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DISCRETE SOBOLEV–POINCARÉ INEQUALITIES FOR VORONOI FINITE VOLUME APPROXIMATIONS*

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Abstract. We prove a discrete Sobolev–Poincaré inequality for functions with arbitrary boundary values on Voronoi finite volume meshes. We use Sobolev's integral representation and estimate weakly singular integrals in the context of finite volumes. We establish the result for star shaped polyhedral domains and generalize it to the finite union of overlapping star shaped domains. In the appendix we prove a discrete Poincaré inequality for space dimensions greater than or equal to two.

Key words. discrete Sobolev inequality, Sobolev integral representation, Voronoi finite volume mesh

AMS subject classifications. 46E35, 46E39, 31B10

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1. Introduction and notation. In this paper we study discrete Sobolev inequalities. In the continuous situation the Sobolev embedding estimates

$$\|u\|_{L^q(\Omega)} \le C_q \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega) \tag{1.1}$$

for $q \in [1, \infty)$ in two space dimensions and for $q \in [1, \frac{2n}{n-2}]$ in $n \ge 3$ space dimensions are well known [1, 10, 14].

For the finite volume discretized situation some results can be found in [3, 6]. But these estimates concern only the case of zero boundary values. The two-dimensional case for admissible finite volume meshes (see [6, Definition 9.1]) is treated in [6, Lemma 9.5]. The corresponding three-dimensional result is proved in [3, Lemma 1]. For $p \in [1, 2]$, a discrete Sobolev inequality estimating the L^{p^*}-norm ($p^* = \frac{np}{n-p}$ if $p < n$ and $p^* < \infty$ if $n = p = 2$) by the discrete $W^{1,p}$-norm is presented in [5, Proposition 2.2]. Moreover, also for the zero boundary value case and $1 \le p < \infty$, the discrete embedding of $W_0^{1,p}$ into L^q for some $q > p$, $1 \le p < \infty$ is established in [7, sect. 5]. A corresponding result for discontinuous Galerkin methods working in the spaces of piecewise polynomial functions on general meshes is obtained in [4, Theorem 6.1]. The idea there is to follow Nirenberg's proof of Sobolev embeddings. Recently in [2], in the context of discontinuous Galerkin finite element methods, broken Sobolev–Poincaré inequalities were proved. There, known classical results in $BV(\Omega)$ and in Sobolev spaces $W^{1,p}(\Omega)$, together with local norm equivalence and global estimates for the reconstruction operator, lead to the desired estimates.

According to our knowledge and to the information of authors of the cited papers concerning finite volume schemes, finding discrete versions of the Sobolev inequality (1.1) for functions with arbitrary boundary values has been an open question up to now. Only a discrete Poincaré inequality ($q = 2$) is available in [6, Lemmas 10.2, 10.3]

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and [9, Lemma 4.2]. But in both papers the second step of the proof is done only for two space dimensions.

The aim of the present paper is to prove a discrete Sobolev–Poincaré inequality for functions with nonzero boundary values on Voronoi finite volume meshes. Such results can be applied to more general boundary value problems, for instance, to problems with inhomogeneous Dirichlet, Neumann, or mixed boundary conditions. The technique used here is an adaptation of Sobolev's integral representation and of the treatment of weakly singular integrals in the context of Voronoi finite volume meshes. The Voronoi property of the mesh essentially comes into play in the proofs of the potential theoretical results, Lemmas 3.1–3.3.

The plan of the paper is as follows. In the remainder of this section we introduce our notation. In section 2 we formulate our assumptions and our main result, the discrete Sobolev–Poincaré inequality for star shaped domains (see Theorems 2.1 and 2.2 for a uniform estimate for a class of Voronoi finite volume meshes having comparable mesh quality). In section 3 we collect three potential theoretical lemmas needed in the proof of our main result, which is contained in section 4. Section 5 is devoted to the proof of the three potential theoretical lemmas. In section 6 we generalize the discrete Sobolev inequality to domains which are a finite union of overlapping star shaped domains (see Theorem 6.1). The last section contains some remarks and open questions. In the appendix we prove a discrete Poincaré inequality for space dimensions greater than or equal to two.

Let $\Omega \subset B(0,\widetilde{R}) \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, be a bounded, open, polyhedral domain, and let $\partial\Omega$ be its boundary. We work with Voronoi finite volume meshes of Ω, and our notation is basically taken from [3, 6]. Moreover, for set valued arguments we write $\operatorname{diam}(\cdot)$ for the diameter of the corresponding set. By $\operatorname{mes}(\cdot)$ and $\operatorname{mes}_d(\cdot)$ we denote the n- and d-dimensional Lebesgue measures, respectively.

A Voronoi finite volume mesh of Ω denoted by $\mathcal{M} = (\mathcal{P}, \mathcal{T}, \mathcal{E})$ is formed by a family of grid points $\mathcal{P}$ in $\bar{\Omega}$, a family $\mathcal{T}$ of Voronoi control volumes, and a family of relatively open parts of hyperplanes in $\mathbb{R}^n$ denoted by $\mathcal{E}$ (which represent the faces of the Voronoi boxes). For a Voronoi mesh we use the following notation; see Figure 1.

For each grid point x_K of the set $\mathcal{P}$ the control volume K of the Voronoi mesh belonging to the point x_K is defined by

$$K = \{x \in \Omega : |x - x_K| < |x - x_L| \quad \forall x_L \in \mathcal{P},\ x_L \neq x_K\}, \quad K \in \mathcal{T}.$$

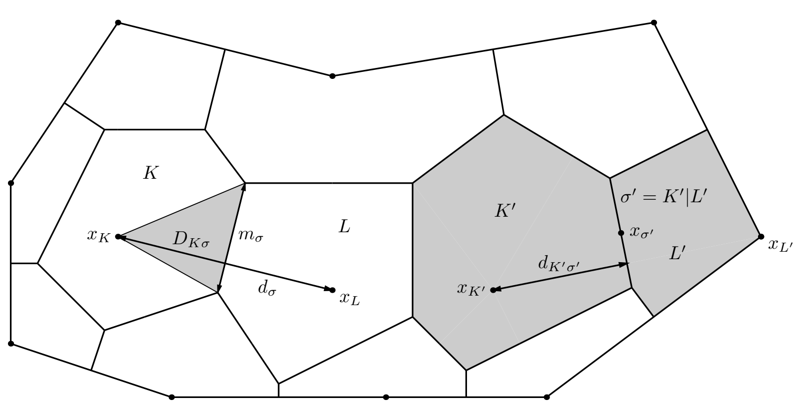

FIG. 1. *Notation of Voronoi finite volume meshes* $\mathcal{M} = (\mathcal{P}, \mathcal{T}, \mathcal{E})$.

For K, $L \in \mathcal{T}$ with $K \neq L$ either the $(n-1)$-dimensional Lebesgue measure of $\bar{K} \cap \bar{L}$ is zero or $\bar{K} \cap \bar{L} = \bar{\sigma}$ for some $\sigma \in \mathcal{E}$. In the latter case the symbol $\sigma = K|L$ denotes the Voronoi surface between K and L. We introduce the following subsets of $\mathcal{E}$. The sets of interior and external Voronoi surfaces are denoted by $\mathcal{E}_{int}$ and $\mathcal{E}_{ext}$, respectively. Additionally, for every $K \in \mathcal{T}$ we call $\mathcal{E}_K$ the subset of $\mathcal{E}$ such that $\partial K = \bar{K} \setminus K = \cup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$. Then $\mathcal{E} = \cup_{K \in \mathcal{T}} \mathcal{E}_K$.

Moreover, for $\sigma \in \mathcal{E}$ we use the following notation: m_σ represents the $(n-1)$-dimensional measure of the Voronoi surface σ, and x_σ corresponds to the coordinates of the center of gravity of σ. For $\sigma = K|L \in \mathcal{E}_{int}$ let d_σ be the Euclidean distance between x_K and x_L.

For $K \in \mathcal{T}$, $\sigma \in \mathcal{E}_K$ we define $d_{K,\sigma}$ to be the Euclidean distance between x_K and the hyperplane containing σ. Then, in the case of (isotropic) Voronoi meshes we have $d_{K,\sigma} = \frac{d_\sigma}{2}$ for $\sigma \in \mathcal{E}_{int}$.

We work with the half-diamonds $D_{K\sigma} = \{t x_K + (1-t) y : t \in (0,1),\ y \in \sigma\}$, where $n \operatorname{mes}(D_{K\sigma}) = m_\sigma d_{K,\sigma}$. Then due to our definitions,

$$n \operatorname{mes}(K) = \sum_{\sigma \in \mathcal{E}_K} m_\sigma d_{K,\sigma} \quad \forall K \in \mathcal{T}.$$

The mesh size is defined by $\operatorname{size}(\mathcal{M}) = \sup_{K \in \mathcal{T}} \operatorname{diam}(K)$.

Definition 1.1. *Let Ω be an open bounded polyhedral subset of $\mathbb{R}^n$, and let $\mathcal{M}$ be a Voronoi finite volume mesh.*

1. *$X(\mathcal{M})$ denotes the set of functions from Ω to $\mathbb{R}$ which are constant on each Voronoi box of the mesh. For $u \in X(\mathcal{M})$ the value in the Voronoi box $K \in \mathcal{T}$ is denoted by u_K.*
2. *For $u \in X(\mathcal{M})$ the discrete H^1-seminorm of u, $|u|_{1,\mathcal{M}}$, is defined by*

$$|u|^2_{1,\mathcal{M}} = \sum_{\sigma \in \mathcal{E}_{int}} \frac{m_\sigma}{d_\sigma} (D_\sigma u)^2,$$

where $D_\sigma u = |u_K - u_L|$, u_K is the value of u in the Voronoi box K, and $\sigma = K|L$.

