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Interpolation for Function Spaces Related to Mixed Boundary Value Problems

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Dedicated to Helga Rothkirch

Abstract. Interpolation theorems are proved for Sobolev spaces of functions on nonsmooth domains with vanishing trace on a part of the boundary.

1. Introduction

During the last years the negatively indexed Sobolev spaces $W^{-1,p}$ proved to be an adequate class for the study of reaction–diffusion equations in nonsmooth situations, see e.g. [14], [6] or [5], as occuring in the mathematical modeling of semiconductor devices and their manufacturing processes. This is due to the fact that in the $W^{-1,p}$ spaces one can cope with jumping coefficients of the differential operators and mixed boundary conditions under weak assumptions on the spatial domain Ω and the Neumann part Γ of its boundary. The concept of regular sets $\Omega \cup \Gamma$, see Gröger [11], turned out to be a powerful tool in the definition of function spaces which are appropriate for the treatment of mixed boundary value problems related to reaction–diffusion equations. This is due to regularity results for the corresponding second order elliptic operators, see [11], [9], [13], [8]. In dealing with evolution equations (but not only in this context) it is desirable to have interpolation results between the function space serving as the domain of the corresponding elliptic operator and the range space, see e.g. [1], [2] and [16] and the references cited there. In the case of smooth boundary operators these things are well elaborated, see [20], [18], [15], but for mixed boundary conditions nothing seems to be established. It is the aim of this paper to investigate the interpolation between spaces of Bessel potentials $H^{s,p}$ including boundary behaviour. The sets $\Omega \subset \mathbb{R}^d$ under consideration are locally defined via Lipschitz diffeomorphisms to so called standard sets, which allow to transform only spaces with derivative order from the interval $s \in [-1, 1]$ and integrability exponent $1 < p < \infty$. Thus, one can only expect results for this subclass.

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2. Notations, Definitions, Preliminaries

2.1. Spatial domain

In the sequel Ω will always be a bounded domain in $\mathbb{R}^d$ and Γ a part of its boundary $\partial\Omega$. The open unit ball in $\mathbb{R}^d$ we denote by B and the halfball $\{y \in B : y_1 < 0\}$ by B^-. For the equatorial plate $\{y \in B : y_1 = 0\}$ we use the symbol Γ_0 and for its half $\{y \in \Gamma_0 : y_2 > 0\}$ the symbol Γ_0^+. Throughout this paper we make the following assumption:

Assumption 2.1. *For every point $x \in \partial\Omega$ there exist two open sets $U, V \subset \mathbb{R}^d$ and a bi–Lipschitz transformation Φ from U onto V such that $x \in U$, and $\Phi(U \cap (\Omega \cup \Gamma))$ coincides with one of the two model sets B and $B^- \cup \Gamma_0$.*

For some of our results we shall replace Assumption 2.1 by the following slightly more restrictive

Assumption 2.2. *For every point $x \in \partial\Omega$ there exist two open sets $U, V \subset \mathbb{R}^d$ and a bi–Lipschitz transformation Φ from U onto V with a. e. constant absolute value of the functional determinant such that $x \in U$, and $\Phi(U \cap (\Omega \cup \Gamma))$ coincides with one of the two model sets B and $B^- \cup \Gamma_0$.*

Remark 2.3. The class of sets mentioned in Assumption 2.1 is in fact the class of regular sets in the sense of Gröger, see [11], [12], [7]. The subclass of regular sets characterized by Assumption 2.2 still contains all Lipschitz domains, see [21, Ch. 1.2, Thm. 2.5], and it is broad enough to cover the domains arising from the applications we have in mind, see §1. For certain technical reasons we do not employ the model sets from Gröger's concept in this work, namely

$$E_1 = B^-, \qquad E_2 = B^- \cup \Gamma_0, \qquad E_3 = B^- \cup \Gamma_0^+.$$

In fact, the two concepts of model sets are equivalent, as follows from our first theorem (for a proof see [10]).

