The Chern character of $\mathcal{O}$-summable Fredholm modules over DGA’s and localization on loop space (arXiv:1901.04721)

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ETH Zürich / Univ. Zürich on November 7, 2019
Alvarez-Gaumé/Atiyah/Bismut/Witten (1980’s):

With $X$ a cpt., even-dim. Riem. spin MF, there (formally!) exists a canonically given even linear map

$$I : \hat{\Omega}(LX) := \prod_{j=0}^{\infty} \Omega^j(LX) \longrightarrow \mathbb{C}, \quad \xi \longmapsto \int_{LX} e^{-E+\omega} \wedge \xi,$$

the *supersymmetric path integral*, such that

1) $I$ is supersymmetric (or equivariantly co-closed):

$$I[(d+\iota_K)\xi] = 0 \quad \text{for all } \xi \in \hat{\Omega}_{S^1}(LX) := \hat{\Omega}(LX) \cap \{\zeta : \iota_K \zeta = 0\}.$$

Above, $K$ is the generator of $S^1 \subset LX$, so $d + \iota_K$ turns $\hat{\Omega}_{S^1}(LX)$ into a super complex ($\sim$ equivariant homology).
For all $\xi \in \hat{\Omega}_{S^1}(LX)$ with $(d + \iota_K)\xi = 0$, one has the Duistermaat-Heckmann localization formula

$$I[\xi] = \int_{X=\text{Fix}(S^1 \subset LX)} \hat{A}(X) \wedge \xi|_X. \quad (1)$$

Given $E = (E, \nabla) \to X$, there exists a canonically given $\text{Bch}(E) \in \hat{\Omega}^+_S(LX)$ such that

- $\text{Bch}(E)|_X = \text{Ch}(E)$,
- $(d + \iota_K)\text{Bch}(E) = 0$,
- $\text{ind}(D^E) = I[\text{Bch}(E)]$.

Why all this?

$$\text{ind}(D^E) = I[\text{Bch}(E)] = \int_X \hat{A}(X) \wedge \text{Bch}(E)|_X = \int_X \hat{A}(X) \wedge \text{Ch}(E).$$
Two very serious (and obviously connected) mathematical problems:

- definition of $I: K(\gamma) = \dot{\gamma} \rightarrow \leftarrow$ Brownian motion
- right choice of observables: growth conditions $\rightarrow \leftarrow$ sufficiently large to carry $\text{Bch}(\mathcal{E})$.

**Program:** construct a natural map $\rho_\varepsilon : C^e(\Omega(X)_\mathbb{T}) \rightarrow \hat{\Omega}(LX)$ of super complexes, and a linear functional $I : C^e(\Omega(X)_\mathbb{T}) \rightarrow \mathbb{C}$ which descends to a linear functional

$$I : \Omega_{\text{int}}(LX) := \text{im}(\rho_\varepsilon) \longrightarrow \mathbb{C}, \quad \text{so that } I \text{ does the job.}$$

**Surprising fact:** Using cyclic homology, $I$ can be constructed in a very general framework, namely, it is the *Chern character of a $\vartheta$-summable Fredholm over a locally convex DGA*....
Def. With $\Omega$ a LC-DGA (locally convex DGA), an (even-dimensional) $\vartheta$ - summable Fredholm module over $\Omega$ is given by a triple $\mathcal{M} = (\mathcal{H}, c, Q)$, such that

- $\mathcal{H}$ is a super Hilbert space,
- $c : \Omega \to \mathcal{L}(\mathcal{H})$ is an even bounded linear map,
- $Q$ an odd self-adjoint (unbounded) linear operator on $\mathcal{H}$ with $e^{-tQ^2}$ trace class for all $t > 0$,

with

$$[Q, c(f)] = c(df), \quad \text{and} \quad c(f\theta) = c(f)c(\theta), \quad c(\theta f) = c(\theta)c(f),$$

for all $f \in \Omega^0$, $\theta \in \Omega$. New: $c$ is not a representation (only on $\Omega^0$)!
Ex. (spin case): We can take $\Omega = \Omega(X)$, $\mathcal{H} = L^2(X, \Sigma)$ with $\Sigma \to X$ spin bundle, $Q = D$ the Dirac operator in $L^2(X, \Sigma)$ and

$$c : \Omega(X) \longrightarrow \mathcal{L}(L^2(X, \Sigma)), \quad c(\alpha)\psi(x) := c(\alpha(x))\psi(x).$$

Def.: The \textit{acyclic extension} of $\Omega$ is the LC-DGA given by $\Omega_T := \Omega[\sigma]$ with $\sigma$ a formal variable of degree $-1$ with $\sigma^2 = 0$, where on $\theta = \theta' + \sigma\theta'' \in \Omega_T$ the differential is $d_T = d - \iota$ with

$$d\theta = d\theta' - \sigma d\theta'' \quad \text{and} \quad \iota\theta = \theta''.$$

Ex. (spin case): Here we have $\Omega(X)_T \cong \Omega(X \times S^1)^{S^1}$ with

$$\iota \leftrightarrow \iota\partial_t.$$
**Def.** For each cont. seminorm $\nu$ on $\Omega_T$ define a seminorm $\epsilon_\nu$ on

$$C(\Omega_T) := \bigoplus_{N=0}^{\infty} \Omega_T \otimes \Omega_T[1]^{\otimes N}$$

by

$$\epsilon_\nu(\theta) := \sum_{N=0}^{\infty} \frac{\nu(\theta_N)}{N!}, \quad \theta = \sum_{N=0}^{\infty} \theta_N \in C(\Omega_T).$$

The completion of $C(\Omega_T)$ with respect to $\epsilon_\nu$'s is denoted by $C^\epsilon(\Omega_T)$ and called the *space of entire chains*.

