v_\tau, \lambda)\, \mathrm{d}\sigma\, (u_\tau - v_\tau) \\
&\quad - \mathcal{B}_\tau \left(\int_0^1 \frac{\partial \mathcal{A}_\tau^\alpha}{\partial u}(\sigma u_\tau + (1-\sigma)v_\tau, \lambda)\, \mathrm{d}\sigma\, (u_\tau - v_\tau), v_\tau^\alpha \right).
\end{aligned}
$$

In other words, the difference $w_\tau = u_\tau - v_\tau \in W_{0E}^\omega(S_\tau; H_0^1(G; \mathbb{R}^m))$ solves the linear system

$$
\left(\mathcal{E}_\tau^\alpha w_\tau^\alpha\right)' + \mathcal{B}_\tau \left(A_\tau^\alpha, w_\tau^\alpha\right) = \mathcal{N}_\tau^\alpha w_\tau, \quad \alpha \in \{1, \ldots, m\}. \tag{4.9}
$$

Here the coefficient functions $A_\tau^\alpha \in L^\infty(S_\tau \times X; \mathbb{R}^{n\times n})$ are defined by $A_\tau^\alpha = \mathcal{A}_\tau^\alpha(u_\tau, \lambda)$, and the linear continuous Volterra operators

$$
\mathcal{N}_\tau^\alpha : C(\overline{S_\tau}; C(\overline{X}; \mathbb{R}^m)) \to L_2^\omega(S_\tau; H^{-1}(G))
$$

are given by

$$
\begin{aligned}
\mathcal{N}_\tau^\alpha w = &\int_0^1 \frac{\partial \mathcal{F}_\tau^\alpha}{\partial u}(\sigma u_\tau + (1-\sigma)v_\tau, \lambda)\, \mathrm{d}\sigma\, w \\
&- \mathcal{B}_\tau \left(\int_0^1 \frac{\partial \mathcal{A}_\tau^\alpha}{\partial u}(\sigma u_\tau + (1-\sigma)v_\tau, \lambda)\, \mathrm{d}\sigma\, w, v_\tau^\alpha \right)
\end{aligned}
$$

for $w \in C(\overline{S_\tau}; C(\overline{X}; \mathbb{R}^m))$. Applying Theorem 3.1 to problem (4.9) we get $w_\tau = u_\tau - v_\tau = 0$, which contradicts to the definition of t^* and, therefore, to the assumption $t^* < t_1$. □

4.4. Additional temporal regularity

In this section, we formulate assumptions on the nonlinearities (2.9) and (2.10), which ensure that the time derivatives of the solutions to (2.7) are Hölder continuous on $I \times \overline{G}$ for any compact subinterval $I \subset S$. To do so, we consider C^∞-isotopies $T : \overline{\Sigma} \times \overline{S} \to \overline{S}$, where $\Sigma = (-\sigma_1, \sigma_1)$ is an open interval. Introducing the notation

$$
T_\sigma(t) = T(\sigma, t) \quad \text{for } \sigma \in \overline{\Sigma} \text{ and } t \in \overline{S},
$$

we assume that both the families $\{T_\sigma\}_{\sigma\in\Sigma}$ and $\{T_\sigma^{-1}\}_{\sigma\in\Sigma}$ of monotone diffeomorphisms from $\overline{S}$ onto itself have uniformly bounded derivatives of arbitrary order. Moreover, we suppose that $T_0 : \overline{S} \to \overline{S}$ is the identity.

In the following, for $\sigma \in \Sigma$ we consider maps, which assign abstract functions $w : \overline{S} \to H$ with values in a Hilbert space H to its temporal transformation

$$t \in \overline{S} \mapsto w(T_\sigma(t)) \in H.$$

As a simple consequence of the change of variables formula and the uniform properties of the above families, these maps generate linear isomorphisms

$$\begin{array}{ll} \mathcal{T}_\sigma^0 \text{ from } L^2(S; L^2(X)) \text{ onto itself}, & \mathcal{T}_\sigma^\Gamma \text{ from } L^2(S; L^2(\Gamma)) \text{ onto itself}, \\ \mathcal{T}_\sigma \text{ from } L^2(S; H_0^1(G)) \text{ onto itself}, & \mathcal{M}_\sigma \text{ from } L^2(S \times X; \mathbb{R}^{n\times n}) \text{ onto itself}, \end{array}$$

as well as their adjoint operators,

$$\begin{array}{ll} \mathcal{T}_0^\sigma \text{ from } L^2(S; L^2(X)) \text{ onto itself}, & \mathcal{T}_\Gamma^\sigma \text{ from } L^2(S; L^2(\Gamma)) \text{ onto itself}, \\ \mathcal{T}^\sigma \text{ from } L^2(S; H^{-1}(G)) \text{ onto itself}, & \mathcal{M}^\sigma \text{ from } L^2(S \times X; \mathbb{R}^{n\times n}) \text{ onto itself}. \end{array}$$

Obviously, $\mathcal{T}_\sigma^0$ maps $C(\overline{S}; C(\overline{X}))$ isomorphically onto itself. Moreover, we get

LEMMA 4.4. *For $\omega \in [0, n+2]$, $\varkappa \in [0, n+1]$, and $\sigma \in \Sigma$ the following holds true:*

1. *$\mathcal{T}_\sigma^0$ and $\mathcal{T}_0^\sigma$ map $L_2^\omega(S; L^2(X))$ isomorphically onto itself.*
2. *$\mathcal{T}_\sigma^\Gamma$ and $\mathcal{T}_\Gamma^\sigma$ map $L_2^\varkappa(S; L^2(\Gamma))$ isomorphically onto itself.*
3. *$\mathcal{T}_\sigma$ maps $L_2^\omega(S; H_0^1(G))$ isomorphically onto itself.*
4. *$\mathcal{T}^\sigma$ maps $L_2^\omega(S; H^{-1}(G))$ isomorphically onto itself.*
5. *Let $E^\alpha \in \mathscr{L}\big(H^1(X); H^{-1}(G)\big)$, $\mathcal{E}^\alpha \in \mathscr{L}\big(L^2(S; H^1(X)); L^2(S; H^{-1}(G))\big)$ be defined as in (2.5) and (2.6). Then, $\mathcal{T}_\sigma$ maps $W_{E^\alpha}^\omega(S; H_0^1(G))$ isomorphically onto itself, and for all $w \in W_{E^\alpha}^\omega(S; H_0^1(G))$ we have the identity*

$$\mathcal{T}^\sigma(\mathcal{E}^\alpha \mathcal{T}_\sigma w)' = (\mathcal{E}^\alpha w)'. \tag{4.10}$$

6. *$\mathcal{M}_\sigma$ and $\mathcal{M}^\sigma$ map $L^\infty(S \times X; \mathbb{R}^{n\times n})$ isomorphically onto itself. Furthermore, we get the transformation rule*

$$\mathcal{T}^\sigma \mathcal{B}(A, \mathcal{T}_\sigma w) = \mathcal{B}(\mathcal{M}^\sigma A, w) \tag{4.11}$$

for all $A \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ and $w \in L_2^\omega(S; H_0^1(G))$.

Proof. Suppose that $L \geq 1$ is a Lipschitz constant for both the transformations T_σ and T_σ^{-1}. Since the map T_σ and its inverse T_σ^{-1} have the same differential and topological properties, for the above isomorphism results it is enough to prove the continuity of the operator under consideration or the continuity of its inverse.

For the proof of the desired results in Morrey spaces we arbitrarily fix some radius $r > 0$ and corresponding subsets

$$S_r \in \left\{S \cap (t - r^2, t) : t \in S\right\}, \quad X_r \in \{X \cap Q_r(x) : x \in X\},$$
$$\Gamma_r \in \{\Gamma \cap Q_r(x) : x \in \Gamma\}, \qquad G_r \in \{G \cap Q_r(x) : x \in G\}.$$

Setting $\delta = Lr$ we can always choose suitable intersections

$$S_\delta \in \left\{S \cap (t - \delta^2, t) : t \in S\right\}, \quad X_\delta \in \{X \cap Q_\delta(x) : x \in X\},$$
$$\Gamma_\delta \in \{\Gamma \cap Q_\delta(x) : x \in \Gamma\}, \qquad G_\delta \in \{G \cap Q_\delta(x) : x \in G\},$$

with $S_r \subset T_\sigma[S_\delta]$, $X_r \subset X_\delta$, $\Gamma_r \subset \Gamma_\delta$, and $G_r \subset G_\delta$. In the following we only derive the essential estimates on these intersections. As usual, the final step to get estimates in Morrey spaces consists of multiplying both sides of the inequality under consideration with radial weights and of taking the suprema over all these radii and intersections on both sides of the inequality.