Theorem 2.4. *There are bi–Lipschitzian transformations Ψ_d and Ψ_d^+ from $\mathbb{R}^d$ onto itself, such that Ψ_d maps the open unit ball B onto the open unit halfball B^- and Ψ_d^+ maps the set $B^- \cup \Gamma_0^+$ onto $B^- \cup \Gamma_0$. Additionally, the mappings Ψ_d and Ψ_d^+ may be taken such that their functional determinant is constant.*

2.2. Function spaces

Throughout the remaining part of the paper $1 < p < \infty$ denotes any number and p' its conjugate defined by $1/p + 1/p' = 1$.

Definition 2.5. (See e.g. [20, Ch. 2.3.3, Ch. 4.2.1] or [19].) For $s \geq 0$ we denote by $H^{s,p}(\Omega)$ the space of complex valued functions which are restrictions to Ω of functions from the space

$$H^{s,p}(\mathbb{R}^d) \overset{\text{def}}{=} \Big\{ \psi \in L^p(\mathbb{R}^d) : \big\| F^{-1}\big((1+|\cdot|_d^2)^{s/2}\, F\psi\big)\big\|_{L^p(\mathbb{R}^d)} < \infty \Big\},$$

where F denotes the Fourier transform and $|\cdot|_d$ the Euclidean norm in $\mathbb{R}^d$. The norm in $H^{s,p}(\Omega)$ is given by the expression

$$\|\psi\|_{H^{s,p}(\Omega)} \overset{\text{def}}{=} \inf \Big\{ \|\varphi\|_{H^{s,p}(\mathbb{R}^d)} : \varphi \in H^{s,p}(\mathbb{R}^d),\ \varphi|_\Omega = \psi \Big\}.$$

Remark 2.6. The space $H^{1,p}(\Omega)$ coincides with the usual Sobolev space

$$W^{1,p}(\Omega) \overset{\text{def}}{=} \Big\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_l} \in L^p(\Omega),\ l \in \{1,\dots,d\} \Big\},$$

see [20, Ch. 4.2.1]. Any domain satisfying Assumption 2.1 is an extension domain for $L^p(\Omega)$ as well as for $W^{1,p}(\Omega)$, see Remark 2.3 and [4, Ch. 3.4, Thm. 3.10]. If Ω is an extension domain simultaneously for both, $L^p(\Omega)$ and $W^{1,p}(\Omega)$, then for $s \in [0,1]$ we could have defined $H^{s,p}(\Omega)$ also as the complex interpolation space $[L^p(\Omega), W^{1,p}(\Omega)]_s$. This follows from the Retraction–Coretraction Theorem, see [20, Ch. 1.2.4], where the restriction operator to Ω acts as retraction and the extension operator to $\mathbb{R}^d$ acts as the coretraction operator.

First we notice that the spaces $H^{s,p}(\Omega)$ allow bi–Lipschitz transformations from one domain to another one.

Theorem 2.7. *Let Ω_1 and Ω_2 be two open subsets of $\mathbb{R}^d$. If Φ is a bi–Lipschitz transformation from Ω_1 onto Ω_2, then, for every $s \in [0,1]$, the mapping*

$$u \longmapsto u \circ \Phi^{-1}, \quad u \in H^{s,p}(\Omega_1),$$

is a topological isomorphism from $H^{s,p}(\Omega_1)$ onto $H^{s,p}(\Omega_2)$.

Proof. For $s = 0$ the claim directly follows from the change of variables formula under Lipschitz transformations, see [3, Ch. 3.3.3, Thm.2]. The case $s = 1$ is proved in [17, Ch. 2.3.3.1, Lem. 3.2]. The remaining cases are obtained via interpolation, see Remark 2.6. □

