The Calderón-Zygmund inequality on noncompact Riemannian manifolds

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Geometric Structures and Spectral Invariants

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This talk is about the paper:


Partially also:

**Batu Güneysu:** *Sequences of Riemannian second order cut-off functions.* Preprint (2014).
For a possibly noncompact smooth Riemannian $m$-manifold $M \equiv (M, g)$, consider the following global problems for second order Sobolev spaces on $M$, on the $L^p$-scale, $1 < p < \infty$:

- **Problem 1 (denseness):** Under which assumptions on $M$ does one have $H^2_0, p(M) = H^2, p(M)$ (without $r_{\text{inj}}(M) > 0$)?

- **Problem 2 (Poisson’s equation):** Under which assumptions on $M$ does one have the implication

  $$f \in L^p(M) \cap C^2(M), \Delta f \in L^p(M) \Rightarrow f \in H^2, p(M)$$

  (that is, $|\text{Hess}(f)| \in L^p(M)$)?

- **Problem 3 (gradient estimate):** Under which assumptions on $M$ does one have an inequality of the form

  $$\|\text{grad}(f)\|_p \leq C(\|\Delta f\|_p + \|f\|_p)$$

  for all $f \in C_c^\infty(M)$?
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As we will see in a moment, there is a \textit{common inequality} behind these types of problems:

\textbf{Definition}

Let $1 < p < \infty$. We say that $M$ satisfies the $L^p$-Calderón-Zygmund inequality (or in short $\text{CZ}(p)$), if there are $C_1 \geq 0$, $C_2 > 0$, such that for all $u \in C_\infty^c(M)$ one has

$$\|\text{Hess} (u)\|_p \leq C_1 \|u\|_p + C_2 \|\Delta u\|_p.$$  \hspace{1cm} (1)

- In $\mathbb{R}^m$, $\text{CZ}(p)$ with $C_1 = 0$ follows e.g. from estimates on singular integral operators that go back to Calderón and Zygmund (1950’s). Note that in $\mathbb{R}^m$: $\|\text{Hess} (u)\|_2 = \|\Delta u\|_2$ ("Bochner’s formula")
- In general, $\text{CZ}(p)$ depends very sensitively on the curvature and there has been no systematic treatment so far
- $\text{CZ}(p)$ always extends automatically from $C_\infty^c(M)$ to $H_0^{2,p}(M)$
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What is the precise connection between $CZ(p)$ and the problems 1, 2, 3?
Definition

a) $M$ is said to admit a sequence $(\chi_n) \subset C_c^\infty(M)$ of Laplacian cut-off functions, if $(\chi_n)$ has the following properties:

$(C1)$ $0 \leq \chi_n(x) \leq 1$ for all $n \in \mathbb{N}$, $x \in M$,

$(C2)$ for all compact $K \subset M$, there is an $n_0(K) \in \mathbb{N}$ such that for all $n \geq n_0(K)$ one has $\chi_n|_K = 1$,

$(C3) \sup_{x \in M} |d\chi_n(x)|_x \to 0$ as $n \to \infty$,

$(C4) \sup_{x \in M} |\Delta \chi_n(x)| \to 0$ as $n \to \infty$.

b) $M$ is said to admit a sequence $(\chi_n) \subset C_c^\infty(M)$ of Hessian cut-off functions, if $(\chi_n)$ has the above properties $(C1)$, $(C2)$, $(C3)$, and in addition

$(C4') \sup_{x \in M} |\text{Hess} \chi_n(x)(x)|_x \to 0$ as $n \to \infty$.

$(C1) \land (C2) \land (C3) \iff$ completeness;

$(C1) \land (C2) \land (C3) \land (C4') \Rightarrow (C1) \land (C2) \land (C3) \land (C4)$
Introduction

CZ(p): Connection to problems 1, 2, 3 and local aspects

CZ(p): Global criteria

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Theorem (G.; G. & P.)

a) If $M$ is complete with $\text{Ric} \geq 0$, then $M$ admits L-C.O.F.’s.

b) If $M$ is complete with $\|R\|_{\infty} < \infty$ (with $R$ the curvature tensor), then $M$ admits H-C.O.F.’s.

No $r_{\text{inj}}(M) > 0$ required! Proofs are subtle and rely on a highly nontrivial rigidity result by Cheeger-Colding (1996) for a), and a smoothing result by B. Chow et. al. (Ricci flow II, 2008) for b).