1. By a change of variables for $u \in L_2^\omega(S; L^2(X))$ and $v \in L_2^\varkappa(S; L^2(\Gamma))$ we get

$$\int_{S_r}\int_{X_r} \left|u\left(T_\sigma^{-1}(s)\right)\right|^2 \mathrm{d}\lambda^n\,\mathrm{d}s \le L \int_{S_\delta}\int_{X_\delta} |u(t)|^2\,\mathrm{d}\lambda^n\,\mathrm{d}t,$$
$$\int_{S_r}\int_{\Gamma_r} \left|v\left(T_\sigma^{-1}(s)\right)\right|^2 \mathrm{d}\lambda_\Gamma\,\mathrm{d}s \le L \int_{S_\delta}\int_{\Gamma_\delta} |v(t)|^2\,\mathrm{d}\lambda_\Gamma\,\mathrm{d}t,$$

which yields the continuity of the map $(\mathcal{T}_\sigma^0)^{-1}$ from $L_2^\omega(S; L^2(X))$ into itself and of the map $(\mathcal{T}_\sigma^\Gamma)^{-1}$ from $L_2^\varkappa(S; L^2(\Gamma))$ into itself.

2. For $u \in L_2^\omega(S; L^2(X))$ and $\varphi \in L^2(S; L^2(X))$, which satisfy $\varphi|(S \setminus S_r) = 0$ and $\varphi(s)|(X \setminus X_r) = 0$ for almost all $s \in S$, we obtain

$$\begin{aligned}\int_{S_r}\int_{X_r} \left(\mathcal{T}_0^\sigma u\right)(s)\,\varphi(s)\,\mathrm{d}\lambda^n\,\mathrm{d}s &= \int_S\int_X \left(\mathcal{T}_0^\sigma u\right)(s)\,\varphi(s)\,\mathrm{d}\lambda^n\,\mathrm{d}s\\ &= \int_S\int_X u(t)\,\left(\mathcal{T}_\sigma^0\varphi\right)(t)\,\mathrm{d}\lambda^n\,\mathrm{d}t\\ &= \int_{S_\delta}\int_{X_\delta} u(t)\,(\mathcal{T}_\sigma^0\varphi)(t)\,\mathrm{d}\lambda^n\,\mathrm{d}t,\end{aligned}$$

which leads to the estimate

$$\int_{S_r}\int_{X_r} \left|\left(\mathcal{T}_0^\sigma u\right)(s)\right|^2 \mathrm{d}\lambda^n\,\mathrm{d}s \le \left\|\mathcal{T}_\sigma^0\right\|^2 \int_{S_\delta}\int_{X_\delta} |u(t)|^2\,\mathrm{d}\lambda^n\,\mathrm{d}s,$$

where $\|\mathcal{T}_\sigma^0\|$ is the norm of the operator $\mathcal{T}_\sigma^0$ mapping $L^2(S; L^2(X))$ into itself. Consequently, $\mathcal{T}_0^\sigma$ is a linear continuous operator from $L_2^\omega(S; L^2(X))$ into itself.

For all $u \in L_2^{\varkappa}(S; L^2(\Gamma))$ and $\varphi \in L^2(S; L^2(\Gamma))$ satisfying $\varphi|(S \setminus S_r) = 0$ and $\varphi(s)|(\Gamma \setminus \Gamma_r) = 0$ for almost all $s \in S$, we have

$$\begin{aligned}\int_{S_r}\int_{\Gamma_r} (\mathcal{T}_\Gamma^\sigma u)(s)\,\varphi(s)\,\mathrm{d}\lambda_\Gamma\,\mathrm{d}s &= \int_S\int_\Gamma (\mathcal{T}_\Gamma^\sigma u)(s)\,\varphi(s)\,\mathrm{d}\lambda_\Gamma\,\mathrm{d}s\\ &= \int_S\int_\Gamma u(t)\,(\mathcal{T}_\sigma^\Gamma \varphi)(t)\,\mathrm{d}\lambda_\Gamma\,\mathrm{d}t\\ &= \int_{S_\delta}\int_{\Gamma_\delta} u(t)\,(\mathcal{T}_\sigma^\Gamma \varphi)(t)\,\mathrm{d}\lambda_\Gamma\,\mathrm{d}t,\end{aligned}$$

which leads to the estimate

$$\int_{S_r}\int_{\Gamma_r} \left|(\mathcal{T}_\Gamma^\sigma u)(s)\right|^2 \mathrm{d}\lambda_\Gamma\,\mathrm{d}s \le \|\mathcal{T}_\sigma^\Gamma\|^2 \int_{S_\delta}\int_{\Gamma_\delta} |u(t)|^2\,\mathrm{d}\lambda_\Gamma\,\mathrm{d}s,$$

where $\|\mathcal{T}_\sigma^\Gamma\|$ is the norm of the operator $\mathcal{T}_\sigma^\Gamma$ mapping $L^2(S; L^2(\Gamma))$ into itself. Hence, $\mathcal{T}_\Gamma^\sigma$ is a linear continuous operator from $L_2^{\varkappa}(S; L^2(\Gamma))$ into itself.

3. Due to a change of variables for all $u \in L_2^{\omega}(S; H_0^1(G))$ we obtain

$$\int_{S_r}\int_{G_r} \left|\nabla u\left(T_\sigma^{-1}(s)\right)\right|^2 \mathrm{d}\lambda^n\,\mathrm{d}s \le L\int_{S_\delta}\int_{G_\delta} |\nabla u(t)|^2\,\mathrm{d}\lambda^n\,\mathrm{d}t.$$

Together with Step 1 this proves the continuity of $\mathcal{T}_\sigma^{-1}$ from $L_2^{\omega}(S; H_0^1(G))$ into itself.

4. For all $f \in L_2^{\omega}(S; H^{-1}(G))$ and $v \in L^2(S_r; H_0^1(G_r))$ the properties of the zero extension map $\mathcal{Z}_{S,G}$ and the localization operators $\mathcal{R}_{S_\delta,G_\delta}$ and $\mathcal{L}_{S_\delta,G_\delta}$ ensure that

$$\begin{aligned}&\int_{S_r} \langle(\mathcal{L}_{S_r,G_r}\mathcal{T}^\sigma f)(s),\, v(s)\rangle_{H_0^1(G_r)}\,\mathrm{d}s\\ &= \int_S \langle(\mathcal{T}^\sigma f)(s),\,(\mathcal{Z}_{S,G}v)(s)\rangle_{H_0^1(G)}\,\mathrm{d}s = \int_S \langle f(t),\,(\mathcal{T}_\sigma\mathcal{Z}_{S,G}v)(t)\rangle_{H_0^1(G)}\,\mathrm{d}t\\ &= \int_{S_\delta} \langle(\mathcal{L}_{S_\delta,G_\delta} f)(t),\,(\mathcal{R}_{S_\delta,G_\delta}\mathcal{T}_\sigma\mathcal{Z}_{S,G}v)(t)\rangle_{H_0^1(G_\delta)}\,\mathrm{d}t\end{aligned}$$

holds true, which yields the estimate

$$\int_{S_r} \left\|(\mathcal{L}_{S_r,G_r}\mathcal{T}^\sigma f)(s)\right\|_{H^{-1}(G_r)}^2 \mathrm{d}s \le \|\mathcal{T}_\sigma\|^2 \int_{S_\delta} \left\|(\mathcal{L}_{S_\delta,G_\delta} f)(t)\right\|_{H^{-1}(G_\delta)}^2 \mathrm{d}t,$$

where $\|\mathcal{T}_\sigma\|$ is the norm of the operator $\mathcal{T}_\sigma$ mapping $L^2(S; H_0^1(G))$ into itself. Hence, $\mathcal{T}^\sigma$ is a linear continuous operator from $L_2^{\omega}(S; H^{-1}(G))$ into itself.

5. If $u = \mathcal{T}_\sigma w \in W_{E^\alpha}^{\omega}(S; H_0^1(G))$, then $w = \mathcal{T}_\sigma^{-1} u$ belongs to $L_2^{\omega}(S; H_0^1(G))$ due to Step 3. Furthermore, for all $\vartheta \in C_0^\infty(S)$ and $\varphi \in H_0^1(G)$ we have

$$\begin{aligned}&\int_S \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}^\alpha\mathcal{T}_\sigma w)(t),\,(\vartheta\varphi)(T_\sigma(t))\right\rangle_{H_0^1(G)} \mathrm{d}t\\ &= -\int_S \left\langle (\mathcal{E}^\alpha w)(T_\sigma(t)),\,\frac{\mathrm{d}\vartheta}{\mathrm{d}s}(T_\sigma(t))\varphi\right\rangle_{H_0^1(G)} \frac{\partial T}{\partial t}(\sigma,t)\,\mathrm{d}t\\ &= -\int_S \left\langle (\mathcal{E}^\alpha w)(s),\,\frac{\mathrm{d}\vartheta}{\mathrm{d}s}(s)\varphi\right\rangle_{H_0^1(G)} \mathrm{d}s.\end{aligned}$$

Hence, Step 4 yields the identity $(\mathcal{E}^\alpha w)' = \mathcal{T}^\sigma(\mathcal{E}^\alpha \mathcal{T}_\sigma w)' \in L_2^\omega(S; H^{-1}(G))$ and, therefore, $w \in W_{E^\alpha}^\omega(S; H_0^1(G))$ with a corresponding norm estimate. If, in addition to that, we have $u(t_0) = 0$, this implies $w(t_0) = 0$.