Definition 2.8. For $s \in [0,1]$ we define $H^{s,p}_\Gamma(\Omega)$ as the closure in $H^{s,p}(\Omega)$ of the set

$$C_c^\infty(\Omega \cup \Gamma) \overset{\text{def}}{=} \Big\{ u|_\Omega \,:\, u \in C_c^\infty(\mathbb{R}^d),\ \mathrm{supp}(u) \cap (\partial\Omega \setminus \Gamma) = \varnothing \Big\}$$

and $H^{-s,p}_\Gamma(\Omega)$ as the dual of $H^{s,p'}_\Gamma(\Omega)$. If Γ is empty and $s \in [-1,1]$, then we write $H^{s,p}_0(\Omega)$ instead of $H^{s,p}_\Gamma(\Omega)$. If Ω is the unit ball B, then we abbreviate $H^{s,p}(B)$ by $H^{s,p}$, $H^{s,p}_0(B)$ by $H^{s,p}_0$, and $L^p(B)$ by L^p.

Remark 2.9. From Assumption 2.1 follows that Γ is relatively open in $\partial\Omega$. Referring to mixed boundary value problems we can identify Γ with the Neumann and $\partial\Omega \setminus \Gamma$ with the Dirichlet part of the boundary $\partial\Omega$.

Theorem 2.10. *For $i \in \{1,2\}$ let $\Omega_i \subset \mathbb{R}^d$ be a bounded domain and Γ_i a part of its boundary such that Assumption 2.1 is satisfied. Moreover, let $\Phi : \Omega_1 \cup \Gamma_1 \longrightarrow \Omega_2 \cup \Gamma_2$ be a bi–Lipschitz transformation. Then, for every $s \in [0,1]$, the mapping*

$$u \longmapsto u \circ \Phi^{-1}, \quad u \in H^{s,p}_{\Gamma_1}(\Omega_1),$$

is a topological isomorphism from $H^{s,p}_{\Gamma_1}(\Omega_1)$ onto $H^{s,p}_{\Gamma_2}(\Omega_2)$.

Proof. If $v \in H^{s,p}_{\Gamma_2}(\Omega_2)$, then there exists a sequence $\{v_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ with

$$\operatorname{supp}(v_j) \cap (\partial\Omega_2 \setminus \Gamma_2) = \varnothing \quad \text{and} \quad \lim_{j\to\infty} \|v - v_j|_{\Omega_2}\|_{H^{s,p}(\Omega_2)} = 0.$$

Theorem 2.7 implies

$$v \circ \Phi,\, v_j \circ \Phi \in H^{s,p}(\Omega_1) \quad \text{and} \quad \lim_{j\to\infty} \|v\circ\Phi - v_j\circ\Phi\|_{H^{s,p}(\Omega_1)} = 0. \tag{2.1}$$

Moreover,

$$\operatorname{supp}(v_j \circ \Phi) \cap (\partial\Omega_1 \setminus \Gamma_1) = \Phi^{-1}\big(\operatorname{supp}(v_j) \cap (\partial\Omega_2 \setminus \Gamma_2)\big) = \varnothing.$$

Because $\partial\Omega_1 \setminus \Gamma_1$ and $\operatorname{supp}(v_j \circ \Phi)$ are compact sets, there must be a positive distance between them, and we denote

$$\delta_j := \frac{1}{4} \operatorname{dist}\big(\partial\Omega_1 \setminus \Gamma_1, \operatorname{supp}(v_j \circ \Phi)\big).$$

Now, $v_j \circ \Phi$ can be extended to a function $u_j \in W^{1,p}(\mathbb{R}^d)$. Applying the convolution with suitable mollifiers to u_j we construct functions $w_{ij} \in C_c^\infty(\mathbb{R}^d)$ fulfilling

$$\operatorname{supp}(w_{ij}) \cap (\partial\Omega_1 \setminus \Gamma_1) = \varnothing \quad \text{for all } i,\, j \in \mathbb{N},\, i > 1/\delta_j$$

and

$$\lim_{i\to\infty} \|v_j \circ \Phi - w_{ij}|_{\Omega_1}\|_{W^{1,p}(\Omega_1)} = 0, \tag{2.2}$$

see [17, Ch. 2.2.2.1, Thm. 2.1]. Relation (2.2) implies that

$$\lim_{i\to\infty} \|v_j \circ \Phi - w_{ij}|_{\Omega_1}\|_{H^{s,p}(\Omega_1)} = 0.$$

Therefore, we have $v_j \circ \Phi \in H^{s,p}_{\Gamma_1}(\Omega_1)$ for all $j \in \mathbb{N}$. Because $H^{s,p}_{\Gamma_1}(\Omega_1)$ is a closed subspace of $H^{s,p}(\Omega_1)$, (2.1) yields $v \circ \Phi \in H^{s,p}_{\Gamma_1}(\Omega_1)$.

