

## Linear Elliptic Boundary Value Problems with Non-smooth Data: Normal Solvability on Sobolev–Campanato Spaces

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*Dedicated to Professor HERBERT GAJEWSKI on his sixtieth birthday*

**Abstract.** In this paper linear elliptic boundary value problems of second order with non-smooth data ( $L^\infty$ -coefficients, Lipschitz domains, regular sets, non-homogeneous mixed boundary conditions) are considered. It is shown that such boundary value problems generate Fredholm operators between appropriate Sobolev–Campanato spaces, that the weak solutions are Hölder continuous up to the boundary and that they depend smoothly (in the sense of a Hölder norm) on the coefficients and on the right-hand sides of the equations and boundary conditions.

### 1. Introduction

In this paper we consider weak solutions to boundary value problems for linear elliptic equations of the type

$$(1.1a) \quad -\nabla \cdot (A\nabla u + bu) + c \cdot \nabla u + du = -\nabla \cdot f + g \quad \text{in } \Omega,$$

$$(1.1b) \quad (A\nabla u + bu) \cdot \nu + eu = f \cdot \nu + h \quad \text{on } \Gamma,$$

$$(1.1c) \quad u = 0 \quad \text{on } \partial\Omega \setminus \Gamma,$$

and for linear elliptic systems the principal part of which is close to be triangular.

In (1.1)  $\Omega \subset \mathbb{R}^N$  is a bounded domain,  $\Gamma$  is a subset of the boundary  $\partial\Omega$ , and  $\nu : \partial\Omega \rightarrow \mathbb{R}^N$  is the unit outward normal vector field on  $\partial\Omega$ . The coefficients  $A, b, c, d$  and  $e$  are bounded measurable maps, defined on  $\Omega$  and  $\Gamma$ , respectively,  $A$  is real symmetric  $(N \times N)$ -matrix valued,  $b$  and  $c$  are  $\mathbb{R}^N$  valued, and  $d$  and  $e$  are scalar valued. By  $\nabla u$  and  $\nabla \cdot f$  we denote the gradient of a function  $u : \Omega \rightarrow \mathbb{R}$  and the divergence of a vector field  $f : \Omega \rightarrow \mathbb{R}^N$ , respectively, and for the Euclidean scalar

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product of two vectors we use a centered dot. Finally, it is supposed that there exists an  $\varepsilon > 0$  such that

$$(1.2) \quad A(x)\xi \cdot \xi \geq \varepsilon \xi \cdot \xi \quad \text{for all } \xi \in \mathbb{R}^N \text{ and for almost all } x \in \Omega.$$

It is well-known that each weak solution to the boundary value problem (1.1) is Hölder continuous up to the boundary if, for example,

$$(1.3) \quad f \in L^p(\Omega; \mathbb{R}^N), \quad g \in L^{p/2}(\Omega) \quad \text{and} \quad h \in L^{p-1}(\Gamma) \quad \text{with} \quad p > N,$$

and if  $\partial\Omega$  and  $\Gamma$  satisfy certain regularity assumptions (see, for instance, GILBARG, TRUDINGER [14] for the case  $\Gamma = \emptyset$ , TROIANIELLO [31] for the case that  $\Gamma$  is open and closed in  $\partial\Omega$  and  $e = h = 0$ , and STAMPACCHIA [30], MURTHY, STAMPACCHIA [22] for more general cases). Moreover, the weak solution to (1.1) – if it is unique – depends continuously in the sense of a Hölder space  $C^{0,\alpha}(\bar{\Omega})$  on the right-hand sides  $f$ ,  $g$  and  $h$  in the sense of the Lebesgue spaces mentioned in (1.3).

In the present paper we will prove, among other things, that the weak solution to (1.1) – if it is unique – depends smoothly in the sense of a Hölder space  $C^{0,\alpha}(\bar{\Omega})$  not only on the right-hand sides  $f$ ,  $g$  and  $h$ , but also on the coefficients  $A$ ,  $b$ ,  $c$ ,  $d$ ,  $e$  in the sense of  $L^\infty$ -norms. This result seems to be new (in case of  $N > 2$ ) even if  $\Gamma = \emptyset$  (pure Dirichlet boundary conditions) or if  $\Gamma = \partial\Omega$  (pure natural boundary conditions). Moreover, it has important consequences because it allows to apply theorems of the differential calculus (Implicit Function Theorem, Sard-Smale Theorem, Lyapunov-Schmidt procedure in bifurcation problems) to quasilinear elliptic boundary value problems with non-smooth data (cf. RECKE [27] and GRIEPENTROG [16]).

The main problem connected with such applications to quasilinear problems consists in the following: On the one hand, one has to work on sufficiently large function spaces such that weak solutions exist. On the other hand, one has to work on sufficiently small function spaces such that the appearing superposition operators are smooth. For example, suppose the coefficient matrix  $A$  in (1.1) to depend on  $u$ , i.e.  $A = A(x, u)$ , let  $A(x, \cdot)$  be smooth for almost all  $x$ , and assume that there exists a constant  $c > 0$  such that  $|A(x, u)| \leq c$  for all  $u$  and almost all  $x$ . Then the corresponding superposition operator

$$u \in W^{1,2}(\Omega) \longmapsto A(\cdot, u(\cdot)) \in L^\infty(\Omega; \mathbb{R}^{N \times N})$$

is *not* continuously differentiable (except that  $N = 1$  or that  $A(x, \cdot)$  is affine for almost all  $x$ ), but its restriction to  $W^{1,2}(\Omega) \cap L^\infty(\Omega)$ , for example, is smooth. For differentiability of superposition operators see, e.g., VALENT [32], APPELL, ZABREJKO [3] and RUNST, SICKEL [29].

In the case of  $N = 2$  the smooth dependence (in the sense of  $C^{0,\alpha}(\bar{\Omega})$ ) of the weak solution of (1.1) on the coefficients follows from the paper of GRÖGER [18]. Moreover, in the case of  $N = 2$  this result holds true for boundary value problems for general elliptic systems, which are not necessarily close to be triangular, as well.

Our paper is closely related with the results of RECKE [28] and XIE [34], which contain some of the results of Sections 5 and 6 of the present paper. [28] concerns the particular case of  $e = h = 0$ . In [34] it is supposed that  $e$  and  $h$  are Lipschitz continuous (in order to absorb the corresponding  $\Gamma$ -integrals in the variational formulation of

(1.1) into  $\Omega$ -integrals via the divergence theorem). Moreover, [34] does not concern the dependence of the weak solutions on the coefficients.

Most of the results of Sections 3 and 4 of the present paper, describing properties of regular sets (introduced by GRÖGER [18], cf. Definition 3.1 below) and trace properties of Sobolev–Campanato functions, seem to be new. The development of these rather technical results is motivated to some extent by the wrong claim of XIE [34, Remark 2.3] that all regular sets have Lipschitz boundaries. Because this claim is used repeatedly (cf., e.g., [34, Remark 4.1]), there are gaps in the proofs of that paper.

In the particular case of  $e = h = 0$  the results of the present paper are essentially due to the second author. The generalizations to the case of nonzero  $e$  and  $h$  (and, especially, the Trace Theorem 4.4) belong to the first author.

The present paper is organized as follows:

In Section 2 we introduce some notation and results related to Sobolev–Campanato spaces.

Section 3 is devoted to the concept and the properties of regular sets.

In Section 4 we define Campanato spaces on the natural boundary part, and we prove a trace theorem for Sobolev–Campanato spaces.

In Section 5 we prove a regularity result for weak solutions to (1.1) for the case of  $b = c = 0$  and  $e = 0$  closely following the methods of TROIANELLO [31].

In Section 6 we show that the operator, associated with the boundary value problem (1.1), is a Fredholm operator (index zero) from  $W_0^{1,2,\omega}(\Omega \cup \Gamma)$  into  $W^{-1,2,\omega}(\Omega \cup \Gamma)$  for all  $\omega \in [0, \bar{\omega})$ , where  $\bar{\omega}$  is a certain number which depends on  $\varepsilon$  and  $\Omega \cup \Gamma$ , only, and which is larger than  $N - 2$ . Moreover, this Fredholm operator depends linearly and continuously (in the sense of the operator norm) on  $A$ ,  $b$ ,  $c$ ,  $d$  and  $e$ . Hence, if it is injective, the Implicit Function Theorem yields that the weak solution to (1.1) depends smoothly on  $A$ ,  $b$ ,  $c$ ,  $d$  and  $e$ . Here,  $W_0^{1,2,\omega}(\Omega \cup \Gamma)$  consists of all elements  $u$  of the Sobolev space  $W_0^{1,2}(\Omega \cup \Gamma)$  such that the gradient  $\nabla u$  belongs to the Campanato space  $\mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)$ , and  $W^{-1,2,\omega}(\Omega \cup \Gamma)$  is the image of  $W_0^{1,2,\omega}(\Omega \cup \Gamma)$  with respect to the duality map of the Hilbert space  $W_0^{1,2}(\Omega \cup \Gamma)$ . Remark that, for  $\omega > N - 2$ , the Sobolev–Campanato space  $W_0^{1,2,\omega}(\Omega \cup \Gamma)$  is continuously embedded into the Hölder space  $C^{0,\alpha}(\bar{\Omega})$  with  $\alpha = (\omega - N + 2)/2$ .

Finally, in Section 7 we show that our results about the boundary value problems for linear elliptic equations of type (1.1) hold for linear elliptic systems the principal part of which is close to be triangular as well.

## 2. Notation and some results on Sobolev–Campanato spaces

By  $\mathfrak{M}_N$  and  $\mathfrak{S}_N$  we denote the spaces of all real  $(N \times N)$ -matrices and real symmetric  $(N \times N)$ -matrices, respectively. The symbol  $|\cdot|$  is used for the absolute value, the Euclidean norm in  $\mathbb{R}^N$  and for the Euclidean operator norm in  $\mathfrak{M}_N$ , respectively, i.e.

$$|\xi| := \sqrt{\xi \cdot \xi} \quad \text{for } \xi \in \mathbb{R}^N,$$

$$|A| := \max \{ |A\xi| : \xi \in \mathbb{R}^N, |\xi| \leq 1 \} \quad \text{for } A \in \mathfrak{M}_N.$$

Let  $\{e_1, \dots, e_N\}$  be the standard orthonormal base in  $\mathbb{R}^N$ . For  $x \in \mathbb{R}^N$  and  $r > 0$  we denote by  $B(x, r) := \{\xi \in \mathbb{R}^N : |\xi - x| < r\}$  the open ball around  $x$  with radius  $r$ .

As usual, for subsets  $G$  of  $\mathbb{R}^N$  we write  $G^\circ$ ,  $\overline{G}$  and  $\partial G$  for the interior, the closure and the (topological) boundary of  $G$ , respectively.

A bijective map  $\Phi$  between two subsets of  $\mathbb{R}^N$  such that  $\Phi$  and  $\Phi^{-1}$  are Lipschitz continuous is called *Lipschitz transformation*.

A subset  $M$  of  $\mathbb{R}^N$  is called *Lipschitz hypersurface in  $\mathbb{R}^N$*  if for each  $x_0 \in M$  there exist open neighborhoods  $U$  of  $x_0$  and  $V$  of zero in  $\mathbb{R}^N$  and a Lipschitz transformation  $\Phi$  from  $U$  onto  $V$  such that  $\Phi(x_0) = 0$  and

$$(2.1) \quad U \cap M = \{x \in U : \Phi_N(x) = 0\}.$$

Here  $\Phi_N : U \rightarrow \mathbb{R}$  is the  $N$ -th component of the map  $\Phi$  (and similar notation will be used later on). The map

$$\varphi : V_N := \{\xi \in V : \xi_N = 0\} \rightarrow M, \quad \varphi(\xi) := \Phi^{-1}(\xi)$$

is called *embedding chart of  $M$  in  $x_0$* . It is a Lipschitz continuous map from the neighborhood  $V_N$  of zero in  $\mathbb{R}^{N-1}$  into  $\mathbb{R}^N$ . Its functional matrix  $D\varphi(\xi)$  exists for  $\lambda^{N-1}$ -almost all  $\xi \in V_N$ , and for such  $\xi$  the absolute value of the corresponding Jacobian  $J\varphi(\xi)$  is defined by

$$(2.2) \quad |J\varphi(\xi)|^2 := \text{sum of squares of all } (N-1) \times (N-1)\text{-subdeterminants of } D\varphi(\xi),$$

cf., e.g., EVANS, GARIEPY [10, Section 3.3.4]. Here  $\lambda^{N-1}$  denotes the  $(N-1)$ -dimensional Lebesgue measure on  $\mathbb{R}^{N-1}$ . Analogously, by  $\lambda^N$  we will denote the  $N$ -dimensional Lebesgue measure on  $\mathbb{R}^N$ .

Let  $M$  be a Lipschitz hypersurface in  $\mathbb{R}^N$ . By  $\lambda_M$  we denote the  $(N-1)$ -dimensional Lebesgue measure on  $M$ . Thus, on the algebra of Lebesgue measurable subsets of  $M$  it is equal to the (suitably normalized)  $(N-1)$ -dimensional Hausdorff measure (cf. [10]). Using embedding charts, we have

$$(2.3) \quad \lambda_M(U \cap M) = \int_{V_N} |J\varphi(\xi)| \, d\lambda^{N-1}(\xi)$$

and, for integrable functions  $u : M \rightarrow \mathbb{R}$ ,

$$(2.4) \quad \int_{U \cap M} u(x) \, d\lambda_M(x) = \int_{V_N} u(\varphi(\xi)) |J\varphi(\xi)| \, d\lambda^{N-1}(\xi).$$

A subset  $M_0$  of  $M$  is called *Lipschitz hypersurface in  $M$*  if for each  $x_0 \in M_0$  there exist open neighborhoods  $U$  of  $x_0$  and  $V$  of zero in  $\mathbb{R}^N$  and a Lipschitz transformation  $\Phi$  from  $U$  onto  $V$  such that  $\Phi(x_0) = 0$  and (2.1) as well as

$$U \cap M_0 = \{x \in U : \Phi_N(x) = \Phi_1(x) = 0\}.$$

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^N$ . We write  $L^\infty(\Omega)$ ,  $L^\infty(\Omega; \mathbb{R}^N)$  and  $L^\infty(\Omega; \mathfrak{S}_N)$  for the spaces of bounded measurable maps from  $\Omega$  into  $\mathbb{R}$ ,  $\mathbb{R}^N$  and  $\mathfrak{S}_N$ , respectively. The norms of these spaces are denoted by  $\|\cdot\|_{L^\infty}$ . Analogously, for

$1 \leq p < \infty$  we write  $\|\cdot\|_{L^p}$  for the norms in the Lebesgue spaces  $L^p(\Omega)$  and  $L^p(\Omega; \mathbb{R}^N)$ , respectively. The Sobolev space  $W^{1,p}(\Omega)$  will be equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \left( \|u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega; \mathbb{R}^N)}^p \right)^{1/p}.$$

For  $1 \leq p < \infty$ ,  $0 \leq \omega \leq N + p$  we denote by  $\mathfrak{L}^{p,\omega}(\Omega)$  the *Campanato space*, i.e. the space of all  $u \in L^p(\Omega)$  such that

$$(2.5) \quad [u]_{\mathfrak{L}^{p,\omega}(\Omega)} := \left( \sup_{\substack{x \in \Omega \\ r > 0}} \left( r^{-\omega} \int_{\Omega(x,r)} |u(y) - u_{\Omega(x,r)}|^p d\lambda^N(y) \right) \right)^{1/p}$$

remains finite. In (2.5) we used the notation

$$(2.6) \quad \Omega(x, r) := \Omega \cap B(x, r), \quad u_{\Omega(x,r)} := \frac{1}{\lambda^N(\Omega(x, r))} \int_{\Omega(x,r)} u(y) d\lambda^N(y).$$

The space  $\mathfrak{L}^{p,\omega}(\Omega)$  is a Banach space with the norm

$$(2.7) \quad \|u\|_{\mathfrak{L}^{p,\omega}(\Omega)} := \left( \|u\|_{L^p(\Omega)}^p + [u]_{\mathfrak{L}^{p,\omega}(\Omega)}^p \right)^{1/p}.$$

Analogously, by  $\mathfrak{L}^{p,\omega}(\Omega; \mathbb{R}^N)$  we denote the space of all  $f \in L^p(\Omega; \mathbb{R}^N)$  with components in  $\mathfrak{L}^{p,\omega}(\Omega)$ , and the norm in  $\mathfrak{L}^{p,\omega}(\Omega; \mathbb{R}^N)$  is defined similarly to (2.7). Finally, for the sake of simplicity, for  $\omega \leq 0$  we will use the notation  $\mathfrak{L}^{p,\omega}(\Omega) := L^p(\Omega)$ .

The following well-known (cf., e.g., TROIANIELLO [31, Section 1.4.1]) property of Campanato spaces will be used repeatedly in our paper: If  $r_0 > 0$  is fixed and if the supremum in (2.5) is taken over  $0 < r < r_0$ , only, then the corresponding  $r_0$ -depending norm, defined analogously to (2.7), is equivalent to the original norm in  $\mathfrak{L}^{p,\omega}(\Omega)$ . Moreover, we will use the following theorem (cf. KUFNER, JOHN, FUČIK [20], GIAQUINTA [13] and TROIANIELLO [31]) that describes embedding and transformation properties of Campanato spaces.

**Theorem 2.1.** (i) *Let  $1 \leq p_1 \leq p_2 < \infty$  and  $\omega_1, \omega_2 \in \mathbb{R}$  such that it holds  $(\omega_1 - N)/p_1 \leq (\omega_2 - N)/p_2$ . Then  $\mathfrak{L}^{p_2,\omega_2}(\Omega)$  is continuously embedded into  $\mathfrak{L}^{p_1,\omega_1}(\Omega)$ .*

(ii) *Let  $\Phi$  be a Lipschitz transformation from  $\Omega$  into  $\mathbb{R}^N$  and  $\omega < N + 2$ . Then there exists a constant  $c > 0$  such that for all  $u \in \mathfrak{L}^{2,\omega}(\Phi(\Omega))$  it holds  $u \circ \Phi \in \mathfrak{L}^{2,\omega}(\Omega)$  and*

$$\|u \circ \Phi\|_{\mathfrak{L}^{2,\omega}(\Omega)} \leq c \|u\|_{\mathfrak{L}^{2,\omega}(\Phi(\Omega))}.$$

For  $0 \leq \omega \leq N + 2$  we denote by  $W^{1,2,\omega}(\Omega)$  the *Sobolev–Campanato space*, i.e. the space of all  $u \in W^{1,2}(\Omega)$  such that  $\nabla u \in \mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)$ . The space  $W^{1,2,\omega}(\Omega)$  is a Banach space with the norm

$$\|u\|_{W^{1,2,\omega}(\Omega)} := \left( \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{\mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)}^2 \right)^{1/2}.$$

In order to formulate further properties of Campanato spaces (equivalence to Morrey and Hölder spaces, multiplier and embedding properties) we have to suppose certain

minimal regularity of the boundary  $\partial\Omega$ . Hence, let us introduce the following usual terminology (using notation (2.6)):

**Definition 2.2.** Let  $a > 0$ . An open set  $\Omega \subset \mathbb{R}^N$  is said to have *property (a)* if for all sufficiently small  $r > 0$  we have  $\lambda^N(\Omega(x, r)) \geq a \lambda^N(B(x, r))$  for all  $x \in \Omega$ .

