On regularity, positivity and long-time behavior of solutions to an evolution system of nonlocally interacting particles

JENS ANDRÉ GRIEPENTROG

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FÜR ANGEWANDTE ANALYSIS
UND STOCHASTIK · BERLIN

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Address:
Weierstraß-Institut für Angewandte Analysis und Stochastik
Mohrenstraße 39, D-10117 Berlin, Germany.

E-mail address:
griepent@wias-berlin.de

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Abstract

An analytical model for multicomponent systems of nonlocally interacting particles is presented. Its derivation is based on the principle of minimization of free energy under the constraint of conservation of particle number and justified by methods established in statistical mechanics. In contrast to the classical Cahn–Hilliard theory with higher order terms, the nonlocal theory leads to an evolution system of second order parabolic equations for the particle densities, weakly coupled by nonlinear and nonlocal drift terms, and state equations which involve both chemical and interaction potential differences. Applying fixed-point arguments and comparison principles we prove the existence of variational solutions in suitable Hilbert spaces for evolution systems. Moreover, using maximal regularity for nonsmooth parabolic boundary value problems in Sobolev–Morrey spaces and comparison principles, we show uniqueness, global regularity and uniform positivity of solutions under minimal assumptions on the regularity of interaction. Applying a refined version of the Łojasiewicz–Simon gradient inequality, this paves the way to the convergence of solutions to equilibrium states. We conclude our considerations with the presentation of simulation results for phase separation processes in ternary systems.

1. Introduction

Description of the physical phenomenon. There are many interesting isothermal drift-diffusion processes taking place in a closed system of nonlocally interacting particles of different type. Generally, in those multicomponent mixtures, all of the configurational changes are the result of processes that try to minimize the free energy of the whole ensemble. This free energy contains the sum of the binding energies between the particles with respect to their type and their distances. The mesoscopic scale of our model is larger than the single particle picture of quantum mechanics but smaller than the continuum mechanical limit: The averaged long-range interaction forces are explicitly described by attractive or repulsive interaction potentials. On the other hand, the short-range repelling forces are accounted for using the logarithmic distribution function of Fermi statistics reflecting the exclusion principle for particles with Fermi-type behavior. These forces are responsible for the diffusion process that enters in competition with the nonlocal drift process. Of course, also noninteracting, and, therefore, purely diffusive components are allowed to take part in the processes under consideration.

Typical applications are transport processes of electrically charged particles in solutions or semiconductor heterostructures, chemotactic aggregation of microorganisms in biological environments, and, especially, phase separation processes in alloys including the case of diffusive damage.

The system of model equations. The justification of our model relies on the methods of statistical mechanics: We assume that particles of different type jump around on a given microscopically scaled lattice following a stochastic exchange process. Here, voids are admissible types of particles, too. Exactly one particle sits on each lattice site (exclusion principle). Two particles of type k and $\ell \in \{0, 1, \dots, m\}$ change their sites x and y with a certain probability, due to diffusion and nonlocal interaction. This process tries to minimize the free energy of the particle ensemble.

To carry over these properties from the discrete microscopic scale to the continuous mesoscopic level, statistical mechanics uses the hydrodynamical limit process: The number of particles in the lattice tends to infinity. As the result, the state of the mesoscopic ensemble is described by densities of particles occupying a spatial domain $X \subset \mathbb{R}^n$ with Lipschitz boundary. Following the pioneering work of GIACOMIN, LEBOWITZ and MARRA in [13, 14] as well as of QUASTEL, REZAKHANLOU and VARADHAN in [33, 34], this limit process leads to an evolution system of $m+1$ conservation laws

$$\left.\begin{aligned} \rho_k' + \nabla \cdot j_k &= 0 && \text{in } (0,\infty) \times X, \\ \mathfrak{n} \cdot j_k &= 0 && \text{on } (0,\infty) \times \partial X, \\ \rho_k(0) &= \rho_k^\circ && \text{in } X, \end{aligned}\right\} \tag{1}$$

with particle densities ρ_0, $\rho_1, \dots, \rho_m$, initial values ρ_0°, $\rho_1^\circ, \dots, \rho_m^\circ$, and current densities j_0, $j_1, \dots, j_m$. Due to the exclusion principle, the particle densities are nonnegative, bounded by a given positive total density Σ, and they sum up to this total density Σ pointwise:

$$0 \le \rho_0, \rho_1, \dots, \rho_m \le \Sigma, \quad \sum_{k=0}^{m} \rho_k = \Sigma. \tag{2}$$

This defines the simplex of admissible states. Note that Σ is allowed to be a nonsmooth function of the space variable to model processes taking place in a fixed heterogeneous environment with a spatially varying total storage capacity Σ. Moreover, the closedness of the system (1) enforces that the sum of all current densities vanishes:

$$\sum_{k=0}^{m} j_k = 0. \tag{3}$$

Consequently, in (1), only m of the $m+1$ equations are independent of each other. Hence, it is convenient to drop out one redundant equation in (1), say the equation for the zero component, and to describe the state of the system by m-component vectors $\rho = (\rho_1, \dots, \rho_m)$, having in mind that $\rho_0 = \Sigma - \sum_{k=1}^m \rho_k$ is a function of ρ.

In our work, the free energy $F(\rho) = \Phi(\rho) + \Psi(\rho)$ of an admissible state ρ is modelled as the sum of the strongly convex chemical energy

$$\Phi(\rho) = \sum_{k=0}^{m} \int_X \rho_k(x) \ln \frac{\rho_k(x)}{\Sigma(x)} \, d\lambda^n(x),$$

and of the (quadratic) potential energy

$$\Psi(\rho) = \frac{1}{2} \sum_{k=0}^{m} \sum_{\ell=0}^{m} \int_X \rho_k(x) (P_{k\ell} \rho_\ell)(x) \, d\lambda^n(x) + \sum_{k=0}^{m} \int_X \rho_k(x) \, \phi_k(x) \, d\lambda^n(x),$$

split into the (possibly nonlocal and nonconvex) energy of self-interaction defined by operators $P_{k\ell} = P_{\ell k}$ and a part representing the potential energy due to external potentials ϕ_k. As mentioned above, the logarithmic chemical energy reflects the Fermi-type behavior of the particles. It prevents the densities ρ to come too close to the boundary of the simplex of admissible states given by (2). This chemical part alone would prefer uniform distributions. To control the behavior of nonlocal self-interaction between particles of type k and $\ell \in \{0, 1, \ldots, m\}$, we have in mind, for example, integral operators given by

$$(P_{k\ell}\rho_\ell)(x) = \int_X p_{k\ell}(x, y)\, \rho_\ell(y)\, d\lambda^n(y).$$

To model, for instance, the phenomenon of phase separation, the (signs of the) entries in the symmetric $(m+1) \times (m+1)$-matrix kernel $(p_{k\ell})$ have to be chosen in such a way that particles of the same type attract and particles of different type repel each other.

Local minimizers of the free energy functional F under the constraints of conservation of particle number and the admissibility conditions (2) are supposed to be physically relevant equilibrium distributions ρ^* of the multicomponent system, and more generally, admissible steady states of the energy-driven evolution system. Correspondingly, they can be found as solutions of the Euler–Lagrange equation as in the work of GRIEPENTROG and GAJEWSKI in [8]. There, we were mainly interested in equilibrium states and we established a descent method to find solutions (ρ^*, v^*) of the Euler–Lagrange equation

$$DF(\rho^*) = v^*, \qquad \int_X \rho^*(x)\, d\lambda^n(x) = \int_X \rho^\circ(x)\, d\lambda^n(x),$$

where $v^* \in \mathbb{R}^m$ denote Lagrange multipliers.

In view of the fact that the Lagrange multipliers v^* should be constant, one assumes their spatial antigradients to be the forces driving the evolutionary process towards equilibrium. The hydrodynamical limit process, see [13, 14, 33, 34], leads to the evolution system (1) with current densities

$$j_k = -\sum_{\ell=1}^{m} m_{k\ell}(\rho)\nabla v_\ell \quad \text{for } k \in \{1, \ldots, m\}, \tag{4}$$

where the symmetric and positive semidefinite mobility matrix $(m_{k\ell}(\rho))$ is the product of the diffusivity matrix $(a_{k\ell}(\rho))$ and the inverse Hessian $D^2\Phi(\rho)^{-1}$ which is nothing else but

$$D^2\Phi(\rho)^{-1} = \operatorname{diag}(\rho) - \rho \otimes \frac{\rho}{\Sigma} = \left(\delta_{k\ell}\rho_k - \rho_k \frac{\rho_\ell}{\Sigma}\right).$$

Note that this positive semidefinite matrix degenerates, if and only if at least one of the densities ρ_k vanishes. The Lagrange multipliers v are thermodynamically identified with the conjugated variables of the densities ρ and denoted as grand chemical potential differences $v = DF(\rho)$. Because of the structure of the free energy, they are written as the sum $v = \zeta + w$ of the chemical potential differences

$$\zeta_k = D_k\Phi(\rho) = \ln \frac{\rho_k}{\Sigma} - \ln \frac{\rho_0}{\Sigma}$$

and the interaction potential differences

$$w_k = D_k\Psi(\rho) = \sum_{\ell=0}^{m}(P_{k\ell}\rho_\ell - P_{0\ell}\rho_\ell) + (\phi_k - \phi_0)$$

for $k \in \{1,\dots,m\}$. We reformulate the above evolution system with gradient structure as a system of drift-diffusion equations with semilinear diffusion and nonlinear nonlocal drift terms, if we rewrite the currents as

$$j_k = -\sum_{\ell=1}^{m} a_{k\ell}(\rho)\nabla\frac{\rho_\ell}{\Sigma} - \sum_{\ell=1}^{m} m_{k\ell}(\rho)\nabla w_\ell \quad \text{for } k \in \{1,\dots,m\}.$$

Both modelling and existence analysis for the case of smooth data and the nondiagonal Stefan–Maxwell diffusion matrix $(a_{k\ell})$ were considered, for instance, in the book [15] of GIOVANGIGLI, in the contribution [1] of BOTHE, and the work [25] of JÜNGEL and STELZER. In contrast to our work, these authors do not consider drift terms to model the interaction processes between the particles. To our knowledge, nonlocal drift terms are not only advantageous for that purpose, but they can also be understood as a natural regularization of nondiagonal terms.

Since our theory of maximal regularity for parabolic boundary value problems with nonsmooth data is restricted to weakly coupled systems, in our work we consider the equidiffusive (diagonal) case $(a_{k\ell}) = (A\Sigma\delta_{k\ell})$ with a diffusion coefficient A, which may depend nonsmoothly on space and time. As in the work [7] of GAJEWSKI and GRÖGER or the paper [10] of GAJEWSKI and SKRYPNIK on the analysis of reaction-drift-diffusion processes of electrically charged particles in semiconductor heterostructures, the total density Σ may depend nonsmoothly on the space variable.

Several authors considered similar models describing nonlocal phase separation processes with a degenerate mobility of the above mentioned type. The case of homogeneous binary systems was justified by GIACOMIN and LEBOWITZ in [14], and analyzed in the pioneering work [11] of GAJEWSKI and ZACHARIAS with the emphasis to existence and uniqueness of solutions. There, the diffusion coefficient A may depend nonsmoothly on the space variable and, in addition to that, strongly monotone and Lipschitz continuous on the gradient ∇v. Based on this, in the papers [28, 29] of LONDEN and PETZELTOVA and in the recent contribution [12] of GAL and GRASSELLI the asymptotic convergence of the trajectory to an equilibrium state were established.

We want to emphasize that in contrast to the above mentioned papers [11, 12, 28, 29], in the present work all the qualitative properties of the solutions, namely, the uniqueness and the uniform regularity as well as the uniform positivity and the asymptotic convergence of the solution, are derived from *minimal* assumptions on the regularity of the gradients ∇w_ℓ of the interaction potentials. We do not need to assume these gradients to be bounded. In fact, it suffices to suppose that for some exponent $\omega_0 > n-2$ and some constant $c_0 > 0$

the Morrey-type condition

$$\sum_{\ell=1}^{m} \int_{X\cap Q(x,r)} |\nabla w_\ell(s)|^2 \, d\lambda^n \le c_0 r^{\omega_0} \tag{5}$$

holds true for every cube $Q(x,r) \subset \mathbb{R}^n$ with center $x \in X$, radius $r > 0$ and every point $s \ge 0$ on the time axis. This enables us to treat a huge variety of interactions, especially the limit case of regularity governed by the nonsmooth data of the problem. An illustrating example is given in Section 7, where we consider nonlocal phase separation processes with interaction potentials defined as solutions to nonsmooth elliptic boundary value problems.

Organization of the work. This work generalizes and a continues the proceedings contribution [20] of the author. In Section 2 we fix both our general assumptions and the functional analytical framework to formulate the problem in heterogeneous environments.

Applying fixed-point arguments and comparison principles, in Section 3 we prove the existence of variational solutions in suitable Hilbert spaces for evolution systems. At the same time, we fill a gap in the proof of [20, Lemma 1].

In Section 4 we collect main results of our theory of maximal regularity for parabolic boundary value problems with nonsmooth data in Sobolev–Morrey and Hölder spaces, see GRIEPENTROG and RECKE [21, 22, 23]. These results are the main tools for our proof of unique solvability and global regularity of solutions, see also [20, Section 4 and 5].

Section 5 is dedicated to the uniform positivity of solutions. Here, the proof relies on a Moser-type iteration procedure, based on a multiplicative Sobolev inequality with Morrey-type weights, which can be found in MAZYA's book [30, Corollary 1.4.7/2], and which exactly corresponds to the regularity assumption (5) on the gradients ∇w_ℓ of the interaction potentials.

Using a refined version of the Łojasiewicz–Simon gradient inequality similar to [12, 28], this paves the way to the proof of asymptotic convergence of solutions to an admissible equilibrium state in Section 6, which turns out to be a solution of the Euler–Lagrange equation under the constraint of conservation of particle number. The free energy decreases monotonically along the trajectory to the corresponding limit.

In Section 7 we conclude our considerations with the presentation of simulation results for phase separation processes in ternary systems including the case of diffusive damage.

2. GENERAL ASSUMPTIONS AND FORMULATION OF THE PROBLEM

The following assumptions are valid for the whole work. Suppose $m \in \mathbb{N}$ to be the number of independent components and $n \in \mathbb{N}$, $n \ge 3$ to be the space dimension. Let $(\mathbb{R}^n, \mathfrak{L}^n, \lambda^n)$ be the σ-finite measure space of n-dimensional Lebesgue-measurable subsets of $\mathbb{R}^n$. For $F \in \mathfrak{L}^n$ and $p \in [1, \infty)$ we denote by $L^p(F;V)$ the set of all Lebesgue p-integrable functions $u : F \to V$ with values in the Banach space $(V, \| \; \|_V)$. The class $L^\infty(F;V)$ consists of all Lebesgue-measurable functions $u : F \to V$ which are essentially bounded.

For every $F \subset \mathbb{R}^n$ we introduce the class $B(F;V)$ of bounded functions $u : F \to V$. We define the set $C(F;V)$ of continuous functions $u : F \to V$ and the subclass $BC(F;V) = B(F;V) \cap C(F;V)$. Moreover, for $\alpha \in (0,1]$ we consider the set $C^{0,\alpha}(F;V)$ of Hölder-continuous functions $u : F \to V$ and the subclass $BC^{0,\alpha}(F;V) = B(F;V) \cap C^{0,\alpha}(F;V)$.

For $k \in \mathbb{N} \cup \{\infty\}$ and open sets $U \subset \mathbb{R}^n$ we denote by $C^k(U;V)$ the set of functions $u : U \to V$ which have continuous derivatives up to the k-th order. The subclass of all these functions with bounded continuous derivatives up to the k-th order forms the set $BC^k(U;V)$. Finally, we introduce the subset $C_0^k(U;V)$ of functions $u \in C^k(U;V)$ with compact support $\operatorname{supp}(u)$ in U.

Here and in what follows we denote, as usual, by $\langle\ ,\ \rangle_H$ and $(\ |\ )_H$ dual pairings and scalar products in Hilbert spaces H, respectively. For subsets Y of $\mathbb{R}^n$ we write $\operatorname{int} Y$, $\operatorname{cl} Y$ and ∂Y for the topological interior, the closure, and the boundary of Y, respectively.

Let $S = (0,T)$ be a time interval of finite length. Considering problems with nonsmooth data, we suppose $X \subset \mathbb{R}^n$ to be a domain with Lipschitz boundary defined in the spirit of GIUSTI or GRÖGER, see [16, 24]:

DEFINITION 1 (Domain with Lipschitz boundary). Let the *open cube* and *half-cube*

$$Q(x,r) = \big\{\xi \in \mathbb{R}^n : \sup_{1\le i\le n} |\xi_i - x_i| < r\big\},$$
$$Q^+(x,r) = \big\{\xi \in \mathbb{R}^n : \sup_{1\le i\le n} |\xi_i - x_i| < r,\ \xi_n - x_n < 0\big\},$$

be defined for $x \in \mathbb{R}^n$, $r > 0$, respectively. A bounded, open and connected set $X \subset \mathbb{R}^n$ is called *domain with Lipschitz boundary* if for each $x \in \partial X$ we find some neighborhood U of x in $\mathbb{R}^n$ and a bi-Lipschitz transformation Λ from U onto $Q(0,1)$ such that $\Lambda[U \cap X] = Q^+(0,1)$ and $\Lambda(x) = 0$.

Hilbert spaces for evolution problems. For the functional analytic formulation of diffusion and drift processes of interacting particles in a fixed environment, its (possibly nonsmooth) spatially heterogeneous storage capacity needs to be measured properly:

ASSUMPTION 1 (Storage capacity). Let $(\mathbb{R}^n, \mathfrak{L}^n, \sigma)$ be a measure space satisfying

$$\sigma_* \lambda^n(F) \le \sigma(F) \le \sigma^* \lambda^n(F) \quad \text{for every set } F \in \mathfrak{L}^n, \tag{6}$$

where $0 < \sigma_* \le \sigma^* < \infty$ are given lower and upper bounds of the total density. We interpret $\sigma(F)$ as the measure of the particle number which can be stored in $F \in \mathfrak{L}^n$.

REMARK 1 (Topological equivalence). Obviously, the density $\Sigma \in L^\infty(\mathbb{R}^n)$ of the measure σ with respect to the Lebesgue measure λ^n satisfies $\sigma_* \le \Sigma(x) \le \sigma^*$ for λ^n-almost all $x \in \mathbb{R}^n$. Hence, in the following we are allowed to introduce all the spaces of integrable functions with respect to the measure space $(\mathbb{R}^n, \mathfrak{L}^n, \sigma)$ without changing their standard topologies. One has only to keep in mind that the constants in the estimates eventually depend on the fixed lower and upper bounds $\sigma_* > 0$ and $\sigma^* > 0$, too.

DEFINITION 2 (Hilbert spaces). 1. We consider the pair of dual Hilbert spaces

$$H = L^2(X;\mathbb{R}^m) \quad \text{and} \quad H^* = [L^2(X;\mathbb{R}^m)]^*,$$

equipped with the nonstandard scalar product given by

$$\langle \rho, h\rangle_H = \langle Jg, h\rangle_H = (g|h)_H = \sum_{k=1}^m \int_X g_k h_k \, d\sigma \quad \text{for } g,\, h \in H.$$

Here, the duality of *intensive* quantities $g \in H$ and the corresponding *extensive* distributions $\rho = Jg \in H^*$ is established by the duality map $J \in \mathcal{L}(H; H^*)$. We introduce the operator $\mathcal{J} \in \mathcal{L}(L^2(S;H); L^2(S;H^*))$ by $(\mathcal{J}u)(s) = Ju(s)$ for $s \in S$ and $u \in L^2(S;H)$.

2. Moreover, we introduce the dual pair of Sobolev spaces

$$V = W^{1,2}(X;\mathbb{R}^m) \quad \text{and} \quad V^* = [W^{1,2}(X;\mathbb{R}^m)]^*,$$

equipped with nonstandard scalar product

$$(u|w)_V = \sum_{k=1}^m \int_X (\nabla u_k \cdot \nabla w_k + u_k w_k)\, d\sigma \quad \text{for } u,\, w \in V.$$

The completely continuous and dense embedding of V into H and its adjoint are denoted by $K \in \mathcal{L}(V;H)$ and $K^* \in \mathcal{L}(H^*;V^*)$. Then, $E = K^*JK \in \mathcal{L}(V;V^*)$ is symmetric and positive semidefinite. Correspondingly, we define the operators $\mathcal{K} : L^2(S;V) \to L^2(S;H)$ and $\mathcal{E} : L^2(S;V) \to L^2(S;V^*)$ by $(\mathcal{K}u)(s) = Ku(s)$ and $(\mathcal{E}u)(s) = Eu(s)$ for $s \in S$ and $u \in L^2(S;V)$ as illustrated in the following commutative diagrams

$$\begin{tikzcd} V \arrow[r, "K"] \arrow[d, "E"] & H \arrow[d, "J"] \\ V^* & H^* \arrow[l, "K^*"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L^2(S;V) \arrow[r, "\mathcal{K}"] \arrow[d, "\mathcal{E}"] & L^2(S;H) \arrow[d, "\mathcal{J}"] \\ L^2(S;V^*) & L^2(S;H^*) \arrow[l, "\mathcal{K}^*"] \end{tikzcd}$$

The generalization of the concept of Sobolev spaces for evolution equations follows the ideas of LIONS and GRÖGER, see DAUTRAY and LIONS [4, Chapter XVIII, §5] and GRIEPENTROG [21, Section 1]:

DEFINITION 3 (Sobolev space). 1. The function $f \in L^2(S;V^*)$ is called *weakly differentiable* if there exists some $f' \in L^2(S;V^*)$ which satisfies

$$\int_S \langle f'(s), v\rangle_V \, \vartheta(s)\, d\lambda(s) = -\int_S \langle f(s), v\rangle_V \, \vartheta'(s)\, d\lambda(s) \quad \text{for all } \vartheta \in C_0^\infty(S),\, v \in V.$$

2. Corresponding to $\mathcal{E} : L^2(S;V) \to L^2(S;V^*)$, which is associated to S and $E \in \mathcal{L}(V;V^*)$ in Definition 2, we define the *Sobolev space*

$$W_E(S;V) = \big\{ u \in L^2(S;V) : (\mathcal{E}u)' \in L^2(S;V^*) \big\},$$

equipped with the scalar product

$$(u|w) = (u|w)_{L^2(S;V)} + ((\mathcal{E}u)'|(\mathcal{E}w)')_{L^2(S;V^*)} \quad \text{for } u,\, w \in W_E(S;V).$$

REMARK 2. The embedding map $\mathcal{K}$ is continuous from $W_E(S;V)$ into $C(\operatorname{cl} S;H)$ and completely continuous from $W_E(S;V)$ into $L^2(S;H)$.