Analogously, it follows that $u \circ \Phi^{-1} \in H^{s,p}_{\Gamma_2}(\Omega_2)$, if $u \in H^{s,p}_{\Gamma_1}(\Omega_1)$. □

Remark 2.11. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and Γ a part of its boundary such that Assumption 2.1 is satisfied. By localization and bi–Lipschitz transformation according to Assumption 2.1 one verifies that $H^{s,p}_0(\Omega) = H^{s,p}_\Gamma(\Omega) = H^{s,p}(\Omega)$, if $0 \le s \le 1/p$, because this is true for the special case $(\Omega, \Gamma) = (B, \varnothing)$ (unit ball with pure Dirichlet boundary) [20, Ch. 4.3.2, Thm. 1].

2.3. Retractions and coretractions

The interpolation between the spaces $H_\Gamma^{s_1,p}(\Omega)$ and $H_\Gamma^{s_2,p}(\Omega)$ can be obtained from the special case $(\Omega,\Gamma) = (B,\varnothing)$ (unit ball with pure Dirichlet boundary). The idea, which enables such a reduction is to define an adequate retraction–coretraction to an n–fold Cartesian product of spaces on the unit ball with homogeneous Dirichlet boundary conditions.

Definition 2.12. We define the continuous operators

$$P : L^{p'} \longrightarrow L^{p'}(B^-) \quad \text{and} \quad Q : L^{p'}(B^-) \longrightarrow L^{p'}$$

setting

$$(Pv)(y) \overset{\text{def}}{=} \frac{1}{2}(v(y) + v(\sigma y)), \qquad y \in B^-,$$

and

$$(Qv)(y) \overset{\text{def}}{=} v(y_-), \qquad y \in B;$$

here and in the sequel $\sigma y \overset{\text{def}}{=} (-y_1, y_2, \ldots, y_d)$ and $y_- \overset{\text{def}}{=} (-|y_1|, y_2, \ldots, y_d)$ for $y = (y_1, \ldots, y_d)$.

Note that P maps $H^{1,p'}$ $\big(\text{resp. } H_0^{1,p'}\big)$ continuously into $H^{1,p'}(B^-)$ $\big(\text{resp. } H_{\Gamma_0}^{1,p'}(B^-)\big)$ and that Q maps $H^{1,p'}(B^-)$ $\big(\text{resp. } H_{\Gamma_0}^{1,p'}(B^-)\big)$ continuously into $H^{1,p'}$ $\big(\text{resp. } H_0^{1,p'}\big)$. As a consequence, for every $s \in [0,1]$, P maps $H^{s,p'}$ continuously into $H^{s,p'}(B^-)$ and Q maps $H^{s,p'}(B^-)$ continuously into $H^{s,p'}$.

Definition 2.13. For Ω and Γ we fix an open covering $U_1, \ldots, U_n$ of $\overline{\Omega}$ and bi–Lipschitz transformations

$$\begin{aligned} \Phi_k : \quad & U_k \cap (\Omega \cup \Gamma) \longrightarrow B^- \cup \Gamma_0 \qquad && \text{if } k \in \{1, \ldots, j\}, \\ \Phi_k : \quad & U_k \cap (\Omega \cup \Gamma) \longrightarrow B && \text{if } k \in \{j+1, \ldots, n\}. \end{aligned}$$