The connection between $\#$-C.O.F.’s, $CZ(p)$ and problems 1, 2 is the following elementary result:

Proposition

Assume that $M$ satisfies $CZ(p)$.

a) If $M$ admits L-C.O.F.’s, then $H^{2,p}_{0}(M) = H^{2,p}(M)$.

b) Assume that $M$ admits H-C.O.F.’s. Then for any $u \in C^{2}(M)$ with $u, |\text{grad}(u)|, \Delta u \in L^{p}(M)$, one has $|\text{Hess}(u)| \in L^{p}(M)$. 

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Assume that $M$ satisfies CZ($p$).

a) If $M$ admits L-C.O.F.’s, then $H^2, p_0^2(M) = H^2, p(M)$.

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Theorem (G.; G. & P.)

a) If $M$ is complete with $\Ric \geq 0$, then $M$ admits $L$-C.O.F.’s.

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The connection between $\#$-C.O.F.’s, $\text{CZ}(p)$ and problems 1, 2 is the following elementary result:

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Assume that $M$ satisfies $\text{CZ}(p)$.

a) If $M$ admits $L$-C.O.F.’s, then $H^2_{0,p}(M) = H^2_{p}(M)$.

b) Assume that $M$ admits $H$-C.O.F.’s. Then for any $u \in C^2(M)$ with $u, |\text{grad}(u)|, \Delta u \in L^p(M)$, one has $|\text{Hess}(u)| \in L^p(M)$. 
The key to problem 3 (and many other results in the sequel!) is the following interpolation result:

**Theorem (G. & P.; Coulhon & Duong 2003 for $1 < p \leq 2$)**

Assume that either $2 \leq p < \infty$, or in case $1 < p < 2$ that either $M$ is complete or a relatively compact open subset of an arbitrary smooth Riemannian manifold. Then there is a $C = C(m, p) > 0$ s.t. for all $\varepsilon > 0$, $u \in C_\infty^c(M)$ one has the **interpolation inequality**

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\|\text{grad}(u)\|_p \leq C\varepsilon^{-1}\|u\|_p + C\varepsilon\|\text{Hess}(u)\|_p,
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in particular, under CZ($p$) one has the **gradient estimate from problem 3**: $\|\text{grad}(u)\|_p \leq C(\|\Delta u\|_p + \|u\|_p)$.

**Proof:** The $2 \leq p < \infty$ case: Apply the divergence theorem to $X := u \cdot \left( |\text{grad}(u)|^2 + \alpha \right)^{\frac{p-2}{2}} \text{grad}(u)$, $\alpha > 0$, and take $\alpha \to 0+$. 
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in particular, under $\text{CZ}(p)$ one has the **gradient estimate from problem 3**:

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**Proof:** The $2 \leq p < \infty$ case: Apply the divergence theorem to

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In what sense do we have $CZ(p)$ locally?
**A Hessian (key-)estimate on balls:** Using harmonic coordinates and an appropriate local elliptic estimate, we get (with $r_{Q,k,\alpha}(x)$ the $C^{k,\alpha}$-harmonic radius at $x$ with $Q^{-1}(\delta_{ij}) \leq (g_{ij}) \leq Q(\delta_{ij})$):

Theorem (G. & P.)

*Fix an arbitrary $x \in M$. Then for all $1 < p < \infty$, all $0 < r < r_{2,1,1/2}(x)/2$, and all real numbers $D$ with $\inf_{B_{r_{2,1,1/2}(x)}(x)} r_{2,1,1/2}(\bullet) \geq D > 0$, there is a $C = C(r, p, m, D) > 0$, such that for all $u \in C_c^\infty(M)$ one has*

\[
\left\| 1_{B_{r/2}(x)} \text{Hess} (u) \right\|_p 
\leq C \left( \left\| 1_{B_2(x)} u \right\|_p + \left\| 1_{B_{2r}(x)} \Delta u \right\|_p + \left\| 1_{B_{2r}(x)} \text{grad} (u) \right\|_p \right).
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Interpolation gives “qualitatively sharp” CZ($p$)’s on balls (first order terms produces the gradient terms in the proof)
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Interpolation gives “qualitatively sharp” $CZ(p)$’s on balls (first order terms produces the gradient terms in the proof)
Theorem (G. & P.)

a) \( \text{CZ}(p) \) holds on any relatively compact open subset \( \Omega \subset M \). Moreover, if \( \Omega \subset M \) is a relatively compact domain with smooth boundary \( \partial \Omega \), then \( \text{CZ}(p) \) holds in the stronger form

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b) Assume that either \( p \geq 2 \) or that \( 1 < p < 2 \) and that \( M \) is complete. If there is a relatively compact domain \( \Omega \subset M \) such that \( \text{CZ}(p) \) holds on \( M \setminus \overline{\Omega} \), then \( \text{CZ}(p) \) also holds on \( M \).