2. Main result. First we formulate our *assumptions on the geometry and the meshes* as follows:

(A1) We assume that the open, polyhedral domain $\Omega \subset B(0, \widetilde{R}) \subset \mathbb{R}^n$ is star shaped with respect to some ball $B(0, R)$.

Let ϱ be the function $\varrho : \mathbb{R}^n \to [0,1]$ given by

$$\varrho(y) = \begin{cases} \exp\left\{ -\frac{R^2}{R^2 - |y|^2} \right\} & \text{if } |y| < R, \\ 0 & \text{if } |y| \geq R. \end{cases}$$

We introduce the piecewise constant approximations $\varrho^{\mathcal{M}} \in X(\mathcal{M})$ as

$$\varrho^{\mathcal{M}}_K(x) = \min_{y \in \bar{K}} \varrho(y) \quad \text{for } x \in K. \tag{2.1}$$

(A2) Let $\mathcal{M} = (\mathcal{P}, \mathcal{T}, \mathcal{E})$ be a Voronoi finite volume mesh of Ω with the property that $\mathcal{E}_K \cap \mathcal{E}_{ext} \neq \emptyset$ implies $x_K \in \partial\Omega$. Moreover, the local mesh size near $B(0, R)$ is assumed to be so small that there exists a constant $\varrho_0 > 0$ such that $\int_\Omega \varrho^{\mathcal{M}}(x)\, dx \geq \varrho_0$.

Let us remark that the assumption concerning ϱ_0 in (A2) can be fulfilled by the demand that for some $r \le R/4$ we suppose $\operatorname{diam}(K) < r$ for all $x_K \in \mathcal{P}$ with $x_K \in B(0,R)$. Then, for almost all $x \in B(0,r)$ we find $\varrho^{\mathcal{M}}(x) \ge \exp\{-\frac{R^2}{R^2-(2r)^2}\} \ge \exp\{-4/3\}$ and

$$\int_\Omega \varrho^{\mathcal{M}}(x)\,dx \ge \operatorname{mes}(B(0,r)) \exp\left\{-\frac{R^2}{R^2-(2r)^2}\right\},$$

which can be taken as ϱ_0 in assumption (A2).

Under assumption (A2) there exist minimal constants $\kappa_1(\mathcal{M}) > 0$, $\kappa_2(\mathcal{M}) \ge 1$ such that the geometric weights fulfill

$$0 < \operatorname{diam}(\sigma) \le \kappa_1(\mathcal{M})\, d_\sigma \quad \forall \sigma \in \mathcal{E}_{int} \tag{2.2}$$

and

$$\max_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{int}} \max_{x\in\overline{\sigma}} |x_K - x| \le \kappa_2(\mathcal{M}) \min_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{int}} d_{K,\sigma} \quad \forall x_K \in \mathcal{P}. \tag{2.3}$$

Having in mind that

$$R_{K,out} := \max_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{int}} \max_{x\in\overline{\sigma}} |x_K - x|, \quad R_{K,inn} := \min_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{int}} d_{K,\sigma}$$

are the smallest radius of a circumscribed ball of K centered at x_K and the greatest radius of a ball fully contained in K and centered at x_K, respectively, the inequality (2.3) implies that

$$R_{K,out} \le \kappa_2(\mathcal{M}) R_{K,inn}.$$

Moreover, inequality (2.3) implies that

$$\max_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{int}} |x_K - x_\sigma| \le \kappa_2(\mathcal{M}) \min_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{int}} d_{K,\sigma} \quad \forall x_K \in \mathcal{P}. \tag{2.4}$$

In this prescribed setting of a Voronoi finite volume mesh we establish the discrete Sobolev–Poincaré inequality.

THEOREM 2.1. *We assume* (A1) *and* (A2). *Let* $q \in (2,\infty)$ *for* $n = 2$ *and* $q \in (2, \frac{2n}{n-2})$ *for* $n \ge 3$, *respectively. Then there exists a constant* $c_q > 0$ *depending only on* n, q, Ω *and the constants* ϱ_0, $\kappa_1(\mathcal{M})$, *and* $\kappa_2(\mathcal{M})$ *such that*

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \le c_q(\mathcal{M})\, |u|_{1,\mathcal{M}} \quad \forall u \in X(\mathcal{M}),$$

where $m_\Omega(u) = \operatorname{mes}(\Omega)^{-1} \int_\Omega u(x)\,dx$.

We prove this theorem in section 4. After deriving the discrete Sobolev inequality for fixed meshes $\mathcal{M}$ and pointing out the dependence of the constants on the quality of the mesh $\mathcal{M}$, we generalize our result to a class of Voronoi finite volume meshes having a unified mesh quality. Namely, we additionally assume the following for the meshes:

(A3) There exist constants $\kappa_1 > 0$ and $\kappa_2 \ge 1$ such that the geometric weights fulfill $0 < \operatorname{diam}(\sigma) \le \kappa_1\, d_\sigma$ for all $\sigma \in \mathcal{E}_{int}$ and $\max_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{int}} |x_K - x_\sigma| \le \kappa_2 \min_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{int}} d_{K,\sigma}$ for all $x_K \in \mathcal{P}$.

Now we can formulate the main theorem of our paper, the discrete Sobolev inequality uniformly on a class of Voronoi finite volume meshes $\mathcal{M}$ characterized by (A2) and (A3).

THEOREM 2.2. *Let Ω be an open bounded polyhedral subset of $\mathbb{R}^n$, and let $\mathcal{M}$ be a Voronoi finite volume mesh such that additionally* (A1)–(A3) *are fulfilled. Let $q \in (2,\infty)$ for $n = 2$ and $q \in (2, \frac{2n}{n-2})$ for $n \geq 3$, respectively. Then there exists a constant $c_q > 0$ depending only on n, q, Ω and the constants in* (A1)–(A3) *such that*

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \leq c_q\, |u|_{1,\mathcal{M}} \quad \forall u \in X(\mathcal{M}).$$

COROLLARY 2.1. *The discrete Sobolev–Poincaré inequalities*

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \leq c_q\, |u|_{1,\mathcal{M}} \quad \forall u \in X(\mathcal{M})$$

for $q \in [1, 2]$ are a direct consequence of Theorem 2.2 *and Hölder's inequality.*

COROLLARY 2.2. *Let Ω be an open bounded polyhedral subset of $\mathbb{R}^n$, and let $\mathcal{M}$ be a Voronoi finite volume mesh such that additionally* (A1)–(A3) *are fulfilled. Let $q \in [1,\infty)$ for $n = 2$ and $q \in [1, \frac{2n}{n-2})$ for $n \geq 3$, respectively. Then there exists a constant $c_q > 0$ depending only on n, q, Ω and the constants in* (A1)–(A3) *such that*

$$\|u\|_{L^q(\Omega)} \leq c_q\, |u|_{1,\mathcal{M}} + \mathrm{mes}(\Omega)^{\frac{1}{q}-1} \left| \int_\Omega u\, dx \right| \quad \forall u \in X(\mathcal{M}).$$

Note that the constant c_q in Theorem 2.2 now depends on the fixed κ_1, κ_2 from assumption (A3) (instead of $\kappa_1(\mathcal{M})$, $\kappa_2(\mathcal{M})$). The dependency on ϱ_0 is of the same quality as in Theorem 2.1. Due to Hölder's inequality, the constant c_q for $q \in [1, 2]$ in Corollary 2.1 can be taken as

$$c_q = \mathrm{mes}(\Omega)^{1/q-1/\widehat{q}}\, c_{\widehat{q}},$$

where $c_{\widehat{q}}$ is the constant from Theorem 2.2 for some $\widehat{q} \in (2, \frac{2n}{n-2})$ if $n > 2$ and $\widehat{q} \in (2,\infty)$ if $n = 2$. Finally, the constant c_q in Corollary 2.2 is the same as in Theorem 2.2 if $q > 2$ and the same as in Corollary 2.1 if $q \in [1, 2]$.

3. Potential theoretical lemmas. In this section we introduce three potential theoretical lemmas which are essential for the proof of the discrete Sobolev–Poincaré inequality, Theorem 2.1. We prove these lemmas in section 5.

LEMMA 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$. We assume* (A1) *and* (A2). *Let $x_{K_0} \in \mathcal{P}$ be a fixed grid point and let $\sigma \in \mathcal{E}_{int}$ be an internal Voronoi surface with gravitational center x_σ. Then*

$$\begin{aligned} &\mathrm{mes}\big(\{x \in B(0,R) :\, [x_{K_0}, x] \cap \sigma \neq \emptyset\}\big) \\ &\leq \frac{1}{n} \max\{2,\, 4\,\kappa_1(\mathcal{M})\}^{n-1} \mathrm{diam}(\Omega)^n \frac{m_\sigma}{|x_{K_0} - x_\sigma|^{n-1}} =: A_n \frac{m_\sigma}{|x_{K_0} - x_\sigma|^{n-1}}. \end{aligned} \tag{3.1}$$

LEMMA 3.2. *Let $n \in \mathbb{N}$, $n \geq 2$. We assume* (A1) *and* (A2). *Let $q \in (2,\infty)$ for $n = 2$ and $q \in (2, \frac{2n}{n-2})$ for $n \geq 3$. Moreover, let β be as given in* (4.5). *Let $x_{K_0} \in \mathcal{P}$ be a fixed grid point. Then*

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} |x_{K_0} - x_\sigma|^{-n+2\beta}\, m_\sigma d_{K,\sigma} \leq n \max\{1+2\kappa_1(\mathcal{M}), 2\}^{n-2\beta} \frac{m_{n-1}}{2\beta} (2\widetilde{R})^{2\beta} =: B_n,$$

where m_{n-1} denotes the measure of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$.