6. Clearly, the operator $\mathcal{M}_\sigma$ maps $L^\infty(S \times X; \mathbb{R}^{n\times n})$ into itself. Using a change of variables, for matrix functions $A \in L^\infty(S \times X; \mathbb{R}^{n\times n})$ and $B \in C(\overline{S} \times \overline{X}; \mathbb{R}^{n\times n})$ we obtain the estimate

$$\int_S \int_X A : (\mathcal{M}_\sigma B)\, \mathrm{d}\lambda^n\, \mathrm{d}t \le \|A\|_\infty\, \|(\mathcal{M}_\sigma B)\|_1 \le L\, \|A\|_\infty\, \|B\|_1.$$

Here we denote by $\|A\|_\infty$ and $\|B\|_1$ the norms of A and B in $L^\infty(S \times X; \mathbb{R}^{n\times n})$ and $L^1(S \times X; \mathbb{R}^{n\times n})$, respectively. A density argument shows that $\mathcal{M}^\sigma$ is a linear continuous operator from $L^\infty(S \times X; \mathbb{R}^{n\times n})$ into itself.

7. By definition for all $A \in L^\infty(S \times X; \mathbb{R}^{n\times n})$, $w \in L_2^\omega(S; H_0^1(G))$, $\vartheta \in C_0^\infty(S)$ and $v \in C_0^\infty(G)$ we get the identity

$$\begin{aligned}
\langle \mathcal{B}(A, \mathcal{T}_\sigma w), \mathcal{T}_\sigma(\vartheta v)\rangle_{L^2(S;H_0^1(G))} &= \int_S \int_X A : \mathcal{M}_\sigma(\nabla w \otimes \nabla(\vartheta v))\, \mathrm{d}\lambda^n\, \mathrm{d}t \\
&= \int_S \int_X (\mathcal{M}^\sigma A) : (\nabla w \otimes \nabla(\vartheta v))\, \mathrm{d}\lambda^n\, \mathrm{d}s \\
&= \langle \mathcal{B}(\mathcal{M}^\sigma A, w), \vartheta v\rangle_{L^2(S;H_0^1(G))}.
\end{aligned}$$

Using a density argument, we see that (4.11) holds true. □

Let W be some neighborhood of $u \in U$ in $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ such that

$$\left(\mathcal{T}_\sigma^0 w^1, \ldots, \mathcal{T}_\sigma^0 w^m\right) \in U \quad \text{for all } w \in W \text{ and } \sigma \in \Sigma.$$

For $\lambda \in V$ the linear isomorphisms introduced above generate transformations

$$\begin{aligned}
\mathcal{A}_\lambda^\alpha &: W \times \Sigma \to L^\infty(S \times X; \mathbb{R}^{n\times n}), \\
\mathcal{F}_\lambda^\alpha &: W \times \Sigma \to L_2^{\omega_0}(S; H^{-1}(G))
\end{aligned}$$

of our nonlinearities $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$ by setting

$$\mathcal{A}_\lambda^\alpha(w, \sigma) = \mathcal{M}^\sigma \mathcal{A}^\alpha(\mathcal{T}_\sigma^0 w^1, \ldots, \mathcal{T}_\sigma^0 w^m, \lambda), \tag{4.12}$$

$$\mathcal{F}_\lambda^\alpha(w, \sigma) = \mathcal{T}^\sigma \mathcal{F}^\alpha(\mathcal{T}_\sigma^0 w^1, \ldots, \mathcal{T}_\sigma^0 w^m, \lambda) \tag{4.13}$$

for $(w, \sigma) \in W \times \Sigma$ and $\alpha \in \{1, \ldots, m\}$. The operators $\mathcal{A}_\lambda^\alpha$ and $\mathcal{F}_\lambda^\alpha$ are Volterra operators with respect to w because $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$ are so. Moreover, $\mathcal{A}_\lambda^\alpha$ and $\mathcal{F}_\lambda^\alpha$ are continuously differentiable with respect to w because of assumptions (2.9) and (2.10) and Lemma 4.4. In the following theorem we suppose that $\mathcal{A}_\lambda^\alpha$ and $\mathcal{F}_\lambda^\alpha$ are continuously differentiable also with respect to σ, which means that

$$\mathcal{A}_\lambda^\alpha \in C^1\left(W \times \Sigma; L^\infty(S \times X; \mathbb{R}^{n\times n})\right), \tag{4.14}$$

$$\mathcal{F}_\lambda^\alpha \in C^1\left(W \times \Sigma; L_2^{\omega_0}(S; H^{-1}(G))\right). \tag{4.15}$$

In the following assertion we claim the fact that, under assumptions (4.14) and (4.15), any solution u to (2.7) has the property that the time derivative of the map

$$t \mapsto \frac{\partial T}{\partial \sigma}(0, t)u(t)$$

belongs to $W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$. Note that, in general, $(\mathcal{E}u)'$ exists as a weak derivative in $L^{\omega}_2(S; H^{-1}(G; \mathbb{R}^m))$ only and, even worse, that u' only exists in the distributional sense as an element of the space $\mathscr{L}\left(C^{\infty}_0(S); H^1_0(G; \mathbb{R}^m)\right)$.

THEOREM 4.5. *Let* $(u, \lambda) \in \left(U \cap W_{0E}(S; H^1_0(G; \mathbb{R}^m))\right) \times V$ *be a solution to* (2.7) *and assume that there exists a constant* $\delta \in (0, 1]$ *such that* $a^{\alpha} \in L^{\infty}(X)$ *and* $\mathcal{A}^{\alpha}(u, \lambda) \in L^{\infty}(S \times X; \mathbb{R}^{n\times n})$ *are* δ*-definite with respect to S and X for all* $\alpha \in \{1, \dots, m\}$. *Suppose that* $\mathcal{A}^{\alpha}_{\lambda}$ *and* $\mathcal{F}^{\alpha}_{\lambda}$ *are continuously differentiable with respect to* σ, *too.*

Then we can find a parameter $\omega \in (n, \omega_0]$, *depending on* δ *and G only, such that the time derivative of the map* $t \mapsto \frac{\partial T}{\partial \sigma}(0, t)u(t)$ *belongs to* $W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))$.

Proof. 1. Because the temporal transformation T_{σ} is close to identity in the norm of $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ for small $\sigma \in \Sigma$, from $u \in W$ it follows that there exists a neighborhood Σ_1 of zero in $\mathbb{R}$ with $\Sigma_1 \subset \Sigma$, such that the temporally transformed function

$w_{\sigma} = \left(\mathcal{T}^{-1}_{\sigma} u^1, \dots, \mathcal{T}^{-1}_{\sigma} u^m\right)$ belongs to $W \cap W_{0E}(S; H^1_0(G; \mathbb{R}^m))$ for every $\sigma \in \Sigma_1$.

If we apply the adjoint operator $\mathcal{T}^{\sigma}$ to the functionals on both sides of

$$(\mathcal{E}^{\alpha} u^{\alpha})' + \mathcal{B}(\mathcal{A}^{\alpha}(u, \lambda), u^{\alpha}) = \mathcal{F}^{\alpha}(u, \lambda), \quad \alpha \in \{1, \dots, m\},$$

then, following Lemma 4.4 and the transformation rules (4.10) and (4.11), for all $\sigma \in \Sigma_1$ and $\alpha \in \{1, \dots, m\}$ we get

$$\begin{aligned}
&\left(\mathcal{E}^{\alpha} w^{\alpha}_{\sigma}\right)' + \mathcal{B}\left(\mathcal{M}^{\sigma} \mathcal{A}^{\alpha}\left(\mathcal{T}_{\sigma} w^1_{\sigma}, \dots, \mathcal{T}_{\sigma} w^m_{\sigma}, \lambda\right), w^{\alpha}_{\sigma}\right) \\
&\quad = \mathcal{T}^{\sigma}(\mathcal{E}^{\alpha} u^{\alpha})' + \mathcal{T}^{\sigma}\mathcal{B}(\mathcal{A}^{\alpha}(u, \lambda), u^{\alpha}) = \mathcal{T}^{\sigma}\mathcal{F}^{\alpha}(u, \lambda) \\
&\quad = \mathcal{T}^{\sigma}\mathcal{F}^{\alpha}(\mathcal{T}_{\sigma} w^1_{\sigma}, \dots, \mathcal{T}_{\sigma} w^m_{\sigma}, \lambda).
\end{aligned}$$