The results, summarized in the following theorem, are classical (cf. CAMPANATO [4, 5, 6, 7], CHEN, WU [8]). Remark, however, that in some references (KUFNER, JOHN, FUČIK [20], NEČAS [23], TROIANIELLO [31] and GIAQUINTA [12, 13]) they are formulated and posed partially under stronger regularity assumptions on  $\partial\Omega$ .

**Theorem 2.3.** *Let  $\Omega$  have property (a). Then the following is true:*

(i) *Let  $0 \leq \omega < N$  and  $u \in L^2(\Omega)$ . Then it holds  $u \in \mathfrak{L}^{2,\omega}(\Omega)$  if and only if*

$$(2.8) \quad \left( \sup_{\substack{x \in \Omega \\ r > 0}} \left( r^{-\omega} \int_{\Omega(x,r)} |u(y)|^2 d\lambda^N(y) \right) \right)^{1/2}$$

*remains finite. In that case (2.8) is an equivalent norm in  $\mathfrak{L}^{2,\omega}(\Omega)$ .*

(ii) *Let  $0 \leq \omega < N$ . Then for all  $u \in \mathfrak{L}^{2,\omega}(\Omega)$  and  $v \in L^\infty(\Omega)$  the product  $uv$  belongs to  $\mathfrak{L}^{2,\omega}(\Omega)$ , again, and there exists a constant  $c > 0$  such that*

$$\|uv\|_{\mathfrak{L}^{2,\omega}(\Omega)} \leq c \|u\|_{\mathfrak{L}^{2,\omega}(\Omega)} \|v\|_{L^\infty(\Omega)}$$

*for all such  $u$  and  $v$ .*

(iii) *Let  $N < \omega \leq N + 2$ . Then  $\mathfrak{L}^{2,\omega}(\Omega)$  is isomorphic to the Hölder space  $C^{0,\alpha}(\bar{\Omega})$  with  $\alpha = (\omega - N)/2$ .*

(iv) *Let  $\omega < N$ . Then  $W^{1,2,\omega}(\Omega)$  is continuously embedded into  $\mathfrak{L}^{2,\omega+2}(\Omega)$ .*

**Remark 2.4.** For more complicated multiplier properties of Campanato functions and applications to interior solution regularity of elliptic equations with unbounded coefficients see DI FAZIO [9] and RAGUSA [24]. Using these results, it should be possible to generalize the results of the present paper to equations with suitable unbounded coefficients  $b, c, d$  and  $e$ .

### 3. Regular sets

To define the concept of regular sets let us denote for  $x_0 \in \mathbb{R}^N$  and  $r > 0$

$$E_1(x_0, r) := \{x \in \mathbb{R}^N : |x - x_0| < r, x_N - x_{0N} < 0\},$$

$$E_2(x_0, r) := \{x \in \mathbb{R}^N : |x - x_0| < r, x_N - x_{0N} \leq 0\},$$

$$E_3(x_0, r) := \{x \in E_2(x_0, r) : x_1 - x_{01} > 0 \text{ or } x_N - x_{0N} < 0\}.$$

Here and later on in the case of  $x_0 = 0$  and  $r = 1$  we shortly write  $E_1, E_2$  and  $E_3$ , respectively. The following terminology is essentially due to GRÖGER [18]:

**Definition 3.1.** A set  $G \subset \mathbb{R}^N$  is called *regular* if it is bounded and if for each  $x \in \partial G$  there exist an open neighborhood  $U$  of  $x$  in  $\mathbb{R}^N$  and a Lipschitz transformation  $\Phi$  of  $U$  into  $\mathbb{R}^N$  such that

$$(3.1) \quad \Phi(U \cap G) \in \{E_1, E_2, E_3\}.$$

**Lemma 3.2.** *Let  $G \subset \mathbb{R}^N$  be regular and  $V \subset \mathbb{R}^N$  be open. Then for each point  $x \in V \cap \partial G$  there exist an open neighborhood  $U$  of  $x$  in  $V$  and a Lipschitz transformation  $\Phi$  of  $U$  into  $\mathbb{R}^N$  with (3.1) and  $\Phi(x) = 0$ .*

*Proof.* By definition, there exist an open neighborhood  $U_0$  of  $x$  in  $\mathbb{R}^N$  and a Lipschitz transformation  $\Phi_0$  of  $U_0$  into  $\mathbb{R}^N$  with  $\Phi_0(U_0 \cap G) = E_k$  and  $k = 1, 2$  or  $3$ . Moreover, the Theorem of Invariance of Domain implies  $\Phi_0(x) \in \partial E_k$ . Hence, in order to prove the lemma it is sufficient to show the following:

$$(3.2) \quad \left\{ \begin{array}{l} \text{For all } l \in \{1, 2, 3\}, \text{ all } y \in \partial E_l \text{ and all sufficiently small } r > 0 \\ \text{there exists a Lipschitz transformation } \Psi \text{ from } B(y, r) \text{ into } \mathbb{R}^N \\ \text{such that } \Psi(y) = 0 \text{ and } \Psi(B(y, r) \cap E_l) \in \{E_1, E_2, E_3\}. \end{array} \right.$$

Indeed, take  $r > 0$  sufficiently small such that  $U := \Phi_0^{-1}(B(y, r)) \subset U_0 \cap V$ , take  $\Psi$  corresponding to (3.2) with  $l = k$ ,  $y = \Phi_0(x)$  and the chosen  $r$ , then  $\Phi := \Psi \circ \Phi_0$  is the Lipschitz transformation sought-after.

Obviously, in order to prove (3.2) it is sufficient to consider only a finite collection of pairs  $(l, y) \in \{1, 2, 3\} \times \partial E_1$  such that for each other pair  $(l^*, y^*) \in \{1, 2, 3\} \times \partial E_1$  the set  $B(y^*, r) \cap E_{l^*}$  is equal to one of the sets  $B(y, r) \cap E_l$  after translations, reflections and rotations in  $\mathbb{R}^N$ . Such a collection is, for example, the following one:

$$(3.3) \quad l = 1, \quad y = 0,$$

$$(3.4) \quad l = 2, \quad y = 0,$$

$$(3.5) \quad l = 3, \quad y = 0,$$

$$(3.6) \quad l = 1, \quad y = -e_N,$$

$$(3.7) \quad l = 1, \quad y = e_1,$$

$$(3.8) \quad l = 2, \quad y = e_1,$$

$$(3.9) \quad l = 3, \quad y = e_2.$$

Assertion (3.2) is obvious in the cases (3.3)–(3.5).

In order to handle the remaining cases (3.6)–(3.9) it is sufficient to show that there exist Lipschitz transformations  $\Psi$  from  $B(0, \frac{1}{2})$  into  $\mathbb{R}^N$  with  $\Psi(0) = 0$ , which map the set  $B(0, \frac{1}{2}) \cap E_1(e_N, 1)$  onto  $E_1$  or the set  $B(0, \frac{1}{2}) \cap E_1(-e_1, 1)$  onto  $E_1$  or the set  $B(0, \frac{1}{2}) \cap E_2(-e_1, 1)$  onto  $E_3$  or the set  $B(0, \frac{1}{2}) \cap E_3(-e_2, 1)$  onto  $E_3$ , respectively. It is not hard to check out the existence of such Lipschitz transformations, but to write down them explicitly is quite complicated.  $\square$

**Remark 3.3.** Let  $G \subset \mathbb{R}^N$  be regular and  $V \subset \mathbb{R}^N$  be open. Then, as a conclusion of the preceding Lemma, for each point  $x \in V \cap \partial G$  there exist an open neighborhood

$U_*$  of  $x$  in  $V$  and a Lipschitz transformation  $\Phi_*$  of  $U_*$  into  $\mathbb{R}^N$  with (3.1) and  $\Phi_*(x) = 0$ . We want to state a slight improvement of this result. Obviously, zero is an inner point of  $\Phi_*(U_*)$ . Therefore, we can choose a small  $0 < r < 1$  such that  $B(0, r) \subset \Phi_*(U_*)$ . Hence,  $U := \Phi_*^{-1}(B(0, r))$  is an open neighborhood of  $x$  in  $V$  and  $\Phi := r^{-1}\Phi_*|_U$  is a Lipschitz transformation of  $U$  but now onto  $B(0, 1)$  with (3.1) and  $\Phi(x) = 0$ .

**Remark 3.4.** It is easy to verify that there exists a Lipschitz transformation of  $\mathbb{R}^N$  onto  $\mathbb{R}^N$  which maps  $E_3$  onto  $E_2$ . Hence, Definition 3.1 would not be changed if one would replace condition (3.1) by

$$(3.10) \quad \Phi(U \cap G) \in \{E_1, E_2\}.$$

But for a regular set  $G \subset \mathbb{R}^N$ , an open set  $V \subset \mathbb{R}^N$  and a point  $x \in V \cap \partial G$  there do not exist an open set  $U \subset V$  with  $x \in U$  and a Lipschitz transformation  $\Phi$  from  $U$  into  $\mathbb{R}^N$  with (3.10) and  $\Phi(x) = 0$ , in general.

In order to simplify subsequent notation we introduce the following

**Definition 3.5.** Let  $G \subset \mathbb{R}^N$  be regular,  $x \in \partial G$ ,  $U \subset \mathbb{R}^N$  be open with  $x \in U$  and  $\Phi$  a Lipschitz transformation of  $U$  onto  $B(0, 1)$  with (3.1) and  $\Phi(x) = 0$ . Then the pair  $(\Phi, U)$  is called a *chart of  $\partial G$  in  $x$* .

**Lemma 3.6.** Let  $G \subset \mathbb{R}^N$  be regular. Then the following is true:

- (i)  $G^\circ$  and  $\overline{G}$  are regular.
- (ii) If  $V$  is an open neighborhood of  $\overline{G}$  in  $\mathbb{R}^N$  and  $\Psi$  a Lipschitz transformation of  $V$  into  $\mathbb{R}^N$  then  $\Psi(G)$  is regular.
- (iii)  $\partial G$  is a Lipschitz hypersurface in  $\mathbb{R}^N$ .
- (iv)  $G^\circ$  satisfies property (a) for some  $a > 0$ .

Proof. (i) Let  $x \in \partial G$ , and let  $(\Phi, U)$  be a chart of  $\partial G$  in  $x$ . By invariance of domain, we have

$$(3.11) \quad \Phi(U \cap G^\circ) = E_1.$$

Hence,  $G^\circ$  is a regular set. By  $\Phi(U) = B(0, 1)$ , we get

$$\Phi(U \cap \overline{G}) = \Phi(U \cap \overline{U \cap G}) = B(0, 1) \cap \overline{\Phi(U \cap G)} = E_2.$$

Thus,  $\overline{G}$  is regular.

(ii) Let  $x \in \partial \Psi(G)$ . Then  $\Psi^{-1}(x) \in \partial G$ , and, hence, there exists a chart  $(\Phi, U)$  of  $\partial G$  in  $\Psi^{-1}(x)$  with  $U \subset V$  (cf. Remark 3.3). Therefore,  $(\Phi \circ \Psi^{-1}, \Psi(U))$  is a chart of  $\partial \Psi(G)$  in  $x$ .

(iii) Let  $x \in \partial G$ , and let  $(\Phi, U)$  be a chart of  $\partial G$  in  $x$ . For small  $0 < r < r_0$  it holds  $B(x, r_0) \subset U$  and

$$(3.12) \quad \Phi(B(x, r) \cap \partial G) = \Phi(B(x, r)) \cap \partial E_1 = \{y \in \Phi(B(x, r)) : y_N = 0\}.$$

Hence,  $\partial G$  is a Lipschitz hypersurface in  $\mathbb{R}^N$ .



(iv) By the compactness of  $\partial G$ , there exist points  $x^{(1)}, \dots, x^{(n)} \in \partial G$  and charts  $(\Phi_l, U_l)$  of  $\partial G$  in  $x^{(l)}$  such that  $\partial G \subset \cup_{l=1}^n U_l$ . Moreover, there exists an  $r_0 > 0$  such that for all  $x \in G^\circ$  it holds  $B(x, r_0) \subset G^\circ$  or  $B(x, r_0) \subset U_l$  for a certain  $l$ .

Let  $x \in G^\circ$ . In the case  $B(x, r_0) \subset G^\circ$  the conclusion is trivial. Otherwise there exists an index  $l \in \{1, \dots, n\}$  such that  $B(x, r_0) \subset U_l$ . Then, because of (3.11), for  $0 < r < r_0$  it holds

$$\begin{aligned} \lambda^N(G^\circ \cap B(x, r)) &\geq \frac{1}{L^N} \lambda^N(\Phi_l(G^\circ \cap B(x, r))) = \frac{1}{L^N} \lambda^N(E_1 \cap \Phi_l(B(x, r))) \\ &\geq \frac{1}{L^N} \lambda^N(E_1 \cap B(\Phi_l(x), r/L)) \geq \frac{1}{2L^N} \lambda^N(B(\Phi_l(x), r/L)) \geq \frac{1}{2L^{2N}} \lambda^N(B(x, r)), \end{aligned}$$

where  $L > 0$  is a common Lipschitz constant for all the maps  $\Phi_l$  and  $\Phi_l^{-1}$ . Hence,  $G^\circ$  has property (a).  $\square$

**Lemma 3.7.** *Let  $G \subset \mathbb{R}^N$  be bounded, and suppose that for each  $x \in \partial G$  there exists an open neighborhood  $V$  of  $x$  in  $\mathbb{R}^N$  such that  $V \cap G$  is regular. Then  $G$  is regular.*

*Proof.* Let  $x \in \partial G$ . Take the open neighborhood  $V$  of  $x$  such that  $V \cap G$  is regular. Because of  $x \in \partial(V \cap G)$  there exists a chart  $(\Phi, U)$  of  $\partial(V \cap G)$  in  $x$  with  $U \subset V$  (cf. Lemma 3.6). Hence,  $\Phi(U \cap G) = \Phi(U \cap V \cap G) \in \{E_1, E_2, E_3\}$ , and  $(\Phi, U)$  is a chart of  $\partial G$  in  $x$ .  $\square$

**Remark 3.8.** Lemma 3.6(iv) shows that the set of all regular subsets in  $\mathbb{R}^N$  is not too large. Nevertheless, Lemma 3.6(i) and (ii) and Lemma 3.7 give a feeling that there exist quite a lot of regular sets. Of course, there exist other sufficient conditions for a set to be regular, for example the following:

If  $\Omega \subset \mathbb{R}^N$  is bounded and open and has a Lipschitz boundary (this condition is stronger than  $\partial\Omega$  to be a Lipschitz hypersurface in  $\mathbb{R}^N$ , see GRISVARD [15, Section 1.2.1]), then  $\Omega$  is regular.

However, the reversal of this claim is not true: There exist open regular subsets of  $\mathbb{R}^N$  which do not have a Lipschitz boundary. This is because the image under a Lipschitz transformation of a bounded open set with Lipschitz boundary can be without Lipschitz boundary (cf. [15, Section 1.2]). Even the claim of XIE [34, Remark 3.1], that such an image has the interior cone property, is wrong, in general. Nevertheless, for regular subsets  $G \subset \mathbb{R}^N$  we have embedding theorems (Theorem 2.3(iv) and Lemma 3.9) and trace theorems (Theorem 4.4).

**Lemma 3.9.** *Let  $G \subset \mathbb{R}^N$  be regular. Then the embedding  $W^{1,2}(G^\circ) \hookrightarrow L^2(G^\circ)$  is completely continuous.*

*Proof.* By the compactness of  $\partial G$ , there exist points  $x^{(1)}, \dots, x^{(m)} \in \partial G$  and charts  $(\Phi_l, U_l)$  of  $\partial G$  in  $x^{(l)}$  such that  $\partial G \subset \cup_{l=1}^m U_l$ . Moreover, there exist balls  $U_{m+1}, \dots, U_{m+n} \subset G^\circ$  such that

$$\bar{G} \subset \bigcup_{l=1}^{m+n} U_l.$$

Let  $\alpha_1, \dots, \alpha_{m+n} \in C_0^\infty(\mathbb{R}^N)$  be a smooth partition of unity subordinate to this covering of  $\overline{G}$ . Further, let  $\{u_k\}_{k \in \mathbb{N}}$  be a bounded sequence in  $W^{1,2}(G^\circ)$ . It holds

$$u_k = \sum_{l=1}^{m+n} \alpha_l u_k \quad \text{for all } k.$$

We have to show that there exists a subsequence  $\{k_i\}_{i \in \mathbb{N}}$  such that for all  $l$  the products  $\alpha_l u_{k_i}$  converge in  $L^2(G^\circ)$  for  $i \rightarrow \infty$ . Taking subsequences of subsequences, it suffices to show that for each  $l$  one can find such a subsequence.

First take  $l \in \{1, \dots, m\}$ . The restrictions to  $U_l$  of the products  $\alpha_l u_k$  form a bounded sequence in  $W^{1,2}(U_l \cap G^\circ)$ . Hence, the functions  $(\alpha_l u_k) \circ \Phi_l^{-1}$  form a bounded sequence in  $W^{1,2}(E_1)$ . Here we used the fact that a Lipschitz coordinate transformation induces a continuous map between the  $W^{1,2}$ -spaces on the corresponding bounded domains, without any requirements concerning the boundaries of the domains (cf., e.g., MORREY [21, Theorem 3.1.7]). By the classical Rellich Embedding Theorem, there exists a subsequence  $\{(\alpha_l u_{k_i}) \circ \Phi_l^{-1}\}_{i \in \mathbb{N}}$  which converges in  $L^2(E_1)$ . Hence, the restrictions to  $U_l$  of the products  $\alpha_l u_{k_i}$  converge in  $L^2(U_l \cap G^\circ)$ . Therefore, the zero extensions to  $G^\circ$  of these restrictions, which are the functions  $\alpha_l u_{k_i}$ , converge in  $L^2(G^\circ)$  for  $i \rightarrow \infty$ .

Now take  $l \in \{m+1, \dots, m+n\}$ . The restrictions to  $U_l$  of the products  $\alpha_l u_k$  form a bounded sequence in  $W^{1,2}(U_l)$ . By the classical Rellich Embedding Theorem, again, there is a subsequence  $\{k_i\}_{i \in \mathbb{N}}$  such that the restrictions to  $U_l$  of  $\alpha_l u_{k_i}$  converge in  $L^2(U_l)$  for  $i \rightarrow \infty$ . Taking the zero extensions to  $G^\circ$  of these restrictions, again, we get the desired result.  $\square$

**Remark 3.10.** In FRAENKEL [11, Theorem 5.3] one can find a similar approach to get a Rellich-type theorem with minimal boundary smoothness assumptions.