Energy functionals. As mentioned in the introduction, the free energy $F = \Phi + \Psi$ is split into a sum of a strongly convex chemical part Φ and a quadratic interaction part Ψ, which are now specified rigorously:

ASSUMPTION 2 (Chemical energy). 1. For every $\gamma \in \left[0, \frac{1}{m+1}\right]$ we introduce

$$U(\gamma) = \left\{h \in L^\infty : \gamma \le h_0, h_1, \ldots, h_m \le 1-\gamma,\ h_0 = 1 - \textstyle\sum_{k=1}^m h_k\right\}$$

as a closed subset of $L^\infty = L^\infty(X;\mathbb{R}^m)$. Here, and in the following, we always have in mind that $h_0 = 1 - \sum_{k=1}^m h_k$ is a function of $h = (h_1, \ldots, h_m)$.

2. The functional $\Phi : H^* \to \mathbb{R} \cup \{+\infty\}$ of *chemical energy* is given by

$$\Phi(\rho) = \begin{cases} \int_X \sum_{k=0}^m h_k \ln h_k \, d\sigma & \text{for } \rho = Jh,\ h \in U(0), \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

The *relative density* $h \in U(0)$ is the density of the corresponding *particle distribution* $\rho = Jh$ with respect to the measure σ. The definition of $h_0 = 1 - \sum_{k=1}^m h_k$ implies the definition of $\rho_0 = \sigma - \sum_{k=1}^m \rho_k$ as a function of $\rho = (\rho_1, \ldots, \rho_m)$.

REMARK 3 (Strong convexity). The functional $\Phi : H^* \to \mathbb{R} \cup \{+\infty\}$ is proper, lower semicontinuous, and strongly convex with the closed and convex effective domain $\operatorname{dom}(\Phi) = J[U(0)]$. We have

$$\delta\Phi(\rho) + (1-\delta)\Phi(\hat\rho) \ge \Phi(\delta\rho + (1-\delta)\hat\rho) + \tfrac{1}{2}(m+1)\delta(1-\delta)\|\rho - \hat\rho\|_{H^*}^2$$

for all ρ, $\hat\rho \in \operatorname{dom}(\Phi)$ and $\delta \in [0,1]$. As a consequence, the subdifferential $\partial\Phi \subset H^* \times H$ is both strongly monotone and maximal monotone.

REMARK 4 (Chemical potentials and real analyticity). 1. For $\gamma \in \left(0, \frac{1}{m+1}\right]$, $h \in U(\gamma)$ and $\rho = Jh \in \operatorname{dom}(\Phi)$ we get

$$\zeta_k = D_k\Phi(\rho) = \ln h_k - \ln h_0 \quad \text{for every } k \in \{1, \ldots, m\}. \tag{8}$$

The components of $\zeta = D\Phi(\rho) \in L^\infty$ are called *chemical potential differences.* There exists a constant $c_1 > 0$ depending on m and γ such that the estimates

$$\langle D\Phi(\rho) - D\Phi(\hat\rho), \rho - \hat\rho\rangle_{H^*} \ge (m+1)\|\rho - \hat\rho\|_{H^*}^2, \tag{9}$$

$$\|D\Phi(\rho) - D\Phi(\hat\rho)\|_H \le c_1\|\rho - \hat\rho\|_{H^*} \tag{10}$$

hold true for all ρ, $\hat\rho \in J[U(\gamma)]$, and the Hessian $D^2\Phi(\rho) \in \mathcal{L}(J[L^\infty];L^\infty)$ is positive definite and has the representation

$$\langle D^2\Phi(\rho)\varrho, \hat\varrho\rangle_{H^*} = \sum_{k=1}^m \sum_{\ell=1}^m \int_X \left(\frac{\delta_{k\ell}}{h_k} + \frac{1}{h_0}\right) g_\ell\, \hat g_k \, d\sigma \quad \text{for } \varrho = Jg,\ \hat\varrho = J\hat g \text{ with } g,\ \hat g \in L^\infty.$$

2. The functional Φ is real analytic in the set $J[U]$, whenever U is open in L^∞ and contained in $U(\gamma)$ for some $\gamma \in \left(0, \frac{1}{m+1}\right]$. Its Fréchet derivative $D\Phi : J[U] \to L^\infty$ is a real analytic operator, and the second derivative $D^2\Phi(\rho) \in \mathcal{L}(J[L^\infty]; L^\infty)$ is an isomorphism for all $\rho \in J[U]$, see GRIEPENTROG and GAJEWSKI [8, Lemma 12].

REMARK 5 (Convex conjugate functional). The Fréchet differentiable *convex conjugate* $\Phi^* : H \to \mathbb{R}$ to the strongly convex functional $\Phi : H^* \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\Phi^*(\zeta) = \int_X \ln\left(1 + \textstyle\sum_{k=1}^m \exp(\zeta_k)\right) d\sigma \quad \text{for every } \zeta \in H,$$

and we obtain $D\Phi^*(\zeta) = Jh \in \mathrm{dom}(\Phi)$, where $h \in U(0)$ is given by the functions

$$h_k = \frac{\exp(\zeta_k)}{1 + \sum_{\ell=1}^m \exp(\zeta_\ell)} \quad \text{for every } k \in \{1, \ldots, m\}. \tag{11}$$

Moreover, the Hessian $D^2\Phi^*(\zeta) \in \mathcal{L}(H; H^*)$ is represented by

$$\langle D^2\Phi^*(\zeta) g, \hat{g}\rangle_H = \sum_{k=1}^m \sum_{\ell=1}^m \int_X (\delta_{k\ell} h_k - h_k h_\ell)\, g_\ell\, \hat{g}_k\, d\sigma \quad \text{for all } g,\, \hat{g} \in H.$$

These preliminary considerations justify the definition of both the diffusion and drift operator discussed in the introduction:

ASSUMPTION 3 (Diffusion). 1. Let the diffusion coefficient $A \in L^\infty(S \times X)$ satisfy

$$\nu \le A \le 1/\nu \quad \lambda^{n+1}\text{-almost everywhere on } S \times X$$

for some constant $\nu \in (0, 1]$. The linear *diffusion operator* $\mathcal{L} : L^2(S; V) \to L^2(S; V^*)$ is defined by

$$\langle \mathcal{L}u, h\rangle_{L^2(S;V)} = \sum_{k=1}^m \int_S \int_X A \nabla u_k \cdot \nabla h_k \, d\sigma\, d\lambda \quad \text{for } u,\, h \in L^2(S; V). \tag{12}$$

2. Let the bilinear operator $\mathcal{B} : L^\infty(S \times X; \mathbb{R}^{m\times m}) \times L^2(S; V) \to L^2(S; V^*)$ for $M \in L^\infty(S \times X; \mathbb{R}^{m\times m})$ and $w,\, h \in L^2(S; V)$ be given by

$$\langle \mathcal{B}(M, w), h\rangle_{L^2(S;V)} = \sum_{k=1}^m \sum_{\ell=1}^m \int_S \int_X A M_{k\ell} \nabla w_\ell \cdot \nabla h_k \, d\sigma\, d\lambda. \tag{13}$$

ASSUMPTION 4 (Drift). Let the *mobility* $M : U(0) \to L^\infty(X; \mathbb{R}^{m\times m})$ be given by the symmetric matrix-valued function

$$Mh = \mathrm{diag}(h) - h \otimes h = (\delta_{k\ell} h_k - h_k h_\ell) \quad \text{for } h \in U(0). \tag{14}$$

Correspondingly, the *drift operator* $\mathcal{M} : \mathrm{dom}(\mathcal{M}) \to L^\infty(S \times X; \mathbb{R}^{m\times m})$ is defined by $(\mathcal{M}u)(s) = Mu(s)$ for $s \in S$, $u \in \mathrm{dom}(\mathcal{M})$ on the domain

$$\mathrm{dom}(\mathcal{M}) = \left\{u \in L^\infty(S \times X; \mathbb{R}^m) :\ 0 \le u_0, u_1, \ldots, u_m \le 1,\ u_0 = 1 - \textstyle\sum_{k=1}^m u_k\right\}.$$

ASSUMPTION 5 (Potential energy of interaction). Let $P_{k\ell} = P_{\ell k}$ be bounded linear operators from $[L^2(X)]^*$ into $W^{1,2}(X)$ representing the *self-interaction* between particles of type $k, \ell \in \{0, 1, \ldots, m\}$ and $\phi_k \in W^{1,2}(X)$ be *external potentials* for $k \in \{0, 1, \ldots, m\}$. The functional $\Psi : H^* \to \mathbb{R}$ of *interaction energy* is given by the quadratic functional

$$\Psi(\rho) = \frac{1}{2} \sum_{k=0}^{m} \sum_{\ell=0}^{m} \int_X h_k P_{k\ell} \rho_\ell \, d\sigma + \sum_{k=0}^{m} \int_X h_k \phi_k \, d\sigma \tag{15}$$

for $\rho = Jh \in H^*$ corresponding to $h \in H$, and having in mind both $\rho_0 = \sigma - \sum_{k=1}^m \rho_k$ and $h_0 = 1 - \sum_{k=1}^m h_k$ as definitions, see Assumption 2.

We define the (possibly nonconvex) functional $F = \Phi + \Psi : H^* \to \mathbb{R} \cup \{+\infty\}$ of *free energy* with the closed and convex effective domain $\operatorname{dom}(F) = \operatorname{dom}(\Phi) = J[U(0)]$.

REMARK 6 (Interaction potentials). Due to Assumption 5 the quadratic functional $\Psi : H^* \to \mathbb{R}$ has the representation

$$\Psi(\rho) = \tfrac{1}{2} \langle \rho, P\rho \rangle_H + \langle \rho, \psi \rangle_H + \Psi(0) \quad \text{for every } \rho \in H^*, \tag{16}$$

where the constant is determined by $\Psi(0) = \frac{1}{2} \int_X P_{00}\sigma \, d\sigma + \int_X \phi_0 \, d\sigma$. The bounded linear operator $P \in \mathcal{L}(H^*; V)$ and the element $\psi \in V$ are given by

$$(P\rho)_k = \sum_{\ell=1}^{m} \big((P_{k\ell} - P_{0\ell}) - (P_{k0} - P_{00}) \big) \rho_\ell \quad \text{and} \quad \psi_k = (P_{k0} - P_{00})\sigma + (\phi_k - \phi_0) \tag{17}$$

for every $k \in \{1, \ldots, m\}$. Note that $KP \in \mathcal{L}(H^*; H)$ is a symmetric and completely continuous operator. The components

$$w_k = D_k \Psi(\rho) = (P\rho)_k + \psi_k \tag{18}$$

of $w = D\Psi(\rho) = P\rho + \psi \in V$ are called *interaction potential differences.* According to Assumption 2, there exists a constant $L > 0$ such that for all $\rho \in \operatorname{dom}(\Phi)$ the estimate

$$\sum_{k=1}^{m} \int_X |\nabla w_k|^2 \, d\sigma \le L$$

holds true, whenever $w = P\rho + \psi \in V$.

Corresponding to P, we define the *self-interaction operator* $\mathcal{P} : L^2(S; H^*) \to L^2(S; V)$ by $(\mathcal{P}\rho)(s) = P\rho(s)$ for $s \in S$, $\rho \in L^2(S; H^*)$.

Now, we can rigorously formulate the concept of a solution of the evolution problem:

DEFINITION 4 (Solution). For a given initial value $a \in U(0)$ we are looking for a *solution* $u \in W_E(S; V) \cap \operatorname{dom}(\mathcal{M})$ *of the evolution system*

$$(\mathcal{E}u)' + \mathcal{L}u + \mathcal{B}(\mathcal{M}u, \mathcal{P}\mathcal{J}u + \psi) = 0, \quad (\mathcal{K}u)(0) = a. \tag{P}$$

3. Existence of solutions

At first we solve a *regularized* problem with truncated nonlinearities. To do so, for $c \in \mathbb{R}$ we define the truncations

$$c^- = -\min\{c, 0\}, \quad c^+ = \max\{c, 0\}, \quad c^\diamond = \min\{\max\{c, 0\}, 1\},$$

and we carry over this setting in the usual way to the concept of truncated functions. The truncation of vector-valued functions is given by the vector of its truncated components.

DEFINITION 5 (Regularization). 1. The *truncated mobility* $R : H \to L^\infty(X; \mathbb{R}^{m\times m})$ is given by the symmetric matrix-valued function

$$Rh = \textstyle\sum_{l=0}^m h_l^\diamond \operatorname{diag}(h^\diamond) - h^\diamond \otimes h^\diamond = \big(\delta_{k\ell} h_k^\diamond \textstyle\sum_{l=0}^m h_l^\diamond - h_\ell^\diamond h_k^\diamond\big) \quad \text{for } h \in H. \tag{19}$$

Correspondingly, the *regularized drift operator* $\mathcal{R} : L^2(S; H) \to L^\infty(S \times X; \mathbb{R}^{m\times m})$ is defined by $(\mathcal{R}u)(s) = Ru(s)$ for $s \in S$, $u \in L^2(S; H)$.

2. The *regularized self-interaction operator* $\mathcal{Q} : L^2(S; H) \to L^2(S; V)$ is given by

$$\mathcal{Q}u = \mathcal{P}\mathcal{J}u^\diamond \quad \text{for } u \in L^2(S; H). \tag{20}$$

LEMMA 1 (Solvability of a regularized problem). *Let the Assumptions 1–5 be satisfied. Then, for every $a \in U(0)$ there exists a solution $u \in W_E(S; V)$ of the regularized problem*

$$(\mathcal{E}u)' + \mathcal{L}u + \mathcal{B}(\mathcal{R}u, \mathcal{Q}u + \psi) = 0, \quad (\mathcal{K}u)(0) = a. \tag{R}$$

Proof. 1. Our proof is based on the application of Schauder's fixed-point principle. Let $\Lambda \geq 0$ be the Lipschitz constant of $\mathcal{Q}$ and $a \in U(0)$ be a fixed initial value. For every $u \in L^2(S; H)$ we have $\mathcal{Q}u \in L^2(S; V)$ and $\mathcal{B}(\mathcal{R}u, \mathcal{Q}u + \psi) \in L^2(S; V^*)$. Hence, there exists a uniquely determined solution $\mathcal{F}u \in W_E(S; V)$ satisfying $\mathcal{K}\mathcal{F}u \in C(\operatorname{cl} S; H)$, see DAUTRAY and LIONS [4, Chapter XVIII, §5] or GRIEPENTROG [21, Section 2], of the evolution problem

$$(\mathcal{E}\mathcal{F}u)' + \mathcal{L}\mathcal{F}u = -\mathcal{B}(\mathcal{R}u, \mathcal{Q}u + \psi), \quad (\mathcal{K}\mathcal{F}u)(0) = a. \tag{21}$$

That means, we have properly defined a fixed-point operator $\mathcal{F} : L^2(S; H) \to L^2(S; H)$. Our aim is to prove that $\mathcal{F} : L^2(S; H) \to L^2(S; H)$ is completely continuous and $\mathcal{F}[\mathcal{C}] \subset \mathcal{C}$ holds for a closed convex set $\mathcal{C} \subset L^2(S; H)$ depending only on the data of the problem.

2. Let $u \in L^2(S; H)$ and $\mathcal{F}u \in W_E(S; V)$ be the solution of problem (21). Applying the test function $\varphi = \mathcal{F}u \in W_E(S; V)$ to (21) and Young's inequality we get the estimate

$$\int_0^t \langle (\mathcal{E}\mathcal{F}u)'(s), (\mathcal{F}u)(s) \rangle_V \, d\lambda(s) + \sum_{k=1}^m \int_0^t \int_X A|\nabla(\mathcal{F}u)_k|^2 \, d\sigma \, d\lambda$$
$$\leq \frac{m}{2\nu} \sum_{k=1}^m \sum_{\ell=1}^m \int_0^t \int_X |(\mathcal{R}u)_{k\ell} \nabla(\mathcal{Q}u + \psi)_\ell|^2 \, d\sigma \, d\lambda + \frac{1}{2} \sum_{k=1}^m \int_0^t \int_X A|\nabla(\mathcal{F}u)_k|^2 \, d\sigma \, d\lambda$$

for all $t \in \operatorname{cl} S$. Using the fact that due to (19) we have $|(\mathcal{R}u)_{k\ell}| \le m$ for $k,\ \ell \in \{0, 1, \dots, m\}$, for all $t \in \operatorname{cl} S$ partial integration yields

$$\int_X |(\mathcal{K}\mathcal{F}u)(t)|^2\, d\sigma + \nu \sum_{k=1}^m \int_0^t \int_X |\nabla(\mathcal{F}u)_k|^2\, d\sigma\, d\lambda \\ \le \sum_{k=1}^m \int_X |a_k|^2\, d\sigma + \frac{m^4}{\nu} \sum_{k=1}^m \int_S \int_X |\nabla(\mathcal{Q}u + \psi)_k|^2\, d\sigma\, d\lambda. \quad (22)$$

To obtain an estimate for $\mathcal{F}u$ in $L^2(S; V)$, we get one contribution by taking $t = T$ on the left hand side of (22). For the other contribution, we integrate both sides of (22) over $t \in S$. Summing up, this yields

$$\int_S \int_X |\mathcal{F}u|^2\, d\sigma\, d\lambda + \nu \sum_{k=1}^m \int_S \int_X |\nabla(\mathcal{F}u)_k|^2\, d\sigma\, d\lambda \\ \le (T+1) \sum_{k=1}^m \int_X |a_k|^2\, d\sigma + \frac{m^4(T+1)}{\nu} \sum_{k=1}^m \int_S \int_X |\nabla(\mathcal{Q}u + \psi)_k|^2\, d\sigma\, d\lambda.$$

Since $\Lambda \ge 0$ is the Lipschitz constant of $\mathcal{Q} : L^2(S; H) \to L^2(S; V)$, this yields

$$\begin{aligned} \nu^2 \|\mathcal{F}u\|^2_{L^2(S;V)} &\le \nu(T+1)\|a\|_H^2 + m^4(T+1)\|\mathcal{Q}u + \psi\|^2_{L^2(S;V)} \\ &\le \nu(T+1)\|a\|_H^2 + 2m^4(T+1)\big(\Lambda \|u^\diamond\|^2_{L^2(S;H)} + T\|\psi\|_V^2\big). \end{aligned}$$

That means, we have $\|\mathcal{F}u\|^2_{L^2(S;V)} \le r^2$ for all $u \in L^2(S; H)$, if we fix $r > 0$ by

$$\nu^2 r^2 = \nu(T+1)\|a\|_H^2 + 2m^4(T+1)\big(m\Lambda T\sigma(X) + T\|\psi\|_V^2\big).$$

Hence, we get $\mathcal{F}[\mathcal{C}] \subset \mathcal{C}$ for the closed ball $\mathcal{C} = \big\{u \in L^2(S; H) : \|u\|_{L^2(S;H)} \le r\big\}$.

Moreover, a similar calculation also shows that

$$\nu^2 \|\mathcal{B}(\mathcal{R}u, \mathcal{Q}u + \psi)\|^2_{L^2(S;V^*)} \le m^4 \|\mathcal{Q}u + \psi\|^2_{L^2(S;V)} \le 2m^4\big(m\Lambda T\sigma(X) + T\|\psi\|_V^2\big)$$

and $\nu^2 \|\mathcal{L}\mathcal{F}u\|^2_{L^2(S;V^*)} \le r^2$ hold true for all $u \in L^2(S; H)$. Consequently, using identity (21), we obtain the boundedness of the set $\mathcal{F}[L^2(S; H)]$ in $W_E(S; V)$. Due to the completely continuous embedding of $W_E(S; V)$ into $L^2(S; H)$, see Remark 2, this yields that the fixed-point map $\mathcal{F} : L^2(S; H) \to L^2(S; H)$ is a compact operator.