LEMMA 3.3. *Let $n \in \mathbb{N}$, $n \geq 2$. We assume* (A1) *and* (A2). *Let $q \in (2, \infty)$ for $n = 2$ and $q \in (2, \frac{2n}{n-2})$ for $n \geq 3$. Moreover, let β be as given in* (4.5). *Let $\sigma \in \mathcal{E}_{int}$ be a fixed inner Voronoi surface, and let x_σ denote its center of gravity. Then*

$$\sum_{K_0 \in \mathcal{T}} \sum_{\sigma_0 \in \mathcal{E}_{K_0}} |x_{K_0} - x_\sigma|^{-n+q\beta} m_{\sigma_0} d_{K_0,\sigma_0}$$
$$\leq n\big(1 + \kappa_2(\mathcal{M})(1 + 2\kappa_1(\mathcal{M}))\big)^{n-q\beta} \frac{m_{n-1}}{q\beta} (2\widetilde{R})^{q\beta} =: D_n.$$

4. Proof of the discrete Sobolev–Poincaré inequality. In this section we give the proof of our main result.

Proof of Theorem 2.1. We adapt the techniques used in [15, 16] to the discretized situation using Voronoi diagrams. We establish some discrete analogue for Sobolev's integral representation (see [15, sect. 116]) and of the treatment of weakly singular integral operators (see [15, sect. 115]).

1. First, let us introduce some notation that we will need in this proof and later on: We denote by

$$[x, y] = \{(1 - s)x + sy : \ s \in [0, 1]\}$$

the line segment connecting the points $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$. Further, for $\sigma \in \mathcal{E}_{int}$ we define the function $\chi_\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \{0, 1\}$ by

$$\chi_\sigma(x, y) = \begin{cases} 1 & \text{if } x, y \in \bar{\Omega} \text{ and } [x, y] \cap \sigma \neq \emptyset, \\ 0 & \text{if } x \notin \bar{\Omega} \text{ or } y \notin \bar{\Omega} \text{ or } [x, y] \cap \sigma = \emptyset. \end{cases} \tag{4.1}$$

Finally, for $u \in X(\mathcal{M})$ and $\sigma = K|L \in \mathcal{E}_{int}$ we introduce the function $\Delta_\sigma u : \Omega \times \Omega \to \mathbb{R}$,

$$(\Delta_\sigma u)(x, y) = \begin{cases} u_L - u_K & \text{if } (1 - s)x + sy \in K \text{ and } (1 - t)x + ty \in L \\ & \text{for some } 0 \leq s < t \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Set $\mathcal{T}_0 = \{K \in \mathcal{T} : \overline{K} \subset B(0, R)\}$, and let $u \in X(\mathcal{M})$ be arbitrarily fixed. Considering $K_0 \in \mathcal{T}$ and $K' \in \mathcal{T}_0$, for almost all $x \in K'$ the intersection $[x_{K_0}, x] \cap \sigma$ consists of at most one point for every $\sigma \in \mathcal{E}_{int}$. Hence, for almost all $x \in K'$ we can substitute $u_{K_0} - m_\Omega(u)$ by the difference

$$u_{K_0} - m_\Omega(u) = (u(x) - m_\Omega(u)) - \sum_{\sigma \in \mathcal{E}_{int}} (\Delta_\sigma u)(x_{K_0}, x)\chi_\sigma(x_{K_0}, x).$$

Multiplying by $\varrho^{\mathcal{M}} \in X(\mathcal{M})$ and integrating over $x \in \Omega$ for every $K_0 \in \mathcal{T}$ we obtain

$$(u_{K_0} - m_\Omega(u)) \int_\Omega \varrho^{\mathcal{M}}(x)\, dx = \int_\Omega (u(x) - m_\Omega(u))\varrho^{\mathcal{M}}(x)\, dx$$
$$- \sum_{K' \in \mathcal{T}_0} \int_{K'} \sum_{\sigma \in \mathcal{E}_{int}} (\Delta_\sigma u)(x_{K_0}, x)\chi_\sigma(x_{K_0}, x)\varrho^{\mathcal{M}}(x)\, dx,$$

which corresponds to a discrete version of Sobolev's integral representation. According to (A2) we estimate

$$|u_{K_0} - m_\Omega(u)| \leq \frac{I_1}{\varrho_0} + \frac{I_2(K_0)}{\varrho_0}, \tag{4.2}$$

where

$$I_1 := \int_\Omega |u(x) - m_\Omega(u)| \varrho^{\mathcal{M}}(x)\, dx$$

and

$$I_2(K_0) := \sum_{K' \in \mathcal{T}_0} \int_{K'} \sum_{\sigma \in \mathcal{E}_{int}} D_\sigma u\, \chi_\sigma(x_{K_0}, x)\, \varrho^{\mathcal{M}}_{K'}\, dx,$$

remembering that $D_\sigma u = |u_K - u_L|$ for $\sigma = K|L \in \mathcal{E}_{int}$.

2. Since $|\varrho^{\mathcal{M}}(y)| < 1$ for almost all $y \in \Omega$ we find

$$I_1 \le \|\varrho^{\mathcal{M}}\|_{L^2(\Omega)} \|u - m_\Omega(u)\|_{L^2(\Omega)} \le \mathrm{mes}(\Omega)^{1/2} \|u - m_\Omega(u)\|_{L^2(\Omega)}.$$

Due to the discrete Poincaré inequality (see Theorem A.1) there is a constant $C_0 > 0$ depending only on Ω such that

$$\|u - m_\Omega(u)\|_{L^2(\Omega)} \le C_0 |u|_{1,\mathcal{M}} \quad \forall u \in X(\mathcal{M}). \tag{4.3}$$

Therefore we obtain

$$I_1 \le \mathrm{mes}(\Omega)^{1/2} C_0\, |u|_{1,\mathcal{M}}. \tag{4.4}$$

3. Now we rearrange the sums in $I_2(K_0)$. We write

$$\begin{aligned} I_2(K_0) &= \sum_{\sigma \in \mathcal{E}_{int}} D_\sigma u \sum_{K' \in \mathcal{T}_0} \int_{K'} \chi_\sigma(x_{K_0}, x)\, \varrho^{\mathcal{M}}_{K'}\, dx \\ &\le \sum_{\sigma \in \mathcal{E}_{int}} D_\sigma u \sum_{K' \in \mathcal{T}_0} \int_{K'} \chi_\sigma(x_{K_0}, x)\, dx \\ &\le \sum_{\sigma \in \mathcal{E}_{int}} D_\sigma u\ \mathrm{mes}\big(\{x \in B(0,R) : \sigma \cap [x_{K_0}, x] \ne \emptyset\}\big); \end{aligned}$$

also see Figure 2.

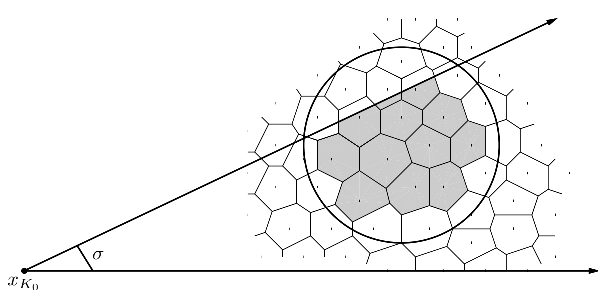

FIG. 2. *Parts of Voronoi boxes included in the ball $B(0,R)$ and shaded by the Voronoi surface σ with respect to the viewpoint x_{K_0}.*

We now use Lemma 3.1 and obtain

$$I_2(K_0) \leq A_n \sum_{\sigma\in\mathcal{E}_{int}} D_\sigma u \, \frac{m_\sigma}{|x_{K_0} - x_\sigma|^{n-1}}.$$

Let $q \in (2, \infty)$ for $n = 2$ and $q \in (2, \frac{2n}{n-2})$ for $n \geq 3$. We introduce the exponent $\beta > 0$ by

$$2\beta = \frac{n}{q} - \frac{n}{2} + 1. \tag{4.5}$$

Applying Hölder's inequality for three factors with $\alpha_1 = q$, $\alpha_2 = 2q/(q-2)$, $\alpha_3 = 2$ we find

$$\begin{aligned}
\frac{I_2(K_0)}{A_n} &\leq \sum_{\sigma\in\mathcal{E}_{int}} |D_\sigma u||x_{K_0} - x_\sigma|^{1-n} m_\sigma \\
&= \sum_{\sigma\in\mathcal{E}_{int}} \Big(|D_\sigma u|^{2/q}|x_{K_0} - x_\sigma|^{-\frac{n}{q}+\beta} d_\sigma^{-\frac{1}{q}}\Big)\Big(|D_\sigma u|^{1-2/q} d_\sigma^{\frac{2-q}{2q}}\Big)\Big(|x_{K_0} - x_\sigma|^{-\frac{n}{2}+\beta} d_\sigma^{\frac{1}{2}}\Big) m_\sigma \\
&\leq \left(\sum_{\sigma\in\mathcal{E}_{int}} |D_\sigma u|^2 |x_{K_0} - x_\sigma|^{-n+q\beta} \frac{m_\sigma}{d_\sigma}\right)^{1/q} \left(\sum_{\sigma\in\mathcal{E}_{int}} |D_\sigma u|^2 \frac{m_\sigma}{d_\sigma}\right)^{\frac{q-2}{2q}} \\
&\quad \times \left(\sum_{\sigma\in\mathcal{E}_{int}} |x_{K_0} - x_\sigma|^{-n+2\beta} \, m_\sigma d_\sigma\right)^{1/2} \\
&\leq \left(\sum_{\sigma\in\mathcal{E}_{int}} |D_\sigma u|^2 |x_{K_0} - x_\sigma|^{-n+q\beta} \frac{m_\sigma}{d_\sigma}\right)^{1/q} \left(\sum_{\sigma\in\mathcal{E}_{int}} |D_\sigma u|^2 \frac{m_\sigma}{d_\sigma}\right)^{\frac{q-2}{2q}} \\
&\quad \times \left(\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} |x_{K_0} - x_\sigma|^{-n+2\beta} \, m_\sigma d_{K,\sigma}\right)^{1/2}.
\end{aligned}$$