The applications of Definition 3.1 to mixed boundary value problems are the motivation for defining abstractly the *Dirichlet boundary part* and the *natural boundary part* of a regular set  $G \subset \mathbb{R}^N$  as well as the corresponding *separating manifold* by

$$(3.13a) \quad \partial_N G := G \cap \partial G,$$

$$(3.13b) \quad \partial_D G := \partial G \setminus \overline{\partial_N G},$$

$$(3.13c) \quad \partial_0 G := \overline{\partial_D G} \cap \overline{\partial_N G}.$$

**Lemma 3.11.** *Let  $G \subset \mathbb{R}^N$  be regular, then the following holds:*

- (i)  $\partial_D G$  and  $\partial_N G$  are relatively open in  $\partial G$ .
- (ii)  $\partial_0 G$  is a Lipschitz hypersurface in  $\partial G$ .

*Proof.* (i)  $\partial_D G$  is relatively open by definition.

Let  $(\Phi, U)$  be a chart of  $\partial G$  in a point  $x \in \partial_N G$ . Then, obviously,  $\Phi(U \cap G) = E_2$ . Moreover, by definition we have  $\partial_N G = G \setminus G^\circ$  and, hence,

$$\Phi(U \cap \partial_N G) = \Phi(U \cap G) \setminus \Phi(U \cap G^\circ) = E_2 \setminus E_1$$

(cf. (3.11)). But  $E_2 \setminus E_1 = E_2 \setminus (E_2)^\circ$  is relatively open in  $\partial E_2$ . Therefore, the set  $\Phi^{-1}(E_2 \setminus (E_2)^\circ) = U \cap \partial_N G$  is relatively open in

$$\Phi^{-1}(\partial E_2) = \partial \Phi^{-1}(E_2) = \partial(U \cap G)$$

and, all the more, in  $U \cap \partial G$ .

(ii) Let  $x \in \partial_0 G$ , and let  $(\Phi, U)$  be a chart of  $\partial G$  in  $x$ . By definition, a point  $\xi \in \partial G$  belongs to  $\partial_0 G$  if and only if for all  $r > 0$

$$B(\xi, r) \cap \partial G \cap G \neq \emptyset \quad \text{and} \quad (B(\xi, r) \cap \partial G) \setminus G \neq \emptyset.$$

Hence, (3.12) yields that a point  $\xi \in U \cap \partial G$  belongs to  $\partial_0 G$  if for all sufficiently small  $r > 0$  we have

$$\{y \in \Phi(B(\xi, r)) \cap \Phi(U \cap G) : y_N = 0\} \neq \emptyset$$

and

$$\{y \in \Phi(B(\xi, r)) \setminus \Phi(U \cap G) : y_N = 0\} \neq \emptyset.$$

This provides  $\Phi(U \cap G) = E_3$  and  $\Phi(U \cap \partial_0 G) = \{y \in \Phi(U) : y_N = y_1 = 0\}$ . Therefore,  $\partial_0 G$  is a Lipschitz hypersurface in  $\partial G$ .  $\square$

Let  $G \subset \mathbb{R}^N$  be a regular set. We will work with the following notation, which is usual in the theory of mixed boundary value problems (cf., e.g., TROIANIello [31], GRÖGER [18]). By  $W_0^{1,2}(G)$  we denote the closure in  $W^{1,2}(G^\circ)$  of the set

$$(3.14) \quad C_0^\infty(G) := \{u|_{G^\circ} : u \in C_0^\infty(\mathbb{R}^N), \text{supp}(u) \cap (\overline{G} \setminus G) = \emptyset\}.$$

In (3.14)  $u|_{G^\circ}$  is the restriction of the function  $u$  to  $G^\circ$ . Furthermore, for  $0 \leq \omega \leq N+2$  we consider subspaces of the Sobolev–Campanato spaces defined as

$$W_0^{1,2,\omega}(G) := W_0^{1,2}(G) \cap W^{1,2,\omega}(G^\circ)$$

and equipped with the norm of  $W^{1,2,\omega}(G^\circ)$ . Finally, let  $W^{-1,2}(G)$  be the dual space to  $W_0^{1,2}(G)$ ,  $\langle \cdot, \cdot \rangle_G$  the dual pairing between these two spaces, and let  $J_G : W_0^{1,2}(G) \rightarrow W^{-1,2}(G)$  be the duality map of  $W_0^{1,2}(G)$ , defined as

$$(3.15) \quad \langle J_G w, v \rangle_G := \int_G (\nabla w \cdot \nabla v + wv) \, d\lambda^N \quad \text{for all } w, v \in W_0^{1,2}(G).$$

By  $W^{-1,2,\omega}(G)$  we denote the subspace of all functionals  $\phi \in W^{-1,2}(G)$ , which belong to the image of the space  $W_0^{1,2,\omega}(G)$  under the duality map  $J_G$ , with the norm

$$(3.16) \quad \|J_G u\|_{W^{-1,2,\omega}(G)} := \|u\|_{W^{1,2,\omega}(G^\circ)} \quad \text{for } u \in W_0^{1,2,\omega}(G).$$

For the sake of simplicity we will denote  $\mathfrak{L}^{2,\omega}(G) := \mathfrak{L}^{2,\omega}(G^\circ)$  and  $\mathfrak{L}^{2,\omega}(G; \mathbb{R}^N) := \mathfrak{L}^{2,\omega}(G^\circ; \mathbb{R}^N)$ .

**Lemma 3.12.** *Let  $G \subset \mathbb{R}^N$  be regular,  $V$  an open neighborhood of  $\overline{G}$  in  $\mathbb{R}^N$  and  $\Psi$  a Lipschitz transformation of  $V$  into  $\mathbb{R}^N$ . Then  $v$  belongs to  $W_0^{1,2}(\Psi(G))$  if and only if  $v \circ \Psi$  is an element of  $W_0^{1,2}(G)$ .*

Proof. By Lemma 3.6(ii),  $H := \Psi(G)$  is a regular set. Let  $v \in W_0^{1,2}(H)$ . Then there exists a sequence  $\{v_k\}_{k \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^N)$  with

$$\text{supp}(v_k) \cap (\overline{H} \setminus H) = \emptyset \quad \text{and} \quad \lim_{k \rightarrow \infty} \|v - v_k|_{H^\circ}\|_{W_0^{1,2}(H)} = 0.$$

Hence, it holds  $v \circ \Psi \in W^{1,2}(G^\circ)$ ,

$$(3.17) \quad \lim_{k \rightarrow \infty} \|v \circ \Psi - v_k \circ \Psi\|_{W^{1,2}(G^\circ)} = 0$$

and

$$\text{supp}(v_k \circ \Psi) \cap (\overline{G} \setminus G) \subset \Psi^{-1}(\text{supp}(v_k) \cap (\overline{H} \setminus H)) = \emptyset.$$

Because  $\overline{G} \setminus G = \partial G \setminus \partial_{\mathcal{N}} G$  and  $\text{supp}(v_k \circ \Psi)$  are closed sets, there must be a positive distance between these two sets. We denote

$$\delta_k := \frac{1}{4} \text{dist}(\overline{G} \setminus G, \text{supp}(v_k \circ \Psi))$$

and extend  $v_k \circ \Psi$  to a function  $u_k \in W^{1,2}(\mathbb{R}^N)$ . Now, we define for  $i, k \in \mathbb{N}$  with  $i > 1/\delta_k$  functions  $w_{ik}$  by convolution with mollifiers  $\zeta_i \in C_0^\infty(\mathbb{R}^N)$

$$w_{ik}(x) = (\zeta_i * u_k)(x) \quad \text{for } x \in \mathbb{R}^N,$$

where

$$\text{supp}(\zeta_i) \subset B(0, 1/i) \quad \text{and} \quad \int_{\mathbb{R}^N} \zeta_i \, d\lambda^N = 1.$$

Obviously, it holds  $w_{ik} \in C_0^\infty(\mathbb{R}^N)$  and  $\text{supp}(w_{ik}) \cap (\overline{G} \setminus G) = \emptyset$  for all  $i > 1/\delta_k$ . Using the convergence properties of convolutions with mollifiers we get

$$\lim_{i \rightarrow \infty} \|v_k \circ \Psi - w_{ik}|_{G^\circ}\|_{W^{1,2}(G^\circ)} = 0,$$

and, therefore,  $v_k \circ \Psi \in W_0^{1,2}(G)$  for all  $k \in \mathbb{N}$ . Because  $W_0^{1,2}(G)$  is a closed subspace of  $W^{1,2}(G^\circ)$ , (3.17) yields  $v \circ \Psi \in W_0^{1,2}(G)$ .

Analogously, it follows  $v \in W_0^{1,2}(H)$ , if we suppose  $v \circ \Psi \in W_0^{1,2}(G)$ .  $\square$

#### 4. Campanato spaces on the natural boundary part

Throughout this section  $G$  is a fixed regular subset of  $\mathbb{R}^N$ .

Because of Lemma 3.6(iii) and Lemma 3.11(i) the natural boundary part  $\partial_{\mathcal{N}} G$  is a Lipschitz hypersurface in  $\mathbb{R}^N$ . Hence, the  $(N-1)$ -dimensional Lebesgue measure  $\lambda_{\partial G}$  can be introduced by (2.2), (2.3) and (2.4), and for  $1 \leq p \leq \infty$  we denote by  $L^p(\partial_{\mathcal{N}} G)$  the corresponding Lebesgue spaces. The norms are defined as

$$\|u\|_{L^p(\partial_{\mathcal{N}} G)} := \left( \int_{\partial_{\mathcal{N}} G} |u(\xi)|^p \, d\lambda_{\partial G}(\xi) \right)^{1/p} \quad \text{for } 1 \leq p < \infty.$$

Further, for  $0 \leq \omega \leq N-1+p$  we denote by  $\mathfrak{L}^{p,\omega}(\partial_{\mathcal{N}}G)$  the corresponding *Campanato space*, i.e. the space of all  $u \in L^p(\partial_{\mathcal{N}}G)$  such that

$$(4.1) \quad [u]_{\mathfrak{L}^{p,\omega}(\partial_{\mathcal{N}}G)} := \left( \sup_{\substack{x \in \partial_{\mathcal{N}}G \\ r > 0}} \left( r^{-\omega} \int_{M(x,r)} |u(y) - u_{M(x,r)}|^p d\lambda_{\partial G}(y) \right) \right)^{1/p}$$

remains finite. In (4.1) we used the notation

$$M(x,r) := B(x,r) \cap \partial_{\mathcal{N}}G, \quad u_{M(x,r)} := \frac{1}{\lambda_{\partial G}(M(x,r))} \int_{M(x,r)} u(y) d\lambda_{\partial G}(y).$$

The space  $\mathfrak{L}^{p,\omega}(\partial_{\mathcal{N}}G)$  is a Banach space with the norm

$$\|u\|_{\mathfrak{L}^{p,\omega}(\partial_{\mathcal{N}}G)} := \left( \|u\|_{L^p(\partial_{\mathcal{N}}G)}^p + [u]_{\mathfrak{L}^{p,\omega}(\partial_{\mathcal{N}}G)}^p \right)^{1/p}.$$

In order to prove certain properties of functions from  $\mathfrak{L}^{2,\omega}(\partial_{\mathcal{N}}G)$  we will work with chart representations. Hence, let us introduce the corresponding terminology.

**Definition 4.1.** (i) Let  $(\Phi, U)$  be a chart of  $\partial G$  in  $x_0 \in \partial G$ . If  $\Phi(U \cap G) = E_k$  ( $k = 1, 2$  or  $3$ ) then  $(\Phi, U)$  is called a *chart of type  $k$* .

(ii) A finite set of charts  $\{(\Phi_l, U_l) : l = 1, \dots, n\}$  such that

$$\partial_{\mathcal{N}}G \subset \bigcup_{l=1}^n U_l$$

is called *atlas of  $\partial_{\mathcal{N}}G$* .

Because for every  $x \in \partial_{\mathcal{N}}G$  there exists a chart  $(\Phi, U)$  of type 2 we can find an atlas  $\{(\Phi_{lk}, U_{lk}) : k = 2, 3; l = 1, \dots, n_k\}$  of  $\partial_{\mathcal{N}}G$ , where the charts  $(\Phi_{lk}, U_{lk})$  are of type  $k$ . Having in mind

$$\partial_{\mathcal{N}}E_2 = \{x \in \mathbb{R}^N : |x| < r, x_N = 0\} \quad \text{and} \quad \partial_{\mathcal{N}}E_3 = \{x \in \partial_{\mathcal{N}}E_2 : x_1 > 0\},$$

we define embedding charts  $\varphi_{lk} : \partial_{\mathcal{N}}E_k \rightarrow \partial_{\mathcal{N}}G$  by  $\varphi_{lk}(\xi) := \Phi_{lk}^{-1}(\xi)$ . Further, let  $\{\alpha_{lk} : k = 2, 3; l = 1, \dots, n_k\} \subset C_0^\infty(\mathbb{R}^N)$  be a smooth partition of unity subordinate to the covering  $\{U_{lk} : k = 2, 3; l = 1, \dots, n_k\}$  of  $\partial_{\mathcal{N}}G$ . Then for all  $u \in L^1(\partial_{\mathcal{N}}G)$  we have

$$(4.2) \quad \int_{\partial_{\mathcal{N}}G} u(x) d\lambda_{\partial G}(x) = \sum_{k=2}^3 \sum_{l=1}^{n_k} \int_{\partial_{\mathcal{N}}E_k} \alpha_{lk}(\varphi_{lk}(\xi)) u(\varphi_{lk}(\xi)) |J\varphi_{lk}(\xi)| d\lambda^{N-1}(\xi).$$

In (4.2)  $J\varphi_{lk}$  is the Jacobian of the embedding chart  $\varphi_{lk}$ , cf. (2.2).

Let  $\Phi$  be a Lipschitz transformation from an open neighborhood of  $\bar{G}$  into  $\mathbb{R}^N$ . Then, because of Lemma 3.6(ii),  $\Phi(G)$  is regular, and  $\partial_{\mathcal{N}}\Phi(G) = \Phi(\partial_{\mathcal{N}}G)$  is a Lipschitz hypersurface in  $\mathbb{R}^N$  (cf. (3.13)). Moreover, (4.2) yields

$$(4.3) \quad \int_{\partial_{\mathcal{N}}G} u(x) |J\Phi_{\mathcal{N}}(x)| d\lambda_{\partial G}(x) = \int_{\partial_{\mathcal{N}}\Phi(G)} u(\Phi_{\mathcal{N}}^{-1}(y)) d\lambda_{\partial\Phi(G)}(y)$$

for all  $u \in L^1(\partial_N G)$ . In (4.3),  $\Phi_N : \partial_N G \rightarrow \partial_N \Phi(G)$  is the restriction of  $\Phi$  to  $\partial_N G$ , and  $J\Phi_N$  is the Jacobian of  $\Phi_N$  in  $x \in \partial_N G$ . By means of embedding charts, we get

$$(4.4) \quad J\Phi_N(\varphi_{lk}(\xi)) \\ = J\psi_{lk}((\psi_{lk})^{-1} \circ \Phi_N \circ \varphi_{lk}(\xi)) J[(\psi_{lk})^{-1} \circ \Phi_N \circ \varphi_{lk}](\xi) [J\varphi_{lk}(\xi)]^{-1}$$

for  $\lambda^{N-1}$ -almost all  $\xi \in \partial_N E_k$ . In (4.4),  $\psi_{lk} : \partial_N E_k \rightarrow \partial_N \Phi(G)$  are embedding charts which are defined as  $\psi_{lk}(\eta) := \Psi_{lk}^{-1}(\eta)$  for  $\eta \in \partial_N E_k$  by an atlas

$$\{(\Psi_{lk}, V_{lk}) : k = 2, 3; l = 1, \dots, n_k\}$$

of  $\Phi(G)$  such that  $(\Psi_{lk}, V_{lk})$  is of type  $k$  and  $\Phi(U_{lk}) \subset V_{lk}$ . Moreover,  $J\psi_{lk}$  is the Jacobian of  $\psi_{lk}$ , defined by (2.2), and  $J[(\psi_{lk})^{-1} \circ \Phi_N \circ \varphi_{lk}](\xi)$  is the determinant of the  $(N-1) \times (N-1)$ -dimensional functional matrix of the map

$$\psi_{lk}^{-1} \circ \Phi_N \circ \varphi_{lk} : \partial_N E_k \rightarrow \partial_N E_k$$

in the point  $\xi \in \partial_N E_k$ . Remark that for fixed  $x \in \partial_N G$  the right-hand side of (4.4) does not depend on the choice of the charts  $(\Phi_{lk}, U_{lk})$  and  $(\Psi_{lk}, V_{lk})$ .

By means of (4.2) and of Theorem 2.3(ii), the following lemma is easy to prove:

**Lemma 4.2.** *Let  $\omega < N - 1$ . Then for all  $u \in \mathfrak{L}^{2,\omega}(\partial_N G)$  and  $v \in L^\infty(\partial_N G)$  the product  $uv$  belongs to  $\mathfrak{L}^{2,\omega}(\partial_N G)$ , and there exists a constant  $c > 0$  such that*

$$\|uv\|_{\mathfrak{L}^{2,\omega}(\partial_N G)} \leq c \|u\|_{\mathfrak{L}^{2,\omega}(\partial_N G)} \|v\|_{L^\infty(\partial_N G)}$$

for all such  $u$  and  $v$ .