Hence, the pair $(w_{\sigma}, \sigma) \in \left(W \cap W_{0E}(S; H^1_0(G; \mathbb{R}^m))\right) \times \Sigma_1$ solves the transformed problem

$$(\mathcal{E}^{\alpha} w^{\alpha})' + \mathcal{B}\left(\mathcal{A}^{\alpha}_{\lambda}(w, \sigma), w^{\alpha}\right) = \mathcal{F}^{\alpha}_{\lambda}(w, \sigma), \quad \alpha \in \{1, \dots, m\}. \tag{4.16}$$

Note, that the pair $(w_{\sigma}, \sigma) = (u, 0)$ is a solution of both the problems (2.7) and (4.16). In view of (4.14) and (4.15) we apply Theorem 4.1 to find some Morrey exponent $\omega \in (n, \omega_0]$ and a neighborhood W_0 of u in $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ with $W_0 \subset W$ such that the following holds true: there exists a neighborhood Σ_0 of zero in $\mathbb{R}$ with $\Sigma_0 \subset \Sigma_1$ and a solution map $\Phi \in C^1\left(\Sigma_0; W^{\omega}_{0E}(S; H^1_0(G; \mathbb{R}^m))\right)$ such that $(w, \sigma) \in W_0 \times \Sigma_0$

is a solution to (4.16) if and only if $w = \Phi(\sigma)$. Because of the above construction this yields $\Phi(\sigma) = w_\sigma = \left(\mathcal{T}_\sigma^{-1}u^1, \dots, \mathcal{T}_\sigma^{-1}u^m\right)$ for all $\sigma \in \Sigma_0$.

2. For every $\alpha \in \{1, \dots, m\}$, $\sigma \in \Sigma_0$ and $\vartheta \in C_0^\infty(S)$ we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\sigma}\int_S u^\alpha(T_\sigma^{-1}(s))\,\vartheta(s)\,\mathrm{d}s &= \int_S u^\alpha(t)\frac{\partial}{\partial\sigma}\left(\vartheta(T_\sigma(t))\frac{\partial T}{\partial t}(\sigma,t)\right)\mathrm{d}t\\ &= \int_S u^\alpha(t)\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}s}(T_\sigma(t))\frac{\partial T}{\partial\sigma}(\sigma,t)\frac{\partial T}{\partial t}(\sigma,t)\right.\\ &\qquad\left.+\,\vartheta(T_\sigma(t))\frac{\partial^2 T}{\partial\sigma\,\partial t}(\sigma,t)\right)\mathrm{d}t\end{aligned}$$

and, furthermore,

$$\int_S u^\alpha(t)\frac{\mathrm{d}\vartheta}{\mathrm{d}s}(T_\sigma(t))\frac{\partial T}{\partial\sigma}(\sigma,t)\frac{\partial T}{\partial t}(\sigma,t)\,\mathrm{d}t = \int_S u^\alpha\left(T_\sigma^{-1}(s)\right)\frac{\partial T}{\partial\sigma}\left(\sigma, T_\sigma^{-1}(s)\right)\frac{\mathrm{d}\vartheta}{\mathrm{d}s}(s)\,\mathrm{d}s.$$

Specifying $\sigma = 0$, from both identities it follows that

$$\int_S\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\sigma}(0)(t) - \frac{\partial^2 T}{\partial\sigma\,\partial t}(0,t)u(t)\right)\vartheta(t)\,\mathrm{d}t = \int_S u(t)\frac{\partial T}{\partial\sigma}(0,t)\frac{\mathrm{d}\vartheta}{\mathrm{d}t}(t)\,\mathrm{d}t$$

for all $\vartheta \in C_0^\infty(S)$. In other words, the map

$$t \mapsto \frac{\partial^2 T}{\partial\sigma\,\partial t}(0,t)u(t) - \frac{\mathrm{d}\Phi}{\mathrm{d}\sigma}(0)(t),$$

which belongs to $W^\omega_{0E}(S; H_0^1(G;\mathbb{R}^m))$ due to Step 1, equals to the weak derivative of the map $t \mapsto \frac{\partial T}{\partial\sigma}(0,t)u(t)$. □

Note that, because of $T(\sigma,t_0) = t_0$ and $T(\sigma,t_1) = t_1$ holds true for all $\sigma \in \Sigma$, the derivative $t \mapsto \frac{\partial T}{\partial\sigma}(0,t)$ vanishes in the endpoints of the time interval S. Nevertheless, as the following example shows, there exist functions $T : \overline{\Sigma} \times \overline{S} \to \mathbb{R}$, for which the derivative $t \mapsto \frac{\partial T}{\partial\sigma}(0,t)$ is bounded from below by a positive constant on every compact subinterval of S.

EXAMPLE. Let $\sigma_1 > 0$ satisfy the condition $2\sigma_1(t_1 - t_0) < 1$. Given an open interval $\Sigma = (-\sigma_1, \sigma_1)$ of parameters, we consider the polynomial $T : \overline{\Sigma} \times \overline{S} \to \mathbb{R}$ and the corresponding family of temporal transformations $T_\sigma : \overline{S} \to \mathbb{R}$ defined by

$$T_\sigma(t) = T(\sigma,t) = t + \sigma(t - t_0)(t_1 - t) \quad \text{for } \sigma \in \overline{\Sigma} \text{ and } t \in \overline{S}. \tag{4.17}$$

Since we have the uniform estimate

$$\frac{\partial T}{\partial t}(\sigma,t) = 1 + \sigma\left((t_1 - t) - (t - t_0)\right) \in \left(\frac{1}{2}, \frac{3}{2}\right) \quad \text{for all } \sigma \in \overline{\Sigma} \text{ and } t \in \overline{S},$$

every transformation T_σ is a monotone diffeomorphism from $\overline{S}$ onto itself and all derivatives of the families $\{T_\sigma\}_{\sigma\in\Sigma}$ and $\{T_\sigma^{-1}\}_{\sigma\in\Sigma}$ are uniformly bounded. Obviously, T_0 is the identity, and

$$\frac{\partial T}{\partial\sigma}(\sigma,t) = (t - t_0)(t_1 - t) \quad \text{for } \sigma \in \Sigma \text{ and } t \in \overline{S}.$$

COROLLARY 4.6. *Let* $(u, \lambda) \in \big(U \cap W_{0E}(S; H_0^1(G; \mathbb{R}^m))\big) \times V$ *be a solution of problem* (2.7). *Suppose that the assumptions of Theorem* 4.5 *are satisfied with respect to the family* $\{T_\sigma\}_{\sigma\in\Sigma}$ *given by* (4.17).

Then, the weak derivative $u' : S \to H_0^1(G; \mathbb{R}^m)$ *exists as a locally integrable function. Moreover, there exists some* $\omega \in (n, \omega_0]$*, depending on* δ *and* G *only, such that the map*

$$t \mapsto (t - t_0)(t_1 - t)u'(t)$$

is an element of $W^{\omega}_{0E}(S; H_0^1(G; \mathbb{R}^m))$. *In particular, on every compact subinterval* I *of* S *the restriction of* u' *to* I *belongs to* $W^{\omega}_{E}(I; H_0^1(G; \mathbb{R}^m))$.

Proof. Due to Theorem 4.5 the function $t \mapsto (t - t_0)(t_1 - t)u(t)$ has a time derivative $v \in W^{\omega}_{0E}(S; H_0^1(G; \mathbb{R}^m))$, where $\omega \in (n, \omega_0]$ depends on δ and G only. Hence, for all $\vartheta \in C_0^\infty(S)$ we get the identity

$$\begin{aligned}\int_S v(t)\,\frac{\vartheta(t)}{(t-t_0)(t_1-t)}\,\mathrm{d}t &= -\int_S (t-t_0)(t_1-t)u(t)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\vartheta(t)}{(t-t_0)(t_1-t)}\,\mathrm{d}t\\ &= \int_S \frac{(t_1-t)-(t-t_0)}{(t-t_0)(t_1-t)}\,u(t)\vartheta(t)\,\mathrm{d}t - \int_S u(t)\frac{\mathrm{d}\vartheta}{\mathrm{d}t}(t)\,\mathrm{d}t.\end{aligned}$$

Consequently, the function given by

$$t \mapsto \frac{1}{(t-t_0)(t_1-t)}\,v(t) - \frac{(t_1-t)-(t-t_0)}{(t-t_0)(t_1-t)}\,u(t),$$

coincides with the weak derivative $u' : S \to H_0^1(G; \mathbb{R}^m)$, since it is locally integrable. Because of $u, v \in W^{\omega}_{0E}(S; H_0^1(G; \mathbb{R}^m))$, the restriction of u' to every compact subinterval $I \subset S$ belongs to $W^{\omega}_{E}(I; H_0^1(G; \mathbb{R}^m))$. Additionally, the map

$$t \mapsto (t-t_0)(t_1-t)u'(t) = v(t) - ((t_1-t)-(t-t_0))u(t)$$

is an element of $W^{\omega}_{0E}(S; H_0^1(G; \mathbb{R}^m))$, too. □

5. Examples of nonlinear operators

In this section, we indicate some classes of nonlinear operators, which are candidates for the leading order coefficient maps $\mathcal{A}^\alpha$ and the right hand sides $\mathcal{F}^\alpha$ occurring in the operator equations in Sects. 2 and 4.