According to the definition of the discrete H^1-seminorm and to Lemma 3.2, we continue our estimate by

$$I_2(K_0) \leq A_n B_n^{1/2} \, |u|_{1,\mathcal{M}}^{1-2/q} \left(\sum_{\sigma\in\mathcal{E}_{int}} |D_\sigma u|^2 |x_{K_0} - x_\sigma|^{-n+q\beta} \, \frac{m_\sigma}{d_\sigma}\right)^{1/q}.$$

This estimate can be obtained for all $K_0 \in \mathcal{T}$. We consider I_2 as an element of $X(\mathcal{M})$ with value $I_2(K_0)$ in $K_0 \in \mathcal{T}$. Taking now the qth power and adding the terms for all Voronoi boxes $K_0 \in \mathcal{T}$ we get

$$\begin{aligned}
\|I_2\|^q_{L^q(\Omega)} &:= \sum_{K_0\in\mathcal{T}} \sum_{\sigma_0\in\mathcal{E}_{K_0}} I_2(K_0)^q \ \mathrm{mes}(D_{K_0\sigma_0}) \\
&\leq A_n^q B_n^{q/2} |u|^{q-2}_{1,\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{int}} |D_\sigma u|^2 \frac{m_\sigma}{d_\sigma} \sum_{K_0\in\mathcal{T}} \sum_{\sigma_0\in\mathcal{E}_{K_0}} |x_{K_0} - x_\sigma|^{-n+q\beta} \, \mathrm{mes}(D_{K_0\sigma_0}).
\end{aligned}$$

Due to Lemma 3.3 we evaluate at first the last two sums on the right-hand side and obtain

$$\begin{aligned}
\|I_2\|^q_{L^q(\Omega)} &\leq \frac{1}{n} A_n^q B_n^{q/2} |u|^{q-2}_{1,\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{int}} |D_\sigma u|^2 \frac{m_\sigma}{d_\sigma} D_n \\
&\leq \frac{1}{n} A_n^q B_n^{q/2} D_n |u|^q_{1,\mathcal{M}}.
\end{aligned} \tag{4.6}$$

4. Because of (4.2), (4.4), and (4.6) we find for $u \in X(\mathcal{M})$ that

$$\begin{aligned}\|u - m_\Omega(u)\|_{L^q(\Omega)} &\le \frac{1}{\varrho_0}\Big[\|I_1\|_{L^q(\Omega)} + \|I_2\|_{L^q(\Omega)}\Big]\\ &\le \frac{1}{\varrho_0}\,\mathrm{mes}(\Omega)^{\frac{1}{q}+\frac{1}{2}} C_0\,|u|_{1,\mathcal{M}} + \frac{A_n}{\varrho_0}\left(\frac{D_n}{n}\right)^{\frac{1}{q}} B_n^{\frac{1}{2}}\,|u|_{1,\mathcal{M}}\end{aligned}$$

with the constants ϱ_0, C_0, A_n, B_n, and D_n from (A2), (4.3), Lemma 3.1, Lemma 3.2, and Lemma 3.3. Taking into account the definition of B_n and D_n in Lemma 3.2 and Lemma 3.3, this estimate yields a constant $c_q > 0$ depending only on n, q, Ω, and the constants ϱ_0, $\kappa_1(\mathcal{M})$, and $\kappa_2(\mathcal{M})$ such that

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \le c_q\,|u|_{1,\mathcal{M}} \quad \forall u \in X(\mathcal{M}),$$

which proves the theorem. ∎

5. Proof of the potential theoretical lemmas. In the proof of Lemma 3.1 we work with the solid angle, which is related to the surface of a sphere in the same way as an ordinary angle is related to the circumference of a circle. The solid angle $\omega^\sigma_{x_{K_0}}$ of the Voronoi surface $\sigma \in \mathcal{E}_{int}$ with respect to the grid point x_{K_0} is the surface area of the projection of σ onto a sphere centered at x_{K_0}, divided by the $(n-1)$th power of the spheres radius. It can be calculated by

$$\omega^\sigma_{x_{K_0}} = \int_\sigma \frac{(x - x_{K_0}|n_\sigma)}{|x - x_{K_0}|^n}\,d\sigma, \tag{5.1}$$

where n_σ denotes the unit vector normal to σ and $(\cdot|\cdot)$ is the scalar product in $\mathbb{R}^n$. This formula results from the following consideration. Let $2\delta = \mathrm{dist}(x_{K_0}, \bar\sigma) > 0$. We denote by

$$\tilde\Omega = \big\{(1-t)x_{K_0} + ty : t \in (0,1),\ y \in \sigma \text{ with } t|y - x_{K_0}| > \delta\big\} \subset \mathbb{R}^n$$

the domain which is traced by σ, the part of the sphere with radius δ and lines passing through x_{K_0} and points of $\partial\sigma$ (see Figure 3). Let at first $n > 2$. Then $x \mapsto |x - x_{K_0}|^{2-n}$ is a harmonic function on $\tilde\Omega$. Denoting by $n(x)$ the outer unit normal at the point x, we obtain by the Gauss theorem

$$0 = \int_{\partial\tilde\Omega} \frac{\partial}{\partial n(x)} \frac{1}{|x - x_{K_0}|^{n-2}}\,d\partial\widetilde\Omega = -(n-2)\int_{\partial\tilde\Omega} \frac{(x - x_{K_0}|n(x))}{|x - x_{K_0}|^n}\,d\partial\widetilde\Omega.$$

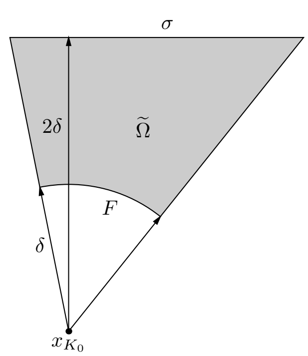

FIG. 3. *Notation for the calculation of the solid angle.*

Having in mind that $(x - x_{K_0}|n(x)) = 0$ on that part of $\partial\widetilde{\Omega}$ which is formed by the rays from x_{K_0} through $\partial\sigma$ and that $(x - x_{K_0}|n(x)) = -|x - x_{K_0}| = -\delta$ on that part F of $\partial\widetilde{\Omega}$ which belongs to the sphere with radius δ, we find that

$$\int_\sigma \frac{(x - x_{K_0}|n(x))}{|x - x_{K_0}|^n}\, d\sigma = \int_F \frac{1}{\delta^{n-1}} \mathrm{d}\partial\widetilde{\Omega} = \omega^\sigma_{x_{K_0}}.$$

If $n = 2$, we start with the harmonic function $x \mapsto \ln \frac{1}{|x - x_{K_0}|}$ on $\tilde{\Omega}$ and apply the Gauss theorem

$$0 = \int_{\partial\tilde{\Omega}} \frac{\partial}{\partial n(x)} \ln \frac{1}{|x - x_{K_0}|}\, d\partial\widetilde{\Omega} = -\int_{\partial\tilde{\Omega}} \frac{(x - x_{K_0}|n(x))}{|x - x_{K_0}|^2}\, d\partial\widetilde{\Omega}.$$

Using arguments similar to those in the higher dimensional case, we obtain (5.1), too. In both cases we find the upper estimate for the solid angle given in (5.1) by

$$\omega^\sigma_{x_{K_0}} \le \int_\sigma \frac{|x - x_{K_0}||n_\sigma|}{|x - x_{K_0}|^n}\, d\sigma \le \int_\sigma \frac{d\sigma}{|x - x_{K_0}|^{n-1}}. \tag{5.2}$$

Proof of Lemma 3.1. 1. At first we calculate the solid angle $\omega^\sigma_{x_{K_0}}$ corresponding to σ and the reference point x_{K_0}. We distinguish two cases.

Case A: $2\,\mathrm{diam}(\sigma) < |x_\sigma - x_{K_0}|$.

For all $x \in \sigma$ we find

$$|x_\sigma - x_{K_0}| \le |x - x_{K_0}| + \mathrm{diam}(\sigma);$$

therefore $|x_\sigma - x_{K_0}| - \mathrm{diam}(\sigma) \le |x - x_{K_0}|$, and in Case A we obtain

$$\frac{1}{2}|x_\sigma - x_{K_0}| \le |x - x_{K_0}| \quad \forall x \in \sigma.$$

Using (5.2) this leads to an upper estimate of the solid angle

$$\omega^\sigma_{x_{K_0}} \le \int_\sigma \frac{d\sigma}{|x - x_{K_0}|^{n-1}} \le \int_\sigma \frac{2^{n-1}\, d\sigma}{|x_\sigma - x_{K_0}|^{n-1}} \le \frac{2^{n-1} m_\sigma}{|x_\sigma - x_{K_0}|^{n-1}}.$$

Case B: $2\,\mathrm{diam}(\sigma) \ge |x_\sigma - x_{K_0}|$.