$$C^\epsilon_+(\Omega_T) \xrightarrow{b+B} C^\epsilon_- (\Omega_T) \xrightarrow{b+B} C^\epsilon_+ (\Omega_T) \quad \text{(cyclic homology)}.$$
Recall that we have fixed a module $\mathcal{M} = (\mathcal{H}, c, Q)$ over $\Omega$. Define for each $N \in \mathbb{N}_0$ a linear map

$$F_\mathcal{M} : C^e(\Omega_T) \rightarrow \{\text{closed operators in } \mathcal{H}\}$$

by

$$F_\mathcal{M}^{(0)} = Q^2,$$

$$F_\mathcal{M}^{(1)}(\theta) = c(d\theta') - [Q, c(\theta')] - c(\theta''),$$

$$F_\mathcal{M}^{(2)}(\theta_1 \otimes \theta_2) = (-1)^{|\theta_1'|}(c(\theta_1\theta_2') - c(\theta_1')c(\theta_2')),$$

$$F_\mathcal{M}^{(N)} = 0 \quad \text{for all } N \geq 3.$$

For $M \leq N$ denote with $\mathcal{P}_{M,N}$ all tuples $I = (I_1, \ldots, I_M)$ of subsets of $\{1, \ldots, N\}$ with $I_1 \cup \cdots \cup I_M = \{1, \ldots, N\}$ and with each element of $I_a$ smaller than each element of $I_b$ whenever $a < b$. 
Given $\theta_1 \otimes \cdots \otimes \theta_N \in \Omega_T^\otimes N$ and $l = (l_1, \ldots, l_M) \in \mathcal{P}_{M,N}$, $1 \leq a \leq M$ set $\theta_{l_a} := (\theta_{i+1}, \ldots, \theta_{i+m})$, if $l_a = \{j \mid i < j \leq i + m\}$ for some $i, m$.

**Thm (G.-Ludewig)** There exists a unique continuous linear functional

$$
\text{Ch}(\mathcal{M}) : \mathcal{C}^\epsilon(\Omega_T) \longrightarrow \mathbb{C},
$$

the Chern Character of $\mathcal{M}$, such that

$$
\langle \text{Ch}(\mathcal{M}), \theta_0 \otimes \cdots \otimes \theta_N \rangle = \sum_{M=1}^{N} (-1)^M \sum_{l \in \mathcal{P}_{M,N}} \int_{\Delta_M} \text{Str} \left( c(\theta_0) e^{-\tau_1 Q^2} F_{\mathcal{M}}(\theta_{l_1}) \right) \times \\
\times e^{-(\tau_2 - \tau_1) Q^2} F_{\mathcal{M}}(\theta_{l_2}) \cdots e^{-(\tau_M - \tau_{M-1}) Q^2} F_{\mathcal{M}}(\theta_{l_M}) e^{-(1-\tau_M) Q^2}) d\tau.
$$
On the properties of $\text{Ch}(\mathcal{M})$:

**Thm (G.-Ludewig)** $\text{Ch}(\mathcal{M})$ is even, co-closed, invariant under homotopies of $\mathcal{M}$, Chen-normalized, and leads to a noncommutative index theorem of the form

$$\text{ind}_{c(p)\mathcal{H}}(c(p)Dc(p)) = \langle \text{Ch}(\mathcal{M}), \text{Ch}(p) \rangle,$$

for all $p = p^2 \in \text{Mat}_n(\Omega^0)$. Here, $\text{Ch}(p) \in C_+^\varepsilon(\Omega_T)$ is the Bismut-Chern character of $p$. 
Consider now $\mathcal{M}_X = (L^2(X, \Sigma), c, D)$ over $\Omega(X)$. The extended Chen integral map

$$\rho_\epsilon : C^\epsilon(\Omega(X)_\mathbb{T}) \longrightarrow \hat{\Omega}(LX),$$

$$\rho(\theta_0, \ldots, \theta_N) = A_{S^1} \int_{\Delta_N} \theta'_0(0) \wedge (\nu_K \theta'_1(\tau_1) + \theta''_1(\tau_1)) \wedge \cdots \wedge (\nu_K \theta'_N(\tau_N) + \theta''_N(\tau_N)) \, d\tau$$

is a continuous map of super complexes (G.-Cacciatori).
Set

\[ I := \text{Ch}(\mathcal{M}_X) : C^\epsilon(\Omega(X)_\mathbb{T}) \to \mathbb{C}, \quad \Omega_{\text{int}}(LX) := \text{im}(\rho_\epsilon) \subset \hat{\Omega}(LX). \]

**Thm (G.-Ludewig)** There exists a unique linear functional

\[ I : \Omega_{\text{int}}(LX) \to \mathbb{C} \quad \text{with} \quad I \circ \rho_\epsilon = I. \]

Moreover, \( I \) is even and equiv. co-closed, and for all \( \xi \in \Omega_{\text{int}}(LX) \) with \( (d + \iota_K)\xi = 0 \) one has the localization formula

\[ I[\xi] = \int_X \hat{A}(X) \wedge \xi|_X. \quad (2) \]

Finally, for all \( \mathcal{E} \to X \) one has \( I[\text{Bch}(\mathcal{E})] = \text{ind}(D_{\mathcal{E}}). \)
Proof:
i) $I$ is well-defined, even, equiv. co-closed $\leftrightarrow I$ Chen normalized, even, co-closed;

ii) $I[Bch(\mathcal{E})] = \text{ind}(D_\mathcal{E}) \leftrightarrow$ non. index theorem and $\rho_\varepsilon(\text{Ch}(p)) = Bch(\text{im}(p))$;

iii) localization formula $\leftrightarrow I$ homotopy invariant + heat kernel methods à la Getzler.

Thank you very much for your attention!