5.1. Leading order coefficients

In place of the maps $\mathcal{A}^\alpha$ and $\mathcal{A}^\alpha_\lambda$ of Sect. 4 we consider superposition operators

$$\mathcal{A}(u, \lambda)(t, x) = A(t, x, u(t, x), \lambda) \qquad \text{for almost all } (t, x) \in S \times X, \tag{5.1}$$

$$\mathcal{C}(u, \lambda, \sigma)(t, x) = A(T_\sigma(t), x, u(t, x), \lambda) \qquad \text{for almost all } (t, x) \in S \times X. \tag{5.2}$$

Here, $A : S \times X \times \Omega \times V \to \mathbb{R}$ is the function, generating the superposition operators, Ω is an open subset in $\mathbb{R}^m$, and V is an open subset of the Banach space Λ. Introduced in Sect. 4, we consider the family of diffeomorphisms $T_\sigma : \overline{S} \to \overline{S}$ with uniform properties with respect to $\sigma \in \Sigma = (-\sigma_1, \sigma_1)$. In view of applications to parabolic systems we will consider vector valued functions $u : S \times X \to \mathbb{R}^m$.

We define U as the subset of all $u \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$, for which we can find a compact set $F \subset \Omega$ such that $u(t,x) \in F$ for all $(t,x) \in S \times X$. Obviously, U is open in $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$. Next, we state conditions on the function A, which ensure that $\mathcal{A} \in C^1(U \times V; L^\infty(S \times X))$ and $\mathcal{C} \in C^1(U \times V \times \Sigma; L^\infty(S \times X))$:

THEOREM 5.1. *We formulate the following C^1-Carathéodory conditions on A:*

(C1) *$(\xi, \lambda) \mapsto A(t,x,\xi,\lambda)$ belongs to $C^1(\Omega \times V)$ for almost all $(t,x) \in S \times X$, and $(t,x) \mapsto \frac{\partial A}{\partial \xi}(t,x,\xi,\lambda)$ and $(t,x) \mapsto \frac{\partial A}{\partial \lambda}(t,x,\xi,\lambda)$ are measurable for all $(\xi,\lambda) \in \Omega \times V$.*

(C2) *For all $\lambda \in V$ and compact sets $F \subset \Omega$ there exists $\varrho > 0$ such that*

$$\left|\frac{\partial A}{\partial \xi}(t,x,\xi,\lambda)\right| + \left\|\frac{\partial A}{\partial \lambda}(t,x,\xi,\lambda)\right\|_{\Lambda^*} + |A(t,x,\xi,\lambda)| \le \varrho$$

for almost all $(t,x) \in S \times X$ and all $\xi \in F$.

(C3) *For all $\lambda \in V$, compact sets $F \subset \Omega$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left|\frac{\partial A}{\partial \xi}(t,x,\xi,\lambda) - \frac{\partial A}{\partial \xi}(t,x,\eta,\mu)\right| + \left\|\frac{\partial A}{\partial \lambda}(t,x,\xi,\lambda) - \frac{\partial A}{\partial \lambda}(t,x,\eta,\mu)\right\|_{\Lambda^*} + |A(t,x,\xi,\lambda) - A(t,x,\eta,\mu)| < \varepsilon,$$

for almost all $(t,x) \in S \times X$ and all $\xi \in F$, $(\eta,\mu) \in F \times V$ with $|\xi - \eta| + \|\lambda - \mu\|_\Lambda < \delta$.

(C4) *For all $\lambda \in V$ and compact sets $F \subset \Omega$ there exists $L > 0$ such that*

$$\left|\frac{\partial A}{\partial \xi}(t,x,\xi,\lambda) - \frac{\partial A}{\partial \xi}(t,x,\eta,\lambda)\right| \le L|\xi - \eta|,$$

for almost all $(t,x) \in S \times X$ and all $\xi, \eta \in F$.

(C5) *$t \mapsto A(t,x,\xi,\lambda)$ belongs to $C^1(\overline{S})$ for almost all $x \in X$ and all $(\xi,\lambda) \in \Omega \times V$, and $x \mapsto \frac{\partial A}{\partial s}(t,x,\xi,\lambda)$ is measurable for all $t \in \overline{S}$ and $(\xi,\lambda) \in \Omega \times V$.*

(C6) *For all $\lambda \in V$ and compact sets $F \subset \Omega$ there exists $\varrho > 0$ such that*

$$\left|\frac{\partial A}{\partial s}(t,x,\xi,\lambda)\right| \le \varrho$$

for almost all $x \in X$, all $t \in \overline{S}$ and $\xi \in F$.

(C7) *For all $\lambda \in V$, compact sets $F \subset \Omega$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left|\frac{\partial A}{\partial \xi}(t,x,\xi,\lambda) - \frac{\partial A}{\partial \xi}(s,x,\eta,\mu)\right| + \left\|\frac{\partial A}{\partial \lambda}(t,x,\xi,\lambda) - \frac{\partial A}{\partial \lambda}(s,x,\eta,\mu)\right\|_{\Lambda^*} + \left|\frac{\partial A}{\partial s}(t,x,\xi,\lambda) - \frac{\partial A}{\partial s}(s,x,\eta,\mu)\right| + |A(t,x,\xi,\lambda) - A(s,x,\eta,\mu)| < \varepsilon$$

for almost all $x \in X$, *all* $s, t \in \overline{S}$ *and all* $\xi \in F$, $(\eta, \mu) \in F \times V$, *which satisfy the conditions* $|s - t| < \delta$ *and* $|\xi - \eta| + \|\lambda - \mu\|_\Lambda < \delta$.

1. *If conditions* (C1), (C2), *and* (C3) *are satisfied, then the operator* $\mathcal{A}$ *defined by* (5.1) *belongs to* $C^1(U \times V; L^\infty(S \times X))$. *Moreover, we have*

$$\left(\frac{\partial \mathcal{A}}{\partial u}(u, \lambda)\, v\right)(t, x) = \frac{\partial A}{\partial \xi}(t, x, u(t, x), \lambda)\, v(t, x), \tag{5.3a}$$

$$\left(\frac{\partial \mathcal{A}}{\partial \lambda}(u, \lambda)\, \mu\right)(t, x) = \frac{\partial A}{\partial \lambda}(t, x, u(t, x), \lambda)\, \mu, \tag{5.3b}$$

for almost all $(t, x) \in S \times X$, *all* $(u, \lambda) \in U \times V$, $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$, *and* $\mu \in \Lambda$.

2. *If, additionally,* (C4) *holds true, then* $u \mapsto \frac{\partial \mathcal{A}}{\partial u}(u, \lambda)$ *is locally Lipschitz continuous from* U *into* $\mathscr{L}\left(C(\overline{S}; C(\overline{X}; \mathbb{R}^m)); L^\infty(S \times X)\right)$ *for all* $\lambda \in V$.
3. *If conditions* (C1), (C2), (C3), *and* (C5), (C6), (C7) *are satisfied, then the operator* $\mathcal{C}$ *defined by* (5.2) *belongs to* $C^1(U \times V \times \Sigma; L^\infty(S \times X))$, *and we get*

$$\left(\frac{\partial \mathcal{C}}{\partial u}(u, \lambda, \sigma)\, v\right)(t, x) = \frac{\partial A}{\partial \xi}(T_\sigma(t), x, u(t, x), \lambda)\, v(t, x), \tag{5.4a}$$

$$\left(\frac{\partial \mathcal{C}}{\partial \lambda}(u, \lambda, \sigma)\, \mu\right)(t, x) = \frac{\partial A}{\partial \lambda}(T_\sigma(t), x, u(t, x), \lambda)\, \mu, \tag{5.4b}$$

$$\frac{\partial \mathcal{C}}{\partial \sigma}(u, \lambda, \sigma)(t, x) = \frac{\partial A}{\partial s}(T_\sigma(t), x, u(t, x), \lambda)\, \frac{\partial T}{\partial \sigma}(\sigma, t), \tag{5.4c}$$

for almost all $(t, x) \in S \times X$ *and all* $(u, \lambda, \sigma) \in U \times V \times \Sigma$, $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$, *and* $\mu \in \Lambda$.