This is possible due to Assumption 2.1. We define linear continuous mappings

$$\begin{aligned} T_k : \quad & L^{p'}(B^-) \longrightarrow L^{p'}(U_k \cap \Omega) \qquad && \text{if } k \in \{1, \ldots, j\}, \\ T_k : \quad & L^{p'} \longrightarrow L^{p'}(U_k \cap \Omega) && \text{if } k \in \{j+1, \ldots, n\}, \end{aligned}$$

by

$$(T_k v)(x) \overset{\text{def}}{=} v(\Phi_k(x)), \qquad x \in \Omega \cap U_k.$$

By Theorem 2.7 the operator T_k maps $H^{1,p'}(B^-)$ $\big(\text{resp. } H^{1,p'}\big)$ continuously and isomorphically onto $H^{1,p'}(U_k \cap \Omega)$. As a consequence, for every $s \in [0,1]$, the operator T_k maps $H^{s,p'}(B^-)$ $\big(\text{resp. } H^{s,p'}\big)$ continuously and isomorphically onto $H^{s,p'}(U_k \cap \Omega)$. By Theorem 2.10 the image of $H_{\Gamma_0}^{s,p'}(B^-)$ $\big(\text{resp. } H_0^{s,p'}\big)$ under T_k is $H_{U_k \cap \Gamma}^{s,p'}(U_k \cap \Omega)$.

Definition 2.14. We fix a C^∞-partition of unity $\eta_1, \dots, \eta_n$ subordinate to the open covering $U_1, \dots, U_n$ of $\overline{\Omega}$ and define the mappings

$$R : \left[H_0^{-1,p}\right]^n \longrightarrow H_\Gamma^{-1,p}(\Omega) \quad \text{and} \quad S : H_\Gamma^{-1,p}(\Omega) \longrightarrow \left[H_0^{-1,p}\right]^n \tag{2.3a}$$

by

$$\langle Rg, u\rangle \stackrel{\text{def}}{=} \sum_{k=1}^{j} \left\langle g_k, QT_k^{-1}(\eta_k u)\right\rangle + \sum_{k=j+1}^{n} \left\langle g_k, T_k^{-1}(\eta_k u)\right\rangle, \qquad g = (g_1, \dots, g_n) \in \left[H_0^{-1,p}\right]^n, \quad u \in H_\Gamma^{1,p'}(\Omega), \tag{2.3b}$$

and

$$\langle Sf, v\rangle \stackrel{\text{def}}{=} \left\langle f, \sum_{k=1}^{j} T_k P v_k + \sum_{k=j+1}^{n} T_k v_k \right\rangle, \qquad f \in H_\Gamma^{-1,p}(\Omega), \quad v = (v_1, \dots, v_n) \in \left[H_0^{1,p'}\right]^n. \tag{2.3c}$$

In (2.3b) $\eta_k u$ is to be regarded as an element of $H^{1,p'}_{U_k \cap \Gamma}(U_k \cap \Omega)$. The functions $T_k P v_k$ and $T_k v_k$ in (2.3c) are to be interpreted in a natural way as elements of $H_\Gamma^{1,p'}(\Omega)$ (extension by zero).

Lemma 2.15. *The mappings R and S defined above have the following properties:*

(1) *RS is the identity mapping of $H_\Gamma^{-1,p}(\Omega)$.*

(2) *For each $s \in [0,1]$ the operator R maps $\left[H_0^{-s,p}\right]^n$ onto $H_\Gamma^{-s,p}(\Omega)$.*

(3) *For each $s \in [0,1]$ the operator S maps $H_\Gamma^{-s,p}(\Omega)$ into $\left[H_0^{-s,p}\right]^n$.*

Proof. The assertions follow immediately from the definitions of the operators R and S and the properties of the operators P, Q and T_k mentioned above. □

Lemma 2.16. *Let Assumption 2.2 be satisfied and let R and S be defined as before but this time by means of bi–Lipschitz transformations Φ_k with a.e. constant absolute value of the functional determinant. Then the operator R maps $\left[H_0^{1,p}\right]^n$ onto $H_\Gamma^{1,p}(\Omega)$, and the operator S maps $H_\Gamma^{1,p}(\Omega)$ into $\left[H_0^{1,p}\right]^n$.*