3. Let $(u_i) \subset L^2(S; H)$ be a sequence which converges to $u \in L^2(S; H)$ in $L^2(S; H)$. For every $i \in \mathbb{N}$ there exists a unique solution $\mathcal{F}u_i \in W_E(S; V)$ of the problem

$$(\mathcal{E}\mathcal{F}u_i)' + \mathcal{L}\mathcal{F}u_i = -\mathcal{B}(\mathcal{R}u_i, \mathcal{Q}u_i + \psi), \quad (\mathcal{K}\mathcal{F}u_i)(0) = a.$$

Because $\mathcal{F}u \in W_E(S; V)$ is the solution of problem (21), for every $i \in \mathbb{N}$ it follows

$$(\mathcal{E}\mathcal{F}u_i - \mathcal{E}\mathcal{F}u)' + \mathcal{L}(\mathcal{F}u_i - \mathcal{F}u) = \mathcal{B}(\mathcal{R}u, \mathcal{Q}u + \psi) - \mathcal{B}(\mathcal{R}u_i, \mathcal{Q}u_i + \psi), \quad (\mathcal{K}\mathcal{F}u_i - \mathcal{K}\mathcal{F}u)(0) = 0.$$

Testing with $\varphi = \mathcal{F}u_i - \mathcal{F}u \in W_E(S;V)$, Young's inequality yields the estimate

$$\begin{aligned}\int_0^t &\langle (\mathcal{E}\mathcal{F}u_i - \mathcal{E}\mathcal{F}u)'(s), (\mathcal{F}u_i - \mathcal{F}u)(s)\rangle_V \, d\lambda(s) \\ &\leq \frac{m}{2\nu}\sum_{k=1}^m\sum_{\ell=1}^m \int_0^t\int_X \left(|(\mathcal{R}u_i)_{k\ell}\nabla(\mathcal{Q}u_i - \mathcal{Q}u)_\ell|^2 + |(\mathcal{R}u - \mathcal{R}u_i)_{k\ell}\nabla(\mathcal{Q}u+\psi)_\ell|^2\right) d\sigma\, d\lambda\end{aligned}$$

for all $t \in \operatorname{cl} S$, $i \in \mathbb{N}$. Having in mind that $|(\mathcal{R}u)_{k\ell}| \leq m$ for $k,\ \ell \in \{0,1,\dots,m\}$, and integrating by parts, for all $t \in \operatorname{cl} S$ we get

$$\begin{aligned}\int_X |(\mathcal{K}\mathcal{F}u_i - \mathcal{K}\mathcal{F}u)(t)|^2\, d\sigma \leq \frac{m}{\nu}\sum_{k=1}^m\sum_{\ell=1}^m \int_S\int_X |(\mathcal{R}u - \mathcal{R}u_i)_{k\ell}\nabla(\mathcal{Q}u+\psi)_\ell|^2\, d\sigma\, d\lambda \\ + \frac{m^4}{\nu}\sum_{k=1}^m \int_S\int_X |\nabla(\mathcal{Q}u_i - \mathcal{Q}u)_k|^2\, d\sigma\, d\lambda,\end{aligned}$$

and, hence, after integration over $t \in S$,

$$\begin{aligned}\int_S\int_X |\mathcal{F}u_i - \mathcal{F}u|^2\, d\sigma\, d\lambda \leq \frac{mT}{\nu}\sum_{k=1}^m\sum_{\ell=1}^m \int_S\int_X |(\mathcal{R}u - \mathcal{R}u_i)_{k\ell}\nabla(\mathcal{Q}u+\psi)_\ell|^2\, d\sigma\, d\lambda \\ + \frac{m^4 T}{\nu}\sum_{k=1}^m \int_S\int_X |\nabla(\mathcal{Q}u_i - \mathcal{Q}u)_k|^2\, d\sigma\, d\lambda.\end{aligned}$$

The integrand of the first part of the right hand side is majorized by $4m^2|\nabla(\mathcal{Q}u+\psi)_\ell|^2$ for every $i \in \mathbb{N}$, and it tends to zero λ^{n+1}-almost everywhere on $S \times X$ in the limit $i \to \infty$ because of the Lipschitz continuity of $\mathcal{R}$ and the convergence $\lim_{i\to\infty}\|u_i - u\|_{L^2(S;H)} = 0$. Hence, applying Lebesgue's theorem, the first part tends to zero. On the other hand,

$$\lim_{i\to\infty}\|\mathcal{Q}u_i - \mathcal{Q}u\|_{L^2(S;V)} \leq \Lambda \lim_{i\to\infty}\|u_i - u\|_{L^2(S;H)} = 0.$$

That means, the second part of the right hand side tends to zero, too. We arrive at $\lim_{i\to\infty}\|\mathcal{F}u_i - \mathcal{F}u\|_{L^2(S;H)} = 0$, in other words, $\mathcal{F} : L^2(S;H) \to L^2(S;H)$ is continuous.

4. Together with the second step of the proof, $\mathcal{F} : L^2(S;H) \to L^2(S;H)$ is a completely continuous operator, which maps the closed convex set $\mathcal{C}$ into itself. Hence, Schauder's fixed-point theorem yields a solution $u \in W_E(S;V) \cap \mathcal{C}$ of the equation $\mathcal{F}u = u$. Consequently, we have found a solution $u \in W_E(S;V)$ of the regularized problem (R). □

THEOREM 2 (Solvability of the original problem). *Suppose the Assumptions 1–5 to be fulfilled. Then, for every $a \in U(0)$ there exists a solution $u \in W_E(S;V) \cap \operatorname{dom}(\mathcal{M})$ of the evolution system* (P).

Proof. 1. Let $a \in U(0)$ be given. Due to Lemma 1 there exists a solution $u \in W_E(S;V)$ of the regularized problem (R). Let us use the notation $w = \mathcal{P}\mathcal{J}u^\diamond + \psi \in L^2(S;V)$.

2. Considering the test function $\varphi = (-u_1^-, \dots, -u_m^-) \in L^2(S;V)$, from (19) it follows

$$\sum_{k=1}^m \sum_{\ell=1}^m (\mathcal{R}u)_{k\ell} \nabla w_\ell \cdot \nabla \varphi_k = -\sum_{k=1}^m \sum_{l=0}^m u_l^\diamond u_k^\diamond \nabla w_k \cdot \nabla u_k^- + \sum_{k=1}^m \sum_{\ell=1}^m u_\ell^\diamond u_k^\diamond \nabla w_\ell \cdot \nabla u_k^- = 0,$$

since for every $k \in \{1, \dots, m\}$ by definition we have $u_k^\diamond \nabla u_k^- = 0$. Hence, testing (R) with φ and having in mind $a_1, \dots, a_m \ge 0$, for all $t \in \operatorname{cl} S$ integration by parts yields

$$\begin{aligned} 0 &= \int_0^t \langle (\mathcal{E}u)'(s), \varphi(s) \rangle_V \, d\lambda(s) + \sum_{k=1}^m \int_0^t \int_X A \nabla u_k \cdot \nabla \varphi_k \, d\sigma \, d\lambda \\ &\ge \frac{1}{2} \sum_{k=1}^m \int_X |u_k^-(t)|^2 \, d\sigma + \nu \sum_{k=1}^m \int_0^t \int_X |\nabla u_k^-|^2 \, d\sigma \, d\lambda \end{aligned}$$

That means, we arrive at $u_1, \dots, u_m \ge 0$, and, hence, $u_0 = 1 - \sum_{k=1}^m u_k \le 1$.

3. Choosing the test function $\varphi = (-u_0^-, \dots, -u_0^-) \in L^2(S;V)$, from (19) we obtain

$$\begin{aligned} \sum_{k=1}^m \sum_{\ell=1}^m (\mathcal{R}u)_{k\ell} \nabla w_\ell \cdot \nabla \varphi_k &= -\sum_{\ell=1}^m \sum_{l=0}^m u_l^\diamond u_\ell^\diamond \nabla w_\ell \cdot \nabla u_0^- + \sum_{k=1}^m \sum_{\ell=1}^m u_\ell^\diamond u_k^\diamond \nabla w_\ell \cdot \nabla u_0^- \\ &= -\sum_{\ell=1}^m u_0^\diamond u_\ell^\diamond \nabla w_\ell \cdot \nabla u_0^- = 0, \end{aligned}$$

because of the relation $u_0^\diamond \nabla u_0^- = 0$. Applying the test function φ to (R) and remembering the facts that $u_0 = 1 - \sum_{k=1}^m u_k$ and $a_0 \ge 0$, for all $t \in \operatorname{cl} S$ integration by parts yields

$$\begin{aligned} 0 &= -\int_0^t \langle (\mathcal{E}u)'(s), \varphi(s) \rangle_V \, d\lambda(s) - \sum_{k=1}^m \int_0^t \int_X A \nabla u_k \cdot \nabla \varphi_k \, d\sigma \, d\lambda \\ &\ge \frac{1}{2} \int_X |u_0^-(t)|^2 \, d\sigma + \nu \int_0^t \int_X |\nabla u_0^-|^2 \, d\sigma \, d\lambda, \end{aligned}$$

in other words, we get $u_0 \ge 0$, and, therefore, $\sum_{k=1}^m u_k = 1 - u_0 \le 1$.

4. It follows from Step 2 and 3 of the proof that for every solution $u \in W_E(S;V)$ of the regularized problem (R) we have $u \in \operatorname{dom}(\mathcal{M})$. By the definition of truncation this yields $u = u^\diamond$. Consequently, $u \in W_E(S;V) \cap \operatorname{dom}(\mathcal{M})$ is a solution of the original problem (P). □

REMARK 7 (Conservation of particle number). If $u \in W_E(S;V) \cap \operatorname{dom}(\mathcal{M})$ is a solution of the evolution system (P) for the initial value $a \in U(0)$, then, taking $\varphi = (1, \dots, 1) \in L^2(S;V)$ as a test function in (P), we get that for every component the number of particles is conserved:

$$\int_X u_k(t) \, d\sigma = \int_X a_k \, d\sigma \quad \text{for all } t \in \operatorname{cl} S \text{ and every } k \in \{0, 1, \dots, m\}.$$

4. UNIQUENESS AND UNIFORM REGULARITY OF THE SOLUTION

Before we prove the uniqueness and the uniform regularity of the solution to problem (P), we collect main results of our theory of maximal regularity for parabolic boundary value problems with nonsmooth data in Sobolev–Morrey and Hölder spaces.

Parabolic Morrey spaces of functions. For $r > 0$ we define a set of subintervals of S by

$$\mathcal{S}^r = \big\{S \cap (t - r^2, t) : t \in S\big\}.$$

Let $G = \operatorname{cl} X$ be the closure of the domain $X \subset \mathbb{R}^n$ with Lipschitz boundary. For $r > 0$ we introduce the set of intersections of X and G with cubes, respectively, by

$$\mathcal{X}_r = \big\{X \cap Q(x, r) : x \in X\big\} \quad \text{and} \quad \mathcal{G}_r = \big\{G \cap Q(x, r) : x \in G\big\}.$$

The following classical definitions go back to CAMPANATO [2] and DA PRATO [3]:

DEFINITION 6. For $\omega \in [0, n+2]$ the *parabolic Morrey space* $L_2^\omega(S; H)$ is the set of all functions $u \in L^2(S; H)$ such that

$$[u]^2_{L_2^\omega(S;H)} = \sup_{r>0} \sup_{I \in \mathcal{S}^r} \sup_{Y \in \mathcal{X}_r} r^{-\omega} \sum_{k=1}^m \int_I \int_Y |u_k|^2 \, d\sigma \, d\lambda$$

remains finite. The norm of $u \in L_2^\omega(S; H)$ is defined by

$$\|u\|^2_{L_2^\omega(S;H)} = \|u\|^2_{L^2(S;H)} + [u]^2_{L_2^\omega(S;H)}.$$

REMARK 8. Note that these spaces are usually denoted by $L^{2,\omega}(S \times X; \mathbb{R}^m)$. Apart from these, later on we use further parabolic Morrey-type function spaces. Hence, we have decided to use a different but integrated naming scheme. The set $L^\infty(S \times X)$ of bounded measurable functions is a space of multipliers for $L_2^\omega(S; H)$.

DEFINITION 7. For $\omega \in [0, n+2]$ we introduce the *parabolic Sobolev–Morrey space*

$$L_2^\omega(S; V) = \big\{u \in L^2(S; V) : (|\nabla u_1|, \dots, |\nabla u_m|) \in L_2^\omega(S; H)\big\},$$

and we define the norm of $u \in L_2^\omega(S; V)$ by

$$\|u\|^2_{L_2^\omega(S;V)} = \|u\|^2_{L^2(S;V)} + [(|\nabla u_1|, \dots, |\nabla u_m|)]^2_{L_2^\omega(S;H)}.$$

Parabolic Sobolev–Morrey spaces of functionals. We define function spaces associated with relative open subsets Y of the closure $G = \operatorname{cl} X$ of the domain $X \subset \mathbb{R}^n$ with Lipschitz boundary. Following the work [24] of GRÖGER, by $W_0^{1,2}(Y; \mathbb{R}^m)$ we denote the closure of

$$C_0^\infty(Y; \mathbb{R}^m) = \big\{u|\operatorname{int} Y : u \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^m),\ \operatorname{supp}(u) \cap (\operatorname{cl} Y \setminus Y) = \varnothing\big\}$$

in $W^{1,2}(\operatorname{int} Y; \mathbb{R}^m)$, and we write $W^{-1,2}(Y; \mathbb{R}^m)$ for the dual space of $W_0^{1,2}(Y; \mathbb{R}^m)$. In particular, $W_0^{1,2}(G; \mathbb{R}^m)$ and $W^{-1,2}(G; \mathbb{R}^m)$ coincide with the classical Sobolev spaces $V = W^{1,2}(X; \mathbb{R}^m)$ and $V^* = [W^{1,2}(X; \mathbb{R}^m)]^*$, respectively. For a shorter notation we introduce the family $\mathfrak{V} = \big\{W_0^{1,2}(Y; \mathbb{R}^m) : Y \text{ relative open subset of } G\big\}$ of these subspaces.

Let I be an open subinterval of S and consider a subspace $V_0 \in \mathfrak{V}$ of V. If $Z_{V_0} : V_0 \to V$ is the zero extension map, then we define $\mathcal{Z}_{I,V_0} : L^2(I;V_0) \to L^2(S;V)$ by

$$(\mathcal{Z}_{I,V_0} u)(s) = \begin{cases} Z_{V_0} u(s) & \text{if } s \in I, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } u \in L^2(I;V_0).$$

Note that $\mathcal{Z}_{I,V_0}$ is a linear isometry from $L^2(I;V_0)$ into $L^2(S;V)$. In the same spirit as the parabolic Morrey spaces of functions, in GRIEPENTROG [21, Section 5] we have constructed a new scale of parabolic Sobolev–Morrey spaces of functionals as subspaces of $L^2(S;V^*)$. To localize a functional $f \in L^2(S;V^*)$ we define the mapping $f \mapsto \mathcal{L}_{I,V_0} f$ from $L^2(S;V^*)$ into $L^2(I;V_0^*)$ as the adjoint operator to the extension map $\mathcal{Z}_{I,V_0} : L^2(I;V_0) \to L^2(S;V)$. That means, we set

$$\langle \mathcal{L}_{I,V_0} f, \varphi \rangle_{L^2(I;V_0)} = \langle f, \mathcal{Z}_{I,V_0} \varphi \rangle_{L^2(S;V)} \quad \text{for } \varphi \in L^2(I;V_0).$$

The isometric property of $\mathcal{Z}_{I,V_0}$ yields

$$\| \mathcal{L}_{I,V_0} f \|_{L^2(I;V_0^*)} \le \| f \|_{L^2(S;V^*)} \quad \text{for all } f \in L^2(S;V^*).$$

DEFINITION 8. For $r > 0$ we consider the subfamily $\mathfrak{V}_r = \{ W_0^{1,2}(Y;\mathbb{R}^m) : Y \in \mathcal{G}_r \}$ of $\mathfrak{V}$. Then, for $\omega \in [0, n+2]$ we define the *parabolic Sobolev–Morrey space* $L_2^\omega(S;V^*)$ *of functionals* as the set of all elements $f \in L^2(S;V^*)$ for which

$$[f]^2_{L_2^\omega(S;V^*)} = \sup_{r>0} \sup_{I \in \mathcal{S}^r} \sup_{V_0 \in \mathfrak{V}_r} r^{-\omega} \, \| \mathcal{L}_{I,V_0} f \|^2_{L^2(I;V_0^*)}$$

has a finite value. We introduce the norm of $f \in L_2^\omega(S;V^*)$ by

$$\| f \|^2_{L_2^\omega(S;V^*)} = \| f \|^2_{L^2(S;V^*)} + [f]^2_{L_2^\omega(S;V^*)}.$$

Then, the following theorem on functionals holds true, see [21, Theorem 5.6]:

THEOREM 3. *If Assumption 1 is satisfied, then the mapping* $(g_0, g) \mapsto f$ *defined by*

$$\langle f, \varphi \rangle_{L^2(S;V)} = \sum_{k=1}^m \int_S \int_X g_{0k} \, \varphi_k \, d\sigma \, d\lambda + \sum_{k=1}^m \int_S \int_X g_k \cdot \nabla \varphi_k \, d\sigma \, d\lambda \quad \textit{for } \varphi \in L^2(S;V),$$

generates a linear continuous operator from $L_2^{\omega-2}(S;H) \times [L_2^\omega(S;H)]^n$ *into* $L_2^\omega(S;V^*)$ *for* $\omega \in [0, n+2]$, *and its norm depends on* m, n *and* X, *only.*

Sobolev–Morrey spaces for evolution equations. Based upon the preceeding definitions, in GRIEPENTROG [21, Section 6] we have constructed a new function class suitable for the regularity theory of second order parabolic boundary value problems with nonsmooth data, see GRIEPENTROG [22, Sections 6 and 7]. In fact, all these results were already established in the doctoral thesis [17] of the author.

Definition 9. Using the same notation as in Definition 2 and 3, for $\omega \in [0, n+2]$ we introduce the *Sobolev–Morrey space for evolution equations*

$$W_E^\omega(S;V) = \big\{u \in W_E(S;V) \cap L_2^\omega(S;V) : (\mathcal{E}u)' \in L_2^\omega(S;V^*)\big\}.$$

The norm is defined by

$$\|u\|^2_{W_E^\omega(S;V)} = \|u\|^2_{L_2^\omega(S;V)} + \|(\mathcal{E}u)'\|^2_{L_2^\omega(S;V^*)} \quad \text{for } u \in W_E^\omega(S;V^*).$$

It turns out to be of great advantage that these spaces are embedded into parabolic Hölder spaces, see Griepentrog [21, Theorem 3.4, Theorem 6.8, Theorem 6.9]:

Theorem 4 (Embedding). *If Assumption 1 holds true, then for the exponents $\omega \in (n, n+2]$ and $\alpha = \frac{1}{2}(\omega - n) \in (0,1]$ the space $W_E^\omega(S;V)$ is continuously embedded into $C^{0,\alpha/2}(\operatorname{cl} S; C(\operatorname{cl} X;\mathbb{R}^m)) \cap C(\operatorname{cl} S; C^{0,\alpha}(\operatorname{cl} X;\mathbb{R}^m))$. For every exponent $\alpha \in \big(0, \frac{1}{2}(\omega-n)\big)$ this embedding is completely continuous.*

The main tools to prove uniqueness and uniform regularity of the solution to problem (P) are maximal regularity results for linear parabolic boundary value problems with nonsmooth data in Sobolev–Morrey spaces, see Griepentrog and Recke in [22, Theorem 6.8] and [23, Theorem 3.1]:

Theorem 5 (Maximal parabolic regularity). *Let the Assumptions 1 and 3 be satisfied.*

1. There exists $\bar\omega \in (n, n+2]$ such that for every $\omega \in [0,\bar\omega)$ the mapping $u \mapsto (\mathcal{E}u)' + \mathcal{L}u$ is a linear isomorphism from $\{u \in W_E^\omega(S;V) : (\mathcal{K}u)(0) = 0\}$ onto $L_2^\omega(S;V^*)$.

2. Assume that $\mathcal{N} : C(\operatorname{cl} S; C(\operatorname{cl} X;\mathbb{R}^m)) \to L_2^{\omega_2}(S;V^*)$ is a bounded linear Volterra operator for some $\omega_2 \in (n, n+2]$. Then there exists an exponent $\bar\omega \in (n, \omega_2]$ such that for every $\omega \in (n, \bar\omega]$ the mapping $u \mapsto (\mathcal{E}u)' + \mathcal{L}u + \mathcal{N}u$ is a linear isomorphism from $\{u \in W_E^\omega(S;V) : (\mathcal{K}u)(0) = 0\}$ onto $L_2^\omega(S;V^*)$.

To apply these results, in addition to Remark 6 it suffices to suppose a regularity assumption on the interaction functional Ψ in terms of a suitable Morrey-type condition. In fact, this universal condition ensures not only uniqueness and uniform regularity, but also uniform positivity and asymptotic convergence of the solution to problem (P).

Definition 10. Let $\omega \in [0, n]$ be some fixed exponent.

1. The *Morrey space* $L^{2,\omega}(X;\mathbb{R}^m)$ is the set of all functions $h \in H$ such that

$$[h]^2_{L^{2,\omega}(X;\mathbb{R}^m)} = \sup_{r>0} \sup_{Y \in \mathcal{X}_r} r^{-\omega} \sum_{k=1}^m \int_Y |h_k|^2 \, d\sigma$$

remains finite. The norm of $h \in L^{2,\omega}(X;\mathbb{R}^m)$ is defined by

$$\|h\|^2_{L^{2,\omega}(X;\mathbb{R}^m)} = \|h\|^2_H + [h]^2_{L^{2,\omega}(X;\mathbb{R}^m)}.$$

2. We define the *Sobolev–Morrey space* by

$$W^{1,2,\omega}(X;\mathbb{R}^m) = \big\{w \in V : (|\nabla w_1|, \dots, |\nabla w_m|) \in L^{2,\omega}(X;\mathbb{R}^m)\big\},$$

and the norm of $w \in W^{1,2,\omega}(X;\mathbb{R}^m)$ is given by

$$\|w\|^2_{W^{1,2,\omega}(X;\mathbb{R}^m)} = \|w\|^2_V + [(|\nabla w_1|, \dots, |\nabla w_m|)]^2_{L^{2,\omega}(X;\mathbb{R}^m)}.$$

Assumption 6 (Regularity of interaction). Introducing the abbreviations

$$L^\infty = L^\infty(X;\mathbb{R}^m) \subset H \quad \text{and} \quad C = C(\operatorname{cl} X;\mathbb{R}^m) \subset L^\infty,$$

as well as

$$C^{0,\alpha} = C^{0,\alpha}(\operatorname{cl} X;\mathbb{R}^m) \subset C \quad \text{and} \qquad W^{1,2,\omega} = W^{1,2,\omega}(X;\mathbb{R}^m) \subset V$$

for $\alpha \in (0,1]$ and $\omega \in [0,n]$, we suppose that $P \in \mathcal{L}(H^*;V)$ and $\psi \in V$ defined in (17) satisfy the regularity properties

$$P|J[L^\infty] \in \mathcal{L}(J[L^\infty]; W^{1,2,\omega_0}) \quad \text{and} \quad \psi \in W^{1,2,\omega_0}$$

for some given Morrey exponent $\omega_0 \in (n-2,n]$. The corresponding Hölder exponent is denoted by $\alpha_0 = \frac{1}{2}(\omega_0 - n + 2) \in (0,1]$.