Proof. For the sake of simplicity we denote by $\|\ \|_C$ the norm in $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$.

1. From the definition of the set U and from conditions (C1) and (C2) it follows that $\mathcal{A}$ maps $U \times V$ into $L^\infty(S \times X)$.

In order to prove (5.3a) we fix a pair $(u, \lambda) \in U \times V$ and $\varepsilon > 0$. Again, the definition of U and conditions (C1) and (C2) yield that the operator, which assigns v to

$$(t, x) \mapsto \frac{\partial A}{\partial \xi}(t, x, u(t, x), \lambda)\, v(t, x),$$

is a linear continuous map from $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ into $L^\infty(S \times X)$. Since U is an open subset of $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$, we can choose a compact set $F \subset \Omega$ and some $\delta > 0$ such that for all $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ with $\|v\|_C < \delta$ we have both $u(t, x) \in F$ and $(u + v)(t, x) \in F$ for almost all $(t, x) \in S \times X$. Taking δ small enough we can assume that this δ corresponds to λ, F and ε with respect to condition (C3). Hence, for all $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ with $\|v\|_C < \delta$ and almost all $(t, x) \in S \times X$ we obtain

$$\left| A(t,x,(u+v)(t,x),\lambda) - A(t,x,u(t,x),\lambda) - \frac{\partial A}{\partial \xi}(t,x,u(t,x),\lambda)\, v(t,x) \right|$$
$$= \left| \int_0^1 \left(\frac{\partial A}{\partial \xi}(t,x,(u+\tau v)(t,x),\lambda) - \frac{\partial A}{\partial \xi}(t,x,u(t,x),\lambda) \right) \mathrm{d}\tau \, v(t,x) \right| \le \varepsilon \|v\|_C,$$

which proves (5.3a) and the differentiability of $\mathcal{A}$ in $U \times V$ with respect to u.

To show that (5.3b) holds true, we fix a pair $(u,\lambda) \in U \times V$ and $\varepsilon > 0$. From the definition of the set U and from conditions (C1) and (C2) it follows that the operator, which assigns μ to

$$(t,x) \mapsto \frac{\partial A}{\partial \lambda}(t,x,u(t,x),\lambda)\,\mu,$$

is a linear continuous map from Λ into $L^\infty(S \times X)$. We choose $F \subset \Omega$ such that $u(t,x) \in F$ for almost all $(t,x) \in S \times X$. Additionally, we take $\delta > 0$ small enough such that $\lambda + \mu \in V$ holds true for all $\mu \in \Lambda$ with $\|\mu\|_\Lambda < \delta$ and we suppose that this δ is suitable for λ, F and $\varepsilon > 0$ from condition (C3). Then, for all $\mu \in \Lambda$ with $\|\mu\|_\Lambda < \delta$ and almost all $(t,x) \in S \times X$ we get

$$\left| A(t,x,u(t,x),\lambda+\mu) - A(t,x,u(t,x),\lambda) - \frac{\partial A}{\partial \lambda}(t,x,u(t,x),\lambda)\,\mu \right|$$
$$= \left| \int_0^1 \left(\frac{\partial A}{\partial \lambda}(t,x,u(t,x),\lambda+\tau\mu) - \frac{\partial A}{\partial \lambda}(t,x,u(t,x),\lambda) \right) \mathrm{d}\tau \, \mu \right| \le \varepsilon \|\mu\|_\Lambda,$$

which leads to (5.3b) and the differentiability of $\mathcal{A}$ in $U \times V$ with respect to λ.

2. In order to prove that $\frac{\partial \mathcal{A}}{\partial u}$ and $\frac{\partial \mathcal{A}}{\partial \lambda}$ are continuous maps, we fix a pair $(u,\lambda) \in U \times V$ and some $\varepsilon > 0$. Because U is open in $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$, we can find a compact set $F \subset \Omega$ and some $\delta > 0$ such that for all $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ with $\|v\|_C < \delta$ we have both $u(t,x) \in F$ and $(u+v)(t,x) \in F$ for almost all $(t,x) \in S \times X$. Taking δ small enough we ensure that this δ corresponds to λ, F and ε with respect to condition (C3) and that $\lambda + \mu \in V$ holds true for all $\mu \in \Lambda$ with $\|\mu\|_\Lambda < \delta$. Hence, for all $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ and $\mu \in \Lambda$ with $\|v\|_C + \|\mu\|_\Lambda < \delta$, for all $\varphi \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ and $\chi \in \Lambda$ and almost all $(t,x) \in S \times X$, condition (C3) yields

$$\left| \left(\frac{\partial \mathcal{A}}{\partial u}(u+v,\lambda+\mu)\,\varphi - \frac{\partial \mathcal{A}}{\partial u}(u,\lambda)\,\varphi \right)(t,x) \right|$$
$$= \left| \left(\frac{\partial A}{\partial \xi}(t,x,(u+v)(t,x),\lambda+\mu) - \frac{\partial A}{\partial \xi}(t,x,u(t,x),\lambda) \right) \varphi(t,x) \right| \le \varepsilon \|\varphi\|_C,$$

and

$$\left| \left(\frac{\partial \mathcal{A}}{\partial \lambda}(u+v,\lambda+\mu)\,\chi - \frac{\partial \mathcal{A}}{\partial \lambda}(u,\lambda)\,\chi \right)(t,x) \right|$$
$$= \left| \left(\frac{\partial A}{\partial \lambda}(t,x,(u+v)(t,x),\lambda+\mu) - \frac{\partial A}{\partial \lambda}(t,x,u(t,x),\lambda) \right) \chi \right| \le \varepsilon \|\chi\|_\Lambda,$$

in other words, $\frac{\partial \mathcal{A}}{\partial u}$ and $\frac{\partial \mathcal{A}}{\partial \lambda}$ are continuous on $U \times V$.

3. Next, we show that $u \mapsto \frac{\partial \mathcal{A}}{\partial u}(u, \lambda)$ is locally Lipschitz continuous, whenever all the conditions (C1), (C2), (C3), and (C4) are satisfied. To do so, we fix $(u, \lambda) \in U \times V$, and, again, we choose a compact set $F \subset \Omega$ and some $\delta > 0$ such that for all $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ with $\|v\|_C < \delta$ we have both $u(t, x) \in F$ and $(u+v)(t, x) \in F$ for almost all $(t, x) \in S \times X$. Let $L > 0$ be the Lipschitz constant, which corresponds to λ and F in condition (C4). Then, for all $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ with $\|v\|_C < \delta$, for all $\varphi \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ and almost all $(t, x) \in S \times X$, we arrive at

$$\left|\left(\frac{\partial \mathcal{A}}{\partial u}(u+v, \lambda)\, \varphi - \frac{\partial \mathcal{A}}{\partial u}(u, \lambda)\, \varphi\right)(t, x)\right|$$
$$= \left|\left(\frac{\partial A}{\partial \xi}(t, x, (u+v)(t, x), \lambda) - \frac{\partial A}{\partial \xi}(t, x, u(t, x), \lambda)\right) \varphi(t, x)\right| \leq L\|v\|_C \|\varphi\|_C,$$

which leads to the local Lipschitz continuity of $u \mapsto \frac{\partial \mathcal{A}}{\partial u}(u, \lambda)$.

4. Analogously to Step 1, we can use the definition of the set U and conditions (C1), (C2), (C3), (C5), (C6), and (C7) to show that $\mathcal{C}$ maps $U \times V \times \Sigma$ into $L^\infty(S \times X)$, that $\mathcal{C}$ is differentiable with respect to u and λ in $U \times V \times \Sigma$, and that (5.4a) and (5.4b) are the corresponding derivatives.

5. To prove the remaining assertions, we make use of the uniform properties of the family of temporal transformations: We take $M > 0$ such that $\left|\frac{\partial T}{\partial \sigma}(\sigma, t)\right| \leq M$ for all $t \in \overline{S}$ and $\sigma \in \Sigma$.