Analogously, from (4.3) and Theorem 2.1(ii) we get

**Lemma 4.3.** *Let  $\Phi$  be a Lipschitz transformation from an open neighborhood of  $\bar{G}$  into  $\mathbb{R}^N$  and  $\omega < N + 1$ . Then, there exists a constant  $c > 0$  such that for all  $u \in \mathfrak{L}^{2,\omega}(\Phi(\partial_N G))$  it holds  $u \circ \Phi \in \mathfrak{L}^{2,\omega}(\partial_N G)$  and*

$$\|u \circ \Phi\|_{\mathfrak{L}^{2,\omega}(\partial_N G)} \leq c \|u\|_{\mathfrak{L}^{2,\omega}(\Phi(\partial_N G))}.$$

The main result of this section is the following theorem about traces of Sobolev–Campanato functions on the natural boundary part of regular sets:

**Theorem 4.4.** *Let  $\omega < N$ . Then there exists a linear bounded operator  $T$  from  $W_0^{1,2,\omega}(G)$  into  $\mathfrak{L}^{2,\omega+1}(\partial_N G)$  such that, for all  $u \in C_0^\infty(G)$ ,  $Tu$  is the restriction of  $u$  to  $\partial_N G$ . Furthermore,  $T$  maps  $W_0^{1,2}(G)$  completely continuous into  $L^2(\partial_N G)$ .*

*Proof. Step 1.* Let  $k \in \{2, 3\}$ ,  $0 < r < 1$ ,  $0 < \varrho < \min\{r, 1-r\}$  and  $x_0 \in \partial_N E_k(0, r)$  be fixed. For  $u \in W_0^{1,2}(E_k)$  we denote by  $u|_{\partial_N E_k} \in L^2(\partial_N E_k)$  the restriction of  $u$  to  $\partial_N E_k$  (in the usual sense of trace of a  $W^{1,2}(E_1)$ -function in  $L^2(\partial E_1)$ ) and by

$$u_{\partial_N E_k} := \frac{1}{\lambda^{N-1}(\partial_N E_k)} \int_{\partial_N E_k} u|_{\partial_N E_k} d\lambda^{N-1}$$

the corresponding mean value. In a similar manner we will use the notation

$$u|_{\partial_{\mathcal{N}}E_k(0,r)\cap B(x_0,\varrho)}, \quad u|_{\partial_{\mathcal{N}}E_k(x_0,\varrho)}, \quad u|_{\partial_{\mathcal{N}}E_k(0,r)\cap B(x_0,\varrho)} \quad \text{and} \quad u|_{\partial_{\mathcal{N}}E_k(x_0,\varrho)}.$$

Finally, for  $v \in W_0^{1,2}(E_k(x_0, \varrho))$  we define  $Hv \in W_0^{1,2}(E_k)$  by

$$(Hv)(x) := v(x_0 + \varrho x) \quad \text{for } \lambda^N\text{-almost all } x \in E_k.$$

The usual trace theorem yields that there exists a constant  $c > 0$  such that for all  $u \in W_0^{1,2}(E_k)$  we have

$$\|u|_{\partial_{\mathcal{N}}E_2} - u|_{\partial_{\mathcal{N}}E_2}\|_{L^2(\partial_{\mathcal{N}}E_2)}^2 \leq c \|\nabla u\|_{L^2(E_1; \mathbb{R}^N)}^2.$$

Hence, for such functions we get

$$\begin{aligned} \|u|_{\partial_{\mathcal{N}}E_2(x_0,\varrho)} - u|_{\partial_{\mathcal{N}}E_2(x_0,\varrho)}\|_{L^2(\partial_{\mathcal{N}}E_2(x_0,\varrho))}^2 &= \varrho^{N-1} \|(Hu)|_{\partial_{\mathcal{N}}E_k} - (Hu)|_{\partial_{\mathcal{N}}E_2}\|_{L^2(\partial_{\mathcal{N}}E_2)}^2 \\ &\leq c\varrho^{N-1} \|\nabla(Hu)\|_{L^2(E_1; \mathbb{R}^N)}^2 \\ &= c\varrho \|\nabla u\|_{L^2(E_1(x_0,\varrho); \mathbb{R}^N)}^2. \end{aligned}$$

Moreover, we have  $\partial_{\mathcal{N}}E_k(0, r) \cap B(x_0, \varrho) \subset \partial_{\mathcal{N}}E_2(x_0, \varrho)$ . Hence, the minimizing property of the mean value yields

$$\begin{aligned} &\|u|_{\partial_{\mathcal{N}}E_k(0,r)\cap B(x_0,\varrho)} - u|_{\partial_{\mathcal{N}}E_k(0,r)\cap B(x_0,\varrho)}\|_{L^2(\partial_{\mathcal{N}}E_2(0,r)\cap B(x_0,\varrho))}^2 \\ &\leq \|u|_{\partial_{\mathcal{N}}E_2(x_0,\varrho)} - u|_{\partial_{\mathcal{N}}E_2(x_0,\varrho)}\|_{L^2(\partial_{\mathcal{N}}E_2(x_0,\varrho))}^2 \leq c\varrho \|\nabla u\|_{L^2(E_1(x_0,\varrho); \mathbb{R}^N)}^2. \end{aligned}$$

Summarizing, we get: There exists a constant  $c > 0$  such that for all  $\omega < N$ ,  $0 < r < 1$ ,  $k \in \{2, 3\}$  and  $u \in W_0^{1,2,\omega}(E_k)$  we have  $u|_{\partial_{\mathcal{N}}E_k(0,r)} \in \mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}}E_k(0,r))$  and

$$\|u|_{\partial_{\mathcal{N}}E_k(0,r)}\|_{\mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}}E_k(0,r))} \leq c \|\nabla u\|_{\mathfrak{L}^{2,\omega}(E_1; \mathbb{R}^N)}.$$

*Step 2.* Let  $\{(\Phi_{lk}, U_{lk}) : l = 1, \dots, n_k, k = 2, 3\}$  be an atlas of  $\partial_{\mathcal{N}}G$ , such that the charts  $(\Phi_{lk}, U_{lk})$  are of type  $k$ . Let  $\{\alpha_{lk} : l = 1, \dots, n_k, k = 2, 3\} \subset C_0^\infty(\mathbb{R}^N)$  be a smooth partition of unity subordinate to the open covering  $\{U_{lk}\}$  of  $\partial_{\mathcal{N}}G$ . Because of the compactness of the support of  $\alpha_{lk}$  we can find a number  $0 < r < 1$  such that

$$\text{supp}(\alpha_{lk}) \subset V_{lk} := \Phi_{lk}^{-1}(B(0, r))$$

and that the sets  $V_{lk}$  still form an open covering of  $\partial_{\mathcal{N}}G$ .

For  $u \in W_0^{1,2,\omega}(G)$  we set

$$u_{lk}(\xi) := \alpha_{lk}(\Phi_{lk}^{-1}(\xi)) u(\Phi_{lk}^{-1}(\xi)) \quad \text{for } \lambda^N\text{-almost all } \xi \in E_1.$$

By construction, Theorems 2.1(ii), 2.3(ii) and Lemma 3.12, for each  $u \in W_0^{1,2,\omega}(G)$  we have  $u_{lk} \in W_0^{1,2,\omega}(E_k)$  and

$$\|\nabla u_{lk}\|_{\mathfrak{L}^{2,\omega}(E_1; \mathbb{R}^N)} \leq c \|\nabla u\|_{\mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)},$$

where the constant  $c > 0$  does not depend on  $u, l$  and  $k$ . Thus, Step 1 implies

$$\|u_{lk}|_{\partial_{\mathcal{N}}E_k(0,r)}\|_{\mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}}E_k(0,r))} \leq c \|\nabla u\|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)}$$

with a modified constant  $c > 0$ . Let

$$v_{lk}(x) := \begin{cases} u_{lk}(\Phi_{lk}(x)) & \text{for } \lambda_{\partial_{\mathcal{N}}G}\text{-almost all } x \in \partial_{\mathcal{N}}G \cap V_{lk}, \\ 0 & \text{for } \lambda_{\partial_{\mathcal{N}}G}\text{-almost all } x \in \partial_{\mathcal{N}}G \setminus V_{lk}. \end{cases}$$

Because of Lemma 4.2 and Lemma 4.3 it holds that  $v_{lk} \in \mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}}G)$  and

$$\|v_{lk}\|_{\mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}}G)} \leq c \|\nabla u\|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)},$$

where the constant  $c > 0$  does not depend on  $u, l$  and  $k$ , again. Finally,

$$u|_{\partial_{\mathcal{N}}G} = \sum_{k=2}^3 \sum_{l=1}^{n_k} v_{lk}$$

yields the sought-for estimate. Hence, the proof is finished.

For the compactness of  $T$  from  $W_0^{1,2}(G)$  into  $L^2(\partial_{\mathcal{N}}G)$  one has to proceed as in the proof of Lemma 3.9 and to use the usual trace theorem.  $\square$

### 5. Admissible sets

The following terminology is essentially due to RECKE [27]:

**Definition 5.1.** Let  $G \subset \mathbb{R}^N$  be regular. (i) For  $0 < \varepsilon \leq 1$  we denote by  $\mathfrak{A}(\varepsilon, G)$  the set of all pairs  $(A, d) \in L^\infty(G; \mathfrak{S}_N) \times L^\infty(G)$ , such that for  $\lambda^N$ -almost all  $x \in G$

$$\varepsilon \leq d(x) \leq \frac{1}{\varepsilon} \quad \text{and} \quad \varepsilon |\xi|^2 \leq A(x)\xi \cdot \xi \leq \frac{1}{\varepsilon} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N.$$

(ii) A regular subset  $G_0$  of  $G$  is called *G-admissible* if for each  $0 < \varepsilon \leq 1$  there exists an  $\bar{\omega} > N - 2$  such that for all  $\omega < \bar{\omega}$ ,  $(A, d) \in \mathfrak{A}(\varepsilon, G)$ ,  $f \in \mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$ ,  $g \in \mathfrak{L}^{2,\omega-2}(G)$ ,  $h \in \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}G)$  and  $u \in W_0^{1,2}(G)$  which satisfy

$$(5.1) \quad \int_G (A \nabla u \cdot \nabla v + duv) \, d\lambda^N = \int_G (f \cdot \nabla v + gv) \, d\lambda^N + \int_{\partial_{\mathcal{N}}G} hv \, d\lambda_{\partial G}$$

for all  $v \in W_0^{1,2}(G)$ , it holds  $\nabla u \in \mathfrak{L}^{2,\omega}(G_0; \mathbb{R}^N)$  and

$$(5.2) \quad \|\nabla u\|_{\mathfrak{L}^{2,\omega}(G_0;\mathbb{R}^N)} \leq c \left( \|f\|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(G)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}G)} + \|u\|_{W^{1,2}(G^o)} \right),$$

where the constant  $c > 0$  in (5.2) depends only on  $G, G_0, N, \varepsilon$  and  $\omega$ .

(iii)  $G$  is called *admissible* if it is *G-admissible*.



**Remark 5.2.** Let  $G \subset \mathbb{R}^N$  be regular. The variational equation (5.1) is the weak formulation of the boundary value problem

$$(5.3a) \quad -\nabla \cdot (A\nabla u) + du = -\nabla \cdot f + g \quad \text{in } G^\circ,$$

$$(5.3b) \quad (A\nabla u) \cdot \nu = f \cdot \nu + h \quad \text{on } \partial_{\mathcal{N}}G,$$

$$(5.3c) \quad u = 0 \quad \text{on } \partial_{\mathcal{D}}G.$$

The Lax-Milgram Lemma yields that for all  $(A, d) \in \mathfrak{A}(\varepsilon, G)$ ,  $f \in L^2(G; \mathbb{R}^N)$ ,  $g \in L^2(G)$  and  $h \in L^2(\partial_{\mathcal{N}}G)$  there exists exactly one weak solution  $u \in W_0^{1,2}(G)$  of the boundary value problem (5.3), and the linear map

$$(f, g, h) \in L^2(G; \mathbb{R}^N) \times L^2(G) \times L^2(\partial_{\mathcal{N}}G) \mapsto u \in W_0^{1,2}(G)$$

is continuous. Hence,  $G$  is admissible if and only if for each  $0 < \varepsilon \leq 1$  there exists an  $\bar{\omega} > N - 2$  such that for all  $\omega < \bar{\omega}$ ,  $(A, d) \in \mathfrak{A}(\varepsilon, G)$ ,  $f \in \mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$ ,  $g \in \mathfrak{L}^{2,\omega-2}(G)$  and  $h \in \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}G)$  the weak solution  $u \in W_0^{1,2}(G)$  to (5.3) belongs to  $W_0^{1,2,\omega}(G)$ , and the linear map

$$(f, g, h) \in \mathfrak{L}^{2,\omega}(G; \mathbb{R}^N) \times \mathfrak{L}^{2,\omega-2}(G) \times \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}G) \mapsto u \in W_0^{1,2,\omega}(G)$$

is continuous. In particular, if  $G$  is admissible, then for all weak solutions  $u$  to (5.3) it holds not only (5.2) with  $G_0 = G$ , but also

$$\|u\|_{W_0^{1,2,\omega}(G)} \leq c (\|f\|_{\mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(G)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}G)})$$

(with another constant  $c > 0$ , possibly).

It is well-known (cf., e.g., TROIANELLO [31, Theorem 2.19]) that bounded open subsets of  $\mathbb{R}^N$  with smooth boundary are admissible. The aim of this section is to prove the following

**Theorem 5.3.** *Each regular subset  $G \subset \mathbb{R}^N$  is admissible.*

In order to prove Theorem 5.3, we need some lemmas concerning localization and transformation properties of admissible sets and the admissibility of the sets  $E_1, E_2$  and  $E_3$ .

**Lemma 5.4.** *Let  $G \subset \mathbb{R}^N$  be regular, and suppose that for each  $x \in \partial G$  there exist open neighborhoods  $U_0$  and  $U$  of  $x$  in  $\mathbb{R}^N$  with  $U_0 \subset U$ , such that  $U_0 \cap G$  is  $(U \cap G)$ -admissible. Then  $G$  is admissible.*

*Proof.* Because of the compactness of  $\partial G$ , there exist open subsets  $U_{0l}, U_l$  in  $\mathbb{R}^N$  ( $l = 1, \dots, n$ ) such that

$$(5.4) \quad U_{0l} \subset U_l, \quad \partial G \subset \bigcup_{l=1}^n U_{0l} \quad \text{and} \quad U_{0l} \cap G \text{ is } (U_l \cap G)\text{-admissible.}$$

Moreover, there exist open balls  $U_{0l}, U_l$  in  $G^\circ$  ( $l = n + 1, \dots, n + m$ ) such that

$$U_{0l} \subset U_l \quad \text{and} \quad \bar{G} \subset \bigcup_{l=1}^{n+m} U_{0l}.$$

Remark that each ball  $U_{0l}$  is  $U_l$ -admissible for  $l = n + 1, \dots, n + m$ .

Let  $\{\alpha_1, \dots, \alpha_{n+m}\} \subset C_0^\infty(\mathbb{R}^N)$  be a smooth partition of unity subordinate to the covering  $\{U_{01}, \dots, U_{0n+m}\}$  of  $\bar{G}$ .

Now, take  $(A, d) \in \mathfrak{A}(\varepsilon, G)$ ,  $f \in \mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$ ,  $g \in \mathfrak{L}^{2,\omega-2}(G)$ ,  $h \in \mathfrak{L}^{2,\omega-1}(\partial_N G)$ , and  $u \in W_0^{1,2}(G)$  such that (5.1) holds. Then, we have for all  $w \in W_0^{1,2}(U_l \cap G)$

$$(5.5) \quad \left\{ \begin{aligned} & \int_{U_l \cap G} (A \nabla(\alpha_l u) \cdot \nabla w + d \alpha_l u w) \, d\lambda^N \\ &= \int_G (A \nabla u \cdot \nabla(\alpha_l w) + d u \alpha_l w + A(u \nabla w - w \nabla u) \cdot \nabla \alpha_l) \, d\lambda^N \\ &= \int_G (f \cdot \nabla(\alpha_l w) + g \alpha_l w) \, d\lambda^N + \int_{\partial_N G} h \alpha_l w \, d\lambda_{\partial G} \\ &\quad + \int_G A(u \nabla w - w \nabla u) \cdot \nabla \alpha_l \, d\lambda^N \\ &= \int_{U_l \cap G} (\alpha_l g + f \cdot \nabla \alpha_l - A \nabla u \cdot \nabla \alpha_l) w \, d\lambda^N \\ &\quad + \int_{U_l \cap G} (\alpha_l f + u A \nabla \alpha_l) \cdot \nabla w \, d\lambda^N + \int_{U_l \cap \partial_N G} \alpha_l h w \, d\lambda_{\partial G}. \end{aligned} \right.$$

In order to apply the assumption (5.4), we use the multiplier properties (Theorem 2.3(ii) and Lemma 4.2), the continuous embedding  $W_0^{1,2}(G) \hookrightarrow \mathfrak{L}^{2,2}(G)$  (cf. Theorem 2.3(iv)) and the trace property  $W_0^{1,2}(G) \hookrightarrow \mathfrak{L}^{2,1}(\partial_N G)$  (cf. Theorem 4.4). Thus, we have for  $\mu = \min\{\omega, 2\}$

$$(5.6a) \quad \alpha_l f + u A \nabla \alpha_l \in \mathfrak{L}^{2,\mu}(U_l \cap G; \mathbb{R}^N),$$

$$(5.6b) \quad \alpha_l g + f \cdot \nabla \alpha_l - A \nabla u \cdot \nabla \alpha_l \in \mathfrak{L}^{2,\mu-2}(U_l \cap G),$$

$$(5.6c) \quad \alpha_l h \in \mathfrak{L}^{2,\mu-1}(U_l \cap \partial_N G).$$

Moreover, we have  $\alpha_l u \in W_0^{1,2}(U_l \cap G)$  (here and later on we use the symbol  $\alpha_l u$  for the restriction of the product  $\alpha_l u$  to  $U_l \cap G$ , too). Hence, it follows from (5.4), (5.5) that there exists an  $N - 2 < \bar{\omega} < N$  such that, if  $\mu < \bar{\omega}$ , it holds  $\nabla(\alpha_l u) \in \mathfrak{L}^{2,\mu}(U_{0l} \cap G; \mathbb{R}^N)$  and the estimate

$$(5.7) \quad \begin{aligned} & \|\nabla(\alpha_l u)\|_{\mathfrak{L}^{2,\mu}(U_{0l} \cap G; \mathbb{R}^N)} \\ & \leq c (\|f\|_{\mathfrak{L}^{2,\mu}(G; \mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\mu-2}(G)} + \|h\|_{\mathfrak{L}^{2,\mu-1}(\partial_N G)} + \|u\|_{W^{1,2}(G^\circ)}). \end{aligned}$$

Finally,  $\nabla(\alpha_l u) \in \mathfrak{L}^{2,\mu}(G; \mathbb{R}^N)$  is the zero extension of  $\nabla(\alpha_l u) \in \mathfrak{L}^{2,\mu}(U_{0l} \cap G; \mathbb{R}^N)$  for every  $l = 1, \dots, n+m$ , and, hence,

$$\nabla u = \sum_{l=1}^{n+m} \nabla(\alpha_l u) \in \mathfrak{L}^{2,\mu}(G; \mathbb{R}^N)$$

and

$$(5.8) \quad \begin{aligned} \|\nabla u\|_{\mathfrak{L}^{2,\mu}(G; \mathbb{R}^N)} &\leq \sum_{l=1}^{n+m} \|\nabla(\alpha_l u)\|_{\mathfrak{L}^{2,\mu}(U_{0l} \cap G; \mathbb{R}^N)} \\ &\leq c \left( \|f\|_{\mathfrak{L}^{2,\mu}(G; \mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\mu-2}(G)} + \|h\|_{\mathfrak{L}^{2,\mu-1}(\partial_N G)} + \|u\|_{W^{1,2}(G^\circ)} \right). \end{aligned}$$

Note, that the constants  $c > 0$  in (5.7), (5.8) do not depend on  $A, d, f, g, h$  and  $u$ .