Remark 9. Since the embedding of $W^{1,2,\omega}$ into $C^{0,\alpha}$ is continuous for $\omega \in (n-2,n]$ and $\alpha = \frac{1}{2}(\omega - n + 2) \in (0,1]$, it is completely continuous for every $\alpha \in \big(0, \frac{1}{2}(\omega - n + 2)\big)$. Hence, $P|J[L^\infty]$ is completely continuous as an operator in $\mathcal{L}(J[L^\infty]; L^\infty)$.

According to Assumption 2 and Remark 6, there exists a constant $L_0 > 0$ such that for all admissible states $\rho \in \operatorname{dom}(\Phi) = J[U(0)]$ the estimate

$$\sum_{\ell=1}^m \int_Y |\nabla w_\ell|^2 \, d\sigma \le L_0 r^{\omega_0} \quad \text{for all } r > 0 \text{ and } Y \in \mathcal{X}_r$$

holds true, whenever $w = P\rho + \psi \in W^{1,2,\omega_0}$ are the corresponding interaction potentials.

Remark 10. Note that Assumption 6 is much weaker compared with Gajewski and Zacharias in [11], Londen and Petzeltova in [28, 29] or Gal and Grasselli in [12]. These authors consider the homogeneous case $\sigma = \lambda^n$ and suppose that $P \in \mathcal{L}(H^*;V)$ and $\psi \in V$ satisfy Lipschitz conditions

$$P|J[L^\infty] \in \mathcal{L}(J[L^\infty]; C^{0,1}) \quad \text{and} \quad \psi \in C^{0,1}$$

which excludes many interactions governed by nonsmooth data. We refer to Section 7 for a comprehensive discussion of an illustrating example for nonlocal phase separation.

Remark 11. Due to Assumptions 5, 6 and Remark 9, for all $r > 0$, $I \in \mathcal{S}^r$ and $Y \in \mathcal{X}_r$ the potentials $w = \mathcal{P}\mathcal{J}u + \psi \in L^2(S;V)$ corresponding to $u \in \operatorname{dom}(\mathcal{M})$ satisfy

$$r^{-\omega_0 - 2} \sum_{k=1}^m \int_I \int_Y |\nabla w_k|^2 \, d\sigma \, d\lambda = r^{-2} \int_I r^{-\omega_0} \sum_{k=1}^m \int_Y |\nabla w_k(s)|^2 \, d\sigma \, d\lambda(s) \le L_0.$$

The mapping $u \mapsto w = \mathcal{P}\mathcal{J}u + \psi$ is a continuous affine operator from $L^\infty(S \times X; \mathbb{R}^m)$ into $L_2^{\omega_2}(S; V)$ for $\omega_2 = \omega_0 + 2 \in (n, n+2]$.

The proof of the uniqueness result presented in this work is based on the global regularity of the difference of two solutions to problem (P).

THEOREM 6 (Unique solvability). *Suppose the Assumptions 1–6 to be fulfilled. Then, for all initial values $a \in U(0)$ and every time interval $S = (0, T)$ the evolution system* (P) *has a unique solution $u \in W_E(S; V) \cap \operatorname{dom}(\mathcal{M})$.*

Proof. 1. Let u, $\hat{u} \in W_E(S; V) \cap \operatorname{dom}(\mathcal{M})$ be two solutions of problem (P). Then, setting

$$f = \mathcal{B}(\mathcal{M}\hat{u}, \mathcal{P}\mathcal{J}\hat{u} + \psi) - \mathcal{B}(\mathcal{M}u, \mathcal{P}\mathcal{J}u + \psi) \in L^2(S, V^*), \tag{23}$$

the difference $c = u - \hat{u} \in W_E(S; V)$ solves the linear parabolic problem

$$(\mathcal{E}c)' + \mathcal{L}c = f, \quad (\mathcal{K}c)(0) = 0.$$

Following Theorem 3 and Remark 11, due to u, $\hat{u} \in \operatorname{dom}(\mathcal{M})$ and $\mathcal{P}\mathcal{J}u + \psi$, $\mathcal{P}\mathcal{J}\hat{u} + \psi \in L_2^{\omega_2}(S; V)$, the right-hand side f even belongs to $L_2^{\omega_2}(S; V^*)$. Now, Theorem 5 on maximal parabolic regularity in Sobolev–Morrey spaces yields $c \in W_E^\omega(S; V)$ for some $\omega \in (n, \omega_2]$. Having in mind Theorem 4, we obtain the continuity property $c \in C(\operatorname{cl} S; C)$.

2. Using the bilinearity of $\mathcal{B}$, from (23) we deduce an alternative representation of the functional $f \in L_2^{\omega_2}(S; V^*)$ in terms of the difference $c = u - \hat{u} \in W_E^\omega(S; V)$:

$$\begin{aligned}
-f &= \mathcal{B}(\operatorname{diag}(u) - u \otimes u, \mathcal{P}\mathcal{J}u + \psi) - \mathcal{B}(\operatorname{diag}(\hat{u}) - \hat{u} \otimes u, \mathcal{P}\mathcal{J}u + \psi) \\
&\quad + \mathcal{B}(\operatorname{diag}(\hat{u}) - \hat{u} \otimes u, \mathcal{P}\mathcal{J}u + \psi) - \mathcal{B}(\operatorname{diag}(\hat{u}) - \hat{u} \otimes u, \mathcal{P}\mathcal{J}\hat{u} + \psi) \\
&\quad + \mathcal{B}(\operatorname{diag}(\hat{u}) - \hat{u} \otimes u, \mathcal{P}\mathcal{J}\hat{u} + \psi) - \mathcal{B}(\operatorname{diag}(\hat{u}) - \hat{u} \otimes \hat{u}, \mathcal{P}\mathcal{J}\hat{u} + \psi) \\
&\quad = \mathcal{B}(\operatorname{diag}(c) - c \otimes u, \mathcal{P}\mathcal{J}u + \psi) + \mathcal{B}(\operatorname{diag}(\hat{u}) - \hat{u} \otimes u, \mathcal{P}\mathcal{J}c) - \mathcal{B}(\hat{u} \otimes c, \mathcal{P}\mathcal{J}\hat{u} + \psi).
\end{aligned}$$

Using this representation and $\omega_2 \in (n, n+2]$, Theorem 3 allows us to define a bounded linear Volterra operator $\mathcal{N} : C(\operatorname{cl} S; C) \to L_2^{\omega_2}(S; V^*)$ by setting

$$\mathcal{N}\hat{c} = \mathcal{B}(\operatorname{diag}(\hat{c}) - \hat{c} \otimes u, \mathcal{P}\mathcal{J}u + \psi) + \mathcal{B}(\operatorname{diag}(\hat{u}) - \hat{u} \otimes u, \mathcal{P}\mathcal{J}\hat{c}) - \mathcal{B}(\hat{u} \otimes \hat{c}, \mathcal{P}\mathcal{J}\hat{u} + \psi)$$

for $\hat{c} \in C(\operatorname{cl} S; C)$. Because we have $\mathcal{N}c = -f$, Step 1 of the proof yields that the difference $c \in W_E(S; V) \cap C(\operatorname{cl} S; C)$ solves the linear parabolic problem

$$(\mathcal{E}c)' + \mathcal{L}c + \mathcal{N}c = 0, \quad (\mathcal{K}c)(0) = 0.$$

By Theorem 5 on maximal parabolic regularity in Sobolev–Morrey spaces, the mapping $c \mapsto (\mathcal{E}c)' + \mathcal{L}c + \mathcal{N}c$ is a linear isomorphism from $\{u \in W_E^\omega(S; V) : (\mathcal{K}u)(0) = 0\}$ onto $L_2^\omega(S; V^*)$ for some $\omega \in (n, \omega_2]$. Hence, we get $c = u - \hat{u} = 0$, which finishes the proof. $\square$

THEOREM 7 (Uniform regularity of the solution). *Let the Assumptions 1–6 be satisfied, and $\theta \in \left(0, \frac{1}{2}\right)$ and $T \geq 1$ be arbitrarily fixed. Suppose $a \in U(0)$ to be an admissible initial value and $u \in W_E(S; V) \cap \operatorname{dom}(\mathcal{M})$ to be the solution of the evolution system* (P).

1. Then, we have $u|(\theta,T) \in W_E^\omega((\theta,T);V)$ for some $\omega \in (n,\omega_2]$ depending on m, n, ν, $\omega_2 \in (n,n+2]$, and X.

2. The norm of $u|(t_0,t_1)$ in the space $W_E^\omega((t_0,t_1);V)$ is uniformly bounded regardless of the choice of $T \geq 1$ or $t_0 \in \big[\theta, T-\frac{1}{2}\big]$, whenever $t_1 = t_0 + \frac{1}{2}$ holds true. In this case the bound depends on m, n, ν, ω_0, θ, L, L_0 and X.

Proof. 1. Since the initial value $a \in U(0)$ may have spatial discontinuities, we can not expect the solution to be regular in the whole interval S. We arbitrarily fix $t_0 \in \big[\theta, T-\frac{1}{2}\big]$ and choose some temporal cut-off function $\vartheta \in C^\infty(\mathbb{R})$ satisfying $0 \leq \vartheta \leq 1$ and

$$\vartheta(s) = 0 \quad \text{for } s \leq t_0 - \theta, \quad \vartheta(s) = 1 \quad \text{for } s \geq t_0, \quad \text{and} \quad |\vartheta'(s)| \leq \frac{2}{\theta} \quad \text{for } s \in \mathbb{R}$$

to consider the regularity properties of the product ϑu.

2. Considering $t_1 \in (t_0,T]$ and testing (P) with the product $\vartheta\varphi$ for functions $\varphi \in L^2(S;V)$ vanishing outside the subinterval $S_0 = (t_0-\theta, t_1)$ of S, we obtain that the restriction $\hat{u} = (\vartheta u)|S_0 \in W_E(S_0;V)$ is the solution to the evolution problem

$$(\mathcal{E}_0\hat{u})' + \mathcal{L}_0\hat{u} = f_0, \quad (\mathcal{K}_0\hat{u})(t_0-\theta) = 0, \tag{24}$$

where the operators $\mathcal{K}_0$, $\mathcal{E}_0$ and $\mathcal{L}_0$ are associated to S_0 and V as in Definition 2 and 3 and Assumption 3, and the right-hand side $f_0 \in L^2(S_0;V^*)$ is represented by

$$\begin{aligned}\langle f_0, \varphi\rangle_{L^2(S_0;V)} &= \sum_{k=1}^m \int_{S_0}\int_X \vartheta' u_k\, \varphi_k \, d\sigma\, d\lambda \\ &\quad - \sum_{k=1}^m\sum_{\ell=1}^m \int_{S_0}\int_X \vartheta A(\mathcal{M}u)_{k\ell} \nabla(\mathcal{P}\mathcal{J}u + \psi)_\ell \cdot \nabla\varphi_k \, d\sigma\, d\lambda\end{aligned}$$

for $\varphi \in L^2(S_0;V)$. In view of $u \in \operatorname{dom}(\mathcal{M})$ and $\mathcal{P}\mathcal{J}u + \psi \in L_2^{\omega_2}(S;V)$ the right-hand side f_0 of (24) belongs to $L_2^{\omega_2}(S_0;V^*)$ due to Theorem 3 and Remark 11, and the norm of f_0 in $L_2^{\omega_2}(S_0;V^*)$ depends on m, n, ω_0, θ, L, L_0 and X.

3. In the case $S_0 = S$, Step 2 and Theorem 5 on maximal parabolic regularity in Sobolev–Morrey spaces yield $\hat{u} = \vartheta u \in W_E^\omega(S;V)$ for some $\omega \in (n,\omega_2]$, which leads to $u|(\theta,T) \in W_E^\omega((\theta,T);V)$.

4. Setting $t_1 = t_0 + \frac{1}{2}$, the subinterval S_0 of S has the fixed length $\theta + \frac{1}{2}$. Together with Step 2 we obtain that the norm of the functional f_0 in $L^{\omega_2}(S_0;V^*)$ is uniformly bounded for all choices of the subinterval $S_0 = \big(t_0-\theta, t_0+\frac{1}{2}\big)$ in S. Due to Theorem 5 this carries over to the norm of the solution $\hat{u} \in W_E^\omega(S_0;V)$ of (24) with respect to the family of subintervals S_0 under consideration. Consequently, the norm of $u|(t_0,t_1)$ in the space $W_E^\omega((t_0,t_1);V)$ is uniformly bounded regardless of the choice of $T \geq 1$ or $t_0 \in \big[\theta, T-\frac{1}{2}\big]$, whenever $t_1 = t_0 + \frac{1}{2}$ holds true. $\square$

5. Uniform positivity of the solution

In this section we restrict ourselves to consider initial values $a \in U(0)$ which have the property that for every $k \in \{0, 1, \dots, m\}$ the k-th component has a positive number $\int_X a_k \, d\sigma > 0$ of particles. In the opposite case $\int_X a_k \, d\sigma = 0$, the k-th component would not give a contribution to the multicomponent mixture and could be dropped out of the system, since the particle number is conserved along the trajectory, see Remark 7.

ASSUMPTION 7 (Strict admissibility). Let $a \in U(0)$ be some admissible initial value for problem (P) and consider the *mean values*

$$\bar{a}_k = \fint_X a_k \, d\sigma = \frac{1}{\sigma(X)} \int_X a_k \, d\sigma \in [0, 1] \quad \text{for } k \in \{0, 1, \dots, m\}.$$

We suppose that $\bar{a}_k \in (0, 1)$ holds true for every $k \in \{0, 1, \dots, m\}$ and set

$$\beta = \tfrac{1}{2} \min\{\bar{a}_0, \bar{a}_1, \dots, \bar{a}_m\} \in \big(0, \tfrac{1}{2}\big).$$

In this case the initial value $a \in U(0)$ is called *strictly admissible*.

DEFINITION 11 (Regularized potentials). Let Assumption 7 be satisfied and set

$$\Gamma_\beta = \big\{\gamma > 0 : \gamma^2 < \min\big\{\tfrac{\beta}{2}, \beta^2\big\}\big\}.$$

For every $\gamma \in \Gamma_\beta$ we consider a convex function $\iota_\gamma \in C^\infty(\mathbb{R})$ with the property

$$\iota_\gamma(z) = \ln \frac{\beta}{z + \gamma^2} \quad \text{for } z \in [0, 1].$$

If $u \in W_E(S; V) \cap \operatorname{dom}(\mathcal{M})$ is the solution of (P), then the functions $\zeta_k = \iota_\gamma \circ u_k \in L^2(S; W^{1,2}(X)) \cap L^\infty(S \times X)$, defined for $k \in \{0, 1, \dots, m\}$, are called *corresponding γ-regularized potentials.*

REMARK 12 (Measure estimate for level sets). Suppose Assumption 7 to be fulfilled. Then for every $k \in \{0, 1, \dots, m\}$ and $\gamma \in \Gamma_\beta$ we consider the *level sets*

$$N_{\beta-\gamma^2}(a_k) = \big\{x \in X : a_k(x) \geq \beta - \gamma^2\big\},$$

and we obtain

$$\begin{aligned} \bar{a}_k \, \sigma(X) &= \int_{N_{\beta-\gamma^2}(a_k)} a_k \, d\sigma + \int_{X \setminus N_{\beta-\gamma^2}(a_k)} a_k \, d\sigma \\ &\leq \sigma(N_{\beta-\gamma^2}(a_k)) + (\beta - \gamma^2)\big(\sigma(X) - \sigma(N_{\beta-\gamma^2}(a_k))\big), \end{aligned}$$

which leads to the following measure estimate

$$\sigma(N_{\beta-\gamma^2}(a_k)) \geq (1 - \beta + \gamma^2) \, \sigma(N_{\beta-\gamma^2}(a_k)) \geq (\bar{a}_k - \beta + \gamma^2) \, \sigma(X) \geq \beta\sigma(X).$$

If $u \in W_E(S; V) \cap \operatorname{dom}(\mathcal{M})$ is the solution of (P), then $u(s) \in V$ satisfies the same relations for every $s \in S$, since the particle number is conserved, see Remark 7.

We apply a variant of Moser-type iteration to get Harnack-type estimates for the lower bound of the relative densities based on the classical results [26, 27, 32] of KRUZHKOV and MOSER.

LEMMA 8 (Basis of Moser-type iteration). *Let the Assumptions 1–7 be satisfied and* $u \in W_E(S;V) \cap \operatorname{dom}(\mathcal{M})$ *be the solution of* (P) *to the strictly admissible initial value* $a \in U(0)$. *Suppose* $\zeta_0, \zeta_1, \ldots, \zeta_m$ *to be the corresponding* γ*-regularized potentials for some* $\gamma \in \Gamma_\beta$ *and* $w = \mathcal{P}\mathcal{J}u + \psi$ *to be the corresponding interaction potentials.*

Then, for all t_0, $t \in S$ *with* $t_0 < t$ *we have the estimate*

$$\frac{\nu}{2}\sum_{k=0}^{m}\int_{t_0}^{t}\int_X |\nabla\zeta_k|^2\,d\sigma\,d\lambda \le (m+1)\,\sigma(X)\ln\frac{\beta}{\gamma^2}\left(\frac{1}{\beta}+\frac{1}{2}\right) + \frac{m(m+1)}{2\nu^3}\sum_{k=1}^{m}\int_{t_0}^{t}\int_X |\nabla w_k|^2\,d\sigma\,d\lambda.$$

Proof. 1. Due to our assumptions on $\iota_\gamma \in C^\infty(\mathbb{R})$ we have

$$\iota_\gamma'(z) = -\frac{1}{z+\gamma^2} \quad \text{and} \quad \iota_\gamma''(z) = \frac{1}{(z+\gamma^2)^2} = |\iota_\gamma'(z)|^2$$

as well as

$$-\ln\left(\frac{1}{\beta}+\frac{1}{2}\right) \le \ln\frac{\beta}{1+\gamma^2} \le \iota_\gamma(z) \le \ln\frac{\beta}{\gamma^2} \quad \text{for all } z \in [0,1].$$

2. We consider the test function $\varphi = \big(\iota_\gamma' \circ u_1, \ldots, \iota_\gamma' \circ u_m\big) \in L^2(S;V) \cap L^\infty(S \times X;\mathbb{R}^m)$ for (P). Then, integrating by parts, for all t_0, $t \in S$ with $t_0 < t$ we get

$$\begin{aligned}\int_{t_0}^{t}\langle(\mathcal{E}u)'(s),\varphi(s)\rangle_V\,d\lambda(s) &= \sum_{k=1}^{m}\int_X \zeta_k(t)\,d\sigma - \sum_{k=1}^{m}\int_X \zeta_k(t_0)\,d\sigma\\ &\ge -m\sigma(X)\ln\frac{\beta}{\gamma^2}\left(\frac{1}{\beta}+\frac{1}{2}\right).\end{aligned}$$

3. Using the properties of ι_γ and the positivity of A, we obtain

$$\langle \mathcal{L}u,\varphi\rangle_{L^2(S;V)} = \sum_{k=1}^{m}\int_{t_0}^{t}\int_X |\iota_\gamma'\circ u_k|^2 A\nabla u_k\cdot\nabla u_k\,d\sigma\,d\lambda \ge \nu\sum_{k=1}^{m}\int_{t_0}^{t}\int_X |\nabla\zeta_k|^2\,d\sigma\,d\lambda.$$

Applying Young's inequality to the right-hand side of the identity

$$\langle \mathcal{B}(\mathcal{M}u,w),\varphi\rangle_{L^2(S;V)} = -\sum_{k=1}^{m}\sum_{\ell=1}^{m}\int_{t_0}^{t}\int_X \frac{u_k(\delta_{k\ell}-u_\ell)}{u_k+\gamma^2}\,A\nabla w_\ell\cdot\nabla\zeta_k\,d\sigma\,d\lambda,$$

and remembering the boundedness of A, we also get the following estimate

$$\langle \mathcal{B}(\mathcal{M}u,w),\varphi\rangle_{L^2(S;V)} \ge -\frac{\nu}{2}\sum_{k=1}^{m}\int_{t_0}^{t}\int_X |\nabla\zeta_k|^2\,d\sigma\,d\lambda - \frac{m^2}{2\nu^3}\sum_{\ell=1}^{m}\int_{t_0}^{t}\int_X |\nabla w_\ell|^2\,d\sigma\,d\lambda.$$

4. Summing up the results of Step 2 and 3, the variational formulation of (P) yields

$$\frac{\nu}{2}\sum_{k=1}^{m}\int_{t_0}^{t}\int_X |\nabla\zeta_k|^2\,d\sigma\,d\lambda \le m\sigma(X)\ln\frac{\beta}{\gamma^2}\left(\frac{1}{\beta}+\frac{1}{2}\right)+\frac{m^2}{2\nu^3}\sum_{\ell=1}^{m}\int_{t_0}^{t}\int_X |\nabla w_\ell|^2\,d\sigma\,d\lambda$$

for all t_0, $t \in S$ with $t_0 < t$.