In order to show that $\frac{\partial \mathcal{C}}{\partial u}$ and $\frac{\partial \mathcal{C}}{\partial \lambda}$ are continuous operators, we fix a triple $(u, \lambda, \sigma) \in U \times V \times \Sigma$ and $\varepsilon > 0$. Since U is an open subset of $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$, there exists a compact set $F \subset \Omega$ and some $\delta > 0$ such that for all $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ with $\|v\|_C < \delta$ we have both $u(t, x) \in F$ and $(u+v)(t, x) \in F$ for almost all $(t, x) \in S \times X$. We choose δ small enough such that it corresponds to λ, F and ε with respect to condition (C7) and that $\lambda + \mu \in V$ and $\sigma + \kappa \in \Sigma$ hold true for all $\mu \in \Lambda$ with $\|\mu\|_\Lambda < \delta$ and all $\kappa \in \mathbb{R}$ with $|\kappa| < \frac{\delta}{M}$. Consequently, for all $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$, $\mu \in \Lambda$, and $\kappa \in \mathbb{R}$ with $\|v\|_C + \|\mu\|_\Lambda < \delta$ and $|\kappa| < \frac{\delta}{M}$, for all $\varphi \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ and $\chi \in \Lambda$, and almost all $(t, x) \in S \times X$, we obtain

$$\left|\left(\frac{\partial A}{\partial \xi}(T_{\sigma+\kappa}(t), x, (u+v)(t, x), \lambda+\mu) - \frac{\partial A}{\partial \xi}(T_\sigma(t), x, u(t, x), \lambda)\right) \varphi(t, x)\right|$$
$$\leq \varepsilon \|\varphi\|_C,$$
$$\left|\frac{\partial A}{\partial \lambda}(T_{\sigma+\kappa}(t), x, (u+v)(t, x), \lambda+\mu)\, \chi - \frac{\partial A}{\partial \lambda}(T_\sigma(t), x, u(t, x), \lambda)\, \chi\right| \leq \varepsilon \|\chi\|_\Lambda,$$

which means,

$$\left|\left(\frac{\partial \mathcal{C}}{\partial u}(u+v, \lambda+\mu, \sigma+\kappa)\, \varphi - \frac{\partial \mathcal{C}}{\partial u}(u, \lambda, \sigma)\, \varphi\right)(t, x)\right| \leq \varepsilon \|\varphi\|_C,$$
$$\left|\left(\frac{\partial \mathcal{C}}{\partial \lambda}(u+v, \lambda+\mu, \sigma+\kappa)\, \chi - \frac{\partial \mathcal{C}}{\partial \lambda}(u, \lambda, \sigma)\, \chi\right)(t, x)\right| \leq \varepsilon \|\chi\|_\Lambda.$$

Hence, $\frac{\partial \mathcal{C}}{\partial u}$ and $\frac{\partial \mathcal{C}}{\partial \lambda}$ are continuous on $U \times V$.

6. For the proof of (5.4c) we fix a triple $(u, \lambda, \sigma) \in U \times V \times \Sigma$ and $\varepsilon > 0$. Because of conditions (C1), (C2), (C5), and (C6) the function

$$(t, x) \mapsto \frac{\partial A}{\partial s}(T_\sigma(t), x, u(t, x), \lambda)\, \frac{\partial T}{\partial \sigma}(\sigma, t),$$

belongs to $L^\infty(S \times X)$. We choose a compact set $F \subset \Omega$ such that $u(t, x) \in F$ for almost all $(t, x) \in S \times X$ and some bound $\varrho > 0$ corresponding to λ and F with respect to (C6). Furthermore, we can find some $\delta > 0$ such that for all $\kappa \in \mathbb{R}$ with $|\kappa| < \frac{\delta}{M}$ we have both $\sigma + \kappa \in \Sigma$ and $\left|T_{\sigma+\kappa}(t) - T_\sigma(t) - \frac{\partial T}{\partial \sigma}(\sigma, t)\kappa\right| < \varepsilon|\kappa|$ for every $t \in \overline{S}$. Simultaneously, we take δ small enough such it corresponds to λ, F and ε from condition (C7). For all $\xi \in F$ and $\kappa \in \mathbb{R}$ with $|\kappa| < \frac{\delta}{M}$ and almost all $(t, x) \in S \times X$ this leads to

$$\begin{aligned}
&A(T_{\sigma+\kappa}(t), x, \xi, \lambda) - A(T_\sigma(t), x, \xi, \lambda) - \frac{\partial A}{\partial s}(T_\sigma(t), x, \xi, \lambda)\, \frac{\partial T}{\partial \sigma}(\sigma, t)\kappa \\
&= \int_0^1 \frac{\partial A}{\partial s}((1-\tau)T_\sigma(t) + \tau T_{\sigma+\kappa}(t), x, \xi, \lambda)\, \mathrm{d}\tau \left(T_{\sigma+\kappa}(t) - T_\sigma(t) - \frac{\partial T}{\partial \sigma}(\sigma, t)\kappa\right) \\
&\quad + \int_0^1 \left(\frac{\partial A}{\partial s}((1-\tau)T_\sigma(t) + \tau T_{\sigma+\kappa}(t), x, \xi, \lambda) - \frac{\partial A}{\partial s}(T_\sigma(t), x, \xi, \lambda)\right) \mathrm{d}\tau\, \frac{\partial T}{\partial \sigma}(\sigma, t)\kappa,
\end{aligned}$$

which yields

$$\begin{aligned}
&\left|\mathcal{C}(u, \lambda, \sigma + \kappa)(t, x) - \mathcal{C}(u, \lambda, \sigma)(t, x) - \frac{\partial A}{\partial s}(T_\sigma(t), x, u(t, x), \lambda)\, \frac{\partial T}{\partial \sigma}(\sigma, t)\kappa\right| \\
&\quad \le (\varrho + M)\varepsilon|\kappa| \quad \text{for all } \kappa \in \mathbb{R} \text{ with } |\kappa| < \frac{\delta}{M} \text{ and almost all } (t, x) \in S \times X.
\end{aligned}$$

This proves (5.4c) and the differentiability of $\mathcal{C}$ in $U \times V \times \Sigma$ with respect to σ.

7. Finally, we show that $\frac{\partial \mathcal{C}}{\partial \sigma}$ is continuous: We fix a triple $(u, \lambda, \sigma) \in U \times V \times \Sigma$ and some $\varepsilon > 0$. Because U is open in $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$, we can find a compact set $F \subset \Omega$ and some $\delta > 0$ such that for all $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ with $\|v\|_C < \delta$ we have $u(t, x) \in F$ and $(u + v)(t, x) \in F$ for almost all $(t, x) \in S \times X$. In view of (C6) we take some bound $\varrho > 0$ depending on λ and F, and we choose δ small enough such that it corresponds to λ, F and ε with respect to condition (C7), that $\lambda + \mu \in V$, $\sigma + \kappa \in \Sigma$ and $\left|\frac{\partial T}{\partial \sigma}(\sigma + \kappa, t) - \frac{\partial T}{\partial \sigma}(\sigma, t)\right| < \varepsilon$ hold true for all $\mu \in \Lambda$ with $\|\mu\|_\Lambda < \delta$ all $\kappa \in \mathbb{R}$ with $|\kappa| < \frac{\delta}{M}$, and all $t \in \overline{S}$. Consequently, for all $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$, $\mu \in \Lambda$, and $\kappa \in \mathbb{R}$ with $\|v\|_C + \|\mu\|_\Lambda < \delta$ and $|\kappa| < \frac{\delta}{M}$, and almost all $(t, x) \in S \times X$,

we get

$$\begin{aligned}
&\left|\left(\frac{\partial \mathcal{C}}{\partial \sigma}(u+v,\lambda+\mu,\sigma+\kappa)-\frac{\partial \mathcal{C}}{\partial \sigma}(u,\lambda,\sigma)\right)(t,x)\right| \\
&\quad \leq \left|\left(\frac{\partial A}{\partial s}(T_{\sigma+\kappa}(t),x,(u+v)(t,x),\lambda+\mu)\right.\right. \\
&\qquad \left.\left. -\ \frac{\partial A}{\partial s}(T_\sigma(t),x,u(t,x),\lambda)\right)\ \frac{\partial T}{\partial \sigma}(\sigma+\kappa,t)\right| \\
&\qquad + \left|\frac{\partial A}{\partial s}(T_\sigma(t),x,u(t,x),\lambda)\left(\frac{\partial T}{\partial \sigma}(\sigma+\kappa,t)-\frac{\partial T}{\partial \sigma}(\sigma,t)\right)\right| \leq (M+\varrho)\varepsilon,
\end{aligned}$$

which finishes the proof. □

5.2. Right hand sides

In this section, we consider operators $\mathcal{F} \in C^1\left(U \times V; L_2^\omega(S; H^{-1}(G))\right)$, which are candidates for the right hand sides $\mathcal{F}^\alpha$ of problem (2.7). Please, remember that the notation $G = X \cup \Gamma$ indicates the decomposition of the regular set $G \subset \mathbb{R}^n$ into its interior $X \subset \mathbb{R}^n$ and its Neumann boundary part $\Gamma \subset \partial G$, as introduced in (2.1).