Now, we again apply assumption (5.4) and the variational equation (5.5). Then we get (5.6) with  $\mu = \min\{\omega, 4\}$ , and, hence,  $\nabla u \in \mathfrak{L}^{2,\mu}(G; \mathbb{R}^N)$  and (5.8) for this new  $\mu$  and a new constant  $c > 0$  (with the same dependencies). Reiterating this procedure as often as necessary we obtain that  $\nabla u$  belongs to  $\mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$ , and it holds (5.8) for each  $\mu = \omega < \bar{\omega}$ . Therefore,  $G$  is admissible.  $\square$

**Lemma 5.5.** *Let  $G_0$  and  $G$  be regular subsets of  $\mathbb{R}^N$ ,  $G_0 \subset G$ , and let  $G_0$  be  $G$ -admissible. Further, let  $U$  be an open neighborhood of  $\bar{G}$  in  $\mathbb{R}^N$  and  $\Phi$  a Lipschitz transformation of  $U$  into  $\mathbb{R}^N$ . Then  $\Phi(G_0)$  is  $\Phi(G)$ -admissible.*

*Proof.* First of all,  $H := \Phi(G)$  and  $H_0 := \Phi(G_0)$  are regular sets of  $\mathbb{R}^N$  (cf. Lemma 3.6), and it holds  $\partial_N H_0 = \Phi(\partial_N G_0)$  and  $\partial_N H = \Phi(\partial_N G)$ .

Let  $D\Phi(x)$  be the functional matrix of  $\Phi$  in  $x$ . By  $J\Phi(x) := \det D\Phi(x)$  we denote the Jacobian of  $\Phi$  in  $x$ . Let  $L \geq 1$  be a common Lipschitz constant of both the transformations  $\Phi$  and  $\Phi^{-1}$ . Then, for  $\lambda^N$ -almost all  $x \in G$  and all  $\xi \in \mathbb{R}$  we have

$$(5.9) \quad L^{-1}|\xi| \leq |D\Phi(x)\xi| \leq L|\xi| \quad \text{and} \quad L^{-N} \leq |J\Phi(x)| \leq L^N.$$

Let  $\Phi_N : \partial_N G \rightarrow \partial_N H$  be the restriction of the Lipschitz transformation  $\Phi$  on the Lipschitz hypersurface  $\partial_N G$ . By  $J\Phi_N(x)$  we denote the Jacobian of  $\Phi_N$  in  $x \in \partial_N G$ , which is defined by (4.4). Then, for  $\lambda_{\partial_N G}$ -almost all  $x \in \partial_N G$  we obtain

$$L^{1-N} \leq |J\Phi_N(x)| \leq L^{N-1}.$$

Now, take  $(A, d) \in \mathfrak{A}(\varepsilon, H)$ ,  $f \in \mathfrak{L}^{2,\omega}(H; \mathbb{R}^N)$ ,  $g \in \mathfrak{L}^{2,\omega-2}(H)$ ,  $h \in \mathfrak{L}^{2,\omega-1}(\partial_N H)$  and  $u \in W_0^{1,2}(H)$  such that

$$(5.10) \quad \int_H (A\nabla u \cdot \nabla v + duv) \, d\lambda^N = \int_H (f \cdot \nabla v + gv) \, d\lambda^N + \int_{\partial_N H} hv \, d\lambda_{\partial_N G}$$

for all  $v \in W_0^{1,2}(H)$ . Because of Lemma 3.12 we have  $u \circ \Phi \in W_0^{1,2}(G)$  and

$$(5.11) \quad \|u \circ \Phi\|_{W_0^{1,2}(G)} \leq c \|u\|_{W_0^{1,2}(H)},$$

where the constant  $c > 0$  in (5.11) does not depend on  $u$ . Moreover, Theorem 2.1(ii) and Lemma 4.3 yield a constant  $c > 0$  such that

$$(5.12a) \quad f \circ \Phi \in \mathfrak{L}^{2,\omega}(G; \mathbb{R}^N), \quad \|f \circ \Phi\|_{\mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)} \leq c \|f\|_{\mathfrak{L}^{2,\omega}(H; \mathbb{R}^N)},$$

$$(5.12b) \quad g \circ \Phi \in \mathfrak{L}^{2,\omega-2}(G), \quad \|g \circ \Phi\|_{\mathfrak{L}^{2,\omega-2}(G)} \leq c \|g\|_{\mathfrak{L}^{2,\omega-2}(H)},$$

$$(5.12c) \quad h \circ \Phi_{\mathcal{N}} \in \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}G), \quad \|h \circ \Phi_{\mathcal{N}}\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}G)} \leq c \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}H)}.$$

Finally, from (5.9) it follows that the map

$$x \in G \mapsto (|J\Phi(x)| D\Phi(x)^{-1} A(\Phi(x)) (D\Phi(x)^{\top})^{-1}, |J\Phi(x)| d(\Phi(x))) \in \mathfrak{S}_N \times \mathbb{R}$$

belongs to  $\mathfrak{A}(\delta, G)$  with  $\delta = \varepsilon L^{-N-2}$ . Therefore, the chain rule for derivatives, the transformation formulas for integrals and the variational equation (5.10) imply that for all  $w \in W_0^{1,2}(G)$  we have

$$(5.13) \quad \left\{ \begin{array}{l} \int_G |J\Phi| ((D\Phi^{-1})(A \circ \Phi)(D\Phi^{-1})^{\top}) \nabla(u \circ \Phi) \cdot \nabla w \, d\lambda^N \\ \quad + \int_G |J\Phi| (d \circ \Phi) \cdot (u \circ \Phi) \cdot w \, d\lambda^N \\ = \int_H (A \nabla u \cdot \nabla(w \circ \Phi^{-1}) + du \cdot (w \circ \Phi^{-1})) \, d\lambda^N \\ = \int_H (f \cdot \nabla(w \circ \Phi^{-1}) + g \cdot (w \circ \Phi^{-1})) \, d\lambda^N \\ \quad + \int_{\partial_{\mathcal{N}}H} h \cdot (w \circ \Phi_{\mathcal{N}}^{-1}) \, d\lambda_{\partial H} \\ = \int_G |J\Phi| (D\Phi^{-1}(f \circ \Phi) \cdot \nabla w + (g \circ \Phi) \cdot w) \, d\lambda^N \\ \quad + \int_{\partial_{\mathcal{N}}G} |J\Phi_{\mathcal{N}}| (h \circ \Phi_{\mathcal{N}}) \cdot w \, d\lambda_{\partial G}. \end{array} \right.$$

Now, Theorem 2.1 and Theorem 2.3(ii) yield that the maps

$$x \in G \mapsto |J\Phi(x)| D\Phi(x)^{-1} (f \circ \Phi)(x) \in \mathbb{R}^N \quad \text{and} \quad x \in G \mapsto |J\Phi(x)| (g \circ \Phi)(x) \in \mathbb{R}$$

belong to  $\mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$  and  $\mathfrak{L}^{2,\omega-2}(G)$ , respectively. Analogously, from the Lemma 4.2 and Lemma 4.3 it follows that the map

$$x \in \partial_{\mathcal{N}}G \mapsto |J\Phi_{\mathcal{N}}(x)| (h \circ \Phi_{\mathcal{N}})(x) \in \mathbb{R}$$

belongs to  $\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}G)$ . Moreover, there is a constant  $c > 0$  such that the norms of

these maps can be estimated by

$$\begin{aligned} \| |J\Phi| D\Phi^{-1}(f \circ \Phi) \|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)} &\leq c \|f\|_{\mathfrak{L}^{2,\omega}(H;\mathbb{R}^N)}, \\ \| |J\Phi| (g \circ \Phi) \|_{\mathfrak{L}^{2,\omega-2}(G)} &\leq c \|g\|_{\mathfrak{L}^{2,\omega-2}(H)}, \\ \| |J\Phi_{\mathcal{N}}| (h \circ \Phi_{\mathcal{N}}) \|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}G)} &\leq c \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}H)}. \end{aligned}$$

Hence, the assumption that  $G_0$  is  $G$ -admissible, (5.12) and (5.13) imply, that there exists  $N - 2 < \bar{\omega} < N$  such that for all  $\omega < \bar{\omega}$  we have  $\nabla(u \circ \Phi) \in \mathfrak{L}^{2,\omega}(G_0;\mathbb{R}^N)$  and

$$(5.14) \quad \begin{aligned} &\| \nabla(u \circ \Phi) \|_{\mathfrak{L}^{2,\omega}(G_0;\mathbb{R}^N)} \\ &\leq c (\|f\|_{\mathfrak{L}^{2,\omega}(H;\mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(H)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}H)} + \|u\|_{W^{1,2}(H^{\circ})}). \end{aligned}$$

Applying again Theorem 2.1 and Theorem 2.3(ii), finally, there exists a constant  $c > 0$  such that  $\nabla u \in \mathfrak{L}^{2,\omega}(H_0;\mathbb{R}^N)$  and

$$(5.15) \quad \begin{aligned} &\| \nabla u \|_{\mathfrak{L}^{2,\omega}(H_0;\mathbb{R}^N)} \\ &\leq c (\|f\|_{\mathfrak{L}^{2,\omega}(H;\mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(H)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}H)} + \|u\|_{W^{1,2}(H^{\circ})}). \end{aligned}$$

Note, that the constants  $c > 0$  in (5.14) and (5.15) does not depend on  $A, d, f, g, h$  and  $u$ . Hence,  $H_0$  is  $H$ -admissible.  $\square$

Now we prove the Theorem 5.3:

Let  $x \in \partial G$ . By Lemma 5.4 and Lemma 5.5, the proof is done, if we have found open neighborhoods  $U_0, U_1$  and  $U_2$  of  $x$  and a Lipschitz transformation  $\Phi$  of  $U_2$  into  $\mathbb{R}^N$  such that

$$(5.16) \quad U_0 \subset U_1, \quad \overline{U_1} \subset U_2, \quad \text{and} \quad \Phi(U_0 \cap G) \text{ is } \Phi(U_1 \cap G)\text{-admissible.}$$

Let  $(\Phi, U)$  be a chart of  $G$  in  $x$  and  $\Phi(U \cap G) = E_k$  with  $k = 1, 2$  or  $3$ . If we take  $0 < r < 1$ , then it holds

$$\begin{aligned} \Phi(\Phi^{-1}(B(0, r/4)) \cap G) &= E_k(0, r/4), \\ \Phi(\Phi^{-1}(B(0, r/2)) \cap G) &= E_k(0, r/2). \end{aligned}$$

Hence, with the choice

$$U_0 = \Phi^{-1}(B(0, r/4)), \quad U_1 = \Phi^{-1}(B(0, r/2)) \quad \text{and} \quad U_2 = \Phi^{-1}(B(0, r)),$$

condition (5.16) is fulfilled if it is shown that for all  $k \in \{1, 2, 3\}$  and  $0 < r < 1$ , the sets  $E_k(0, r)$  are  $E_k$ -admissible.

In order to show this, we have to prove some lemmas, again, and to use the following notation: For  $x_0 \in \mathbb{R}^N$ ,  $r > 0$  and  $\delta \in \mathbb{R}$  we denote by

$$B(x_0, r, \delta) := \{x \in B(x_0, r) : x_N - x_{0N} = \delta r\}$$

the intersection of the hyperplane  $\{x \in \mathbb{R}^N : x_N - x_{0N} = \delta r\}$  with the open ball  $B(x_0, r)$ .

**Lemma 5.6.** *For all  $0 < \varepsilon \leq 1$  there exists an  $\bar{\omega} > N - 2$  such that for all  $\omega < \bar{\omega}$ ,  $\delta \in \mathbb{R}$ ,  $0 < \varrho \leq r < 1$  and  $x_0 \in \mathbb{R}^N$  the following holds: Let  $A \in L^\infty(B(x_0, r); \mathfrak{S}_N)$ ,  $f \in L^2(B(x_0, r); \mathbb{R}^N)$ ,  $g \in L^2(B(x_0, r))$ ,  $h \in L^2(B(x_0, r, \delta))$  and  $u \in W^{1,2}(B(x_0, r))$  satisfy*

$$(5.17) \quad \int_{B(x_0, r)} A \nabla u \cdot \nabla v \, d\lambda^N = \int_{B(x_0, r)} (f \cdot \nabla v + gv) \, d\lambda^N + \int_{B(x_0, r, \delta)} hv \, d\lambda^{N-1}$$

for all  $v \in W_0^{1,2}(B(x_0, r))$ , where  $A(x)\xi \cdot \xi \geq \varepsilon |\xi|^2$  for all  $\xi \in \mathbb{R}^N$  and for  $\lambda^N$ -almost all  $x \in B(x_0, r)$ . Then

$$(5.18) \quad \|\nabla u\|_{L^2(B(x_0, \varrho); \mathbb{R}^N)}^2 \leq c \left( \left( \frac{\varrho}{r} \right)^\omega \|\nabla u\|_{L^2(B(x_0, r); \mathbb{R}^N)}^2 + \|f\|_{L^2(B(x_0, r); \mathbb{R}^N)}^2 \right) \\ + c \left( r^2 \|g\|_{L^2(B(x_0, r))}^2 + r \|h\|_{L^2(B(x_0, r, \delta))}^2 \right),$$

where the constant  $c > 0$  in (5.18) depends only on  $N, \varepsilon$  and  $\omega$ .

*Proof.* Considering the transformation  $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$  defined as  $\Phi(y) = x_0 + ry$ , we get for all  $v \in W_0^{1,2}(B(x_0, r))$

$$\int_{B(x_0, r)} |v|^2 \, d\lambda^N = r^N \int_{B(0,1)} |v \circ \Phi|^2 \, d\lambda^N, \\ \int_{B(x_0, r, \delta)} |v|^2 \, d\lambda^{N-1} = r^{N-1} \int_{B(0,1, \delta)} |v \circ \Phi|^2 \, d\lambda^{N-1}.$$

Due to Poincaré's inequality and to the classical trace theorem, there exists a constant  $c > 0$ , depending on  $N$  only, such that for all such  $v$  we have

$$\int_{B(0,1)} |v \circ \Phi|^2 \, d\lambda^N \leq c \int_{B(0,1)} |\nabla(v \circ \Phi)|^2 \, d\lambda^N, \\ \int_{B(0,1, \delta)} |v \circ \Phi|^2 \, d\lambda^{N-1} \leq c \int_{B(0,1)} |\nabla(v \circ \Phi)|^2 \, d\lambda^N.$$

Applying the chain rule and the transformation formulas again it follows that

$$(5.19a) \quad \int_{B(x_0, r)} |v|^2 \, d\lambda^N \leq cr^2 \int_{B(x_0, r)} |\nabla v|^2 \, d\lambda^N,$$

$$(5.19b) \quad \int_{B(x_0, r, \delta)} |v|^2 \, d\lambda^{N-1} \leq cr \int_{B(x_0, r)} |\nabla v|^2 \, d\lambda^N.$$

Now, we consider the uniquely determined function  $u_0 \in W_0^{1,2}(B(x_0, r))$  satisfying the variational equation (5.17). Taking  $v = u_0$  as a test function in (5.17), we get

$$\varepsilon \|\nabla u_0\|_{L^2(B(x_0, r))}^2 \leq \|f\|_{L^2(B(x_0, r); \mathbb{R}^N)} \|\nabla u_0\|_{L^2(B(x_0, r); \mathbb{R}^N)} \\ + \|g\|_{L^2(B(x_0, r))} \|u_0\|_{L^2(B(x_0, r))} \\ + \|h\|_{L^2(B(x_0, r, \delta))} \|u_0\|_{L^2(B(x_0, r, \delta))}.$$

Combining this with (5.19), we state

$$(5.20) \quad \|\nabla u_0\|_{L^2(B(x_0, r); \mathbb{R}^N)}^2 \leq c \left( \|f\|_{L^2(B(x_0, r); \mathbb{R}^N)}^2 + r^2 \|g\|_{L^2(B(x_0, r))}^2 + r \|h\|_{L^2(B(x_0, r, \delta))}^2 \right)$$

with a new constant  $c > 0$ , depending on  $N$  and  $\varepsilon$ , only. Now, the difference  $w = u - u_0 \in W^{1,2}(B(x_0, r))$  satisfies

$$\int_{B(x_0, r)} A \nabla w \cdot \nabla v \, d\lambda^N = 0 \quad \text{for all } v \in W_0^{1,2}(B(x_0, r)).$$

But Lemma 5.6 is true in the case of vanishing  $f$ ,  $g$  and  $h$ , see TROIANIELLO [31, Lemma 2.15]. Hence, for  $0 < \varrho \leq r < 1$  it holds

$$\|\nabla w\|_{L^2(B(x_0, \varrho); \mathbb{R}^N)}^2 \leq c \left( \frac{\varrho}{r} \right)^\omega \|\nabla w\|_{L^2(B(x_0, r); \mathbb{R}^N)}^2$$

with a constant  $c > 0$ , which depends only on  $N$ ,  $\varepsilon$  and  $\omega$ . Using  $u = w + u_0$  and (5.20), we get (5.18).  $\square$

For  $0 < \varepsilon \leq 1$  let us denote by  $\bar{\omega}(\varepsilon)$  the supremum of all  $N - 2 < \bar{\omega} < N$  such that the assertion of Lemma 5.6 is true. Obviously,  $\bar{\omega}(\varepsilon)$  depends only on  $\varepsilon$  and the space dimension  $N$ , and the map  $\varepsilon \mapsto \bar{\omega}(\varepsilon)$  is non-decreasing.

**Lemma 5.7.** *Let  $0 < \varepsilon \leq 1$ ,  $\omega < \bar{\omega}(\varepsilon)$ ,  $0 < R < 1$ ,  $(A, d) \in \mathfrak{A}(\varepsilon, B(0, 1))$ ,  $f \in \mathfrak{L}^{2, \omega}(B(0, 1); \mathbb{R}^N)$ ,  $g \in \mathfrak{L}^{2, \omega-2}(B(0, 1))$ ,  $h \in \mathfrak{L}^{2, \omega-1}(B(0, 1, 0))$  and  $u \in W^{1,2}(B(0, 1))$  satisfy*

$$(5.21) \quad \int_{B(0,1)} (A \nabla u \cdot \nabla v + duv) \, d\lambda^N = \int_{B(0,1)} (f \cdot \nabla v + gv) \, d\lambda^N + \int_{B(0,1,0)} hv \, d\lambda^{N-1}$$

for all  $v \in W_0^{1,2}(B(0, 1))$ . Then it holds that  $\nabla u|_{B(0, R)} \in \mathfrak{L}^{2, \omega}(B(0, R); \mathbb{R}^N)$  and

$$(5.22) \quad \|\nabla u\|_{\mathfrak{L}^{2, \omega}(B(0, R); \mathbb{R}^N)}^2 \leq c \left( \|u\|_{W^{1,2}(B(0,1))} + \|f\|_{\mathfrak{L}^{2, \omega}(B(0,1); \mathbb{R}^N)} \right) + c \left( \|g\|_{\mathfrak{L}^{2, \omega-2}(B(0,1))} + \|h\|_{\mathfrak{L}^{2, \omega-1}(B(0,1,0))} \right),$$

where the constant  $c > 0$  in (5.22) depends only on  $N$ ,  $\varepsilon$ ,  $\omega$  and  $R$ .