Proof. 1. If $g \in \left[H_0^{1,p}\right]^n$, then

$$
\begin{aligned}
&\langle Rg, u\rangle \\
&\quad = \int_B \Big\{ \sum_{k=1}^{j} g_k(y)(\eta_k u)\big(\Phi_k^{-1}(y_-)\big) + \sum_{k=j+1}^{n} g_k(y)(\eta_k u)\big(\Phi_k^{-1}(y)\big)\Big\}\,\mathrm{d}y \\
&\quad = \sum_{j=1}^{k} \int_{B^-} \big(g_k(y) + g_k(\sigma y)\big)(\eta_k u)\big(\Phi_k^{-1}(y)\big)\,\mathrm{d}y + \sum_{k=j+1}^{n} \int_B g_k(y)(\eta_k u)\big(\Phi_k^{-1}(y)\big)\,\mathrm{d}y \\
&\quad = \sum_{j=1}^{k} \int_{U_k\cap\Omega} 2(Pg_k)(\Phi_k(x))(\eta_k u)(x) d_k\,\mathrm{d}x + \sum_{k=j+1}^{n} \int_{U_k\cap\Omega} g_k(\Phi_k(x))(\eta_k u)(x) d_k\,\mathrm{d}x \\
&\quad = \int_\Omega \Big\{ \sum_{j=1}^{k} 2 d_k \eta_k(x)(T_k P g_k)(x) + \sum_{k=j+1}^{n} d_k \eta_k(x)(T_k g_k)(x)\Big\} u(x)\,\mathrm{d}x.
\end{aligned}
$$

Here d_k denotes the constant absolute value of the functional determinant of the transformation Φ_k and $\eta_k T_k P g_k$ (resp. $\eta_k T_k g_k$) is to be regarded as a function on Ω vanishing outside $U_k \cap \Omega$. Thus, the functional Rg is represented by the function

$$
\sum_{k=1}^{j} 2 d_k \eta_k T_k P g_k + \sum_{k=j+1}^{n} d_k \eta_k T_k g_k \in H_\Gamma^{1,p}(\Omega).
$$

2. If $f \in H_\Gamma^{1,p}(\Omega)$, then

$$
\begin{aligned}
&\langle Sf, v\rangle \\
&\quad = \frac{1}{2}\sum_{k=1}^{j} \int_{U_k\cap\Omega} f(x)\big(v_k(\Phi_k(x)) + v_k(\sigma\Phi_k(x))\big)\,\mathrm{d}x + \sum_{k=j+1}^{n} \int_{U_k\cap\Omega} f(x) v_k(\Phi_k(x))\,\mathrm{d}x \\
&\quad = \frac{1}{2}\sum_{k=1}^{j} \int_{B^-} f\big(\Phi_k^{-1}(y)\big)\big(v_k(y) + v_k(\sigma y)\big) d_k^{-1}\,\mathrm{d}y + \sum_{k=j+1}^{n} \int_B f\big(\Phi_k^{-1}(y)\big) v_k(y) d_k^{-1}\,\mathrm{d}y \\
&\quad = \int_B \Big\{ \frac{1}{2}\sum_{k=1}^{j} f\big(\Phi_k^{-1}(y_-)\big) v_k(y) d_k^{-1} + \sum_{k=j+1}^{n} f\big(\Phi_k^{-1}(y)\big) v_k(y) d_k^{-1}\Big\}\,\mathrm{d}y \\
&\quad = \int_B \Big\{ \frac{1}{2}\sum_{k=1}^{j} d_k^{-1}\big(Q T_k^{-1} f\big)(y) v_k(y) + \sum_{k=j+1}^{n} d_k^{-1}\big(T_k^{-1} f\big)(y) v_k(y)\Big\}\,\mathrm{d}y.
\end{aligned}
$$