5. We repeat the above considerations with the test function $\varphi = \left(\iota'_\gamma \circ u_0, \dots, \iota'_\gamma \circ u_0\right) \in L^2(S;V) \cap L^\infty(S \times X;\mathbb{R}^m)$ for (P). Integrating by parts, for all t_0, $t \in S$ with $t_0 < t$ we obtain the estimate

$$-\int_{t_0}^{t}\langle(\mathcal{E}u)'(s),\varphi(s)\rangle_V\,d\lambda(s) = \int_X \zeta_0(t)\,d\sigma - \int_X \zeta_0(t_0)\,d\sigma \ge -\sigma(X)\ln\frac{\beta}{\gamma^2}\left(\frac{1}{\beta}+\frac{1}{2}\right).$$

6. Similarly to Step 3 we get

$$-\langle\mathcal{L}u,\varphi\rangle_{L^2(S;V)} = \int_{t_0}^{t}\int_X |\iota'_\gamma \circ u_0|^2 A\nabla u_0\cdot\nabla u_0\,d\sigma\,d\lambda \ge \nu\int_{t_0}^{t}\int_X |\nabla\zeta_0|^2\,d\sigma\,d\lambda.$$

Using Young's inequality to estimate the right-hand side of the identity

$$-\langle\mathcal{B}(\mathcal{M}u,w),\varphi\rangle_{L^2(S;V)} = \sum_{\ell=1}^{m}\int_{t_0}^{t}\int_X \frac{u_0 u_\ell}{u_0+\gamma^2}\,A\nabla w_\ell\cdot\nabla\zeta_0\,d\sigma\,d\lambda,$$

and having in mind the boundedness of A, we obtain

$$-\langle\mathcal{B}(\mathcal{M}u,w),\varphi\rangle_{L^2(S;V)} \ge -\frac{\nu}{2}\int_{t_0}^{t}\int_X |\nabla\zeta_0|^2\,d\sigma\,d\lambda - \frac{m}{2\nu^3}\sum_{\ell=1}^{m}\int_{t_0}^{t}\int_X |\nabla w_\ell|^2\,d\sigma\,d\lambda.$$

7. Summing up the results of Step 5 and 6, the variational formulation of (P) yields

$$\frac{\nu}{2}\int_{t_0}^{t}\int_X |\nabla\zeta_0|^2\,d\sigma\,d\lambda \le \sigma(X)\ln\frac{\beta}{\gamma^2}\left(\frac{1}{\beta}+\frac{1}{2}\right)+\frac{m}{2\nu^3}\sum_{\ell=1}^{m}\int_{t_0}^{t}\int_X |\nabla w_\ell|^2\,d\sigma\,d\lambda$$

for all t_0, $t \in S$ with $t_0 < t$. In view of Step 4, this finishes the proof. □

Since for every domain $X \subset \mathbb{R}^n$ with Lipschitz boundary there exists a bounded linear extension operator from $W^{1,2}(X)$ to $W^{1,2}(\mathbb{R}^n)$, see GIUSTI [16], the following generalized Sobolev embedding theorem of MAZYA [30, Corollary 1.4.7/2] is applicable. This will be our main tool to prove the uniform positivity of solutions under Assumption 6 on the regularity of the interaction operator, which is a much more general and natural assumption compared with that of LONDEN and PETZELTOVA in [28, Section 3] or GAL and GRASSELLI in [12, Section 4], see Remark 10.

THEOREM 9 (Embedding). *Let $X \subset \mathbb{R}^n$ be a domain with Lipschitz boundary. Suppose that μ is a Radon measure with support in* $\operatorname{cl} X$ *which satisfies*

$$\mu(\operatorname{cl} X \cap Q(x,r)) \le c_1 r^\omega \quad \textit{for all } x \in \mathbb{R}^n \textit{ and } 0 < r \le 1,$$

and some constants $c_1 > 0$ and $\omega \in (n-2, n]$. Then there exists some constant $c_2 > 0$ depending on c_1, n, ω and X, such that for all $v \in W^{1,2}(X)$ the multiplicative inequality

$$\int_{\operatorname{cl} X} |v|^2 \, d\mu \le c_2 \left(\int_X \left(|\nabla v|^2 + |v|^2 \right) d\lambda^n \right)^{1-\alpha} \left(\int_X |v|^2 \, d\lambda^n \right)^{\alpha}$$

holds true, where $\alpha = \frac{1}{2}(\omega - n + 2) \in (0, 1]$ is the corresponding Hölder exponent.

LEMMA 10 (Inductive step of Moser-type iteration). *Suppose the Assumptions 1–7 to be satisfied and $u \in W_E(S; V) \cap \operatorname{dom}(\mathcal{M})$ to be the solution of* (P) *to the strictly admissible initial value $a \in U(0)$. Let $\zeta_0, \zeta_1, \ldots, \zeta_m$ be the corresponding γ-regularized potentials for some $\gamma \in \Gamma_\beta$ and $w = \mathcal{P}\mathcal{J}u + \psi$ be the corresponding interaction potentials.*

Let $\varkappa = 1 + 2/n$ and $\alpha_0 = \frac{1}{2}(\omega_0 - n + 2) \in (0, 1]$ be the Hölder exponent corresponding to $\omega_0 \in (n-2, n]$. Let $t_0 \in [0, T)$ and $\theta \in \left(0, \frac{1}{2}\right)$ with $[t_0, t_0 + \theta] \subset [0, T)$ and consider the sequence $(t_i) \subset [t_0, t_0 + \theta)$ of points defined by $t_i = t_0 + \theta(1 - 2^{-i})$ for $i \in \mathbb{N}$.

Then there exists some constant $c > 0$ depending only on m, n, ν, α_0, β, θ, L_0, and X such that for every $p \ge 2$, $i \in \mathbb{N}$, and $t \in [t_0 + \theta, T]$ the following estimate holds true:

$$\sum_{k=0}^{m} \int_{t_{i+1}}^{t} \int_X |\zeta_k^+|^{\varkappa p} \, d\sigma \, d\lambda \le \left(2^{i+1} c \, p^{2/\alpha_0} \int_{t_i}^{t} \int_X \left(\sum_{k=0}^{m} |\zeta_k^+|^p + \sum_{k=1}^{m} |\nabla w_k|^2 \right) d\sigma \, d\lambda \right)^{\varkappa}.$$

Proof. 1. Let $i \in \mathbb{N}$ and $p \ge 2$ be fixed. Having in mind $t_i = t_0 + \theta(1 - 2^{-i})$, we choose some cut-off function $\vartheta \in C^\infty(\mathbb{R})$ satisfying $0 \le \vartheta \le 1$ and

$$\vartheta(s) = 0 \quad \text{for } s \le t_i, \quad \vartheta(s) = 1 \quad \text{for } s \ge t_{i+1}, \quad \text{and} \quad |\vartheta'(s)| \le \frac{2^{i+2}}{\theta} \quad \text{for } s \in \mathbb{R},$$

and we define functions $v_0, v_1, \ldots, v_m \in L^2(S; W^{1,2}(X)) \cap L^\infty(S \times X)$ by

$$v_k = -\frac{p \, |\zeta_k^+|^{p-1}}{u_k + \gamma^2} \, \vartheta^2 \chi_{[t_i, t]} \quad \text{for } k \in \{0, 1, \ldots, m\} \text{ and some } t \in (t_i, T],$$

where $\chi_{[t_i,t]} : \mathbb{R} \to [0, 1]$ is the indicator function of the interval $[t_i, t]$.

2. We take $\varphi = (v_1, \ldots, v_m) \in L^2(S; V) \cap L^\infty(S \times X; \mathbb{R}^m)$ as a test function for problem (P). Integrating by parts and applying the chain rule, for all $t \in (t_i, T]$ we get

$$\begin{aligned} \int_S \langle (\mathcal{E}u)'(s), \varphi(s) \rangle_V \, d\lambda(s) &= \sum_{k=1}^{m} \int_X |\zeta_k^+(t)|^p \, \vartheta^2(t) \, d\sigma - \sum_{k=1}^{m} \int_{t_i}^{t} \int_X |\zeta_k^+|^p \, 2\vartheta'\vartheta \, d\sigma \, d\lambda \\ &\ge \sum_{k=1}^{m} \int_X |\zeta_k^+(t)|^p \, \vartheta^2(t) \, d\sigma - \frac{2^{i+3}}{\theta} \sum_{k=1}^{m} \int_{t_i}^{t} \int_X |\zeta_k^+|^p \, d\sigma \, d\lambda. \end{aligned}$$

3. Due to our assumptions on $\iota_\gamma \in C^\infty(\mathbb{R})$ we have

$$\iota'_\gamma(z) = -\frac{1}{z + \gamma^2} \quad \text{and} \quad \iota''_\gamma(z) = \frac{1}{(z + \gamma^2)^2} = |\iota'_\gamma(z)|^2 \quad \text{for all } z \in [0, 1].$$

Hence, the chain rule yields

$$\nabla v_k = -\frac{p|\zeta_k^+|^{p-1} + p(p-1)|\zeta_k^+|^{p-2}}{u_k + \gamma^2}\, \vartheta^2 \chi_{[t_i,t]} \nabla\zeta_k^+ \quad \text{for every } k \in \{0,1,\dots,m\}.$$

Therefore, using the positivity of A, we get the estimate

$$\langle \mathcal{L}u, \varphi\rangle_{L^2(S;V)} \geq \nu \sum_{k=1}^m \int_{t_i}^t \int_X p\big(|\zeta_k^+|^{p-1} + (p-1)|\zeta_k^+|^{p-2}\big)\vartheta^2\, |\nabla\zeta_k^+|^2\, d\sigma\, d\lambda.$$

4. Using Young's inequality and the boundedness of A we obtain

$$\begin{aligned}\langle \mathcal{B}(\mathcal{M}u, w), \varphi\rangle_{L^2(S;V)} \geq{} & -\frac{\nu}{2} \sum_{k=1}^m \int_{t_i}^t \int_X p\big(|\zeta_k^+|^{p-1} + (p-1)|\zeta_k^+|^{p-2}\big)\vartheta^2\, |\nabla\zeta_k^+|^2\, d\sigma\, d\lambda \\ & - \frac{m}{2\nu^3} \sum_{k=1}^m \sum_{\ell=1}^m \int_{t_i}^t \int_X p\big(|\zeta_k^+|^{p-1} + (p-1)|\zeta_k^+|^{p-2}\big)\vartheta^2\, |\nabla w_\ell|^2\, d\sigma\, d\lambda.\end{aligned}$$

5. Having in mind another variant of Young's inequality, namely, $b^\delta d^{1-\delta} \leq \delta b + (1-\delta)d$ for every b, $d \geq 0$ and $0 \leq \delta \leq 1$, for $k \in \{0,1,\dots,m\}$ we get the polynomial estimate

$$p|\zeta_k^+|^{p-1} + p(p-1)|\zeta_k^+|^{p-2} \leq \big((p-1)|\zeta_k^+|^p + 1\big) + \big((p-1)(p-2)|\zeta_k^+|^p + 2(p-1)\big),$$

and, hence,

$$\tfrac{1}{2}p^2\, |\zeta_k^+|^{p-2} \leq p(p-1)|\zeta_k^+|^{p-2} \leq p|\zeta_k^+|^{p-1} + p(p-1)|\zeta_k^+|^{p-2} \leq p^2\big(|\zeta_k^+|^p + 1\big).$$

Together with Step 3 and 4 this yields

$$\begin{aligned}\langle \mathcal{L}u + \mathcal{B}(\mathcal{M}u, w), \varphi\rangle_{L^2(S;V)} \geq{} & \frac{\nu}{4} \sum_{k=1}^m \int_{t_i}^t \int_X p^2\, |\zeta_k^+|^{p-2}\, \vartheta^2\, |\nabla\zeta_k^+|^2\, d\sigma\, d\lambda \\ & - \frac{m}{2\nu^3} \sum_{k=1}^m \sum_{\ell=1}^m \int_{t_i}^t \int_X p^2\big(|\zeta_k^+|^p + 1\big)\vartheta^2\, |\nabla w_\ell|^2\, d\sigma\, d\lambda.\end{aligned}$$

6. If we define the functions $\pi_0, \pi_1, \dots, \pi_m \in L^2(S; W^{1,2}(X)) \cap L^\infty(S \times X)$ by

$$\pi_k = |\zeta_k^+|^{p/2} \quad \text{for } k \in \{0,1,\dots,m\},$$

the chain rule yields $4\, |\nabla\pi_k|^2 = p^2\, |\zeta_k^+|^{p-2}\, |\nabla\zeta_k^+|^2$. Summing up the results of Step 2 and 5, the variational formulation of (P) leads to the following estimate for every $t \in (t_i, T]$:

$$\begin{aligned}&\sum_{k=1}^m \int_X |\pi_k(t)|^2\, \vartheta^2(t)\, d\sigma + \nu \sum_{k=1}^m \int_{t_i}^t \int_X |\nabla\pi_k|^2\, \vartheta^2\, d\sigma\, d\lambda \\ &\qquad \leq \frac{2^{i+3}}{\theta} \sum_{k=1}^m \int_{t_i}^t \int_X |\pi_k|^2\, d\sigma\, d\lambda + \frac{m}{2\nu^3} \sum_{k=1}^m \sum_{\ell=1}^m \int_{t_i}^t \int_X p^2\big(|\pi_k|^2 + 1\big)\vartheta^2\, |\nabla w_\ell|^2\, d\sigma\, d\lambda.\end{aligned}$$

7. We repeat the above arguments for the test function $\varphi = (v_0, \dots, v_0) \in L^2(S;V) \cap L^\infty(S \times X;\mathbb{R}^m)$. Integrating by parts and using the chain rule, for every $t \in (t_i, T]$ we obtain

$$\begin{aligned} -\int_S \langle (\mathcal{E}u)'(s), \varphi(s)\rangle_V \, d\lambda(s) &= \int_X |\zeta_0^+(t)|^p \, \vartheta^2(t) \, d\sigma - \int_{t_i}^t \int_X |\zeta_0^+|^p \, 2\vartheta'\vartheta \, d\sigma \, d\lambda \\ &\geq \int_X |\zeta_0^+(t)|^p \, \vartheta^2(t) \, d\sigma - \frac{2^{i+3}}{\theta} \int_{t_i}^t \int_X |\zeta_0^+|^p \, d\sigma \, d\lambda. \end{aligned}$$

8. Similarly to Step 3 and 4, the positivity and the boundedness of A yields the estimates

$$-\langle \mathcal{L}u, \varphi\rangle_{L^2(S;V)} \geq \nu \int_{t_i}^t \int_X p\big(|\zeta_0^+|^{p-1} + (p-1)|\zeta_0^+|^{p-2}\big)\vartheta^2 \, |\nabla \zeta_0^+|^2 \, d\sigma \, d\lambda$$

as well as

$$\begin{aligned} -\langle \mathcal{B}(\mathcal{M}u, w), \varphi\rangle_{L^2(S;V)} \geq &-\frac{\nu}{2} \int_{t_i}^t \int_X p\big(|\zeta_0^+|^{p-1} + (p-1)|\zeta_0^+|^{p-2}\big)\vartheta^2 \, |\nabla \zeta_0^+|^2 \, d\sigma \, d\lambda \\ &- \frac{m}{2\nu^3} \sum_{\ell=1}^m \int_{t_i}^t \int_X p\big(|\zeta_0^+|^{p-1} + (p-1)|\zeta_0^+|^{p-2}\big)\vartheta^2 \, |\nabla w_\ell|^2 \, d\sigma \, d\lambda. \end{aligned}$$

Analogously to Step 5, this leads to

$$\begin{aligned} -\langle \mathcal{L}u + \mathcal{B}(\mathcal{M}u, w), \varphi\rangle_{L^2(S;V)} \geq &\frac{\nu}{4} \int_{t_i}^t \int_X p^2 \, |\zeta_0^+|^{p-2} \, \vartheta^2 \, |\nabla \zeta_0^+|^2 \, d\sigma \, d\lambda \\ &- \frac{m}{2\nu^3} \sum_{\ell=1}^m \int_{t_i}^t \int_X p^2 \big(|\zeta_0^+|^p + 1\big)\vartheta^2 \, |\nabla w_\ell|^2 \, d\sigma \, d\lambda. \end{aligned}$$

9. Summing up the estimates of Step 7 and 8 as in Step 6 and having in mind $\pi_0 = |\zeta_0^+|^{p/2}$ and $4\,|\nabla \pi_0|^2 = p^2 \, |\zeta_0^+|^{p-2} \, |\nabla \zeta_0^+|^2$, the variational formulation of (P) yields the estimate

$$\begin{aligned} &\int_X |\pi_0(t)|^2 \, \vartheta^2(t) \, d\sigma + \nu \int_{t_i}^t \int_X |\nabla \pi_0|^2 \, \vartheta^2 \, d\sigma \, d\lambda \\ &\qquad \leq \frac{2^{i+3}}{\theta} \int_{t_i}^t \int_X |\pi_0|^2 \, d\sigma \, d\lambda + \frac{m}{2\nu^3} \sum_{\ell=1}^m \int_{t_i}^t \int_X p^2 \big(|\pi_0|^2 + 1\big)\vartheta^2 \, |\nabla w_\ell|^2 \, d\sigma \, d\lambda. \end{aligned}$$

Together with Step 6, for every $t \in (t_i, T]$ we end up with

$$\begin{aligned} &\sum_{k=0}^m \int_X |\pi_k(t)|^2 \, \vartheta^2(t) \, d\sigma + \nu \sum_{k=0}^m \int_{t_i}^t \int_X |\nabla \pi_k|^2 \, \vartheta^2 \, d\sigma \, d\lambda \\ &\qquad \leq \frac{2^{i+3}}{\theta} \sum_{k=0}^m \int_{t_i}^t \int_X |\pi_k|^2 \, d\sigma \, d\lambda + \frac{m}{2\nu^3} \sum_{k=0}^m \sum_{\ell=1}^m \int_{t_i}^t \int_X p^2 \big(|\pi_k|^2 + 1\big)\vartheta^2 \, |\nabla w_\ell|^2 \, d\sigma \, d\lambda. \end{aligned}$$

10. Following Assumption 6 and Remark 9, there exists a uniform bound $L_0 > 0$ for the Morrey seminorm

$$\sum_{\ell=1}^{m} \int_{X\cap Q(x,r)} |\nabla w_\ell(s)|^2 \, d\sigma \leq L_0 r^{\omega_0} \quad \text{for all } x \in X,\ r > 0 \text{ and } \lambda\text{-almost all } s \in S.$$

Due to Remark 12, the set where the function $\pi_k(s) \in V$ vanishes, for λ-almost all $s \in S$ and every $k \in \{0, 1, \dots, m\}$ has at least σ-measure $\beta\sigma(X)$. Applying Theorem 9, Lemma 18, and Young's inequality $(\delta b)^{1-\alpha_0}(\delta^{1-1/\alpha_0} d)^{\alpha_0} \leq (1-\alpha_0)\delta b + \alpha_0 \delta^{1-1/\alpha_0} d$ for real numbers b, $d \geq 0$, $\delta > 0$, we find some constant $c_0 > 0$, depending on m, n, α_0, β, L_0, and X, such that for λ-almost all $s \in S$ and $\delta > 0$ we have

$$\sum_{k=0}^{m} \sum_{\ell=1}^{m} \int_X |\pi_k(s)|^2 \, |\nabla w_\ell(s)|^2 \, d\sigma \leq c_0 \sum_{k=0}^{m} \int_X \left(\delta \, |\nabla \pi_k(s)|^2 + \delta^{1-1/\alpha_0} \, |\pi_k(s)|^2 \right) d\sigma.$$

Specifying $\delta > 0$ such that $c_0 m p^2 \delta = \nu^4$ holds true, Step 9 yields

$$\begin{aligned} &\sum_{k=0}^{m} \int_X |\pi_k(t)|^2 \, \vartheta^2(t) \, d\sigma + \frac{\nu}{2} \sum_{k=0}^{m} \int_{t_i}^{t} \int_X |\nabla \pi_k|^2 \, \vartheta^2 \, d\sigma \, d\lambda \\ &\qquad \leq \int_{t_i}^{t} \int_X \left(\left(\frac{2^{i+3}}{\theta} + \frac{(c_0 m p^2)^{1/\alpha_0}}{2\nu^{4/\alpha_0 - 1}} \right) \sum_{k=0}^{m} |\pi_k|^2 + \frac{m(m+1)p^2}{2\nu^3} \sum_{k=1}^{m} |\nabla w_k|^2 \right) \vartheta^2 \, d\sigma \, d\lambda \end{aligned}$$

for every $t \in (t_i, T]$. Consequently, using the properties of the cut-off function ϑ, there exists some constant $c_1 > 0$ depending on m, n, ν, α_0, β, θ, L_0, and X such that

$$\begin{aligned} &\sup_{s\in[t_{i+1},t]} \sum_{k=0}^{m} \int_X |\pi_k(s)|^2 \, d\sigma + \sum_{k=0}^{m} \int_{t_{i+1}}^{t} \int_X |\nabla \pi_k|^2 \, d\sigma \, d\lambda \\ &\qquad\qquad \leq 2^{i+1} c_1 p^{2/\alpha_0} \int_{t_i}^{t} \int_X \left(\sum_{k=0}^{m} |\pi_k|^2 + \sum_{k=1}^{m} |\nabla w_k|^2 \right) d\sigma \, d\lambda \end{aligned}$$

for every $t \in [t_0 + \theta, T]$.