For $x \in \sigma = K|L$ we have $x \in \overline{K}$, and due to the definition of Voronoi boxes, (2.2), and the situation in Case B, we can estimate

$$|x - x_{K_0}| \ge |x - x_K| \ge d_{K,\sigma} \ge \frac{1}{2\kappa_1(\mathcal{M})}\mathrm{diam}(\sigma) \ge \frac{1}{4\kappa_1(\mathcal{M})}|x_\sigma - x_{K_0}| \quad \forall x \in \sigma.$$

According to (5.2) this yields

$$\omega^\sigma_{x_{K_0}} \le \int_\sigma \frac{d\sigma}{|x - x_{K_0}|^{n-1}} \le \int_\sigma \frac{(4\,\kappa_1(\mathcal{M}))^{n-1}\, d\sigma}{|x_\sigma - x_{K_0}|^{n-1}} \le \frac{(4\,\kappa_1(\mathcal{M}))^{n-1} m_\sigma}{|x_\sigma - x_{K_0}|^{n-1}}.$$

Therefore, in Cases A and B the solid angle $\omega^\sigma_{x_{K_0}}$ of σ with respect to the grid point x_{K_0} can be estimated by

$$\omega^\sigma_{x_{K_0}} \le \max\{2, 4\,\kappa_1(\mathcal{M})\}^{n-1} \frac{m_\sigma}{|x_\sigma - x_{K_0}|^{n-1}}. \tag{5.3}$$

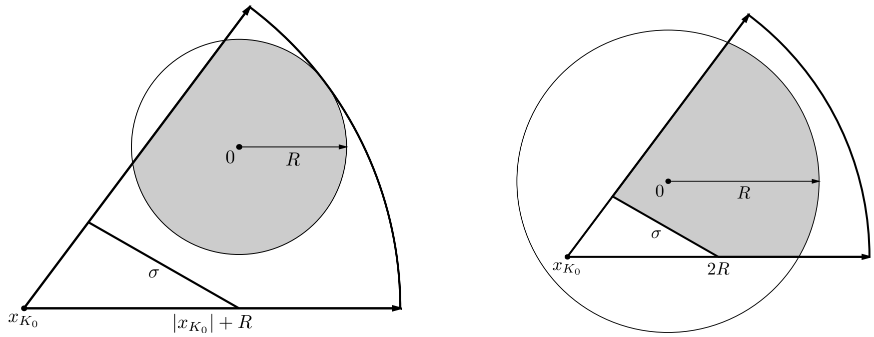

FIG. 4. *Subsets $\{x \in B(0,R) : [x_{K_0}, x] \cap \sigma \neq \emptyset\}$ of the ball $B(0,R)$ shaded by the Voronoi surface σ with respect to the viewpoint x_{K_0}.* (a) *Far-point case: $x_{K_0} \notin B(0,R)$; shaded set included in a sector with radius $|x_{K_0}| + R$.* (b) *Near-point case: $x_{K_0} \in B(0,R)$; shaded set belongs to a sector with radius $2R$.*

2. We estimate the measure of the subset

$$\{x \in B(0,R) : [x_{K_0}, x] \cap \sigma \neq \emptyset\}$$

of points, which are shaded by the beams starting from the viewpoint x_{K_0} and passing through the Voronoi surface σ. To do so, first we discuss the far-point case $x_{K_0} \notin B(0,R)$: In that case the above subset is included in the sector of the ball $B(x_{K_0}, |x_{K_0}| + R)$ with solid angle $\omega^\sigma_{x_{K_0}}$; see Figure 4. Using Fubini's theorem we obtain

$$\operatorname{mes}\big(\{x \in B(0,R) : [x_{K_0}, x] \cap \sigma \neq \emptyset\}\big) \le \int_0^{|x_{K_0}|+R} \omega^\sigma_{x_{K_0}} r^{n-1}\, dr = \frac{1}{n}\, \omega^\sigma_{x_{K_0}} (|x_{K_0}|+R)^n.$$

In the near-point case we have $x_{K_0} \in B(0,R)$. Here, the shaded subset under consideration is part of the sector of the ball $B(x_{K_0}, 2R)$ with solid angle $\omega^\sigma_{x_{K_0}}$; see Figure 4 again. By the same argument as before, we get

$$\operatorname{mes}\big(\{x \in B(0,R) : [x_{K_0}, x] \cap \sigma \neq \emptyset\}\big) \le \int_0^{2R} \omega^\sigma_{x_{K_0}} r^{n-1}\, dr = \frac{1}{n}\, \omega^\sigma_{x_{K_0}} (2R)^n.$$

In view of $|x_{K_0}| + R \le \operatorname{diam}(\Omega)$ and $2R \le \operatorname{diam}(\Omega)$, from the discussion of the two cases in step 2 and the estimate of the solid angle in step 1, we obtain the desired result. ◻

Proof of Lemma 3.2. 1. The idea is to prove that

$$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} |x_{K_0} - x_\sigma|^{-n+2\beta}\, m_\sigma d_{K,\sigma} \le c \int_\Omega |x_{K_0} - x|^{-n+2\beta}\, dx,$$

where the right-hand side is known to be finite for $\beta > 0$, which is fulfilled for $q \in (2,\infty)$ if $n = 2$ and for $q \in (2, \frac{2n}{n-2})$ if $n \ge 3$. The factor $c > 0$ in front depends on n, q, $\kappa_1(\mathcal{M})$. We estimate the integrand pointwise.

2. Let $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$ be given. Since σ belongs to the closure of the Voronoi box K, we have, due to the definition of the Voronoi boxes, that $|x_K - x_\sigma| \le |x_L - x_\sigma|$ for all $L \in \mathcal{T}$ and also for $L = K_0$. Thus

$$|x_K - x_\sigma| \le |x_{K_0} - x_\sigma|.$$

For $x \in D_{K\sigma}$ we estimate $|x_{K_0} - x|$ from above. Let x_i, $i \in I_\sigma$, denote the set of vertices of σ. Due to (2.2) and $|x_K - x_\sigma| \geq d_{K,\sigma}$, for the points x_i, $i \in I_\sigma$, and x_K we can estimate

$$\begin{aligned} |x_{K_0} - x_i| &\leq |x_{K_0} - x_\sigma| + \operatorname{diam}(\sigma) \\ &\leq |x_{K_0} - x_\sigma| + 2\kappa_1(\mathcal{M}) d_{K,\sigma} \leq (1 + 2\kappa_1(\mathcal{M}))\, |x_{K_0} - x_\sigma|, \quad i \in I_\sigma, \end{aligned}$$

$$|x_{K_0} - x_K| \leq |x_{K_0} - x_\sigma| + |x_\sigma - x_K| \leq 2|x_{K_0} - x_\sigma|.$$

Since all $x \in D_{K\sigma}$ are convex combinations of x_K, x_i, $i \in I_\sigma$, we find

$$|x_{K_0} - x| \leq \max\big\{1 + 2\kappa_1(\mathcal{M}), 2\big\} |x_{K_0} - x_\sigma| \quad \forall x \in D_{K\sigma}.$$

3. Now we derive the desired estimate. We apply the estimate from step 2,

$$\begin{aligned} &\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} |x_{K_0} - x_\sigma|^{-n+2\beta} m_\sigma d_{K,\sigma} \\ &\quad = \sum_{K\in\mathcal{T},\, \sigma\in\mathcal{E}_K,\, |x_{K_0}-x_\sigma|\geq|x_K-x_\sigma|} |x_{K_0} - x_\sigma|^{-n+2\beta}\, m_\sigma d_{K,\sigma} \\ &\quad \leq n \sum_{K\in\mathcal{T},\, \sigma\in\mathcal{E}_K,\, |x_{K_0}-x_\sigma|\geq|x_K-x_\sigma|} \max\big\{1+2\kappa_1(\mathcal{M}), 2\big\}^{n-2\beta} \int_{D_{K\sigma}} |x_{K_0} - x|^{-n+2\beta}\, dx \\ &\quad = n \max\big\{1 + 2\kappa_1(\mathcal{M}), 2\big\}^{n-2\beta} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \int_{D_{K\sigma}} |x_{K_0} - x|^{-n+2\beta}\, dx \\ &\quad \leq n \max\big\{1 + 2\kappa_1(\mathcal{M}), 2\big\}^{n-2\beta} \int_\Omega |x_{K_0} - x|^{-n+2\beta}\, dx. \end{aligned}$$

Hence, the result follows from $\int_\Omega |x_{K_0} - x|^{-n+2\beta}\, dx \leq \frac{m_{n-1}}{2\beta} (2\widetilde{R})^{2\beta}$. ∎

Proof of Lemma 3.3. 1. Similarly to the proof of Lemma 3.2, we now look for an inequality

$$\sum_{K_0\in\mathcal{T}} \sum_{\sigma_0\in\mathcal{E}_{K_0}} |x_{K_0} - x_\sigma|^{-n+q\beta}\, m_{\sigma_0} d_{K_0,\sigma_0} \leq c \int_\Omega |x - x_\sigma|^{-n+q\beta}\, dx.$$

We estimate the integrand pointwise.