Thus, the functional Sf is represented by

$$
\left(\frac{1}{2} d_1^{-1} Q T_1^{-1} f, \ldots, \frac{1}{2} d_j^{-1} Q T_j^{-1} f, d_{j+1}^{-1} T_{j+1}^{-1} f, \ldots, d_n^{-1} T_n^{-1} f\right) \in \left[H_0^{1,p}\right]^n.
$$

3. The fact that R maps $\left[H_0^{1,p}\right]^n$ *onto* $H_\Gamma^{1,p}(\Omega)$ follows from the preceding steps of the proof and the relation $RSf = f$ for $f \in H_\Gamma^{1,p}(\Omega)$ (cf. Lemma 2.15). □

3. Interpolation spaces

We are now going to formulate our interpolation results.

Theorem 3.1. *Let* $0 < s < 1$ *and* $1 < p_0, p_1 < \infty$. *Suppose that there holds* $1/p = (1-s)/p_0 + s/p_1$ *and* $s \neq 1/p'$. *Then*

$$H_\Gamma^{-s,p}(\Omega) = \left[L^{p_0}(\Omega), H_\Gamma^{-1,p_1}(\Omega)\right]_s \quad \textit{and} \quad H_\Gamma^{s,p'}(\Omega) = \left[L^{p_0'}(\Omega), H_\Gamma^{1,p_1'}(\Omega)\right]_s.$$

Proof. The second result is well known if $(\Omega, \Gamma) = (B, \varnothing)$, see [20, Ch. 4.3.2, Thm. 1 and 2]. The first result for the same case $(\Omega, \Gamma) = (B, \varnothing)$ is a consequence of the corresponding second result and the Duality Theorem for complex interpolation, see [20, Ch. 1.11.3]. The proof of the first result for the general case follows from the special case $(\Omega, \Gamma) = (B, \varnothing)$ by the Retraction–Coretraction Theorem, see [20, Ch. 1.2.4]. It is just the result of Lemma 2.15 which shows that the Retraction–Coretraction Theorem is applicable and leads to the desired result. The general case of the second assertion then follows from the first assertion by another application of the Duality Theorem for complex interpolation. □

Theorem 3.2. *Let* $s_0, s_1 \in [0, 1]$, $1 < p_0, p_1 < \infty$ *and* $s_i \neq 1/p_i'$, $i \in \{0, 1\}$. *Furthermore, suppose that there holds* $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $s = (1-\theta)s_0 + \theta s_1 \neq 1/p'$. *Then*

$$\left[H_\Gamma^{-s_0,p_0}(\Omega), H_\Gamma^{-s_1,p_1}(\Omega)\right]_\theta = H_\Gamma^{-s,p}(\Omega) \quad \textit{and} \quad \left[H_\Gamma^{s_0,p_0'}(\Omega), H_\Gamma^{s_1,p_1'}(\Omega)\right]_\theta = H_\Gamma^{s,p'}(\Omega).$$

Proof. The theorem is an immediate consequence of Theorem 3.1 and the Reiteration Theorem for complex interpolation, see [20, Ch. 1.9.3, Rem. 1]. □

In the following part of this section we shall deal with the interpolation of spaces $H_\Gamma^{s_0,p_0}(\Omega)$ and $H_\Gamma^{s_1,p_1}(\Omega)$ for $s_0, s_1 \in [-1, 1]$, $1 < p_0, p_1 < \infty$. In the sequel, we suppose that Assumption 2.2 is always satisfied.

Theorem 3.3. *Let* $1 < p_0, p_1 < \infty$, $1/p_i - 1 < s_i \leq 1$ *and* $s_i \neq 1/p_i$, $i \in \{0, 1\}$. *Moreover, suppose that* $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $s = (1-\theta)s_0 + \theta s_1 \neq 1/p$. *Then*

$$\left[H_\Gamma^{s_0,p_0}(\Omega), H_\Gamma^{s_1,p_1}(\Omega)\right]_\theta = H_\Gamma^{s,p}(\Omega). \tag{3.1}$$

Proof. 1. In case that $(\Omega, \Gamma) = (B, \varnothing)$ the assertion (3.1) follows immediately from [20, Ch. 4.3.2, Thm. 1 and 2].