11. In view of Remark 12, the set where the function $\pi_k(s) \in V$ vanishes, for λ-almost all $s \in S$ and every $k \in \{0, 1, \dots, m\}$ has at least σ-measure $\beta\sigma(X)$. Hence, Lemma 19 and Young's inequality yields a constant $c_2 > 0$ depending on β, m, n and X such that

$$\sum_{k=0}^{m} \int_{t_{i+1}}^{t} \int_X |\pi_k|^{2\varkappa} \, d\sigma \, d\lambda \leq c_2 \left(\sup_{s\in[t_{i+1},t]} \sum_{k=0}^{m} \int_X |\pi_k(s)|^2 \, d\sigma + \sum_{k=0}^{m} \int_{t_{i+1}}^{t} \int_X |\nabla \pi_k|^2 \, d\sigma \, d\lambda \right)^{\varkappa}$$

holds true for every $t \in [t_0 + \theta, T]$. Using the definition of π_k and Step 10, we end up with

$$\sum_{k=0}^{m} \int_{t_{i+1}}^{t} \int_X |\zeta_k^+|^{\varkappa p} \, d\sigma \, d\lambda \leq c_2 \left(2^{i+1} c_1 p^{2/\alpha_0} \int_{t_i}^{t} \int_X \left(\sum_{k=0}^{m} |\zeta_k^+|^p + \sum_{k=1}^{m} |\nabla w_k|^2 \right) d\sigma \, d\lambda \right)^{\varkappa},$$

which finishes the proof. □

THEOREM 11 (Uniform positivity). *Let the Assumptions 1–7 be satisfied and let* $u \in W_E(S;V) \cap \operatorname{dom}(\mathcal{M})$ *solve problem* (P) *for the strictly admissible initial value* $a \in U(0)$. *Then, for every* $\theta \in \big(0, \frac{1}{2}\big)$ *there exists a lower bound* $\gamma \in \big(0, \frac{1}{2}\big)$ *depending on* m, n, ν, α_0, β, θ, L, L_0, *and* X, *but not on* $T \geq 1$, *such that* $u(t) \in U(\gamma)$ *for all* $t \in [\theta, T]$.

Proof. 1. Let $\varkappa = 1 + 2/n$ and $\alpha_0 = \frac{1}{2}(\omega_0 - n + 2) \in (0,1]$ be the Hölder exponent corresponding to $\omega_0 \in (n-2, n]$. Fixing a time shift $\theta \in \big(0, \frac{1}{2}\big)$, we consider $t_0 \in [0, T-\theta)$ and the sequence $(t_i) \subset [t_0, t_0 + \theta) \subset [t_0, T)$ defined by $t_i = t_0 + \theta(1 - 2^{-i})$ for $i \in \mathbb{N}$.

Let $a \in U(0)$ be strictly admissible and $u \in W_E(S;V) \cap \operatorname{dom}(\mathcal{M})$ be the solution of (P). Suppose $\zeta_0, \zeta_1, \ldots, \zeta_m$ to be the corresponding γ-regularized potentials for some parameter $\gamma \in \Gamma_\beta$ which will be determined later, and $w = \mathcal{P}\mathcal{J}u + \psi$ to be the corresponding interaction potentials.

Now, Lemma 10 plays the role of the inductive step of a Moser-type iteration: There exists some constant $c_1 \geq 1$ depending on m, n, ν, α_0, β, θ, L_0, and X such that for every $t \in [t_0 + \theta, T]$ the sequence $(b_i) \subset [1, \infty)$ of quantities

$$b_i = 1 + \int_{t_i}^{t} \int_X \left(\sum_{k=0}^{m} |\zeta_k^+|^{2\varkappa^i} + \sum_{k=1}^{m} |\nabla w_k|^2 \right) d\sigma \, d\lambda \quad \text{for } i \in \mathbb{N}, \tag{25}$$

satisfies the recursive estimate

$$b_{i+1} \leq b_i + \big(2^{i+1} (2\varkappa^i)^{2/\alpha_0} c_1 b_i\big)^{\varkappa} \quad \text{for every } i \in \mathbb{N}.$$

Hence, we find a constant $c_2 > 0$ depending on m, n, ν, α_0, β, θ, L_0, and X such that

$$b_{i+1} \leq c_2^{i+1} b_i^{\varkappa} \quad \text{for all } i \in \mathbb{N}.$$

Applying this estimate recursively for $j \in \{0, 1, \ldots, i-1\}$, we get

$$b_i \leq c_2^{p_i(\varkappa)} b_0^{\varkappa^i} \quad \text{for all } i \in \mathbb{N},$$

where we have introduced the polynomial $p_i(\varkappa) = \sum_{j=0}^{i-1} (i-j)\varkappa^j$ for $i \in \mathbb{N}$. Because of

$$\varkappa^{-i} p_i(\varkappa) = \sum_{j=0}^{i-1} (i-j) \varkappa^{j-i} = \sum_{j=1}^{i} j \varkappa^{-j} \leq \frac{\varkappa}{(\varkappa - 1)^2} \quad \text{for all } i \in \mathbb{N},$$

there exists some constant $c_3 > 0$ depending on m, n, ν, α_0, β, θ, L_0, and X such that

$$b_i^{\varkappa^{-i}} \leq c_3 b_0 \quad \text{for every } i \in \mathbb{N}.$$

Having in mind the definition (25) of (b_i), for every $i \in \mathbb{N}$ this yields the estimate

$$\left(\sum_{k=0}^{m} \int_{t_i}^{t} \int_X |\zeta_k^+|^{2\varkappa^i} \, d\sigma \, d\lambda \right)^{\varkappa^{-i}} \leq c_3 \left(1 + \int_{t_0}^{t} \int_X \left(\sum_{k=0}^{m} |\zeta_k^+|^2 + \sum_{k=1}^{m} |\nabla w_k|^2 \right) d\sigma \, d\lambda \right).$$

Passing to the limit $i \to \infty$, for all $t \in [t_0+\theta, T]$, $\tau \in [t_0+\theta, t]$, $k \in \{0,1,\dots,m\}$ we get

$$\|\zeta_k^+(\tau)\|^2_{L^\infty(X)} \le c_3 \left(1 + \int_{t_0}^{t}\int_X \left(\sum_{k=0}^{m} |\zeta_k^+|^2 + \sum_{k=1}^{m} |\nabla w_k|^2\right) d\sigma\, d\lambda\right).$$

2. Remembering Remark 12, the set where the function $\zeta_k^+(s) \in V$ vanishes, for almost all $s \in S$ and every $k \in \{0,1,\dots,m\}$ has at least σ-measure $\beta\sigma(X)$. Therefore, using Lemma 18, Lemma 8 as the basis of the Moser-type iteration, and, finally, Remark 6, for $t = \min\{t_0+1, T\}$, every $k \in \{0,1,\dots,m\}$ and $\tau \in [t_0+\theta, t]$, it follows that we have

$$\begin{aligned}\|\zeta_k^+(\tau)\|^2_{L^\infty(X)} &\le c_4 \left(1 + \int_{t_0}^{t}\int_X \left(\sum_{k=0}^{m} |\nabla\zeta_k^+|^2 + \sum_{k=1}^{m} |\nabla w_k|^2\right) d\sigma\, d\lambda\right)\\ &\le c_5 \left(1 + \ln\frac{\beta}{\gamma^2}\left(\frac{1}{\beta}+\frac{1}{2}\right) + \sum_{k=1}^{m}\int_{t_0}^{t}\int_X |\nabla w_k|^2\, d\sigma\, d\lambda\right) \le c_6 \ln\frac{3\beta}{\gamma^2}\end{aligned}$$

where the constants c_4, c_5, $c_6 > 0$ depend on m, n, ν, α_0, β, θ, L, L_0, and X, since the length of all the time intervals (t_0, t) under consideration is uniformly bounded.

3. The properties of logarithmic and quadratic functions yield some $\gamma \in \Gamma_\beta$ depending on m, n, ν, α_0, β, θ, L, L_0, and X such that

$$c_6\big(\ln 3\beta - \ln\gamma^2\big) < (\ln\beta - \ln\gamma)^2.$$

Using Step 2, for all $k \in \{0,1,\dots,m\}$ and $\tau \in [t_0+\theta, \min\{t_0+1, T\}]$ this yields

$$\left(\ln\frac{\beta}{u_k(\tau)+\gamma^2}\right)^2 \le c_6 \ln\frac{3\beta}{\gamma^2} \le \left(\ln\frac{\beta}{\gamma}\right)^2 \quad \text{on } X \setminus N_{\beta-\gamma^2}(u_k(\tau)).$$

Therefore, for all $k \in \{0,1,\dots,m\}$ and $\tau \in [t_0+\theta, \min\{t_0+1, T\}]$ we obtain

$$u_k(\tau) \ge \gamma - \gamma^2 \quad \text{on } X \setminus N_{\beta-\gamma^2}(u_k(\tau)).$$

But, obviously, for every $k \in \{0,1,\dots,m\}$ and $\tau \in [t_0+\theta, \min\{t_0+1, T\}]$ we also get

$$u_k(\tau) \ge \beta - \gamma^2 \ge \gamma - \gamma^2 \quad \text{on } N_{\beta-\gamma^2}(u_k(\tau)).$$

Since $t_0 \in [0, T-\theta)$ was arbitrarily fixed at the beginning, for every $t_0 \in [0, T-\theta)$ and $\tau \in [t_0+\theta, \min\{t_0+1, T\}]$ we end up with $u_k(\tau) \ge \gamma(1-\gamma) > 0$ on X for all $k \in \{0,1,\dots,m\}$. Because $\theta \in \left(0, \frac{1}{2}\right)$ yields $\cup_{t_0\in[0,T-\theta)}[t_0+\theta, \min\{t_0+1, T\}] = [\theta, T]$, the proof is finished. □

6. Asymptotic convergence of the solution

Throughout the whole section we suppose that all the Assumptions 1–7 are satisfied. Let $u : (0,\infty) \to V$ be the complete trajectory of the solution to problem (P). That means, for every finite interval $S = (0, T)$ the restriction $u|S \in W_E(S;V) \cap \mathrm{dom}(\mathcal{M})$ is the solution of system (P) to the strictly admissible initial value $a \in U(0)$. We will study the asymptotic convergence of the trajectory to some stationary point.

REMARK 13. For $h \in U(\gamma)$ and $\rho = Jh \in \operatorname{dom}(F)$ the components of the vector-valued function $v = DF(\rho) \in L^\infty$ are called *grand chemical potential differences.*

Due to Remark 4 and 9 the functional F is real analytic in $J[U]$, whenever U is open in L^∞ and contained in $U\big(\frac{\gamma}{2}\big)$. Furthermore, its Fréchet derivative $DF : J[U] \to L^\infty$ is a real analytic operator.

REMARK 14 (Regularity of potentials). Let $\theta \in \big(0, \frac{1}{2}\big)$ be arbitrarily fixed. We apply the uniform regularity and positivity results established in Section 4 and 5 to the complete trajectory $u : (0, \infty) \to V$ on subintervals of $[\theta, T]$ for $T \geq 1$:

1. Due to Theorem 4 and 7 we find some $\alpha \in (0, 1]$ depending on m, n, ν, ω_0, θ, L, L_0, and X such that the norm of the restriction $u|S_0$ in $C^{0,\alpha/2}(\operatorname{cl} S_0; C) \cap C(\operatorname{cl} S_0; C^{0,\alpha})$ is uniformly bounded for all $S_0 = (t_0, t_0 + \frac{1}{2})$ with $t_0 \geq \theta$.
2. Following Theorem 11 there exists some bound $\gamma \in \big(0, \frac{1}{2}\big)$ depending on m, n, ν, ω_0, θ, β, L, L_0, and X such that $u(t) \in U(\gamma)$ for all $t \geq \theta$.
3. Having in mind Remark 4, 6, 13 and that the particle densities $\rho : (0, \infty) \to H^*$ are given by $\rho(s) = JKu(s)$ for $s \in (0, \infty)$, the chemical potentials $\zeta = D\Phi(\rho) : (0, \infty) \to V$, the interaction potentials $w = D\Psi(\rho) : (0, \infty) \to V$, and, therefore, the grand chemical potentials $v = DF(\rho) : (0, \infty) \to V$, are correctly defined on the open interval $(0, \infty)$ by
$$\zeta_k = \ln u_k - \ln u_0, \quad w_k = (P\rho)_k + \psi_k \quad \text{and} \quad v_k = \zeta_k + w_k \quad \text{for } k \in \{1, \dots, m\}.$$
4. Moreover, the norms of $\zeta|S_0$, $w|S_0$ and $v|S_0$ in $C^{0,\alpha/2}(\operatorname{cl} S_0; C) \cap C(\operatorname{cl} S_0; C^{0,\alpha})$ are uniformly bounded for all $S_0 = (t_0, t_0 + \frac{1}{2})$ with $t_0 \geq \theta$, too.

REMARK 15 (Reformulation of the problem). Remarks 4, 5 and 14 enable us to reformulate the problem in *relative densities* $u : (0, \infty) \to V$ and *grand chemical potentials* $v : (0, \infty) \to V$. For all $\theta \in \big(0, \frac{1}{2}\big)$ and $T \geq 1$ we get that
$$u|(\theta, T) \in W_E((\theta, T); V) \quad \text{and} \quad v|(\theta, T) \in L^2((\theta, T); V)$$
satisfy the identity
$$\int_\theta^T \langle (\mathcal{E}u)'(s), \varphi(s) \rangle_V \, d\lambda(s) + \sum_{k=1}^m \sum_{\ell=1}^m \int_\theta^T \int_X A(\mathcal{M}u)_{k\ell} \nabla v_\ell \cdot \nabla \varphi_k \, d\sigma \, d\lambda = 0 \qquad (26)$$
for all $\varphi \in L^2((\theta, T); V)$.

REMARK 16. Let $L^\infty = L^\infty(X; \mathbb{R}^m)$ be the space of essentially bounded and measurable functions and $J[L^\infty] \subset H^*$ be its topological image under the duality map $J \in \mathcal{L}(H; H^*)$. Moreover, consider the Hilbert sum decomposition $H = H_1 + H_0$ into the m-dimensional subspace $H_1 \subset V \cap L^\infty$ of constant functions and the closed subspace
$$H_0 = \big\{h \in H : \textstyle\int_X h \, d\sigma = 0\big\}.$$
The annihilator of H_1, which coincides with the subspace $J[H_0]$ of $J[L^\infty]$, is defined by
$$H_1^0 = \big\{\rho \in H^* : \langle \rho, v \rangle_H = 0 \text{ for all } v \in H_1\big\}.$$

We start with preliminary norm estimates and, more important, with the decay property of the free energy functional F along trajectories, which, in a first step, ensures the convergence of the solution along a discrete sequence of points on the time axis.

LEMMA 12 (Norm equivalence). *There exist two constants c_1, $c_2 > 0$ depending on m, n, ν, α, β, γ, θ, and X such that for all t, $\tau \in [\theta, T]$ with $t < \tau$ the following norm estimate holds true:*

$$c_1 \int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}^2 \, d\lambda(s) \le \sum_{k=1}^m \int_t^\tau \int_X |\nabla v_k|^2 \, d\sigma \, d\lambda \le c_2 \int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}^2 \, d\lambda(s). \tag{27}$$

Proof. 1. To prove the first inequality of (27) we consider the integral identity

$$\int_t^\tau \langle (\mathcal{E}u)'(s), \varphi(s) \rangle_V \, d\lambda(s) + \sum_{k=1}^m \sum_{\ell=1}^m \int_t^\tau \int_X A(\mathcal{M}u)_{k\ell} \nabla v_\ell \cdot \nabla \varphi_k \, d\sigma \, d\lambda = 0, \tag{28}$$

and take the supremum over all test functions $\varphi \in L^2((t,\tau); V)$ in (28) which satisfy the condition $\|\varphi\|_{L^2((t,\tau);V)} = 1$. Then the uniform boundedness of the matrix-valued function $\mathcal{M}u$ gives the result.

2. Due to Remark 14 and 15, the sum of $\zeta = D\Phi(\rho)$ and $w = D\Psi(\rho)$ yields an admissible test function $v = \zeta + w$ for (28). Introducing the mean value $\bar{v}(s) = \fint_X v(s) \, d\sigma \in H_1$ for $s \in [t, \tau]$ and testing (28) with $\varphi = v - \bar{v} \in L^2((t,\tau); V)$ we get

$$\begin{aligned} \sum_{k=1}^m \sum_{\ell=1}^m \int_t^\tau \int_X A(\mathcal{M}u)_{k\ell} \nabla v_\ell \cdot \nabla v_k \, d\sigma \, d\lambda &= -\int_t^\tau \langle (\mathcal{E}u)'(s), v(s) - \bar{v}(s) \rangle_V \, d\lambda(s) \\ &\le \int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*} \|v(s) - \bar{v}(s)\|_V \, d\lambda(s). \end{aligned}$$

Using Young's and Poincaré's inequality (43) and the uniform positive definiteness of A and the matrix-valued function $\mathcal{M}u$ on $[\theta, T] \times X$, see Theorem 11, this yields

$$\begin{aligned} \sum_{k=1}^m \int_t^\tau \int_X |\nabla v_k|^2 \, d\sigma \, d\lambda &\le \frac{c_1}{2\varepsilon} \int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}^2 \, d\lambda(s) + \frac{c_1 \varepsilon}{2} \int_t^\tau \|v(s) - \bar{v}(s)\|_V^2 \, d\lambda(s) \\ &\le \frac{c_1}{2\varepsilon} \int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}^2 \, d\lambda(s) + c_2 \varepsilon \sum_{k=1}^m \int_t^\tau \int_X |\nabla v_k|^2 \, d\sigma \, d\lambda \end{aligned}$$

for all $\varepsilon > 0$, where c_1, $c_2 > 0$ are constants depending on m, n, ν, α, β, γ, θ and X. Hence, we obtain the second inequality of (27) if we take $\varepsilon > 0$ sufficiently small. □

THEOREM 13 (Decay of free energy). *The free energy F is bounded, continuous and decreasing along the trajectory: For all t, $\tau \in [\theta, T]$ with $t < \tau$ we have*

$$F(\rho(t)) - F(\rho(\tau)) = \sum_{k=1}^m \sum_{\ell=1}^m \int_t^\tau \int_X A(\mathcal{M}u)_{k\ell} \nabla v_\ell \cdot \nabla v_k \, d\sigma \, d\lambda. \tag{29}$$

Consequently, the free energy F converges asymptotically and monotonously decreasing to an infimum $F_* \in \mathbb{R}$ along the trajectory.

Proof. 1. Let t, $\tau \in [\theta, T]$ with $t < \tau$ be arbitrarily fixed. Following Remark 15, the sum of $\zeta = D\Phi(\rho)$ and $w = D\Psi(\rho)$ yields an admissible test function $v = \zeta + w$ for (26), which leads to

$$\int_t^\tau \langle (\mathcal{E}u)'(s), \zeta(s) + w(s)\rangle_V \, d\lambda(s) + \sum_{k=1}^m \sum_{\ell=1}^m \int_t^\tau \int_X A(\mathcal{M}u)_{k\ell} \nabla v_\ell \cdot \nabla v_k \, d\sigma \, d\lambda = 0. \quad (30)$$

2. To prove (29) it remains to calculate the first integral: We apply the chain rule and integrate by parts to get

$$\int_t^\tau \langle (\mathcal{E}u)'(s), \zeta(s)\rangle_V \, d\lambda(s) = \int_t^\tau \langle (\mathcal{K}^*\rho)'(s), D\Phi(\rho(s))\rangle_V \, d\lambda(s) = \Phi(\rho(\tau)) - \Phi(\rho(t)). \quad (31)$$

3. Now, assume that $u \in C_0^\infty(\mathbb{R}; V)$. Then, the mapping $s \mapsto \rho(s) = JKu(s)$ belongs to $C_0^\infty(\mathbb{R}; H^*)$. Since $(\mathcal{E}u)'(s) = Eu'(s) = K^*\rho'(s)$ holds true for all $s \in \mathbb{R}$, we get

$$\int_t^\tau \langle (\mathcal{E}u)'(s), (\mathcal{P}\rho)(s)\rangle_V \, d\lambda(s) = \int_t^\tau \langle K^*\rho'(s), P\rho(s)\rangle_V \, d\lambda(s) = \int_t^\tau \langle \rho'(s), KP\rho(s)\rangle_H \, d\lambda(s).$$

We use the symmetry of $KP \in \mathcal{L}(H^*; H)$ and integrate by parts to obtain

$$\begin{aligned} 2\int_t^\tau \langle (\mathcal{E}u)'(s), (\mathcal{P}\rho)(s)\rangle_V \, d\lambda(s) &= \int_t^\tau \big(\langle \rho'(s), KP\rho(s)\rangle_H + \langle \rho(s), KP\rho'(s)\rangle_H\big) \, d\lambda(s) \\ &= \langle \rho(\tau), KP\rho(\tau)\rangle_H - \langle \rho(t), KP\rho(t)\rangle_H. \end{aligned}$$

Since the set of restrictions $u|(t,\tau)$ and $\rho|(t,\tau)$ of smooth functions $u \in C_0^\infty(\mathbb{R}; V)$ and $\rho \in C_0^\infty(\mathbb{R}; H^*)$ are dense in $W_E((t,\tau); V)$ and $L^2((t,\tau); H^*)$, respectively, the identity

$$2\int_t^\tau \langle (\mathcal{E}u)'(s), (\mathcal{P}\rho)(s)\rangle_V \, d\lambda(s) = \langle \rho(\tau), KP\rho(\tau)\rangle_H - \langle \rho(t), KP\rho(t)\rangle_H$$

remains true for the solution of problem (26). Additionally, we have

$$\int_t^\tau \langle (\mathcal{E}u)'(s), \psi\rangle_V \, d\lambda(s) = \langle \rho(\tau), K\psi\rangle_H - \langle \rho(t), K\psi\rangle_H$$

and in view of the representation

$$\Psi(\rho) = \tfrac{1}{2}\langle \rho, KP\rho\rangle_H + \langle \rho, K\psi\rangle_H + \Psi(0), \quad w = D\Psi(\rho) = P\rho + \psi \quad \text{for } \rho \in H^*,$$

see Remark 6, this yields

$$\int_t^\tau \langle (\mathcal{E}u)'(s), w(s)\rangle_V \, d\lambda(s) = \int_t^\tau \langle (\mathcal{E}u)'(s), (\mathcal{P}\rho)(s) + \psi\rangle_V \, d\lambda(s) = \Psi(\rho(\tau)) - \Psi(\rho(t)).$$

Together with (30) and (31) this finishes the proof of (29). □

COROLLARY 14. *If there are t, $\tau \in [\theta, T]$ with $t < \tau$ and $F(\rho(t)) = F(\rho(\tau))$, then there exists a pair $(u^*, v^*) \in U(\gamma) \times H_1$ such that $\rho^* = Ju^*$ solves the stationary problem $DF(\rho^*) = v^*$ and $(u(s), v(s)) = (u^*, v^*)$ holds true for all $s \in [t, T]$.*

Proof. Let t, $\tau \in [\theta, T]$ with $t < \tau$ and $F(\rho(t)) = F(\rho(\tau))$ be given. Then the decay property (29) and the norm equivalence (27) yields

$$\sum_{k=1}^{m} \int_t^\tau \int_X |\nabla v_k|^2 \, d\sigma \, d\lambda = \int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}^2 \, d\lambda(s) = 0.$$

Consequently, both u and v are constant in time on the interval $[t, \tau]$. Since the spatial gradients ∇v_k also vanish, v is even constant in time and space on $[t, \tau] \times X$. Hence, we have found a pair $(u^*, v^*) \in U(\gamma) \times H_1$, which satisfies $(u(s), v(s)) = (u^*, v^*)$ for all $s \in [t, \tau]$. Since $DF(\rho(s)) = v(s)$ and $\rho(s) = Ju(s)$ for $s \in [t, \tau]$, this yields $DF(\rho^*) = v^*$ for $\rho^* = Ju^*$. Obviously, (u^*, v^*) is a solution of the evolution system (26) on the time interval $[t, T]$, too. Due to the uniqueness of the solution to the problem, this means that the trajectory rests at this stationary point $(u^*, v^*) \in U(\gamma) \times H_1$. □

Theorem 15 (Convergence of a subsequence). *There exists an increasing sequence* $(t_l) \subset \mathbb{N}$ *such that* $(u(t_l), v(t_l))$ *converges for* $l \to \infty$ *to* $(u^*, v^*) \in U(\gamma) \times H_1$ *in the sense*

$$\lim_{l\to\infty} \|u(t_l) - u^*\|_{L^\infty} = 0, \quad \lim_{l\to\infty} \|v(t_l) - v^*\|_{L^\infty} = 0, \quad \lim_{t\to\infty} F(\rho(t)) = F(\rho^*),$$

where $\rho^* = Ju^*$ *solves the stationary problem* $DF(\rho^*) = v^*$.