2. Let $K_0 \in \mathcal{T}$ and $\sigma_0 \in \mathcal{E}_{K_0} \cap \mathcal{E}_{int}$ be given, and let the half-diamond $D_{K_0\sigma_0}$ be described by its vertices x_i, $i \in I_{\sigma_0}$, and x_{K_0}. Taking into account (2.2), (2.4), and $\operatorname{diam}(\sigma_0) \leq 2\kappa_1(\mathcal{M})|x_{\sigma_0} - x_{K_0}|$, we can estimate

$$\begin{aligned} |x_\sigma - x_i| &\leq |x_\sigma - x_{K_0}| + |x_{K_0} - x_i| \\ &\leq |x_\sigma - x_{K_0}| + |x_{K_0} - x_{\sigma_0}| + \operatorname{diam}(\sigma_0) \\ &\leq |x_\sigma - x_{K_0}| + (1 + 2\kappa_1(\mathcal{M}))|x_{\sigma_0} - x_{K_0}| \\ &\leq |x_\sigma - x_{K_0}| + \kappa_2(\mathcal{M})(1 + 2\kappa_1(\mathcal{M})) \min_{\tilde\sigma_0\in\mathcal{E}_{K_0}\cap\mathcal{E}_{int}} d_{K_0,\tilde\sigma_0}. \end{aligned}$$

Since x_σ is the gravitational center of some internal Voronoi surface, we have

$$|x_\sigma - x_{K_0}| \geq \min_{\tilde\sigma_0\in\mathcal{E}_{K_0}\cap\mathcal{E}_{int}} d_{K_0,\tilde\sigma_0}.$$

Hence, we get

$$|x_\sigma - x_i| \leq \big(1 + \kappa_2(\mathcal{M})(1 + 2\kappa_1(\mathcal{M}))\big)|x_\sigma - x_{K_0}|, \qquad i \in I_{\sigma_0}.$$

Since all $x \in D_{K_0\sigma_0}$ are convex combinations of x_i, $i \in I_{\sigma_0}$, and x_{K_0}, we obtain

$$|x_\sigma - x| \le \big(1 + \kappa_2(\mathcal{M})(1 + 2\kappa_1(\mathcal{M}))\big)|x_\sigma - x_{K_0}| \quad \forall x \in D_{K_0\sigma_0}.$$

3. Due to $n - q\beta > 0$ and the estimates in step 2, for all $K_0 \in \mathcal{T}$, $\sigma_0 \in \mathcal{E}_{K_0}$, we have

$$\frac{1}{|x_\sigma - x_{K_0}|^{n-q\beta}} \le \big(1 + \kappa_2(\mathcal{M})(1 + 2\kappa_1(\mathcal{M}))\big)^{n-q\beta} \frac{1}{|x_\sigma - x|^{n-q\beta}} \quad \forall x \in D_{K_0\sigma_0}.$$

Therefore

$$\begin{aligned}
&\sum_{K_0\in\mathcal{T}} \sum_{\sigma_0\in\mathcal{E}_{K_0}} |x_{K_0} - x_\sigma|^{-n+q\beta}\, m_{\sigma_0} d_{K_0,\sigma_0} \\
&\qquad \le n\big(1 + \kappa_2(\mathcal{M})(1 + 2\kappa_1(\mathcal{M}))\big)^{n-q\beta} \sum_{K_0\in\mathcal{T}} \sum_{\sigma_0\in\mathcal{E}_{K_0}} \int_{D_{K_0\sigma_0}} \frac{1}{|x_\sigma - x|^{n-q\beta}}\, dx \\
&\qquad = n\big(1 + \kappa_2(\mathcal{M})(1 + 2\kappa_1(\mathcal{M}))\big)^{n-q\beta} \int_\Omega \frac{1}{|x_\sigma - x|^{n-q\beta}}\, dx.
\end{aligned}$$

Because of $n - q\beta > 0$ and $\int_\Omega |x_\sigma - x|^{-n+q\beta}\, dx \le \frac{m_{n-1}}{q\beta}(2\widetilde{R})^{q\beta}$ this finishes the proof. ∎

6. Discrete Sobolev–Poincaré inequalities for more general domains. In this section we discuss how the results of Theorems 2.1 and 2.2, which hold true for star shaped domains Ω, can be used to obtain assertions for a more general situation. In the nondiscretized situation the result can be carried over to domains Ω, which are a finite union of star shaped domains Ω_i (see [15, sect. 118], [16, pp. 69–70]). In our discretized situation we assume the following:

(A4) The open, connected, polyhedral domain $\Omega \subset B(0, \widetilde{R})$ is a finite union of open, polyhedral Ω_i, $i = 1, \dots, N$, and there are $\delta > 0$, $R > 0$, and points $z^i \in \Omega$ such that Ω_i, as well as the set $\Omega_{i\delta} := \Omega_i \cup \cup_{j\neq i}\{x \in \Omega_j : \operatorname{dist}(x, \Omega_i) < \delta\}$, is star shaped with respect to the ball $B(z^i, R)$, $i = 1, \dots, N$.

We introduce the functions

$$\varrho_i : \mathbb{R}^n \to [0, 1], \quad \varrho_i(y) = \begin{cases} \exp\big\{ -\frac{R^2}{R^2 - |y - z^i|^2}\big\} & \text{if } |y - z^i| < R, \\ 0 & \text{if } |y - z^i| \ge R \end{cases}$$

and their piecewise constant approximations $\varrho_i^{\mathcal{M}} \in X(\mathcal{M})$. Concerning the mesh, we assume the following:

(A5) Let $\mathcal{M} = (\mathcal{P}, \mathcal{T}, \mathcal{E})$ be a Voronoi finite volume mesh of Ω with the property that $\mathcal{E}_K \cap \mathcal{E}_{ext} \neq \emptyset$ implies $x_K \in \partial\Omega$. Moreover, the local mesh size near $B(z^i, R)$, $i = 1, \dots, N$, is assumed to be so small that there exists a constant $\varrho_0 > 0$ such that $\int_\Omega \varrho_i^{\mathcal{M}}(x)\, dx \ge \varrho_0$, $i = 1, \dots, N$.

Then the discrete Sobolev–Poincaré inequalities remain true also for finite unions of δ-overlapping star shaped domains.

THEOREM 6.1. *We assume* (A3)–(A5). *Let $q \in (2, \infty)$ for $n = 2$ and $q \in (2, \frac{2n}{n-2})$ for $n \ge 3$, respectively. Then there exists a constant $C_q > 0$ depending only on n, q, Ω, and the constants in* (A3)–(A5) *such that*

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \le C_q\, |u|_{1,\mathcal{M}} \quad \forall u \in X(\mathcal{M}).$$

Proof. We illustrate the idea of the proof for a composition of Ω by only two star shaped subdomains Ω_1 and Ω_2. We introduce

$$\mathcal{T}_i := \{K \in \mathcal{T} : K \subset \Omega_{i\delta},\, K \cap \Omega_i \neq \emptyset\}, \quad \mathcal{T}_{i0} := \{K \in \mathcal{T} : \overline{K} \subset B(z^i, R)\}, \quad i = 1, 2,$$

and apply the estimates of steps 1 and 2 from the proof of Theorem 2.1 for each subdomain separately. For each $K_0 \in \mathcal{T}_i$, we write

$$\begin{aligned}\int_\Omega (u_{K_0} - m_\Omega(u))\varrho_i^{\mathcal{M}}(x)\,dx &= \int_\Omega (u(x) - m_\Omega(u))\varrho_i^{\mathcal{M}}(x)\,dx \\ &\quad - \sum_{K' \in \mathcal{T}_{i0}} \int_{K'} \sum_{\sigma \in \mathcal{E}_{int}} (\Delta_\sigma u)(x_{K_0}, x)\chi_\sigma(x_{K_0}, x)\varrho_i^{\mathcal{M}}(x)\,dx,\end{aligned}$$

and find

$$|u_{K_0} - m_\Omega(u)| \le \frac{I_1^i}{\varrho_0} + \frac{I_2^i(K_0)}{\varrho_0}, \quad K_0 \in \mathcal{T}_i,$$

where

$$\begin{aligned}I_1^i &:= \int_\Omega \big|u(y) - m_\Omega(u)\big|\varrho_i^{\mathcal{M}}(y)\,dy = \sum_{K' \in \mathcal{T}_{i0}} \int_{K'} \big|u(x) - m_\Omega(u)\big|\varrho_{iK'}^{\mathcal{M}}\,dx, \\ I_2^i(K_0) &:= \sum_{K' \in \mathcal{T}_{i0}} \int_{K'} \sum_{\sigma \in \mathcal{E}_{int}} D_\sigma u\, \chi_\sigma(x_{K_0}, x)\, \varrho_{iK'}^{\mathcal{M}}\,dx.\end{aligned}$$

Here, since the discrete Poincaré inequality (A.1) works on Ω,

$$I_1^i \le \operatorname{mes}(\Omega)^{1/2}\, \|u - m_\Omega(u)\|_{L^2(\Omega)} \le C_0 \operatorname{mes}(\Omega)^{1/2}\, |u|_{1,\mathcal{M}}.$$

The expression for $I_2^i(K_0)$ can be estimated according to step 3 of the proof of Theorem 2.1 by

$$I_2^i(K_0) \le A_n^i (B_n^i)^{1/2}\, |u|_{1,\mathcal{M}}^{1-2/q} \left(\sum_{\sigma \in \mathcal{E}_{int}} |D_\sigma u|^2 |x_{K_0} - x_\sigma|^{-n+q\beta}\, \frac{m_\sigma}{d_\sigma} \right)^{1/q}, \quad K_0 \in \mathcal{T}_i,$$

where the constants A_n^i, B_n^i now contain the geometric data from $\mathcal{T}_i$, $i = 1, 2$, which can be estimated from above by those of $\mathcal{T}$. Following the estimates in (4.6), we get

$$\sum_{i=1}^2 \sum_{K_0 \in \mathcal{T}_i} \sum_{\sigma_0 \in \mathcal{E}_{K_0}} I_2^i(K_0)^q m_{\sigma_0} d_{K_0,\sigma_0} \le \sum_{i=1}^2 (A_n^i)^q (B_n^i)^{q/2} D_n^i |u|_{1,\mathcal{M}}^q.$$

Note that in A_n^i, B_n^i, and D_n^i now the constants κ_1 and κ_2 (see (A3)) are used. Then estimates like in step 4 of the proof of Theorem 2.1 give the desired result. This technique can be generalized to a finite union of δ-overlapping star shaped domains. □

Since Theorem 2.2, Corollary 2.1, and Corollary 2.2 are direct consequences of Theorem 2.1 (with fixed κ_1, κ_2), these three statements also remain true for more general domains characterized by (A4).