*Proof.* Let  $\{r_k\}_{k \in \mathbb{N}}$  be the following decreasing sequence

$$r_k := R + 2^{-k}(1 - R).$$

For all  $0 < r \leq 4^{-N} \min\{R, 1 - R\}$ ,  $x_0 \in \overline{B(0, r_1)}$  and  $v \in W_0^{1,2}(B(x_0, r))$  from (5.21) it follows that

$$\int_{B(x_0, r)} A \nabla u \cdot \nabla v \, d\lambda^N = \int_{B(x_0, r)} (f \cdot \nabla v + (g - du)v) \, d\lambda^N + \int_{B(x_0, r, -x_{0N}/r)} hv \, d\lambda^{N-1}.$$

Hence, Lemma 5.6 yields that for  $\mu < \bar{\omega} < \bar{\omega}(\varepsilon)$  and  $0 < \varrho \leq r \leq 4^{-N} \min\{R, 1 - R\}$  we have (5.18) with  $\delta = -x_{0N}/r$  and therefore

$$\begin{aligned} \|\nabla u\|_{L^2(B(x_0, \varrho); \mathbb{R}^N)}^2 &\leq c \left(\frac{\varrho}{r}\right)^{\bar{\omega}} \|\nabla u\|_{L^2(B(x_0, r); \mathbb{R}^N)}^2 + c \|u\|_{L^2(B(x_0, r))}^2 \\ &\quad + c r^\mu \left( \|f\|_{\mathfrak{L}^{2, \mu}(B(0, 1); \mathbb{R}^N)}^2 + \|g\|_{\mathfrak{L}^{2, \mu-2}(B(0, 1))}^2 + \|h\|_{\mathfrak{L}^{2, \mu-1}(B(0, 1, 0))}^2 \right). \end{aligned}$$

For  $\mu \leq \omega$  set

$$\begin{aligned} \kappa_\mu(f, g, h, u) &:= \|u\|_{W^{1, 2}(B(0, 1))}^2 + \|f\|_{\mathfrak{L}^{2, \mu}(B(0, 1), \mathbb{R}^N)}^2 \\ &\quad + \|g\|_{\mathfrak{L}^{2, \mu-2}(B(0, 1))}^2 + \|h\|_{\mathfrak{L}^{2, \mu-1}(B(0, 1, 0))}^2, \end{aligned}$$

and let  $\omega < \bar{\omega}$ . Because of the continuous embedding  $W^{1, 2}(B(0, 1)) \hookrightarrow \mathfrak{L}^{2, 2}(B(0, 1))$  we can find a constant  $c > 0$  depending only on  $\mu, \varepsilon, R$  and  $N$  such that

$$\|u\|_{\mathfrak{L}^{2, \mu}(B(0, 1))}^2 \leq c \kappa_\mu(f, g, h, u) \quad \text{for } \mu = \min\{\omega, 2\}.$$

This yields for such  $\mu$  the estimate

$$\|\nabla u\|_{L^2(B(x_0, \varrho); \mathbb{R}^N)}^2 \leq c \left(\frac{\varrho}{r}\right)^{\bar{\omega}} \|\nabla u\|_{L^2(B(x_0, r); \mathbb{R}^N)}^2 + c r^\mu \kappa_\mu(f, g, h, u).$$

Now, we apply a fundamental lemma of CAMPANATO (cf. [7, Lemma 1.1], [13, Section 3.2]): For  $0 < \varrho \leq r \leq 4^{-N} \min\{R, 1 - R\}$  and  $\mu = \min\{\omega, 2\}$  we obtain

$$\|\nabla u\|_{L^2(B(x_0, \varrho); \mathbb{R}^N)}^2 \leq c \left(\frac{\varrho}{r}\right)^\mu \|\nabla u\|_{L^2(B(x_0, r); \mathbb{R}^N)}^2 + c \varrho^\mu \kappa_\mu(f, g, h, u),$$

where the constant  $c > 0$  depends only on  $\mu, \varepsilon, R$  and  $N$ . Dividing by  $\varrho^\mu$  and specifying  $r = 4^{-N} \min\{R, 1 - R\}$ , we get  $\nabla u|_{B(0, r_1)} \in \mathfrak{L}^{2, \mu}(B(0, r_1); \mathbb{R}^N)$  and

$$\|\nabla u\|_{\mathfrak{L}^{2, \mu}(B(0, r_1); \mathbb{R}^N)}^2 \leq c \kappa_\mu(f, g, h, u) \quad \text{for } \mu = \min\{\omega, 2\}.$$

Hence, again Theorem 2.3(iv) yields  $u|_{B(0, r_1)} \in \mathfrak{L}^{2, \mu}(B(0, r_1); \mathbb{R}^N)$  and

$$\|u\|_{\mathfrak{L}^{2, \mu}(B(0, r_1))}^2 \leq c \kappa_\mu(f, g, h, u) \quad \text{for } \mu = \min\{\omega, 4\}.$$

Repeating the same arguments as above we can prove

$$u|_{B(0, r_2)} \in \mathfrak{L}^{2, \mu}(B(0, r_2)) \quad \text{for } \mu = \min\{\omega, 6\}$$

with a corresponding norm estimate. After less than  $N$  steps of this iteration we arrive at  $\mu = \omega$  and the claim of the lemma, because it holds  $R < r_k \leq 1$  and  $4^{-N} \min\{R, 1 - R\} < r_k - r_{k-1}$  for all  $k = 1, \dots, N$ .  $\square$

**Lemma 5.8.** *Let  $0 < r < 1$ . Then  $E_1(0, r)$  is  $E_1$ -admissible, and  $E_2(0, r)$  is  $E_2$ -admissible.*



Proof. Using the notation  $E_0 := B(0, 1)$ , for  $k \in \{1, 2\}$  and  $u \in L^2(E_1)$  we define  $S_k u \in L^2(E_0)$  by

$$(S_k u)(x) := \begin{cases} u(x) & \text{for } x \in E_k, \\ (-1)^k u(\hat{x}, -x_N) & \text{for } x = (\hat{x}, x_N) \in E_0 \setminus E_k. \end{cases}$$

Thus,  $S_1 u$  and  $S_2 u$  are the extensions of  $u$  to  $E_0$  by *anti-reflection* and by *reflection*, respectively. It is well-known that  $u \in W_0^{1,2}(E_k)$  if and only if  $S_k u \in W_0^{1,2}(E_0)$  and in this case

$$\|S_k u\|_{W^{1,2}(E_0)} = \sqrt{2} \|u\|_{W^{1,2}(E_1)}.$$

Moreover, for  $0 \leq \omega < N$  we have  $u \in \mathfrak{L}^{2,\omega}(E_1)$  if and only if  $S_k u \in \mathfrak{L}^{2,\omega}(E_0)$  and, in this case it holds

$$\|S_k u\|_{\mathfrak{L}^{2,\omega}(E_0)} \leq \sqrt{2} \|u\|_{\mathfrak{L}^{2,\omega}(E_1)} \leq \sqrt{2} \|S_k u\|_{\mathfrak{L}^{2,\omega}(E_0)},$$

cf. TROIANELLO [31, Lemma 1.16 and Remark after the proof of Theorem 1.17].

Now, we extend elements  $f \in L^2(E_1; \mathbb{R}^N)$  to  $R_k f \in L^2(E_0; \mathbb{R}^N)$  and elements  $(A, d) \in \mathfrak{A}(\varepsilon, E_1)$  to  $(R_k A, R_k d) \in \mathfrak{A}(\varepsilon, E_0)$  by

$$(R_k f)_j(\hat{x}, x_N) := \begin{cases} (-1)^k f_j(\hat{x}, -x_N) & \text{for } j < N, \\ (-1)^{k+1} f_N(\hat{x}, -x_N) & \text{for } j = N, \end{cases}$$

$$(R_k A)_{ij}(\hat{x}, x_N) := \begin{cases} A_{ij}(\hat{x}, -x_N) & \text{for } i, j < N \text{ or } i = j = N, \\ -A_{ij}(\hat{x}, -x_N) & \text{otherwise,} \end{cases}$$

and

$$(R_k d)(\hat{x}, x_N) := d(\hat{x}, -x_N)$$

for  $(\hat{x}, x_N) \in E_0 \setminus E_k$ . Then, we get  $S_k(du) = (R_k d)(S_k u)$ ,  $R_k(Af) = (R_k A)(R_k f)$  and  $R_k(\nabla u) = \nabla(S_k u)$  for all  $(A, d) \in \mathfrak{A}(\varepsilon, E_1)$ ,  $f \in L^2(E_1; \mathbb{R}^N)$  and  $u \in W_0^{1,2}(E_1)$ .

Finally, for  $k \in \{1, 2\}$ ,  $v \in W_0^{1,2}(E_0)$  and  $h \in L^2(B(0, 1, 0))$  we define  $T_k v \in W_0^{1,2}(E_k)$  by

$$(T_k v)(\hat{x}, x_N) := v(\hat{x}, x_N) + (-1)^k v(\hat{x}, -x_N) \quad \text{for } (\hat{x}, x_N) \in E_k$$

and  $T_1 h = 0$ ,  $T_2 h = 2h$ , respectively. The functions  $T_1 v$  and  $T_2 v$  are the restrictions of the *antisymmetric* and *symmetric* part of  $2v$  to  $E_k$ , respectively, and we have

$$\begin{aligned} & \int_{E_0} ((R_k f) \cdot \nabla v + (S_k g)v) \, d\lambda^N + \int_{E_k \setminus E_1} (T_k h)v \, d\lambda^{N-1} \\ &= \int_{E_1} (f \cdot \nabla(T_k v) + g(T_k v)) \, d\lambda^N + \int_{E_k \setminus E_1} h(T_k v) \, d\lambda^{N-1} \end{aligned}$$

for all  $f \in L^2(E_1; \mathbb{R}^N)$ ,  $g \in L^2(E_1)$ ,  $h \in L^2(B(0, 1, 0))$  and  $v \in W_0^{1,2}(E_0)$ .

Now, consider  $(A, d) \in \mathfrak{A}(\varepsilon, E_1)$ ,  $f \in \mathfrak{L}^{2,\omega}(E_1; \mathbb{R}^N)$  and  $g \in \mathfrak{L}^{2,\omega-2}(E_1)$ ,  $h \in \mathfrak{L}^{2,\omega-1}(B(0, 1, 0))$  and take  $u \in W_0^{1,2}(E_k)$  such that for all  $v \in W_0^{1,2}(E_k)$  we have

$$\int_{E_1} (A \nabla u \cdot \nabla v + duv) d\lambda^N = \int_{E_1} (f \cdot \nabla v + gv) d\lambda^N + \int_{E_k \setminus E_1} hv d\lambda^{N-1}.$$

Then, for all  $w \in W_0^{1,2}(E_0)$  it follows that

$$\begin{aligned} & \int_{E_0} ((R_k A) \nabla(S_k u) \cdot \nabla w + (R_k d)(S_k u)w) d\lambda^N \\ &= \int_{E_1} (A \nabla u \cdot \nabla(T_k w) + du(T_k w)) d\lambda^N \\ &= \int_{E_1} (f \cdot \nabla(T_k w) + g(T_k w)) d\lambda^N + \int_{E_k \setminus E_1} h(T_k w) d\lambda^{N-1} \\ &= \int_{E_0} ((R_k f) \cdot \nabla w + (S_k g)w) d\lambda^N + \int_{E_k \setminus E_1} (T_k h)w d\lambda^{N-1}. \end{aligned}$$

Hence, Lemma 5.7 yields the following: For  $0 < r < 1$  and  $0 \leq \omega < \bar{\omega}(\varepsilon)$  we have

$$\nabla(S_k u)|_{B(0,r)} = R_k(\nabla u)|_{B(0,r)} \in \mathfrak{L}^{2,\omega}(B(0, r); \mathbb{R}^N)$$

and

$$\begin{aligned} \|\nabla u\|_{\mathfrak{L}^{2,\omega}(E_1(0,r); \mathbb{R}^N)} &\leq \|\nabla(S_k u)\|_{\mathfrak{L}^{2,\omega}(B(0,r); \mathbb{R}^N)} \\ &\leq c_1 (\|R_k f\|_{\mathfrak{L}^{2,\omega}(E_0; \mathbb{R}^N)} + \|S_k g\|_{\mathfrak{L}^{2,\omega-2}(E_0)}) \\ &\quad + c_1 (\|T_k h\|_{\mathfrak{L}^{2,\omega-1}(B(0,1,0))} + \|S_k u\|_{W^{1,2}(E_0)}). \end{aligned}$$

Therefore,

$$\begin{aligned} & \|\nabla u\|_{\mathfrak{L}^{2,\omega}(E_1(0,r); \mathbb{R}^N)} \\ & \leq c_2 (\|f\|_{\mathfrak{L}^{2,\omega}(E_1; \mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(E_1)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(B(0,1,0))} + \|u\|_{W^{1,2}(E_1)}), \end{aligned}$$

where the constants  $c_1 > 0$  and  $c_2 > 0$  do not depend on  $A$ ,  $d$ ,  $f$ ,  $g$ ,  $h$  and  $u$ .  $\square$

**Remark 5.9.** From Lemma 5.4 to Lemma 5.8 it follows that Theorem 5.3 is proved if  $G$  is open (i.e.  $\partial G = \partial_{\mathcal{D}} G$ ) or if  $G$  is closed (i.e.  $\partial G = \partial_{\mathcal{N}} G$ ), because in that cases there exists an atlas of  $\partial G$ , consisting of charts of type 1 and 2, only.

In order to finish the proof of Theorem 5.3, it suffices to prove the following

**Lemma 5.10.**  $E_3$  is admissible.

*Proof.* Again, denote  $E_0 := B(0, 1)$ , and let  $\overline{E_0}$  be its closure. Further, we introduce the set  $E_5 := \{x \in \overline{E_0} : x_N < 0\}$ .

There exists a Lipschitz transformation of  $\mathbb{R}^N$  onto  $\mathbb{R}^N$  which maps  $E_3$  onto  $E_5$ . Hence, because of Lemma 5.5, it is sufficient to show that  $E_5$  is admissible.

In order to do this, we define suitable extensions to  $\overline{E_0}$  of  $(A, d) \in \mathfrak{A}(\varepsilon, E_5)$ ,  $f \in \mathfrak{L}^{2,\omega}(E_5; \mathbb{R}^N)$ ,  $g \in \mathfrak{L}^{2,\omega-2}(E_5)$  and  $u \in W_0^{1,2}(E_5)$  as in the proof of Lemma 5.8.

Further, we extend  $h \in \mathfrak{L}^{2,\omega-1}(\partial_N E_5)$  to  $S_5 h \in \mathfrak{L}^{2,\omega-1}(\partial_N \overline{E_0})$  by *anti-reflection*

$$(S_5 h)(x) := \begin{cases} h(x) & \text{for } x \in \partial_N E_5, \\ -h(\hat{x}, -x_N) & \text{for } x = (\hat{x}, x_N) \in \partial_N \overline{E_0} \setminus \partial_N E_5. \end{cases}$$

After that we proceed as in the proof of Lemma 5.8, using the fact that  $\overline{E_0}$  is admissible (according to Remark 5.9 above).  $\square$

## 6. Solution regularity and Fredholm property

In this section  $G$  is a fixed regular subset of  $\mathbb{R}^N$ . Again, by  $\langle \cdot, \cdot \rangle_G$  and  $J_G$  we denote the dual pairing and the duality map of  $W_0^{1,2}(G)$ , respectively (cf. (3.15)). By definition (cf. (3.16)),  $J_G$  is an isometric isomorphism from  $W_0^{1,2,\omega}(G)$  onto  $W^{-1,2,\omega}(G)$ .

For  $0 < \varepsilon \leq 1$  let  $\overline{\omega}(\varepsilon, G)$  be the supremum of all  $0 < \overline{\omega} < N$  such that for all  $\omega < \overline{\omega}$  the following is true: For each  $(A, d) \in \mathfrak{A}(\varepsilon, G)$ ,  $f \in \mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$ ,  $g \in \mathfrak{L}^{2,\omega-2}(G)$ ,  $h \in \mathfrak{L}^{2,\omega-1}(\partial_N G)$  and  $u \in W_0^{1,2}(G)$  with

$$\int_G (A \nabla u \cdot \nabla v + d u v) \, d\lambda^N = \int_G (f \cdot \nabla v + g v) \, d\lambda^N + \int_{\partial_N G} h v \, d\lambda_{\partial G}$$

for all  $v \in W_0^{1,2}(G)$ , it holds  $\nabla u \in \mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$  and

$$\|\nabla u\|_{\mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)} \leq c (\|f\|_{\mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(G)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_N G)}),$$

where the constant  $c > 0$  depends only on  $G$ ,  $N$ ,  $\varepsilon$  and  $\omega$ . Because of Theorem 5.3 we have

$$N - 2 < \overline{\omega}(\varepsilon, G) \leq \overline{\omega}(1, G).$$

Let  $F : L^2(G; \mathbb{R}^N) \times L^2(G) \times L^2(\partial_N G) \mapsto W^{-1,2}(G)$  be the linear bounded operator which is defined by

$$\langle F(f, g, h), v \rangle_G := \int_G (f \cdot \nabla v + g v) \, d\lambda^N + \int_{\partial_N G} h v \, d\lambda_{\partial G} \quad \text{for all } v \in W_0^{1,2}(G).$$

**Remark 6.1.** Let  $\omega < \overline{\omega}(1, G)$ . Then Theorem 5.3 yields that  $F$  is a bounded operator from  $\mathfrak{L}^{2,\omega}(G; \mathbb{R}^N) \times \mathfrak{L}^{2,\omega-2}(G) \times \mathfrak{L}^{2,\omega-1}(\partial_N G)$  onto  $W^{-1,2,\omega}(G)$ . In other words: A functional  $\phi \in W^{-1,2}(G)$  belongs to  $W^{-1,2,\omega}(G)$  if and only if there exist  $f \in \mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$ ,  $g \in \mathfrak{L}^{2,\omega-2}(G)$  and  $h \in \mathfrak{L}^{2,\omega-1}(\partial_N G)$  such that  $\phi = F(f, g, h)$ .

In GRIESENTROG [17] and GRÖGER, RECKE [19] are derived other, more direct criteria for a functional  $\phi \in W^{-1,2}(G)$  to belong to  $W^{-1,2,\omega}(G)$  and corresponding expressions for a norm in  $W^{-1,2,\omega}(G)$  which are equivalent to (3.16). In RAKOTOSON [25, 26] analogous characterizations are given for elements of  $W_{\text{loc}}^{-1,2,\omega}(G)$ .

For  $A \in L^\infty(G; \mathfrak{S}_N)$ ,  $b, c \in L^\infty(G; \mathbb{R}^N)$ ,  $d \in L^\infty(G)$  and  $e \in L^\infty(\partial_N G)$  we denote by  $L(A, b, c, d, e)$  the linear bounded operator from  $W_0^{1,2}(G)$  into  $W^{-1,2}(G)$  which is defined by

$$\langle L(A, b, c, d, e)u, v \rangle_G := \int_G ((A\nabla u + bu) \cdot \nabla v + (c \cdot \nabla u + du)v) \, d\lambda^N + \int_{\partial_N G} euv \, d\lambda_{\partial G}$$

for  $u, v \in W_0^{1,2}(G)$ . Furthermore, let  $L_\omega(A, b, c, d, e)$  be the restriction of  $L(A, b, c, d, e)$  to  $W_0^{1,2,\omega}(G)$ .