Proof. 1. Let $\bar{v}(s) = \fint_X v(s) \, d\sigma \in H_1$ denote the mean value of v for $s \geq \theta$. We will prove by contradiction that the following convergence result holds true:

$$\lim_{l\to\infty} \|v(l) - \bar{v}(l)\|_H = 0. \tag{32}$$

Otherwise we could find some $\varepsilon > 0$ and an increasing sequence $(l_i) \subset \mathbb{N}$ with

$$\|v(l_i) - \bar{v}(l_i)\|_H^2 \geq 2\varepsilon \quad \text{for all } i \in \mathbb{N}.$$

Since $v - \bar{v}$ is Hölder continuous in time, see Theorem 4 and 7, we have

$$\|[v(s_1) - \bar{v}(s_1)] - [v(s_2) - \bar{v}(s_2)]\|_H \leq c_0 |s_1 - s_2|^{\alpha/2} \quad \text{for all } s_1,\, s_2 \geq \theta,\, |s_1 - s_2| \leq \tfrac{1}{2},$$

and some $c_0 > 0$. Therefore, we could find some $\tau \in \left(0, \frac{1}{2}\right)$ satisfying

$$\|v(s) - \bar{v}(s)\|_H^2 \geq \varepsilon \quad \text{for all } s \in [l_i, l_i + \tau] \text{ and } i \in \mathbb{N}.$$

Hence, integrating over the time interval $(l_i, l_i + \tau)$ and applying Poincaré's inequality (43), the decay property (29) for all $i \in \mathbb{N}$ would give

$$F(\rho(l_i)) - F(\rho(l_i + \tau)) = \sum_{k=1}^{m} \sum_{\ell=1}^{m} \int_{l_i}^{l_i+\tau} \int_X A(\mathcal{M}u)_{k\ell} \nabla v_\ell \cdot \nabla v_k \, d\sigma \, d\lambda \geq c_1 \varepsilon \tau,$$

where $c_1 > 0$ is some suitable constant depending m, n, ν, α, β, γ, θ and X. Having in mind that $l_i + \tau \leq l_{i+1}$ and summing up over $i \in \{1, \ldots, j\}$ this would lead to

$$F(\rho(l_1)) - F(\rho(l_j + \tau)) \geq \sum_{i=1}^{j} [F(\rho(l_i)) - F(\rho(l_i + \tau))] \geq c_1 \varepsilon \tau j \quad \text{for all } j \in \mathbb{N},$$

which contradicts to the boundedness of the range of the free energy functional F on its effective domain $\mathrm{dom}(F) = J[U(0)]$. Hence, (32) holds true.

2. Since the sequence $(u(l)) \subset U(\gamma)$ is bounded in $C^{0,\alpha}$ and, therefore, precompact in L^∞, we find some accumulation point $u^* \in U(\gamma)$ and an increasing subsequence $(t_l) \subset \mathbb{N}$ such that $\lim_{l\to\infty} \|u(t_l)-u^*\|_{L^\infty} = 0$. We get $\lim_{l\to\infty} \|\rho(t_l)-\rho^*\|_{J[L^\infty]} = 0$ for $\rho^* = Ju^*$ and $\lim_{l\to\infty} F(\rho(t_l)) = F(\rho^*)$. This yields $\lim_{t\to\infty} F(\rho(t)) = F(\rho^*) = F_*$ applying Theorem 13.

Using $v(t_l) = DF(\rho(t_l)) \in L^\infty$ for $l \in \mathbb{N}$ and setting $v^* = DF(\rho^*) \in L^\infty$ we also have $\lim_{l\to\infty} \|v(t_l) - v^*\|_{L^\infty} = 0$. Remembering (32) and the notation $\bar{v}^* = \fint_X v^*\, d\sigma \in H_1$, in

$$\|v^* - \bar{v}^*\|_H \le \|v^* - v(t_l)\|_H + \|v(t_l) - \bar{v}(t_l)\|_H + \|\bar{v}(t_l) - \bar{v}^*\|_H,$$

each of the three terms on the right hand side tends to zero, when passing to the limit $l \to \infty$, which means that $v^* \in H_1$ is constant. □

REMARK 17 (Regularity of stationary states). Let $(u^*, v^*) \in U(\gamma) \times H_1$ be a pair such that $\rho^* = Ju^*$ solves the stationary problem $v^* = DF(\rho^*)$. Following Assumption 6 we obtain $w^* = D\Psi(\rho^*) \in W^{1,2,\omega_0}$ and, hence, $v^* - w^* = D\Phi(\rho^*) \in W^{1,2,\omega_0}$. Using the representation (11) of the relative densities

$$u_k^* = \frac{\exp(v_k^* - w_k^*)}{1 + \sum_{\ell=1}^m \exp(v_\ell^* - w_\ell^*)} \quad \text{for every } k \in \{1, \dots, m\},$$

see Remark 5, this yields $u^* \in W^{1,2,\omega_0}$ as a regularity result.

In the case of strong convexity of the functional F the whole trajectory (u, v) converges to the uniquely determined limit $(u^*, v^*) \in U(\gamma) \times H_1$. However, in general F is not convex, and we cannot apply this standard argument. Instead of this we follow the ideas of MIRANVILLE and ROUGIREL [31, Theorem 2.1, Lemma 2.1] using the differential properties, especially, the real analyticity, of the free energy functional collected in Remark 4, 9 and 13. The proof is based on a refined version of the Łojasiewicz–Simon gradient inequality established by GAJEWSKI and GRIEPENTROG [8, Theorem 6]. There, the gradient inequality found in FEIREISL, ISSARD-ROCH and PETZELTOVA [5] was generalized to the case of minimization problems for analytic functionals with affine constraints to bring into play the conservation of particle number:

THEOREM 16 (Łojasiewicz–Simon gradient inequality). *Let the set U be open in L^∞ and contained in $U\left(\frac{\gamma}{2}\right)$ and assume that $(\rho^*, v^*) \in J[U] \times H_1$ is a solution of the stationary problem $DF(\rho^*) = v^*$. Then, we find constants δ, $\lambda > 0$ and $\vartheta \in \left(0, \frac{1}{2}\right]$ such that for all $\rho \in J[U]$ which satisfy $\rho - \rho^* \in H_1^0$ and $\|\rho - \rho^*\|_{H^*} \le \delta$ we have*

$$|F(\rho) - F(\rho^*)|^{1-\vartheta} \le \lambda \inf\left\{\|DF(\rho) - \tilde{v}\|_H : \tilde{v} \in H_1\right\}.$$

Having in mind Remark 10, we impose the less restrictive Assumption 6 on the interaction functional Ψ compared with the work of LONDEN and PETZELTOVA in [28, Section 5] or GAL and GRASSELLI in [12, Section 4]. Nevertheless, due to Theorem 4, 7, and 11, we have at hand the uniform Hölder continuity and the uniform positivity of the solution to (26), which enables us to give an elementary convergence proof in the spirit of MIRANVILLE and ROUGIREL [31, Theorem 2.1, Lemma 2.1].

THEOREM 17 (Convergence of the whole trajectory). *The solution* (u, v) *of the evolution system* (26) *converges for* $t \to \infty$ *to a limit* $(u^*, v^*) \in U(\gamma) \times H_1$ *in the sense*

$$\lim_{t\to\infty} \|u(t) - u^*\|_{C^{0,\alpha'}} = 0, \quad \lim_{t\to\infty} \|v(t) - v^*\|_{C^{0,\alpha'}} = 0, \quad \lim_{t\to\infty} F(\rho(t)) = F(\rho^*),$$

where $\alpha' \in (0, \alpha)$ *and* $\rho^* = Ju^*$ *solves the stationary problem* $DF(\rho^*) = v^*$.

Proof. 1. Let the set U be open in L^∞ and $U(\gamma) \subset U \subset U(\frac{\gamma}{2})$. Applying Theorem 15 we choose an increasing sequence $(t_l) \subset \mathbb{N}$ such that $(u(t_l), v(t_l))$ converges for $l \to \infty$ to a pair $(u^*, v^*) \in U(\gamma) \times H_1$ in the sense

$$\lim_{l\to\infty} \|u(t_l) - u^*\|_{L^\infty} = 0, \quad \lim_{l\to\infty} \|v(t_l) - v^*\|_{L^\infty} = 0, \quad \lim_{t\to\infty} F(\rho(t)) = F(\rho^*),$$

where $\rho^* = Ju^*$ solves the stationary problem $DF(\rho^*) = v^*$ and, hence, $u^* \in W^{1,2,\omega_0}$ due to Remark 17. To prove the convergence of the whole trajectory we consider two cases:

2. If there are points t, $\tau \in [\theta, \infty)$ with $t < \tau$ and $F(\rho(t)) = F(\rho(\tau))$, then Corollary 14 yields $(u(s), v(s)) = (u^*, v^*)$ for all $s \geq t$, in other words, the trajectory has arrived at the stationary point in finite time.

3. To consider the alternative case, from now on we assume that F strictly decreases along the trajectory. That means,

$$F(\rho(t)) > F(\rho(\tau)) > F(\rho^*) \quad \text{for all } t,\ \tau \in [\theta, \infty) \text{ with } t < \tau. \tag{33}$$

Due to Remark 4, 9 and 13 and Theorem 11 the Łojasiewicz–Simon gradient inequality is applicable to the solution of problem (26): Using Theorem 16, there are constants δ, $\lambda > 0$ and $0 < \vartheta \leq \frac{1}{2}$ such that for every $s \geq \theta$ with $\|\rho(s) - \rho^*\|_{H^*} \leq \delta$ we have

$$|F(\rho(s)) - F(\rho^*)|^{1-\vartheta} \leq \lambda \inf \big\{ \|v(s) - \tilde{v}\|_H : \tilde{v} \in H_1 \big\} = \lambda \|v(s) - \bar{v}(s)\|_H, \tag{34}$$

since the infimum over $\tilde{v} \in H_1$ is attained at the mean value $\bar{v}(s) = ⨍_X v(s)\, d\sigma \in H_1$. Moreover, the condition $\rho(s) - \rho^* \in H_1^0$ is satisfied for all $s \geq \theta$, since the particle number is conserved due to Remark 7.

4. The next step is to find some constant $c_1 > 0$ depending on m, n, ν, α, β, γ, θ, ϑ and X such that the estimate

$$\int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}\, d\lambda(s) \leq c_1 [F(\rho(t)) - F(\rho^*)]^\vartheta \tag{35}$$

holds true, whenever the points t, $\tau \in [\theta, \infty)$ with $t < \tau$ fulfil the condition

$$\|\rho(s) - \rho^*\|_{H^*} \leq \delta \quad \text{for all } s \in [t, \tau]. \tag{36}$$

To do so, assume that (36) is satisfied for t, $\tau \in [\theta, \infty)$ with $t < \tau$. Starting with $s_0 = \theta$, we define an increasing sequence of points $s_l \geq \theta$ with $\lim_{l\to\infty} s_l = \infty$ by

$$F(\rho(s_l)) - F(\rho^*) = 2[F(\rho(s_{l+1})) - F(\rho^*)] \quad \text{for } l \in \mathbb{N}, \tag{37}$$

which consecutively halfs the remaining excess of free energy. This is possible due to the continuity of F along the trajectory. To prove (35) we distinguish between two cases:

5. Let t, $\tau \in [s_l, s_{l+1}]$ with $t < \tau$ for some $l \in \mathbb{N}$. Applying Poincaré's inequality (43) to (34), the norm equivalence (27) leads to

$$|F(\rho(s)) - F(\rho^*)|^{1-\vartheta} \leq \lambda\|v(s) - \bar{v}(s)\|_H \leq c_2\|(\mathcal{E}u)'(s)\|_{V^*} \quad \text{for } \lambda\text{-almost all } s \in [t, \tau],$$

where $c_2 > 0$ is some suitable constant m, n, ν, α, β, γ, θ and X. Together with (33) and $0 < \vartheta \leq \frac{1}{2}$ this yields

$$\begin{aligned}\int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}\, d\lambda(s) &\leq c_2 \int_t^\tau [F(\rho(s)) - F(\rho^*)]^{\vartheta-1}\|(\mathcal{E}u)'(s)\|_{V^*}^2\, d\lambda(s) \\ &\leq c_2[F(\rho(\tau)) - F(\rho^*)]^{\vartheta-1} \int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}^2\, d\lambda(s).\end{aligned}$$

Using (33) and (37) we estimate the energy difference in the first factor by

$$F(\rho(\tau)) - F(\rho^*) \geq F(\rho(s_{l+1})) - F(\rho^*) = \tfrac{1}{2}[F(\rho(s_l)) - F(\rho^*)] \geq \tfrac{1}{2}[F(\rho(t)) - F(\rho^*)].$$

Having in mind $0 < \vartheta \leq \frac{1}{2}$ we get

$$\int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}\, d\lambda(s) \leq 2c_2[F(\rho(t)) - F(\rho^*)]^{\vartheta-1} \int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}^2\, d\lambda(s).$$

Using (27) and (29) we find some constant $c_3 > 0$ depending on m, n, ν, α, β, γ, θ and X such that the integral on the right hand side can be estimated by

$$\int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}^2\, d\lambda(s) \leq c_3[F(\rho(t)) - F(\rho(\tau))],$$

which yields

$$\int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}\, d\lambda(s) \leq 2c_2c_3[F(\rho(t)) - F(\rho^*)]^{\vartheta-1}[F(\rho(t)) - F(\rho(\tau))].$$

Applying the elementary inequality $\vartheta b^{\vartheta-1}(b - d) \leq b^\vartheta - d^\vartheta$ to $b = F(\rho(t)) - F(\rho^*)$ and $d = F(\rho(\tau)) - F(\rho^*)$, we obtain

$$\int_t^\tau \|(\mathcal{E}u)'(s)\|_{V^*}\, d\lambda(s) \leq \frac{2c_2c_3}{\vartheta}\big([F(\rho(t)) - F(\rho^*)]^\vartheta - [F(\rho(\tau)) - F(\rho^*)]^\vartheta\big). \tag{38}$$

6. Otherwise we find l, $j \in \mathbb{N}$ with $t \in [s_l, s_{l+1})$ and $\tau \in (s_{l+j}, s_{l+j+1}]$. Considering the finite decomposition

$$[t, \tau] = [t, s_{l+1}] \cup \cdots \cup [s_i, s_{i+1}] \cup \cdots \cup [s_{l+j}, \tau],$$

the result of Step 5 holds true for each of these subintervals. Summing up consecutively, we get estimate (38) for the whole interval $[t, \tau]$, too. In both cases, the desired estimate (35) follows immediately, whenever t, $\tau \in [\theta, \infty)$ satisfy $t < \tau$ and condition (36).

7. Let $\varepsilon \in \big(0, \frac{\delta}{2}\big)$ be arbitrarily fixed. Following Step 1 there is some $l(\varepsilon) \in \mathbb{N}$ with

$$\|\rho(t_l) - \rho^*\|_{H^*} \leq \varepsilon \quad \text{for all } l \in \mathbb{N},\, l \geq l(\varepsilon).$$

In view of $\rho \in BC([0, \infty); H^*)$, for every $l \in \mathbb{N}$, $l \geq l(\varepsilon)$ we define $t_l^*(\varepsilon) \in [\theta, \infty]$ by

$$t_l^*(\varepsilon) = \sup\big\{t^* \geq \theta : \|\rho(t) - \rho^*\|_{H^*} \leq \varepsilon \text{ for all } t \in [t_l, t^*]\big\}.$$

We will prove by contradiction that there exists some $l^*(\varepsilon) \in \mathbb{N}$, $l^*(\varepsilon) \geq l(\varepsilon)$ such that $t_l^*(\varepsilon) = \infty$ holds true for all $l \in \mathbb{N}$ with $l \geq l^*(\varepsilon)$: Otherwise we could find an increasing sequence $(\tau_l) \subset [\theta, \infty)$ of points $\tau_l > t_l$ satisfying $\lim_{l\to\infty} \tau_l = \infty$ and

$$\|\rho(\tau_l) - \rho^*\|_{H^*} > \varepsilon, \quad \|\rho(t) - \rho^*\|_{H^*} \leq \delta \quad \text{for all } t \in [t_l, \tau_l] \text{ and } l \geq l(\varepsilon). \tag{39}$$

Hence, condition (36) holds true on the interval $[t_l, \tau_l]$. Together with the relation

$$\|K^*\rho(\tau_l) - K^*\rho(t_l)\|_{V^*} \leq \int_{t_l}^{\tau_l} \|(\mathcal{E}u)'(s)\|_{V^*}\, d\lambda(s),$$

for $l \geq l(\varepsilon)$ estimate (35) yields

$$\begin{aligned}\|K^*\rho(\tau_l) - K^*\rho^*\|_{V^*} &\leq \|K^*\rho(\tau_l) - K^*\rho(t_l)\|_{V^*} + \|K^*\rho(t_l) - K^*\rho^*\|_{V^*} \\ &\leq c_1[F(\rho(t_l)) - F(\rho^*)]^\vartheta + \|K^*\rho(t_l) - K^*\rho^*\|_{V^*}.\end{aligned}$$

Due to $K^* \in \mathcal{L}(H^*; V^*)$ and, applying Step 1, $\lim_{l\to\infty} \|\rho(t_l) - \rho^*\|_{H^*} = 0$, we obtain $\lim_{l\to\infty} \|K^*\rho(\tau_l) - K^*\rho^*\|_{V^*} = 0$. Following Theorem 4 and 7, the sequence $(u(\tau_l)) \subset U(\gamma)$ is bounded in $C^{0,\alpha}$. Hence, $(\rho(\tau_l))$ is precompact in H^*, and there exists a subsequence of $(\rho(\tau_l))$ converging to some limit $\hat{\rho} \in H^*$ in H^*. Using again $K^* \in \mathcal{L}(H^*; V^*)$ this yields $K^*\hat{\rho} = K^*\rho^* \in V^*$, which means

$$\langle \hat{\rho}, K\varphi\rangle_H = \langle K^*\hat{\rho}, \varphi\rangle_V = \langle K^*\rho^*, \varphi\rangle_V = \langle \rho^*, K\varphi\rangle_H \quad \text{for all } \varphi \in V.$$

Since $K \in \mathcal{L}(V; H)$ is the continuous and dense embedding of V in H, we obtain $\hat{\rho} = \rho^*$. Hence, the above mentioned subsequence of $(\rho(\tau_l))$ converges to ρ^* in H^* in contradiction to assumption (39). Therefore, we have shown that for every $\varepsilon \in \big(0, \frac{\delta}{2}\big)$ there exists some $l^*(\varepsilon) \in \mathbb{N}$ such that

$$\|\rho(t) - \rho^*\|_{H^*} \leq \varepsilon \quad \text{for all } t \geq t_l \text{ and } l \in \mathbb{N},\, l \geq l^*(\varepsilon),$$

in other words, the desired result $\lim_{t\to\infty} \|u(t) - u^*\|_H = 0$.