7. Remarks and open questions. Finally, we motivate possible applications of discrete Sobolev inequalities and give an outlook to further problems and questions concerning possible generalizations of our results.

Applications of discrete Sobolev inequalities. A functional analytic tool like a discrete Sobolev–Poincaré inequality enables us to treat discretized boundary value problems similarly to the corresponding continuous boundary value problems. Especially, if the embedding constants hold true for a class of meshes, uniform results with respect to the mesh can be obtained which can be used, for instance, for convergence results, too.

We were forced to prove the discrete Sobolev–Poincaré inequality by the analytical and numerical treatment of (nonlinear) reaction-diffusion systems. For the considered nondiscretized systems the free energy decays exponentially to its equilibrium value. We introduced a discretization scheme (Voronoi finite volume in space and fully implicit in time) which has the special property that it preserves the main features of the continuous problem, namely, positivity, dissipativity, and flux conservation (see [13]). For each fixed mesh we proved the exponential decay of the discretized free energy, too (see [11]). This proof works with the finite dimensional quantities.

To obtain uniform decay rates for a class of Voronoi finite volume meshes we had to translate the quantities from the finite dimensional discretized problems into expressions of functions from $X(\mathcal{M})$ being defined on Ω and being constant on Voronoi boxes of the corresponding meshes, and we had to consider limits of such functions belonging to sequences of Voronoi finite volume meshes to find a contradiction in the indirect proof of an estimate of the free energy by the dissipation rate (see [12, Theorem 3.2]). The essential ingredient in that proof is the discrete Sobolev–Poincaré inequality, Theorem 2.2.

For the application of discrete versions of Sobolev's inequality in the case of homogeneous Dirichlet boundary conditions we refer to [7, sect. 5]. Moreover, this inequality in the discrete $W_0^{1,p}$-setting ($p \in (1,\infty)$) comes into play in the discretization of nonlinear elliptic problems of the form

$$-\mathrm{div}\ a(x, \nabla u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

on general polyhedral meshes in n space dimensions. In [8, sect. 3] it is used to obtain an estimate of the approximate solution. In the prescribed setting the Caratheodory function $a : \Omega \times \mathbb{R}^n$ fulfills, with suitable positive constants c_1, c_2, and $d \in L^{p'}(\Omega)$,

$$\begin{aligned}
&a(x,\zeta) \cdot \zeta \geq c_1 |\zeta|^p \quad \text{for almost all } x \in \Omega,\ \forall \zeta \in \mathbb{R}^n,\\
&(a(x,\zeta) - a(x,\chi)) \cdot (\zeta - \chi) > 0 \quad \text{for almost all } x \in \Omega,\ \forall \zeta \neq \chi \in \mathbb{R}^n,\\
&|a(x,\zeta)| \leq d(x) + c_2 |\zeta|^{p-1} \quad \text{for almost all } x \in \Omega,\ \forall \zeta \in \mathbb{R}^n.
\end{aligned}$$

The anisotropic setting. Let H be a positive definite symmetric $n \times n$ matrix, and let κ_3, $\kappa_4 > 0$ be the smallest and largest eigenvalue of H, that is,

$$\kappa_3 |y|^2 \leq (Hy|y) \leq \kappa_4 |y|^2 \quad \forall y \in \mathbb{R}^n.$$

Then the inverse matrix H^{-1} is positive definite and symmetric, too, and by means of the matrix H^{-1} we define the modified distance function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$d(x,y) := \sqrt{(x-y|H^{-1}(x-y))}.$$

We consider corresponding anisotropic Voronoi finite volume meshes $\mathcal{M}^a = (\mathcal{P}, \mathcal{T}^a, \mathcal{E}^a)$. Belonging to the grid points $x_K \in \mathcal{P}$, the set $\mathcal{T}^a$ of anisotropic Voronoi boxes is given by

$$K^a := \{x \in \Omega : d(x,x_K) < d(x,x_L)\ \forall x_L \in \mathcal{P},\ x_L \neq x_K\}, \quad K^a \in \mathcal{T}^a.$$

The set K^a then is traced by Voronoi surfaces $\sigma^a \in \mathcal{E}_{K^a}$. mes$(\sigma^a)$ is denoted by m_{σ^a}. Note that in this anisotropic context the face $\sigma^a = K^a|L^a$ in general is no longer perpendicular to the line $[x_K, x_L]$. But, if n_{σ^a} denotes the unit normal vector to σ^a, then Hn_{σ^a} is parallel to the line $[x_K, x_L]$. For the Euclidean distance d_{K,σ^a} of x_K to the hyperplane containing σ^a, we find that

$$d_{K,\sigma^a} = \frac{d_{\sigma^a}}{2}\left(n_{\sigma^a} \middle| \frac{Hn_{\sigma^a}}{|Hn_{\sigma^a}|} \right), \tag{7.1}$$

where $d_{\sigma^a} = |x_K - x_L|$. For the corresponding (anisotropic) half-diamonds $D_{K\sigma^a} := \{tx_K + (1-t)y,\ t \in (0,1),\ y \in \sigma^a\}$ we obtain

$$n \text{ mes}\,(D_{K\sigma^a}) = m_{\sigma^a} d_{K,\sigma^a} = m_{\sigma^a} \frac{d_{\sigma^a}}{2} \left(n_{\sigma^a} \middle| \frac{Hn_{\sigma^a}}{|Hn_{\sigma^a}|} \right). \tag{7.2}$$

Let $X(\mathcal{M}^a)$ denote the set of functions from Ω to $\mathbb{R}$ which are constant on each anisotropic Voronoi box of the mesh. For $u \in X(\mathcal{M}^a)$ the value at the box K^a is denoted by u_K again. For $u \in X(\mathcal{M}^a)$ we define a discrete (anisotropic) H^1-seminorm by

$$|u|^2_{1,\mathcal{M}^a} = \sum_{\sigma^a \in \mathcal{E}^a_{int}} \frac{m_{\sigma^a}}{d_{\sigma^a}} |Hn_{\sigma^a}| (D_{\sigma^a} u)^2, \tag{7.3}$$

where $D_{\sigma^a} u = |u_K - u_L|$ and $\sigma^a = K^a|L^a$.

For this anisotropic setting, and more general boundary conditions, one has to prove a discrete (anisotropic) Poincaré inequality using the discrete H^1-seminorm defined in (7.3) by modifying the proof of the discrete isotropic Poincaré inequality; see Theorem A.1. Namely, taking into account that in the anisotropic situation we have

$$\sum_{\sigma^a \in \mathcal{E}^a_{int}} d_{\sigma^a} c_{\sigma^a, y-x} \chi_{\sigma^a}(x,y) \le \frac{\kappa_4}{\kappa_3} \text{diam}(\Omega)$$

instead of (A.4) and $\kappa_3 \le |Hn_{\sigma^a}|$, we get the discrete anisotropic Poincaré inequality

$$\|u - m_\Omega(u)\|^2_{L^2(\Omega)} \le \frac{\kappa_4}{\kappa_3^2}\, C_0^2\, |u|^2_{1,\mathcal{M}^a} \quad \forall u \in X(\mathcal{M}^a),$$

where C_0 is the constant in the isotropic Poincaré inequality (A.1).

Moreover, one has to prove anisotropic versions of Lemmas 3.1, 3.2, and 3.3, respectively. In the proof of the anisotropic versions, in the distinction of cases we now have to use the distance function introduced by the anisotropy. For the space integration, now (7.2) has to be applied. Having in mind all these changes, an anisotropic discrete Sobolev inequality of the form

$$\|u - m_\Omega(u)\|_{L^q(\Omega)} \le \widetilde{c}_q |u|_{1,\mathcal{M}^a} \quad \forall u \in X(\mathcal{M}^a)$$

can be proved, too.

Such estimates are of interest, for example, in the treatment of finite volume discretized electro-reaction-diffusion systems, where for each species a different anisotropic mobility should be taken into account. Such problems can be found in [11, 13].

Critical exponent. For $n \geq 3$, the discrete version of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$ (the critical Sobolev exponent) cannot be obtained by the presented technique using only the Sobolev integral representation. This is exactly the same situation as for the continuous case (see [10, Chap. 7.8], [15, sect. 114–116]), [16, sect. 8].

More general finite volume meshes. Lemma 1 in [3] gives a discrete Sobolev inequality for functions with zero boundary values for finite volume meshes more general than Voronoi diagrams. There the class of admissible finite volume meshes is restricted by the demand that for some $\zeta > 0$ it has to be fulfilled that

$$d_{K,\sigma} > \zeta\, d_\sigma, \quad d_{K,\sigma} > \zeta \operatorname{diam}(K) \quad \forall \sigma \in \mathcal{E}_K \;\forall K \in \mathcal{T}. \tag{7.4}$$

In [6], Lemma 9.5 (for space dimension $n = 2$) uses only the first inequality in (7.4). It raises the question of generalizing our result of Theorem 2.1 for functions with nonzero boundary values to more general finite volume meshes.

Appendix. Discrete Poincaré inequality for functions with nonzero boundary values. The discrete Poincaré inequality for functions with nonzero boundary values can be found in [6, Lemma 10.2], [9, Lemma 4.2]. In [6] and [9] the proof is decomposed in three steps, but the second step works only for two space dimensions. We give here an alternative proof which works for higher space dimensions, too.