THEOREM 5.2. *If $\omega \in [0, n+2]$ and*

$$\begin{aligned}
\mathcal{G}^\ell &\in C^1\left(U \times V; L_2^\omega(S; L^2(X))\right), \\
\mathcal{G}^0 &\in C^1\left(U \times V; L_2^{\omega-2}(S; L^2(X))\right), \\
\mathcal{G}^\Gamma &\in C^1\left(U \times V; L_2^{\omega-1}(S; L^2(\Gamma))\right),
\end{aligned}$$

are Volterra operators for $\ell \in \{1,\ldots,n\}$, then the following statements hold true:

1. *The map $\mathcal{F}$, defined by*

$$\begin{aligned}
\langle \mathcal{F}(u,\lambda),\varphi\rangle_{L^2(S;H_0^1(G))} = &\int_S\int_X \sum_{\ell=1}^n \mathcal{G}^\ell(u,\lambda)(s)\,\frac{\partial \varphi}{\partial x_\ell}(s)\,\mathrm{d}\lambda^n\,\mathrm{d}s \\
&+ \int_S\int_X \mathcal{G}^0(u,\lambda)(s)\,\varphi(s)\,\mathrm{d}\lambda^n\,\mathrm{d}s \\
&+ \int_S\int_\Gamma \mathcal{G}^\Gamma(u,\lambda)(s)\,\varphi(s)\,\mathrm{d}\lambda_\Gamma\,\mathrm{d}s \qquad (5.5)
\end{aligned}$$

   for $(u,\lambda) \in U\times V$ and $\varphi \in L^2(S; H_0^1(G))$, belongs to $C^1\left(U\times V; L_2^\omega(S; H^{-1}(G))\right)$ and admits the Volterra property.

2. *If, for certain $\lambda \in V$ and all $\ell \in \{1,\ldots,n\}$ the maps*

$$u \in U \mapsto \frac{\partial \mathcal{G}^\ell}{\partial u}(u,\lambda) \in \mathscr{L}\left(C(\overline{S}; C(\overline{X};\mathbb{R}^m)); L_2^\omega(S; L^2(X))\right), \qquad (5.6a)$$

$$u \in U \mapsto \frac{\partial \mathcal{G}^0}{\partial u}(u,\lambda) \in \mathscr{L}\left(C(\overline{S}; C(\overline{X};\mathbb{R}^m)); L_2^{\omega-2}(S; L^2(X))\right), \qquad (5.6b)$$

$$u \in U \mapsto \frac{\partial \mathcal{G}^\Gamma}{\partial u}(u,\lambda) \in \mathscr{L}\left(C(\overline{S}; C(\overline{X};\mathbb{R}^m)); L_2^{\omega-1}(S; L^2(\Gamma))\right), \qquad (5.6c)$$

are locally Lipschitz continuous, then the operator

$$u \in U \mapsto \tfrac{\partial \mathcal{F}}{\partial u}(u,\lambda) \in \mathscr{L}\Big(C(\overline{S}; C(\overline{X}; \mathbb{R}^m)); L_2^\omega(S; H^{-1}(G))\Big), \tag{5.7}$$

is also locally Lipschitz continuous.

3. *Let W be an open set in $C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ such that $\left(\mathcal{T}_\sigma^0 w^1, \ldots, \mathcal{T}_\sigma^0 w^m\right) \in U$ holds true for every $w \in W$ and $\sigma \in \Sigma$, and assume that for some $\lambda \in V$ and all $\ell \in \{1, \ldots, n\}$ the assignments*

$$(w,\sigma) \mapsto \mathcal{T}_0^\sigma \mathcal{G}^\ell(\mathcal{T}_\sigma^0 w^1, \ldots, \mathcal{T}_\sigma^0 w^m, \lambda), \tag{5.8a}$$
$$(w,\sigma) \mapsto \mathcal{T}_0^\sigma \mathcal{G}^0(\mathcal{T}_\sigma^0 w^1, \ldots, \mathcal{T}_\sigma^0 w^m, \lambda), \tag{5.8b}$$
$$(w,\sigma) \mapsto \mathcal{T}_\Gamma^\sigma \mathcal{G}^\Gamma(\mathcal{T}_\sigma^0 w^1, \ldots, \mathcal{T}_\sigma^0 w^m, \lambda), \tag{5.8c}$$

generate continuously differentiable operators

$$\mathcal{G}_\lambda^\ell \in C^1\Big(W \times \Sigma; L_2^\omega(S; L^2(X))\Big),$$
$$\mathcal{G}_\lambda^0 \in C^1\Big(W \times \Sigma; L_2^{\omega-2}(S; L^2(X))\Big),$$
$$\mathcal{G}_\lambda^\Gamma \in C^1\Big(W \times \Sigma; L_2^{\omega-1}(S; L^2(\Gamma))\Big),$$

respectively. Then the map $(w,\sigma) \mapsto \mathcal{T}^\sigma \mathcal{F}(\mathcal{T}_\sigma^0 w^1, \ldots, \mathcal{T}_\sigma^0 w^m, \lambda)$ defines a Volterra operator $\mathcal{F}_\lambda \in C^1\left(W \times \Sigma; L_2^\omega(S; H^{-1}(G))\right)$.

Proof. 1. Due to (2.2), (2.3) the assignment $(g, g_0, g_\Gamma) \mapsto \Psi(g, g_0, g_\Gamma)$, defined by

$$\begin{aligned}\langle \Psi(g, g_0, g_\Gamma), \varphi\rangle_{L^2(S; H_0^1(G))} = &\int_S \int_X g(s) \cdot \nabla\varphi(s)\, \mathrm{d}\lambda^n\, \mathrm{d}s \\ &+ \int_S \int_X g_0(s)\, \varphi(s)\, \mathrm{d}\lambda^n\, \mathrm{d}s \\ &+ \int_S \int_\Gamma g_\Gamma(s)\, \varphi(s)\, \mathrm{d}\lambda_\Gamma\, \mathrm{d}s\end{aligned} \tag{5.9}$$

for $\varphi \in L^2(S; H_0^1(G))$, generates a linear continuous operator

$$\Psi : L_2^\omega(S; L^2(X; \mathbb{R}^n)) \times L_2^{\omega-2}(S; L^2(X)) \times L_2^{\omega-1}(S; L^2(\Gamma)) \to L_2^\omega(S; H^{-1}(G)),$$

and its norm depends on n and G, only. Since (5.5) holds true, we obtain

$$\mathcal{F}(u,\lambda) = \Psi\Big(\mathcal{G}^1(u,\lambda), \ldots, \mathcal{G}^n(u,\lambda), \mathcal{G}^0(u,\lambda), \mathcal{G}^\Gamma(u,\lambda)\Big) \quad \text{for all } (u,\lambda) \in U \times V.$$

Hence, as a superposition of continuously differentiable operators, the map $\mathcal{F}$ belongs to $C^1\left(U \times V; L_2^\omega(S; H^{-1}(G))\right)$. Moreover, for all $(u,\lambda) \in U \times V$ and $v \in C(\overline{S}; C(\overline{X}; \mathbb{R}^m))$ we get the identity

$$\frac{\partial \mathcal{F}}{\partial u}(u,\lambda)\, v = \Psi\left(\frac{\partial \mathcal{G}^1}{\partial u}(u,\lambda)\, v, \ldots, \frac{\partial \mathcal{G}^n}{\partial u}(u,\lambda)\, v, \frac{\partial \mathcal{G}^0}{\partial u}(u,\lambda)\, v, \frac{\partial \mathcal{G}^\Gamma}{\partial u}(u,\lambda)\, v\right). \tag{5.10}$$

2. If for some $\lambda \in V$ and all $\ell \in \{1, \ldots, n\}$ the maps $u \mapsto \frac{\partial \mathcal{G}^\ell}{\partial u}(u, \lambda)$, $u \mapsto \frac{\partial \mathcal{G}^0}{\partial u}(u, \lambda)$, and $u \mapsto \frac{\partial \mathcal{G}^\Gamma}{\partial u}(u, \lambda)$ are locally Lipschitz continuous in the sense of (5.6), then it is an easy consequence of (5.10) that the operator $u \mapsto \frac{\partial \mathcal{F}}{\partial u}(u, \lambda)$ given by (5.7) is locally Lipschitz continuous, too.

3. Because of (5.8) and (5.9) for all $(w, \sigma) \in W \times \Sigma$ we obtain

$$\mathcal{F}_\lambda(w, \sigma) = \Psi\left(\mathcal{G}^1_\lambda(w, \sigma), \ldots, \mathcal{G}^n_\lambda(w, \sigma), \mathcal{G}^0_\lambda(w, \sigma), \mathcal{G}^\Gamma_\lambda(w, \sigma)\right).$$

As a superposition of continuously differentiable maps, the operator $\mathcal{F}_\lambda$ belongs to $C^1\left(W \times \Sigma; L^\omega_2(S; H^{-1}(G))\right)$. □

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