8. Due to Remark 14 the set $\{u(t) : t \geq \theta\} \subset U(\gamma)$ is bounded in $C^{0,\alpha}$ and, therefore, precompact in $C^{0,\alpha'}$ for every exponent $\alpha' \in (0, \alpha)$. Assume that we could find some $\varepsilon > 0$ and an increasing sequence $(\tau_l) \subset [\theta, \infty)$ which satisfies $\lim_{l\to\infty} \tau_l = \infty$ and

$$\|u(\tau_l) - u^*\|_{C^{0,\alpha'}} \geq \varepsilon \quad \text{for all } l \in \mathbb{N}. \tag{40}$$

Because the sequence $(u(\tau_l))$ is precompact in $C^{0,\alpha'}$ and converges to u^* in H due to Step 7, we could find a subsequence of $(u(\tau_l))$ converging to u^* in $C^{0,\alpha'}$, which contradicts to assumption (40). In view of Remark 14 we end up with

$$\lim_{t\to\infty} \|u(t) - u^*\|_{C^{0,\alpha'}} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|v(t) - v^*\|_{C^{0,\alpha'}} = 0,$$

since we have $v(t) = DF(\rho(t)) \in C^{0,\alpha'}$ for $t \geq \theta$ and $v^* = DF(\rho^*) \in C^{0,\alpha'}$. Finally, due to Assumption 6 we get $P \in \mathcal{L}(J[L^\infty]; W^{1,2,\omega_0})$ and $w^* = D\Psi(\rho^*) = P\rho^* + \psi \in W^{1,2,\omega_0}$ which yields $\lim_{t\to\infty} \|w(t) - w^*\|_{W^{1,2,\omega_0}} = 0$. □

7. Simulation results for phase separation processes

We present simulation results for phase separation processes in ternary systems of colored particles occupying a domain $X \subset \mathbb{R}^n$ with Lipschitz boundary. For the sake of numerical simplicity, we consider the special case, where the nonlocal potential operator of self-interaction and the contribution, due to external forces, can be described by means of the inverse of a linear second order elliptic operator having appropriate regularity properties. For nonsmooth coefficients b belonging to the set

$$\mathfrak{B} = \big\{ b \in L^\infty(X) : \ \nu \leq b(x) \leq 1/\nu \quad \text{for } \lambda^n\text{-almost all } x \in X \big\},$$

we consider the family of operators $L(b) \in \mathcal{L}\big(W^{1,2}(X); [W^{1,2}(X)]^*\big)$ given by

$$\langle L(b)w, \varphi \rangle_{W^{1,2}(X)} = \int_X \big(b\nabla w \cdot \nabla\varphi + w\varphi \big)\, d\lambda^n \quad \text{for } \varphi \in W^{1,2}(X). \tag{41}$$

For b, $d \in \mathfrak{B}$, according to Assumption 5, we introduce the elliptic operators

$$P_{k\ell} = \kappa_{k\ell} L(b)^{-1} \in \mathcal{L}\big([W^{1,2}(X)]^*; W^{1,2}(X)\big) \quad \text{for } k,\ \ell \in \{0,1,2\},$$

and the external potentials $\phi_k = L(d)^{-1} f_k \in W^{1,2}(X)$ for right-hand sides $f_k \in [W^{1,2}(X)]^*$ defined by

$$\langle f_k, \varphi \rangle_{W^{1,2}(X)} = \int_X g_k \varphi \, d\lambda^n + \int_{\partial X} h_k \varphi \, d\lambda_{\partial X} \quad \text{for } \varphi \in W^{1,2}(X),\ k \in \{0,1,2\}, \tag{42}$$

where $g_k \in L^\infty(X)$ and $h_k \in L^\infty(\partial X)$ are external volume and boundary forces.

Using the above setting, we prescribe constant intensities $\kappa_{k\ell} = \kappa_{\ell k}$ of interaction forces between particles of type k and $\ell \in \{0,1,2\}$. The cases $\kappa_{k\ell} > 0$, $\kappa_{k\ell} < 0$, and $\kappa_{k\ell} = 0$ represent the repulsive interaction, attractive interaction, and no interaction, respectively.

According to Remark 6, we consider the corresponding operator $P \in \mathcal{L}(H^*; V)$ and the element $\psi \in V$, where $H = L^2(X; \mathbb{R}^2)$ and $V = W^{1,2}(X; \mathbb{R}^2)$. In general, in the nonsmooth situation described above the interaction potentials $w = P\rho + \psi \in V$ are not Lipschitz continuous. Hence, the assumptions formulated in the work [11] of Gajewski and Zacharias, in the papers [28, 29] of Londen and Petzeltova and in the recent contribution [12] of Gal and Grasselli are too strong.

In fact, as a sharp result, the regularity theory for nonsmooth elliptic boundary value problems of Griepentrog and Recke, see [18, 19], applied to (41) and (42), just

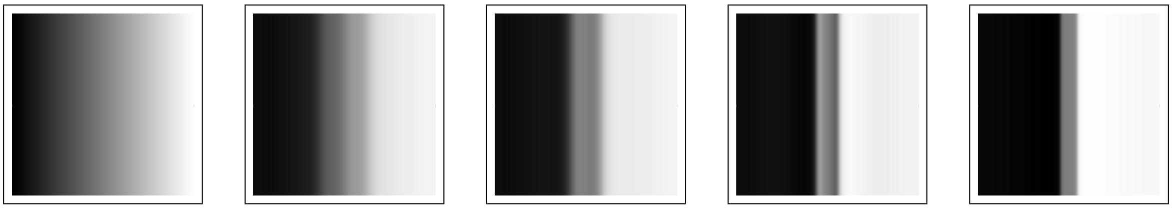

FIGURE 1. Phase separation process in a ternary system for an initial value which is constant in vertical direction. The stripe pattern is preserved during the whole evolution. The final state is a local minimizer of the free energy under the constraint of conservation of particle number, see Fig. 3.

FIGURE 2. Phase separation process in a ternary system for a mirror-symmetric, slightly different initial value. There occur metastable states. Finally, the phases are separated by a straight line and two circular arcs, joining in a triple point. The final state is a global minimizer of the free energy under the constraint of conservation of particle number, see Fig. 3.

ensure that there exists a constant $\omega_0 > n-2$ such that the restriction of P is a bounded linear operator from $J[L^\infty]$ into the Sobolev–Morrey space $W^{1,2,\omega_0}$ and that the external potentials ψ belongs to $W^{1,2,\omega_0}$. Our example represents a natural and desirable situation of nonsmooth data. At the same time this is the limit case, where Assumption 6 on the regularity of interaction is just satisfied, which proves this assumption to be minimal. Note that the natural regularity of solutions to elliptic boundary value problems of the above type was implicitly used already in the papers [9, 10] of GAJEWSKI and SKRYPNIK.

Again, for the sake of numerical simplicity, only, we consider the case of constant coefficients $b = r^2$ and $d = \varrho^2$, where Green's functions to the corresponding elliptic operators are rapidly decreasing kernels of Bessel type, which decay exponentially outside their effective ranges $r > 0$ and $\varrho > 0$, respectively.

For our simulations we have used the dissipative discretization scheme of GÄRTNER and GAJEWSKI, see [6]. It combines a Crank–Nicholson-type discretization in time with a Voronoi finite volume scheme on boundary-conforming Delaunay meshes in space.

Phase separation in ternary systems. Figures 1 and 2 show numerical results for two simulations of three-component phase separation processes in a square with uniform interaction intensities $\kappa_{kk} = -\kappa < 0$ (attraction), $\kappa_{k\ell} = \kappa > 0$ (repulsion) for $k, \ell \in \{0, 1, 2\}$ with $k \neq \ell$ in a homogeneous environment without external forces. This models the phase

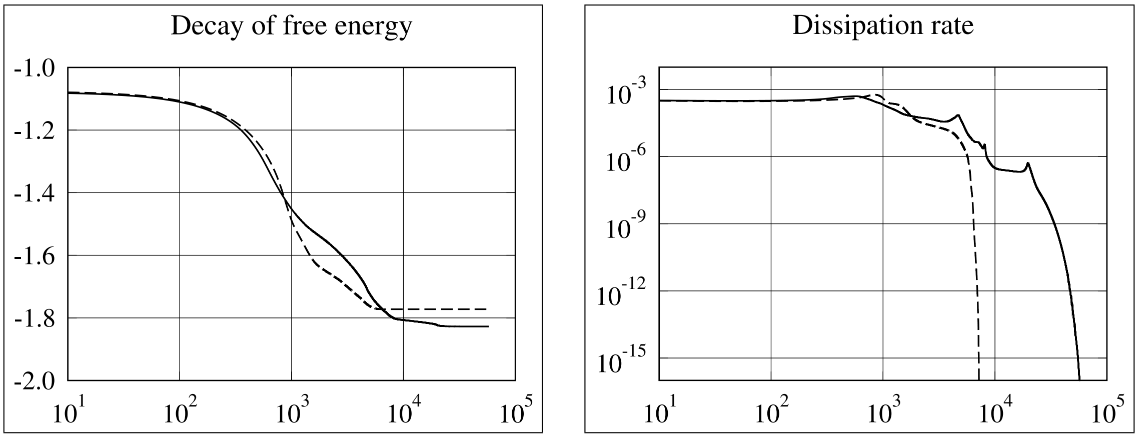

FIGURE 3. Comparison of both the three-component phase separation processes with regard to the decay of free energy (left) and the dissipation rate along the trajectory (right). The free energy approaches a local minimum under the constraint of conservation of particle number at the final state shown in Fig. 1 (dashed lines), whereas it reaches its global minimum at the final state depicted in Fig. 2 (solid lines).

separation in an *incompressible* body consisting of a ternary system of equally treated components.

Note that in both cases the slightly different initial configurations contain equal numbers of black, white, and gray particles, respectively. Obviously, the final states do not depend only on these integral quantities. After initial diffusion, the particles start to agglomerate or to grain until they reach fully separated states. Going on further, we see coarsening of phases or occurence of metastable states still being far from equilibrium. The final states are reached after quite different periods of time depending on the symmetry of the initial value. A comparison of both simulations in Fig. 3 shows major differences with respect to the decay of the free energy and the dissipation rate along the trajectories.

Phase separation in a binary system with damage diffusion. The evolution process changes completely in Fig. 4, when the initial configuration is a randomly chosen distribution of 10 % black ($k = 0$), 45 % white ($k = 1$), and 45 % gray ($k = 2$) particles, and the (black) voids neither interact with themselves nor with white or gray particles; we have modified $\kappa_{0\ell} = \kappa_{k0} = 0$ for k, $\ell \in \{0, 1, 2\}$ in the above setting. Here, we describe the phase separation of white and gray particles in a *compressible* body, which has 90 % of the unit density. The rest is filled up with voids. We have further modified the regime by applying stationary external forces at two boundary parts. The lower half of the left and the upper half of the right part are loaded equally to press the white and gray particles inwards. According to (42), this corresponds to $g_0 = 0$, $g_1 = 0$, $g_2 = 0$, and $h_0 = 0$, $h_1 \neq 0$, $h_2 \neq 0$.

During the evolution, after initial diffusion, both the white and gray components show agglomeration, graining, and slight denting at the pressure zones. White and gray particles

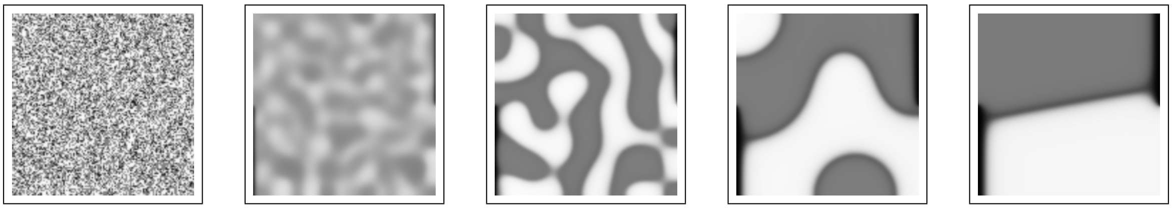

FIGURE 4. Phase separation process in a binary system with inward pressure for an initial value of randomly distributed particles. The lower half of the left and the upper half of the right part are loaded equally to press the white and gray particles inwards. The process starts with diffusion, graining of particles, and a slight denting at the pressure zones. After the phases are fully separated, voids concentrate at interfaces between the phases. The coarsening and hardening of phases is accompanied by the thickening and concentration of damage channels. Finally, both phases are separated by a straight channel of shear damage which connects both the pressure zones.

reach the state of full separation and compression. They leave room for the voids to concentrate as damage channels at the interfaces between white and gray phases, which show strong resistance, obstructing inward pressure. Further coarsening and hardening of phases leads to the thickening of damage channels and significant denting at the pressure zones. The process arrives at metastable states, still being far from equilibrium. Whenever the number of connected phases is reduced by the separation process, there occurs a narrow peak in the dissipation rate, see Fig. 5. In the final state both the white and the gray phases are completely separated from each other by a straight channel of shear damage which connects both the pressure zones.

APPENDIX A. SOME VARIANTS OF SOBOLEV–POINCARÉ INEQUALITIES

Here we collect variants of Sobolev inequalities for functions vanishing on a given subset F of the domain $X \subset \mathbb{R}^n$ with Lipschitz boundary.

LEMMA 18. *Let $X \subset \mathbb{R}^n$ be a domain with Lipschitz boundary. Then, there exists a constant $c_1 > 0$ depending on n and X such that for every $\beta \in (0, 1]$ and all measurable subsets $F \subset X$ satisfying $\sigma(F) \geq \beta\sigma(X)$, the following inequality holds true for all $v \in W^{1,2}(X)$ vanishing σ-almost everywhere on F:*

$$\int_X |v|^2 \, d\sigma \leq \frac{c_1}{\beta} \int_X |\nabla v|^2 \, d\sigma.$$

Proof. 1. The norm $\| \ \|_V$ in $V = W^{1,2}(X)$ is defined by $\|v\|_V^2 = \|v\|_H^2 + [v]_V^2$ for all $v \in V$, where the norm $\| \ \|_H$ in $H = L^2(X)$ and the seminorm $[\ ]_V$ in V are given by

$$\|v\|_H^2 = \int_X |v|^2 \, d\sigma \quad \text{for } v \in H, \quad [v]_V^2 = \int_X |\nabla v|^2 \, d\sigma \quad \text{for } v \in V.$$

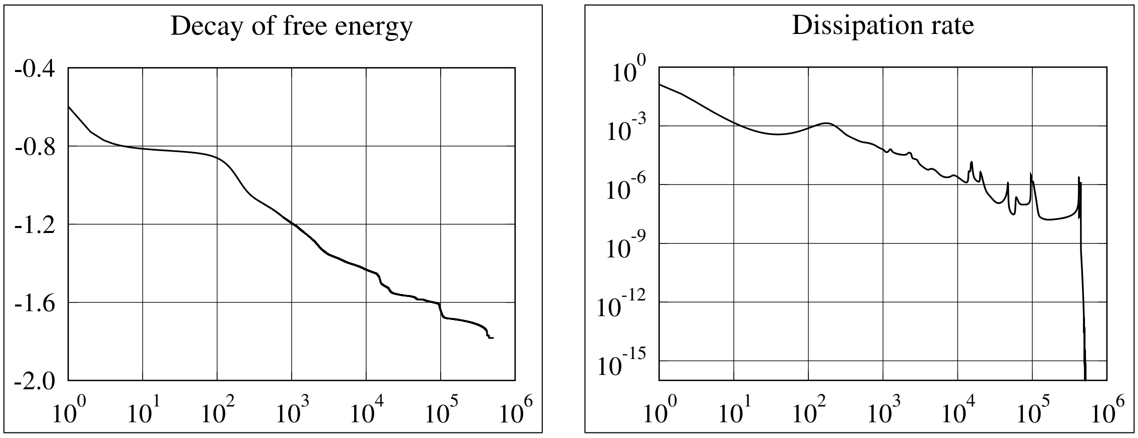

FIGURE 5. Decay of free energy (left) and dissipation rate along the trajectory (right) of the phase separation process in the binary system with damage diffusion shown in Fig. 4. Due to the randomly chosen initial configuration the final state is reached after quite a long time, and there occur many metastable states. The disappearance of these states shows up in the dissipation rate with narrow peaks.

Note that the subspace $V_1 = \{v \in V : [v]_V = 0\}$ of V is the space of constant functions, because $X \subset \mathbb{R}^n$ is connected. Since for every domain $X \subset \mathbb{R}^n$ with Lipschitz boundary there exists a bounded linear extension operator from $W^{1,2}(X)$ to $W^{1,2}(\mathbb{R}^n)$, see GIUSTI [16], the embedding of V into H is completely continuous. Consequently, due to a result [35, Lemma 4.1.3] of ZIEMER, we find a constant $c_1 > 0$ depending on n and X such that for every projector $\Pi \in \mathcal{L}(V; V_1)$ from V onto V_1 we have the generalized Sobolev–Poincaré inequality

$$\|v - \Pi v\|_H \le c_1 \|\Pi\|_{\mathcal{L}(V;V_1)} [v]_V \quad \text{for all } v \in V.$$

2. Let $\beta \in (0, 1]$ and some measurable subset $F \subset X$ satisfying $\sigma(F) \ge \beta\sigma(X)$ be given. We consider the projector $\Pi \in \mathcal{L}(V; V_1)$ from V to its subspace V_1 of constant functions defined by the mean value

$$\Pi v = \fint_F v \, d\sigma = \frac{1}{\sigma(F)} \int_F v \, d\sigma \quad \text{for } v \in V.$$

Then, for every $v \in V$ Cauchy's inequality yields

$$\int_X |\Pi v|^2 \, d\sigma = \int_X \left| \frac{1}{\sigma(F)} \int_F v \, d\sigma \right|^2 d\sigma \le \frac{\sigma(X)}{\sigma(F)} \int_X |v|^2 \, d\sigma.$$

Hence, we obtain $\|\Pi\|^2_{\mathcal{L}(V;V_1)} \le 1/\beta$ and together with Step 1 this leads to

$$\|v - \Pi v\|_H^2 \le \frac{c_1^2}{\beta} [v]_V^2 \quad \text{for all } v \in V. \tag{43}$$

Having in mind the definition of Π, this finishes the proof. □

Lemma 19. *Let $S \subset \mathbb{R}$ be an open interval and $X \subset \mathbb{R}^n$ be a domain with Lipschitz boundary. Then for every $\beta \in (0,1]$ there exists a constant $c_2 > 0$ depending on β, n and X such that for all functions $v \in L^2(S; W^{1,2}(X)) \cap L^\infty(S; L^2(X))$ with the property that $v(s) \in W^{1,2}(X)$ for λ-almost all $s \in S$ vanishes on a measurable subset $F_s \subset X$ satisfying $\sigma(F_s) \geq \beta\sigma(X)$, the inequality*

$$\int_S \int_X |v|^{2\varkappa}\, d\sigma\, d\lambda \leq c_2 \left(\operatorname*{ess\,sup}_{s\in S} \int_X |v(s)|^2\, d\sigma \right)^{\varkappa-1} \int_S \int_X |\nabla v|^2\, d\sigma\, d\lambda$$

holds true, where $\varkappa = 1 + 2/n$.

Proof. 1. For all $v \in L^2(S; W^{1,2}(X)) \cap L^\infty(S; L^2(X))$ Hölder's inequality yields

$$\int_S \int_X |v|^{2\varkappa}\, d\sigma\, d\lambda \leq \int_S \left(\int_X |v(s)|^{2n/(n-2)}\, d\sigma \right)^{(n-2)/n} \left(\int_X |v(s)|^2\, d\sigma \right)^{2/n} d\lambda(s).$$

Due to the continuous embedding of $W^{1,2}(X)$ into $L^{2n/(n-2)}(X)$ we find a constant $c_0 > 0$ depending on n and X such that

$$\left(\int_X |v(s)|^{2n/(n-2)}\, d\sigma \right)^{(n-2)/n} \leq c_0 \int_X \left(|v(s)|^2 + |\nabla v(s)|^2 \right) d\sigma$$

holds true for λ-almost all $s \in S$, which yields

$$\int_S \int_X |v|^{2\varkappa}\, d\sigma\, d\lambda \leq c_0 \int_S \left(\int_X \left(|v(s)|^2 + |\nabla v(s)|^2 \right) d\sigma \right) \left(\int_X |v(s)|^2\, d\sigma \right)^{\varkappa-1} d\lambda(s),$$

and, therefore,

$$\int_S \int_X |v|^{2\varkappa}\, d\sigma\, d\lambda \leq c_0 \left(\operatorname*{ess\,sup}_{s\in S} \int_X |v(s)|^2\, d\sigma \right)^{\varkappa-1} \int_S \int_X \left(|v|^2 + |\nabla v|^2 \right) d\sigma\, d\lambda.$$

2. If, additionally, there exists some $\beta > 0$ such that the function $v(s) \in V$ for λ-almost all $s \in S$ vanishes on a measurable subset $F_s \subset X$ with $\sigma(F_s) \geq \beta\sigma(X)$, then, applying Lemma 18, we end up with

$$\int_S \int_X |v|^{2\varkappa}\, d\sigma\, d\lambda \leq c_0 \left(1 + \frac{c_1}{\beta} \right) \left(\operatorname*{ess\,sup}_{s\in S} \int_X |v(s)|^2\, d\sigma \right)^{\varkappa-1} \int_S \int_X |\nabla v|^2\, d\sigma\, d\lambda,$$

which gives the desired result. □

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