THEOREM A.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be open, bounded, polyhedral, and connected, and let $\Omega_1, \ldots, \Omega_r \subset \mathbb{R}^n$ be nonempty, open, convex sets with $\Omega = \cup_{i=1}^r \Omega_i$. Then there exists a constant $C_0 > 0$ depending only on $\Omega_1, \ldots, \Omega_r$ such that for all Voronoi finite volume meshes $\mathcal{M}$,*

$$\|u - m_\Omega(u)\|_{L^2(\Omega)} \leq C_0\, |u|_{1,\mathcal{M}} \quad \forall u \in X(\mathcal{M}), \tag{A.1}$$

where $m_\Omega(u) = \operatorname{mes}(\Omega)^{-1} \int_\Omega u\, dx$.

Proof. We decompose the proof into two steps. If Ω is convex itself, the proof results from step 1 alone.

1. Estimation on a nonempty, open, convex subset $\omega \subset \mathbb{R}^n$ of Ω: We show that there exists a constant $C_\Omega > 0$ such that

$$\|u - m_{\omega'}(u)\|^2_{L^2(\omega)} \leq \frac{C_\Omega}{\operatorname{mes}(\omega')} |u|^2_{1,\mathcal{M}} \qquad \forall u \in X(\mathcal{M}) \tag{A.2}$$

whenever $\omega' \subset \mathbb{R}^n$ is a measurable subset of ω with $\operatorname{mes}(\omega') > 0$. Here $m_{\omega'}(u)$ denotes the mean value of u on ω'. Because of

$$\int_\omega |u(x) - m_{\omega'}(u)|^2\, dx \leq \frac{1}{\operatorname{mes}(\omega')} \int_\omega \int_{\omega'} |u(x) - u(y)|^2\, dy\, dx$$

it suffices to prove

$$\int_\omega \int_{\omega'} |u(x) - u(y)|^2\, dy\, dx \leq C_\Omega |u|^2_{1,\mathcal{M}}.$$

Using the convexity of $\omega \subset \mathbb{R}^n$ we have

$$|u(x) - u(y)|^2 \leq \left| \sum_{\sigma \in \mathcal{E}_{int}} D_\sigma u\, \chi_\sigma(x, y) \right|^2$$

for almost all $x \in \omega$ and $y \in \omega'$, where the function χ_σ is as defined in (4.1). We apply the Cauchy–Schwarz inequality to obtain

$$|u(x) - u(y)|^2 \leq \sum_{\sigma \in \mathcal{E}_{int}} \frac{|D_\sigma u|^2}{d_\sigma c_{\sigma,y-x}} \chi_\sigma(x,y) \sum_{\sigma \in \mathcal{E}_{int}} d_\sigma c_{\sigma,y-x} \chi_\sigma(x,y) \tag{A.3}$$

for almost all $x \in \omega$ and $y \in \omega'$, where $c_{\sigma,\eta} = |(\frac{\eta}{|\eta|}|n_\sigma)|$ is defined for $\eta \in \mathbb{R}^n \setminus \{0\}$ and n_σ is a unit vector normal to $\sigma \in \mathcal{E}_{int}$. Since $x_K - x_L = \pm d_\sigma n_\sigma$ for $\sigma = K|L \in \mathcal{E}_{int}$ we find for some K^* and L^* (depending on $x \in \omega$, $y \in \omega'$) that

$$\sum_{\sigma \in \mathcal{E}_{int}} d_\sigma c_{\sigma,y-x} \chi_\sigma(x,y) = \left| \left(\frac{y-x}{|y-x|} \middle| x_{K^*} - x_{L^*} \right) \right| \leq \operatorname{diam}(\Omega). \tag{A.4}$$

Integration over $x \in \omega$ and $y \in \omega'$ in (A.3) yields

$$\int_\omega \int_{\omega'} |u(x) - u(y)|^2 \, dy \, dx \leq \operatorname{diam}(\Omega) \int_\omega \int_{\omega'} \sum_{\sigma \in \mathcal{E}_{int}} \frac{|D_\sigma u|^2}{d_\sigma c_{\sigma,y-x}} \chi_\sigma(x,y) \, dy \, dx.$$

By a change of variables, $y = x + z$, we obtain

$$\int_{\omega'} \int_\omega |u(x) - u(y)|^2 \, dx \, dy \leq \operatorname{diam}(\Omega) \int_{\mathbb{R}^n} \sum_{\sigma \in \mathcal{E}_{int}} \frac{|D_\sigma u|^2}{d_\sigma c_{\sigma,z}} \int_\omega \chi_\sigma(x, x+z) \, dx \, dz.$$

Because for all $x \in \omega$ we have $\chi_\sigma(x, x+z) = 0$ if $z \in \mathbb{R}^n$, $|z| > \operatorname{diam}(\Omega)$, and

$$\int_\omega \chi_\sigma(x, x+z) \, dx \leq m_\sigma |z| c_{\sigma,z} \quad \forall z \in \mathbb{R}^n,$$

we end up with

$$\begin{aligned} \int_{\omega'} \int_\omega |u(x) - u(y)|^2 \, dx \, dy &\leq \operatorname{diam}(\Omega)^2 \int_{B(0,\operatorname{diam}(\Omega))} \sum_{\sigma \in \mathcal{E}_{int}} \frac{|D_\sigma u|^2}{d_\sigma} m_\sigma dz \\ &\leq \operatorname{diam}(\Omega)^{n+2} \operatorname{mes}(B(0,1)) |u|^2_{1,\mathcal{M}}, \end{aligned}$$

which means that we can choose $C_\Omega = \operatorname{diam}(\Omega)^{n+2} \operatorname{mes}(B(0,1))$.

2. Estimate (A.1) for the general case: We consider the intersections $\Omega_{ij} = \Omega_i \cap \Omega_j$ for $i, j \in \{1, \ldots, r\}$ and set

$$B := \{(i,j) \in \{1, \ldots, r\}^2 : i \neq j, \, \Omega_{ij} \neq \emptyset\}.$$

Then, following (A.2) for $\omega = \Omega_i$, $\omega' = \Omega_i$, and $\omega' = \Omega_{ij}$, respectively, in step 1 we get

$$\|u - m_{\Omega_i}(u)\|^2_{L^2(\Omega_i)} \leq \frac{C_\Omega}{\operatorname{mes}(\Omega_i)} |u|^2_{1,\mathcal{M}}, \quad i = 1, \ldots, r, \tag{A.5}$$

$$\|u - m_{\Omega_{ij}}(u)\|^2_{L^2(\Omega_i)} \leq \frac{C_\Omega}{\operatorname{mes}(\Omega_{ij})} |u|^2_{1,\mathcal{M}} \quad \forall (i,j) \in B.$$

Hence, for every $(i,j) \in B$ we obtain

$$\begin{aligned}
|m_{\Omega_i}(u) - m_{\Omega_{ij}}(u)|^2 \mathrm{mes}(\Omega_i) \\
&\le 2\int_{\Omega_i} |u - m_{\Omega_i}(u)|^2\,dx + 2\int_{\Omega_i} |u - m_{\Omega_{ij}}(u)|^2\,dx \\
&\le \left(\frac{2C_\Omega}{\mathrm{mes}(\Omega_i)} + \frac{2C_\Omega}{\mathrm{mes}(\Omega_{ij})}\right)|u|^2_{1,\mathcal{M}}.
\end{aligned} \tag{A.6}$$

Since $\Omega = \cup_{i=1}^r \Omega_i$ is both connected and a finite union of bounded, open, convex sets $\Omega_1, \dots, \Omega_r$, for every pair $(i,j) \in \{1,\dots,r\}^2$ with $i \neq j$ we find some $\ell \in \mathbb{N}$, $2 \le \ell \le r$ and pairwise disjoint indices $k_1, \dots, k_\ell \in \{1,\dots,r\}$ with $k_1 = i$, $k_\ell = j$, and $(k_l, k_{l+1}) \in B$ for all $l = 1, \dots, \ell-1$. Hence, using the triangle inequality and (A.6) we can find some constant $M > 0$ depending only on $\Omega_1, \dots, \Omega_r$ such that

$$|m_{\Omega_i}(u) - m_{\Omega_j}(u)| \le M|u|_{1,\mathcal{M}} \quad \forall (i,j) \in \{1,\dots,r\}^2. \tag{A.7}$$

Introducing the averaged quantity

$$\bar{m}(u) = \frac{\sum_{j=1}^r m_{\Omega_j}(u)\,\mathrm{mes}(\Omega_j)}{\sum_{k=1}^r \mathrm{mes}(\Omega_k)}$$

we see that for every $i = 1, \dots, r$ we have

$$|\bar{m}(u) - m_{\Omega_i}(u)| \le \sum_{j=1}^r |m_{\Omega_j}(u) - m_{\Omega_i}(u)| \frac{\mathrm{mes}(\Omega_j)}{\sum_{k=1}^r \mathrm{mes}(\Omega_k)}.$$

Because of (A.7) we obtain

$$|\bar{m}(u) - m_{\Omega_i}(u)| \le M|u|_{1,\mathcal{M}}, \quad i = 1, \dots, r.$$

Together with (A.5) we find some constant $c_0 > 0$ depending only on $\Omega_1, \dots, \Omega_r$ such that

$$\|u - \bar{m}(u)\|^2_{L^2(\Omega_i)} \le c_0 |u|^2_{1,\mathcal{M}}, \quad i = 1, \dots, r.$$

Summing up, this yields

$$\|u - \bar{m}(u)\|^2_{L^2(\Omega)} \le \sum_{i=1}^r \|u - \bar{m}(u)\|^2_{L^2(\Omega_i)} \le r\,c_0\,|u|^2_{1,\mathcal{M}}.$$

Since $\alpha = m_\Omega(u) \in \mathbb{R}$ minimizes the function $\alpha \mapsto \|u - \alpha\|^2_{L^2(\Omega)}$, the assertion of the theorem follows. ∎

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