**Lemma 6.2.** *Let  $\omega < \bar{\omega}(1, G)$ . Then  $L_\omega(A, b, c, d, e)$  maps  $W_0^{1,2,\omega}(G)$  continuously into  $W^{-1,2,\omega}(G)$ . Moreover, it depends continuously in the sense of the operator norm in  $\mathcal{L}(W_0^{1,2,\omega}(G); W^{-1,2,\omega}(G))$  on  $A, b, c, d$  and  $e$ .*

*Proof.* Let  $A \in L^\infty(G; \mathfrak{S}_N)$ ,  $b, c \in L^\infty(G; \mathbb{R}^N)$ ,  $d \in L^\infty(G)$ ,  $e \in L^\infty(\partial_N G)$  and  $u \in W_0^{1,2,\omega}(G)$ . Because of the Theorem 2.3(ii)  $A\nabla u + bu$  belongs to  $\mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$  and depends bilinearly and continuously on  $(A, b)$  and  $u$ . Analogously,  $c \cdot \nabla u + du$  belongs to  $\mathfrak{L}^{2,\omega-2}(G)$  and depends bilinearly and continuously on  $(c, d)$  and  $u$ . Finally, because of Theorem 4.4 and of Lemma 4.2, the product  $eu$  belongs to  $\mathfrak{L}^{2,\omega+1}(\partial_N G)$  and depends bilinearly and continuously on  $e$  and  $u$ . Hence,

$$L(A, b, c, d, e)u = F(A\nabla u + bu, c \cdot \nabla u + du, eu)$$

belongs to  $W^{-1,2,\omega}(G)$  and depends bilinearly and continuously on  $(A, b, c, d, e)$  and  $u$ , cf. Remark 6.1.  $\square$

The main result of this section is the following

**Theorem 6.3.** *Let  $A \in L^\infty(G; \mathfrak{S}_N)$  and  $0 < \varepsilon \leq 1$  such that for  $\lambda^N$ -almost all  $x \in G$  it holds*

$$\varepsilon |\xi|^2 \leq A(x)\xi \cdot \xi \leq \frac{1}{\varepsilon} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N,$$

and let  $\omega < \bar{\omega}(\varepsilon, G)$ . Then the following is true:

- (i) *If  $L(A, b, c, d, e)u \in W^{-1,2,\omega}(G)$  for a certain  $u \in W_0^{1,2}(G)$ , then  $u \in W_0^{1,2,\omega}(G)$ .*
- (ii)  *$L_\omega(A, b, c, d, e)$  is a Fredholm operator of index zero from  $W_0^{1,2,\omega}(G)$  into  $W^{-1,2,\omega}(G)$ .*

*Proof.* (i) We have  $L(A, b, c, d, e)u = J_G w$  for a certain  $w \in W_0^{1,2,\omega}(G)$ . Therefore it holds

$$(6.1) \quad L(A, 0, 0, 1, 0)u = F(\nabla w - bu, w - c \cdot \nabla u - (d-1)u, -eu).$$

Now we proceed as in the proof of Lemma 5.4. We use the multiplier properties Theorem 2.3(ii) and Lemma 4.2, the continuous embedding  $W_0^{1,2}(G) \hookrightarrow \mathfrak{L}^{2,2}(G)$  and the trace property  $W_0^{1,2}(G) \hookrightarrow \mathfrak{L}^{2,1}(\partial_N G)$  to get for  $\mu = \min\{\omega, 2\}$

$$(6.2a) \quad \nabla w - bu \in \mathfrak{L}^{2,\mu}(G; \mathbb{R}^N),$$

$$(6.2b) \quad w - c \cdot \nabla u - (d-1)u \in \mathfrak{L}^{2,\mu-2}(G),$$

$$(6.2c) \quad -eu \in \mathfrak{L}^{2,\mu-1}(\partial_N G).$$

If  $\mu < \bar{\omega}(\varepsilon, G)$ , then Theorem 5.3 yields that  $u \in W_0^{1,2,\mu}(G)$ . Hence, we get (6.2) with  $\mu = \min\{\omega, 4\}$ , and so on until we arrive at  $\mu = \omega$ .

(ii) First we show that that  $L(A, b, c, d, e)$  is a Fredholm operator from  $W_0^{1,2}(G)$  into  $W^{-1,2}(G)$ . We have

$$L(A, b, c, d, e) = L(A, 0, 0, 1, 0) + L(0, b, c, d - 1, e).$$

Because of the Lax-Milgram Lemma,  $L(A, 0, 0, 1, 0)$  is an isomorphism from  $W_0^{1,2}(G)$  onto  $W^{-1,2}(G)$ . But  $L(0, b, c, d - 1, e)$  is completely continuous from  $W_0^{1,2}(G)$  into  $W^{-1,2}(G)$  because of Lemma 3.9 and Theorem 4.4, so the claim is proved.

Now let us prove assertion (ii) of the theorem. We have, because of the claim above,

$$(6.3) \quad W^{-1,2}(G) = \text{im } L(A, b, c, d, e) \oplus \ker (L(A, c, b, d, e) \circ J_G^{-1})$$

and  $\dim \ker L(A, b, c, d, e) = \dim \ker L(A, c, b, d, e) < \infty$ . Here we used that the operator  $L(A, c, b, d, e)$  is the adjoint to the operator  $L(A, b, c, d, e)$ . Assertion (i) implies

$$\ker L(A, b, c, d, e) = \ker L_\omega(A, b, c, d, e),$$

$$\ker L(A, c, b, d, e) = \ker L_\omega(A, c, b, d, e),$$

and, hence,  $\dim \ker L_\omega(A, b, c, d, e) = \dim \ker L_\omega(A, c, b, d, e) < \infty$ . Further, from assertion (i) follows

$$W^{-1,2,\omega}(G) \cap \text{im } L(A, b, c, d, e) = \text{im } L_\omega(A, b, c, d, e).$$

Therefore, (6.3) yields

$$W^{-1,2,\omega}(G) = \text{im } L_\omega(A, b, c, d, e) \oplus \ker (L_\omega(A, c, b, d, e) \circ J_G^{-1}).$$

It remains to show that  $\text{im } L_\omega(A, b, c, d, e)$  is closed in  $W^{-1,2,\omega}(G)$ . Thus, let  $L_\omega(A, b, c, d, e)u_k \rightarrow \phi$  in  $W^{-1,2,\omega}(G)$  for  $k \rightarrow \infty$  (with  $u_k \in W_0^{1,2,\omega}(G)$ ). Then there exists an  $u \in W_0^{1,2}(G)$  such that  $L(A, b, c, d, e)u = \phi$ , because  $\text{im } L(A, b, c, d, e)$  is closed in  $W^{-1,2}(G)$ . Now assertion (i) works again. We get  $u \in W_0^{1,2,\omega}(G)$  and, hence,  $\phi \in \text{im } L_\omega(A, b, c, d, e)$ .  $\square$

**Corollary 6.4.** *Let  $\mathfrak{J}(\varepsilon, G)$  be the set of all  $(A, b, c, d, e)$  such that  $L(A, b, c, d, e)$  is injective and that it holds*

$$(6.4) \quad \varepsilon |\xi|^2 < \text{essinf}_{x \in G} (A(x)\xi \cdot \xi) \quad \text{and} \quad \text{esssup}_{x \in G} (A(x)\xi \cdot \xi) < \frac{1}{\varepsilon} |\xi|^2$$

for all  $\xi \in \mathbb{R}^N \setminus \{0\}$ . Then the following is true:

(i) *The set  $\mathfrak{J}(\varepsilon, G)$  is open in  $L^\infty(G; \mathfrak{S}_N) \times L^\infty(G; \mathbb{R}^N)^2 \times L^\infty(G) \times L^\infty(\partial_N G)$ .*

(ii) *Let  $\omega < \bar{\omega}(\varepsilon, G)$ . Then  $L(A, b, c, d, e)^{-1}F(f, g, h)$  depends analytically, in the norm of the space  $W_0^{1,2,\omega}(G)$ , on  $(A, b, c, d, e) \in \mathfrak{J}(\varepsilon, G)$ ,  $f \in \mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$ ,  $g \in \mathfrak{L}^{2,\omega-2}(G)$  and  $h \in \mathfrak{L}^{2,\omega-1}(\partial_N G)$ .*

Proof. (i) Let  $L(A, b, c, d, e)$  be injective. Then, because of its Fredholm property, it is an isomorphism from  $W_0^{1,2}(G)$  onto  $W^{-1,2}(G)$ . But the set of isomorphisms is open in  $\mathcal{L}(W_0^{1,2}(G); W^{-1,2}(G))$ . Hence, if  $(A_0, b_0, c_0, d_0, e_0)$  is close to  $(A, b, c, d, e)$  in  $L^\infty(G; \mathfrak{S}_N) \times L^\infty(G; \mathbb{R}^N)^2 \times L^\infty(G) \times L^\infty(\partial_N G)$ , then  $L(A_0, b_0, c_0, d_0, e_0)$  is injective, too. Moreover, (6.4) is an open condition in  $L^\infty(G; \mathfrak{S}_N)$ .

(ii)  $L(A, b, c, d, e)$  and  $F(f, g, h)$  depend linearly and continuously (in the norms of  $\mathcal{L}(W_0^{1,2,\omega}(G), W^{-1,2,\omega}(G))$  and  $W^{-1,2,\omega}(G)$ , respectively) and, hence, analytically on  $(A, b, c, d, e)$  and  $(f, g, h)$ , respectively. Therefore,  $L(A, b, c, d, e)^{-1}F(f, g, h)$  depends analytically (in the norm of  $W_0^{1,2,\omega}(G)$ ) on  $(A, b, c, d, e, f, g, h)$ , too. Remark that the assumptions  $\omega < \bar{\omega}(\varepsilon, G)$  and  $(A, b, c, d, e) \in \mathfrak{J}(\varepsilon, G)$  imply that  $L(A, b, c, d, e)$  is an isomorphism from  $W_0^{1,2,\omega}(G)$  onto  $W^{-1,2,\omega}(G)$  (because of Theorem 6.3 (ii)).  $\square$

**Remark 6.5.** Consider the boundary value problem (1.1) with  $(A, b, c, d, e) \in \mathfrak{J}(\varepsilon, \Omega \cup \Gamma)$ , and suppose  $\Omega \cup \Gamma$  to be regular. Then Theorem 2.3 (iii) and (iv) and Corollary 6.4 imply that, for all  $N - 2 < \omega < \bar{\omega}(\varepsilon, \Omega \cup \Gamma)$ , the weak solution to (1.1) depends analytically, in the norm of the Hölder space  $C^{0,\alpha}(\bar{\Omega})$  with

$$\alpha = \frac{\omega - N + 2}{2} > 0$$

on  $(A, b, c, d, e) \in \mathfrak{J}(\varepsilon, \Omega \cup \Gamma)$ ,  $f \in \mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)$ ,  $g \in \mathfrak{L}^{2,\omega-2}(\Omega)$  and  $h \in \mathfrak{L}^{2,\omega-1}(\Gamma)$ .

Now, consider (1.1) with right-hand sides  $f \in L^p(\Omega; \mathbb{R}^N)$  with  $p > N$  and  $g = 0$  and  $h = 0$ . Then, because of the continuous embeddings  $L^p(\Omega; \mathbb{R}^N) \hookrightarrow \mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)$  (cf. Theorem 2.1(i)) and  $W_0^{1,2,\omega}(\Omega \cup \Gamma) \hookrightarrow C^{0,(\omega-N+2)/2}(\bar{\Omega})$  with

$$\omega = \min \left\{ \bar{\omega}(\varepsilon, \Omega \cup \Gamma), N - \frac{2N}{p} \right\},$$

the weak solution to (1.1) depends analytically, in the norm of  $C^{0,\alpha}(\bar{\Omega})$ , on  $(A, b, c, d, e) \in \mathfrak{J}(\varepsilon, \Omega \cup \Gamma)$  and  $f \in L^p(\Omega; \mathbb{R}^N)$ .

Analogously, suppose  $f = 0$  and  $g = 0$  and  $h \in L^{p-1}(\Gamma)$  with  $p > N$  and  $p \geq 3$ . Then, because of the continuous embeddings  $L^{p-1}(\Gamma) \hookrightarrow \mathfrak{L}^{2,\omega-1}(\Gamma)$  and, furthermore,  $W_0^{1,2,\omega}(\Omega \cup \Gamma) \hookrightarrow C^{0,(\omega-N+2)/2}(\bar{\Omega})$  with

$$\omega = \min \left\{ \bar{\omega}(\varepsilon, \Omega \cup \Gamma), N - \frac{2(N-1)}{p-1} \right\},$$

the weak solution to (1.1) depends analytically, in the norm of  $C^{0,\alpha}(\bar{\Omega})$ , on  $(A, b, c, d, e) \in \mathfrak{J}(\varepsilon, \Omega \cup \Gamma)$  and  $h \in L^{p-1}(\Gamma)$ .

Finally, suppose  $f = 0$  and  $h = 0$  and  $g \in L^{p/2}(\Omega)$  with  $p > N$  and  $p \geq 4$ . Then, because of the continuous embeddings  $L^{p/2}(\Omega) \hookrightarrow \mathfrak{L}^{2,\omega-2}(\Omega)$  and, furthermore,  $W_0^{1,2,\omega}(\Omega \cup \Gamma) \hookrightarrow C^{0,(\omega-N+2)/2}(\bar{\Omega})$  with

$$\omega = \min \left\{ \bar{\omega}(\varepsilon, \Omega \cup \Gamma), N + 2 - \frac{4N}{p} \right\},$$

the weak solution to (1.1) depends analytically, in the norm of  $C^{0,\alpha}(\bar{\Omega})$ , on  $(A, b, c, d, e) \in \mathfrak{J}(\varepsilon, \Omega \cup \Gamma)$  and  $g \in L^{p/2}(\Omega)$ .

Remark that our approach does not cover the case  $N = 3$  and  $g \in L^{p/2}(\Omega)$  with  $3 < p < 4$ . Nevertheless, in GRIEPENTROG [17] these problems will be solved by using a more direct criterion for a functional to belong to the space  $W^{-1,2,\omega}(\Omega \cup \Gamma)$ . In fact, there will be shown, that Corollary 6.4 remains true under weaker assumptions on  $f$ ,  $g$  and  $h$ , namely,

$$f \in \mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N), \quad g \in \mathfrak{L}^{2N/(N+2),\omega N/(N+2)}(\Omega) \quad \text{and} \quad h \in \mathfrak{L}^{2(N-1)/N,\omega(N-1)/N}(\Gamma).$$

## 7. Generalizations to elliptic systems

In this last section we consider weak solutions to boundary value problems for linear elliptic systems of the type

$$(7.1a) \quad -\nabla \cdot (A_{ik} \nabla u_k + b_{ik} u_k) + c_{ik} \cdot \nabla u_k + d_{ik} u_k = -\nabla \cdot f_i + g_i \quad \text{in } \Omega,$$

$$(7.1b) \quad (A_{ik} \nabla u_k + b_{ik} u_k) \cdot \nu + e_{ik} u_k = f_i \cdot \nu + h_i \quad \text{on } \Gamma_i,$$

$$(7.1c) \quad u_i = 0 \quad \text{on } \partial\Omega \setminus \Gamma_i.$$

In (7.1) and in the sequel (if no other setting, as in (7.2) below, is prescribed), the summation over repeated subscripts is understood, and free subscripts vary from 1 to  $n$ . Further,  $\Omega \subset \mathbb{R}^N$  is a bounded domain,  $\Gamma_i$  are subsets of  $\partial\Omega$ ,  $\nu : \partial\Omega \rightarrow \mathbb{R}^N$  is the unit outward normal vector field on  $\partial\Omega$ , and  $A_{ik} \in L^\infty(\Omega; \mathfrak{M}_N)$ ,  $b_{ik}, c_{ik} \in L^\infty(\Omega; \mathbb{R}^N)$ ,  $d_{ik} \in L^\infty(\Omega)$  and  $e_{ik} \in L^\infty(\Gamma_i)$ . We assume the regularity of the sets

$$G_i := \Omega \cup \Gamma_i \quad \text{for all } i = 1, \dots, n.$$

Moreover, it is supposed that there exists an  $\varepsilon \in (0, 1]$  such that for all  $\xi \in \mathbb{R}^N$  and  $\lambda^N$ -almost all  $x \in \Omega$  we have

$$(7.2) \quad \varepsilon |\xi|^2 \leq A_{ii}(x) \xi \cdot \xi \quad \text{and} \quad A_{ii}(x) \in \mathfrak{S}_N \quad (\text{no summation over } i)$$

and

$$(7.3) \quad \|A_{ik}\|_{L^\infty(\Omega; \mathfrak{M}_N)} \leq \frac{1}{\varepsilon}.$$

The results of this section follow from the results of the previous Section 6 in a straightforward way. We will formulate them not in the language of operators, like in Theorem 6.3, but in the language of weak solutions to (7.1).

A weak solution to (7.1) is, by definition, a tuple  $u = (u_1, \dots, u_n)$  such that  $u_i \in W_0^{1,2}(G_i)$  and

$$\begin{aligned} & \int_{\Omega} ((A_{ik} \nabla u_k + b_{ik} u_k) \cdot \nabla v + (c_{ik} \cdot \nabla u_k + d_{ik} u_k) v) \, d\lambda^N + \int_{\Gamma_i} e_{ik} u_k v \, d\lambda_{\partial\Omega} \\ &= \int_{\Omega} (f_i \cdot \nabla v + g_i v) \, d\lambda^N + \int_{\Gamma_i} h_i v \, d\lambda_{\partial\Omega} \quad \text{for all } v \in W_0^{1,2}(G_i) \text{ and } i = 1, \dots, n. \end{aligned}$$

**Theorem 7.1.** For all  $\omega < \min \{\bar{\omega}(\varepsilon, G_i) : 1 \leq i \leq n\}$  there exists a constant  $\delta > 0$  such that, if

$$(7.4) \quad \|A_{ik}\|_{L^\infty(\Omega; \mathfrak{M}_N)} \leq \delta \quad \text{for } i > k,$$

the following holds:

(i) If  $u$  is a weak solution to (7.1) with  $f_i \in \mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)$ ,  $g_i \in \mathfrak{L}^{2,\omega-2}(\Omega)$  and  $h_i \in \mathfrak{L}^{2,\omega-1}(\Gamma_i)$ , then  $u \in W^{1,2,\omega}(\Omega)^n$ .

(ii) Suppose that  $u = 0$  is the only weak solution to (7.1) with  $f_i = 0$ ,  $g_i = 0$  and  $h_i = 0$ , i.e. to the homogeneous system, corresponding to (7.1). Then, for arbitrary  $f_i \in \mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)$ ,  $g_i \in \mathfrak{L}^{2,\omega-2}(\Omega)$  and  $h_i \in \mathfrak{L}^{2,\omega-1}(\Gamma_i)$ , there exists exactly one weak solution to (7.1), and this solution depends analytically (in the sense of the space  $W^{1,2,\omega}(\Omega)^n$ ) on the coefficients  $A_{ik}$ ,  $b_{ik}$ ,  $c_{ik}$ ,  $d_{ik}$  and  $e_{ik}$  (in the sense of the corresponding  $L^\infty$ -spaces) and on the right-hand sides  $f_i$ ,  $g_i$  and  $h_i$  (in the sense of the corresponding Campanato spaces).