2. Applying the Retraction-Coretraction Theorem (see [20, Ch. 1.2.4]), Lemma 2.15, and Lemma 2.16 to the special case $(\Omega, \Gamma) = (B, \varnothing)$ we arrive at the desired result for the general case. □

Lemma 3.4. *There holds*

$$\left[H_\Gamma^{1,p}(\Omega), H_\Gamma^{-1,p}(\Omega)\right]_{1/2} = L^p(\Omega). \tag{3.2}$$

Proof. 1. In case that $(\Omega,\Gamma) = (B,\varnothing)$ the assertion (3.2) is an easy consequence of results presented in [20, Ch. 4.3.3, Ch. 4.9.2].

2. The general case follows from the special case $(\Omega,\Gamma) = (B,\varnothing)$ by means of the Retraction-Coretraction Theorem (see [20, Ch. 1.2.4]), Lemma 2.15, and Lemma 2.16. □

Theorem 3.5. *Let* $s_0, s_1 \in [-1,1]$, $0<\theta<1$ *and* $s = (1-\theta)s_0 + \theta s_1$. *Suppose that*

$$s_0, s_1, s \notin \{1/p, -1/p'\}. \tag{3.3}$$

Then $\left[H^{s_0,p}_\Gamma(\Omega), H^{s_1,p}_\Gamma(\Omega)\right]_\theta = H^{s,p}_\Gamma(\Omega)$.

Proof. Because of Theorem 3.2 we may assume that $s_0 > 0$ and $s_1 < 0$. From Theorem 3.1 and Lemma 3.4 one obtains by means of the Reiteration Theorem, see e.g. [20, Ch. 1.9.3 Rem. 1],

$$\begin{cases} H^{s_0,p}_\Gamma(\Omega) &= \left[L^p(\Omega), H^{1,p}_\Gamma(\Omega)\right]_{s_0} \\ &= \left[\left[H^{1,p}_\Gamma(\Omega), H^{-1,p}_\Gamma(\Omega)\right]_{1/2}, H^{1,p}_\Gamma(\Omega)\right]_{s_0} \\ &= \left[H^{1,p}_\Gamma(\Omega), H^{-1,p}_\Gamma(\Omega)\right]_{(1-s_0)/2}. \end{cases} \tag{3.4}$$

Analogously one finds

$$H^{s_1,p}_\Gamma(\Omega) = \left[H^{1,p}_\Gamma(\Omega), H^{-1,p}_\Gamma(\Omega)\right]_{(1-s_1)/2}. \tag{3.5}$$

Hence, using once more the Reiteration Theorem, we get

$$\begin{aligned} &\left[H^{s_0,p}_\Gamma(\Omega), H^{s_1,p}_\Gamma(\Omega)\right]_\theta \\ &= \left[\left[H^{1,p}_\Gamma(\Omega), H^{-1,p}_\Gamma(\Omega)\right]_{(1-s_0)/2}, \left[H^{1,p}_\Gamma(\Omega), H^{-1,p}_\Gamma(\Omega)\right]_{(1-s_1)/2}\right]_\theta \\ &= \left[H^{1,p}_\Gamma(\Omega), H^{-1,p}_\Gamma(\Omega)\right]_{(1-\theta)(1-s_0)/2+\theta(1-s_1)/2} \\ &= \left[H^{1,p}_\Gamma(\Omega), H^{-1,p}_\Gamma(\Omega)\right]_{(1-s)/2} \\ &= H^{s,p}_\Gamma(\Omega). \end{aligned}$$

In the last stage we made use of (3.4) or (3.5), depending on the sign of s. The condition (3.3) has been imposed to avoid forbidden indices in the above calculations. □

Remark 3.6. Real interpolation between the $H^{s,p}$ spaces leads to the usual Besov spaces including trace conditions: one uses Lemma 3.4, an iteration formula between complex and real interpolation, see [20, Ch. 1.10.3, Thm. 2], and quite similar considerations as carried out in Theorem 3.1.

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