**Proof.** For the matrix functions

$$\begin{aligned} A &= [A_{ik}]_{i,k=1}^n \in L^\infty(\Omega; \mathfrak{M}_N)^{n \times n}, \\ b &= [b_{ik}]_{i,k=1}^n \in L^\infty(\Omega; \mathbb{R}^N)^{n \times n}, \\ c &= [c_{ik}]_{i,k=1}^n \in L^\infty(\Omega; \mathbb{R}^N)^{n \times n}, \\ d &= [d_{ik}]_{i,k=1}^n \in L^\infty(\Omega)^{n \times n}, \\ e &= [e_{ik}]_{i,k=1}^n \in L^\infty(\Gamma_1)^n \times \cdots \times L^\infty(\Gamma_n)^n \end{aligned}$$

we denote by

$$L(A, b, c, d, e) : W_0^{1,2}(G_1) \times \cdots \times W_0^{1,2}(G_n) \rightarrow W^{-1,2}(G_1) \times \cdots \times W^{-1,2}(G_n)$$

the linear bounded operator, the  $i$ -th component of which is defined by

$$\begin{aligned} &\langle [L(A, b, c, d, e)u]_i, v \rangle_{G_i} \\ &:= \int_{\Omega} ((A_{ik} \nabla u_k + b_{ik} u_k) \cdot \nabla v + (c_{ik} \cdot \nabla u_k + d_{ik} u_k) v) \, d\lambda^N + \int_{\Gamma_i} e_{ik} u_k v \, d\lambda_{\Gamma_i} \end{aligned}$$

for all  $v \in W_0^{1,2}(G_i)$  (summation over  $k$ , but not over  $i$ ). Further, for

$$\begin{aligned} f &= [f_i]_{i=1}^n \in L^2(\Omega; \mathbb{R}^N)^n, \\ g &= [g_i]_{i=1}^n \in L^2(\Omega)^n, \\ h &= [h_i]_{i=1}^n \in L^2(\Gamma_1) \times \cdots \times L^2(\Gamma_n) \end{aligned}$$

we define  $F(f, g, h) \in W^{-1,2}(G_1) \times \cdots \times W^{-1,2}(G_n)$  by

$$\langle [F(f, g, h)]_i, v \rangle_{G_i} := \int_{\Omega} (f_i \cdot \nabla v + g_i v) \, d\lambda^N + \int_{\Gamma_i} h_i v \, d\lambda_{\Gamma_i} \quad \text{for all } v \in W_0^{1,2}(G_i).$$



Obviously,  $u$  is a weak solution to (7.1) if and only if

$$(7.5) \quad L(A, b, c, d, e)u = F(f, g, h).$$

As in the previous section (see Remark 6.1 and Lemma 6.2) one shows that the operator  $F$  maps the spaces  $\mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)^n \times \mathfrak{L}^{2,\omega-2}(\Omega)^n \times \mathfrak{L}^{2,\omega-1}(\Gamma_1) \times \dots \times \mathfrak{L}^{2,\omega-1}(\Gamma_n)$  into the spaces  $W^{-1,2,\omega}(G_1) \times \dots \times W^{-1,2,\omega}(G_n)$  and the operator  $L(A, b, c, d, e)$  maps  $W_0^{1,2,\omega}(G_1) \times \dots \times W_0^{1,2,\omega}(G_n)$  into  $W^{-1,2,\omega}(G_1) \times \dots \times W^{-1,2,\omega}(G_n)$ , respectively. Moreover, by the same argument the functional  $F(f, g, h)$  depends continuously (in the sense of  $W^{-1,2,\omega}(G_1) \times \dots \times W^{-1,2,\omega}(G_n)$ ) on  $f \in \mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)^n$ ,  $g \in \mathfrak{L}^{2,\omega-2}(\Omega)^n$  and  $h \in \mathfrak{L}^{2,\omega-1}(\Gamma_1) \times \dots \times \mathfrak{L}^{2,\omega-1}(\Gamma_n)$ , and  $L(A, b, c, d, e)$  depends continuously (in the sense of  $\mathcal{L}(W_0^{1,2,\omega}(G_1) \times \dots \times W_0^{1,2,\omega}(G_n); W^{-1,2,\omega}(G_1) \times \dots \times W^{-1,2,\omega}(G_n))$ ) on  $A, b, c, d$  and  $e$ .

Let  $I_n$  be the unit  $n \times n$ -matrix.

*Step 1.* We show that for all  $\varepsilon \in (0, 1]$  and all  $\omega < \min\{\bar{\omega}(\varepsilon, G_i) : 1 \leq i \leq n\}$  there exists a constant  $\delta > 0$  such that, if (7.2), (7.3) and (7.4) are satisfied, the operator  $L(A, 0, 0, I_n, 0)$  is an isomorphism from  $W_0^{1,2,\omega}(G_1) \times \dots \times W_0^{1,2,\omega}(G_n)$  onto  $W^{-1,2,\omega}(G_1) \times \dots \times W^{-1,2,\omega}(G_n)$ . For that it suffices to show that for all  $\varepsilon \in (0, 1]$  and all  $\omega < \min\{\bar{\omega}(\varepsilon, G_i) : 1 \leq i \leq n\}$  the following holds: If (7.2), (7.3) and

$$(7.6) \quad A_{ik} = 0 \quad \text{for } i > k$$

are satisfied, then  $L(A, 0, 0, I_n, 0)$  is bijective from  $W_0^{1,2,\omega}(G_1) \times \dots \times W_0^{1,2,\omega}(G_n)$  onto  $W^{-1,2,\omega}(G_1) \times \dots \times W^{-1,2,\omega}(G_n)$ , and there exists a constant  $c > 0$ , which depends on  $\varepsilon$  and  $\omega$  only, such that

$$\|L(A, 0, 0, I_n, 0)^{-1}\|_{\mathcal{L}(W^{-1,2,\omega}(G_1) \times \dots \times W^{-1,2,\omega}(G_n); W_0^{1,2,\omega}(G_1) \times \dots \times W_0^{1,2,\omega}(G_n))} \leq c.$$

Thus, take  $\omega < \min\{\bar{\omega}(\varepsilon, G_i) : 1 \leq i \leq n\}$  and, furthermore,  $(\phi_1, \dots, \phi_n) \in W^{-1,2,\omega}(G_1) \times \dots \times W^{-1,2,\omega}(G_n)$ , suppose (7.2), (7.3) and (7.6) to be satisfied, and consider the equation

$$(7.7) \quad L(A, 0, 0, I_n, 0)(u_1, \dots, u_n) = (\phi_1, \dots, \phi_n).$$

This equation is equivalent to a system of  $n$  variational equations, the last one of which has the form

$$(7.8) \quad \int_{\Omega} (A_{nn} \nabla u_n \cdot \nabla v + u_n v) \, d\lambda^N = \langle \phi_n, v \rangle_{G_n} \quad \text{for all } v \in W_0^{1,2}(G_n).$$

Because of Theorem 6.3, there exists exactly one  $u_n \in W_0^{1,2,\omega}(G_n)$  which satisfies (7.8), and

$$\|u_n\|_{W_0^{1,2,\omega}(G_n)} \leq c \|\phi_n\|_{W^{-1,2,\omega}(G_n)},$$

where the constant  $c > 0$  depends only on  $\varepsilon$  and  $\omega$ . Next to the last variational equation of the system equivalent to (7.7), for all  $v \in W_0^{1,2}(G_{n-1})$  we obtain

$$(7.9) \quad \int_{\Omega} ((A_{n-1n-1} \nabla u_{n-1} + A_{n-1n} \nabla u_n) \cdot \nabla v + u_{n-1} v) \, d\lambda^N = \langle \phi_{n-1}, v \rangle_{G_{n-1}}.$$

Because of Theorem 6.3 again, there exists exactly one  $u_{n-1} \in W_0^{1,2,\omega}(G_{n-1})$  such that  $u_n$  and  $u_{n-1}$  satisfy (7.8), (7.9) and

$$\|u_{n-1}\|_{W_0^{1,2,\omega}(G_{n-1})} \leq c (\|\phi_n\|_{W^{-1,2,\omega}(G_n)} + \|\phi_{n-1}\|_{W^{-1,2,\omega}(G_{n-1})})$$

with a new constant  $c > 0$ , which depends only on  $\varepsilon$  and  $\omega$  again. Here we used assumption (7.3). Continuing this procedure we get finally the claim of the first step.

*Step 2.* To prove assertion (i) of the theorem, take  $\omega < \min \{\bar{\omega}(\varepsilon, G_i) : 1 \leq i \leq n\}$ ,  $f_i \in \mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)$ ,  $g_i \in \mathfrak{L}^{2,\omega-2}(\Omega)$  and  $h_i \in \mathfrak{L}^{2,\omega-1}(\Gamma_i)$ , suppose (7.2), (7.3) and (7.4) (with the constant  $\delta$  from the first step), and let  $u$  be a solution to (7.5). Then

$$L(A, 0, 0, I_n, 0)u = F(f - bu, g - c \cdot \nabla u - (d - I_n)u, h - eu).$$

(For  $u \in W^{1,2}(\Omega)^n$ , we denote by  $bu$ ,  $c \cdot \nabla u$ ,  $du$  and  $eu$  the elements of  $L^2(\Omega; \mathbb{R}^N)^n$ ,  $L^2(\Omega)^n$  and  $L^2(\Gamma_1) \times \cdots \times L^2(\Gamma_n)$  with components  $b_{ik}u_k$ ,  $c_{ik} \cdot \nabla u_k$ ,  $d_{ik}u_k$  and  $e_{ik}u_k$ , respectively.) Now we proceed as in the proof of Theorem 6.3 (i) (cf. (6.1)). Using the isomorphism property from Step 1, we get the desired result.

*Step 3.* Finally, let us prove assertion (ii) of the theorem. We have to show that  $L(A, b, c, d, e)$  is a Fredholm operator from  $W_0^{1,2,\omega}(G_1) \times \cdots \times W_0^{1,2,\omega}(G_n)$  into  $W^{-1,2,\omega}(G_1) \times \cdots \times W^{-1,2,\omega}(G_n)$ . For  $\omega = 0$  this is true, because we have

$$L(A, b, c, d, e) = L(A, 0, 0, I_n, 0) + L(0, b, c, d - I_n, e),$$

$L(A, 0, 0, I_n, 0)$  is an isomorphism from  $W_0^{1,2}(G_1) \times \cdots \times W_0^{1,2}(G_n)$  onto the space  $W^{-1,2}(G_1) \times \cdots \times W^{-1,2}(G_n)$  (Step 1), and  $L(0, b, c, d - I_n, e)$  is completely continuous from  $W_0^{1,2}(G_1) \times \cdots \times W_0^{1,2}(G_n)$  into  $W^{-1,2}(G_1) \times \cdots \times W^{-1,2}(G_n)$  (because of the completely continuous embeddings  $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$  and  $W_0^{1,2}(G_k) \hookrightarrow L^2(\partial_N G_k)$ ).

Now we can proceed as in the proof of Theorem 6.3 (ii), using the assertion (i) just proved in Step 2. Here, we have to use the fact that the adjoint to the operator  $L(A, b, c, d, e)$  is an operator of the type  $L(A_0, b_0, c_0, d_0, e_0)$ , where (because of (7.4))

$$\|A_{0ik}\|_{L^\infty(\Omega; \mathfrak{M}_N)} \leq \delta \quad \text{for } i < k,$$

and that for such close to triangular operators the assertion (i) holds true, too.  $\square$

**Remark 7.2.** In this section we did not suppose any ellipticity condition apart from (7.2). Especially, we did not assume the differential operator in (7.1) to be *strongly elliptic*, i.e. we did not suppose that there exists a constant  $c > 0$  such that

$$(A_{ik}(x)\xi \cdot \xi)v_i v_k \geq c|\xi|^2|v|^2$$

for all  $\xi \in \mathbb{R}^N$  and all  $v = (v_1, \dots, v_n) \in \mathbb{R}^n$  and  $\lambda^N$ -almost all  $x \in \Omega$ .

If the constant  $\delta$  in (7.4) is sufficiently small, then this differential operator is elliptic in the sense of AGMON, DOUGLAS, NIRENBERG [1] (see also [33, Chapter 9.2]), i.e.

$$(7.10) \quad \det[A_{ik}(x)\xi \cdot \xi]_{i,k=1}^n \neq 0$$

for all  $\xi \in \mathbb{R}^N \setminus \{0\}$  and  $\lambda^N$ -almost all  $x \in \Omega$ . Moreover, in this case the differential operator is even *normally elliptic* in the sense of AMANN [2], i.e. all eigenvalues of the matrix in (7.10) have positive real parts (because this matrix is close to a triangular one with positive diagonal elements).

## References

- [1] S. AGMON, A. DOUGLAS, L. NIRENBERG, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II, *Comm. Pure Appl. Math.* **17** (1964), 35–92.
- [2] H. AMANN, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in: *Function Spaces, Differential Operators and Nonlinear Analysis* (ed. by H.-J. SCHMEISSER, H. TRIEBEL), Teubner-Texte zur Math. **133**, Teubner Verlag, Stuttgart, 1993, 9–126.
- [3] J. APPELL, P. P. ZABREJKO, *Nonlinear Superposition Operators*, Cambridge University Press, Cambridge, 1993.
- [4] S. CAMPANATO, Proprietà di inclusione per spazi di Morrey, *Ricerche Mat.* **12** (1963), 67–86.
- [5] S. CAMPANATO, Proprietà di hölderianità di alcune classi di funzioni, *Ann. Scuola Norm. Sup. Pisa* **17** (1963), 175–188.
- [6] S. CAMPANATO, Equazioni ellittiche del II° ordine e spazi  $\mathcal{L}^{(2,\lambda)}$ , *Ann. Mat. Pura Appl.* **69** (1965), 321–382.
- [7] S. CAMPANATO, Sistemi ellittici in forma divergenza. Regolarità all'interno, *Quaderni della Scuola Norm. Sup. Pisa*, Pisa, 1980.
- [8] YA-ZHE CHEN, LAN-CHENG WU, *Second Order Elliptic Equations and Elliptic Systems*, *Translations of Mathematical Monographs* **174**, Providence, 1998 (Chinese Original: Beijing, 1991).
- [9] G. DI FAZIO, On Dirichlet problem in Morrey spaces, *Differential Integral Equations* **6** (1993), 383–391.
- [10] L. C. EVANS, R. F. GARIEPY, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.
- [11] L. E. FRAENKEL, On regularity of the boundary in the theory of Sobolev spaces, *Proc. London Math. Soc.* **39** (1979), 385–427.
- [12] M. GIAQUINTA, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, *Annals of Mathematics, Studies* **105**, Princeton University Press, Princeton, 1983.
- [13] M. GIAQUINTA, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, *Lectures in Mathematics ETH Zürich*, Birkhäuser Verlag, Basel, 1993.
- [14] D. GILBARG, N.S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, *Grundlehren der mathematischen Wissenschaften* **224**, Springer-Verlag, New York, 1983.
- [15] P. GRISVARD, *Elliptic Problems in Nonsmooth Domains*, *Monographs and Studies in Mathematics* **24**, Pitman, Boston, 1985.
- [16] J. A. GRIEPENTROG, An application of the Implicit Function Theorem to an energy model of the semiconductor theory, *Z. Angew. Math. Mech.* **79** (1999), 43–51.
- [17] J. A. GRIEPENTROG, Linear elliptic boundary value problems with nonsmooth data: Campanato spaces of functionals, *Math. Nachr.* **243** (2002), 19–42.
- [18] K. GRÖGER, A  $W^{1,p}$ -estimate for solutions to mixed boundary problems for second order elliptic differential equations, *Math. Ann.* **283** (1989), 679–687.
- [19] K. GRÖGER, L. RECKE, Preduals of Campanato spaces and Sobolev–Campanato spaces: A general construction, *Math. Nachr.* **230** (2001), 45–72.
- [20] A. KUFNER, O. JOHN, S. FUČIK, *Function Spaces*, Academia, Prague, 1977.
- [21] CH. B. MORREY JR., *Multiple Integrals in the Calculus of Variations*, *Grundlehren der mathematischen Wissenschaften* **130**, Springer-Verlag, New York, 1966.
- [22] M.K. MURTHY, G. STAMPACCHIA, A variational inequality with mixed boundary value conditions, *Israel. J. Math.* **13** (1972), 188–224.
- [23] J. NEČAS, *Introduction to the Theory of Nonlinear Elliptic Equations*, Teubner-Text zur Mathematik **52**, Teubner-Verlagsgesellschaft, Leipzig, 1983.

- [24] M. A. RAGUSA, Regularity for weak solutions to the Dirichlet problem in Morrey space, Riv. Mat. Univ. Parma **3** (1994), 355–369.
- [25] J.-M. RAKOTOSON, Equivalence between the growth of  $\int_{B(x,r)} |\nabla u|^p dy$  and  $T$  in the equation  $P[u] = T$ , J. Differential Equations **86** (1990), 102–122.
- [26] J.-M. RAKOTOSON, Quasilinear equations and spaces of Campanato–Morrey type, Comm. Partial Differential Equations **16** (1991), 1155–1182.
- [27] L. RECKE, Applications of the Implicit Function Theorem to quasi-linear elliptic boundary value problems with non-smooth data, Comm. Partial Differential Equations **20** (1995), 1457–1479.
- [28] L. RECKE, Solvability properties of linear elliptic boundary value problems with non-smooth data, Preprint **94-3** (1994), Fachbereich Mathematik der Humboldt-Universität zu Berlin.
- [29] TH. RUNST AND W. SICKEL, Sobolev Spaces of Fractal Order, Nemycki Operators and Nonlinear Partial Differential Equations, De Gruyter Series, Nonlinear Anal. Appl. **3**, De Gruyter, Berlin, 1996.
- [30] G. STAMPACCHIA, Problemi al contorno ellitici, con dati discontinui, dotati di soluzioni hölderiane, Ann. Mat. Pura Appl. **51** (1960), 1–38.
- [31] G. M. TROIANIELLO, Elliptic Differential Equations and Obstacle Problems, Plenum Press, New York, London, 1987.
- [32] T. VALENT, Boundary Value Problems of Finite Elasticity: Local Theorems of Existence, Uniqueness and Analytic Dependence on Data, Springer Tracts in Natural Philosophy **31**, Springer Verlag, New York, 1988.
- [33] J. T. WLOKA, B. ROWLEY, B. LAWRUK, Boundary Value Problems for Elliptic Systems, Cambridge University Press, Cambridge, 1995.
- [34] H. XIE,  $\mathfrak{L}^{2,\mu}(\Omega)$  estimate to the mixed boundary value problem for second order elliptic equations and its application in the thermistor problem, Nonlinear Anal. **24** (1995), 9–27.

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