The weak \( \text{CZ}(p) \) of part a) follows e.g. from the previous Hessian estimate on balls and interpolation. The strong form follows from the weak form and the (resulting!) gradient estimate, and elliptic regularity.

The topological stability from part b) follows from part a) and the interpolation result: we pick up gradient terms from gluing!
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Which noncompact $M$’s admit $\text{CZ}(p)$?
One can give an essentially complete answer for the Hilbert space case $p = 2$:

**Theorem (G. & P.)**

a) Assume that $\text{Ric} \geq -C^2$. Then $CZ(2)$ holds in the following "infinitesimal" way: For every $\varepsilon > 0$ and every $u \in C_c^\infty(M)$ one has

$$\|Hess(u)\|_2^2 \leq \frac{C\varepsilon^2}{2} \|u\|_2^2 + \left(1 + \frac{C^2}{2\varepsilon^2}\right) \|\Delta u\|_2^2.$$  

b) There exists a smooth 2-dimensional, complete Riemannian manifold $N$ with unbounded curvature, such that $CZ(2)$ fails on $N$.

Part a) is really just Bochner’s formula (remember the $\mathbb{R}^m$ case!):

$$\int |Hess(u)|^2 = \int (du, d\Delta u) - \int \text{Ric}(\text{grad}(u), \text{grad}(u)).$$

Part b) is rather complicated ($\sim$ parabolic model surfaces).
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Part b) is rather complicated ($\rightsquigarrow$ parabolic model surfaces).
A result for arbitrary $p$, but positive injectivity radius:

**Theorem (G. & P.)**

Let $1 < p < \infty$ and assume $\|\text{Ric}\|_{\infty} < \infty$, $r_{\text{inj}}(M) > 0$. Then there is a $C = C(m, p, \|\text{Ric}\|_{\infty}, r_{\text{inj}}(M)) > 0$ such that for all $u \in C_c^\infty(M)$ one has

$$\|\text{Hess}(u)\|_p \leq C(\|u\|_p + \|\Delta u\|_p).$$

Idea of proof: By harmonic radius estimates there is a $D = D(m, r_{\text{inj}}(M), \|\text{Ric}\|_{\infty}) > 0$ such that $r_{2,1,1/2}(M) \geq D$. Let $r := D/2$. By the Hessian estimate on balls we have a $c = c(r, p, m, D) > 0$ such that, for all $(x_i)$, all $u \in C_c^\infty(M)$,

$$\int_{B_{r/2}(x_i)} |\text{Hess}(u)|^p \leq c \int_{B_{2r}(x_i)} (|\Delta u|^p + |\text{grad}(u)|^p + |u|^p).$$

Take appropriate sequence of points $(x_i)$, sum over $i$ and use interpolation.
A result for arbitrary $p$, but positive injectivity radius:

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Take appropriate sequence of points $(x_i)$, sum over $i$ and use interpolation.
The latter theorem applies to give:

**Theorem**

Let $f : M \to N$ be an isometric immersion of $M$ into a smooth complete, simply connected Riemannian manifold $N$ with $-\tilde{C}^2 \leq \text{Sec}_N \leq 0$. If $\|\|f\|\|_{\infty} < \infty$, then $CZ(p)$ holds on $M$ for every $1 < p < \infty$. 
A result for small $p$, but without positive injectivity radius assumption

**Theorem (G. & P.)**

Let $1 < p \leq 2$, and assume that $M$ is complete, that

$$
\max(\|R\|_\infty, \|\nabla R\|_\infty) < \infty,
$$

and that there are $D \geq 1$, $0 \leq \delta < 2$ with the volume growth

$$
\text{vol}(B_{tr}(x)) \leq Dt^D e^{t\delta + r^\delta} \text{vol}(B_r(x)) \quad \text{for all } x \in M, r > 0, t \geq 1.
$$

Then there is a

$$
C = C(m, p, \|R\|_\infty, \|\nabla R\|_\infty, D, \delta) > 0,
$$

such that for all $u \in C^\infty_c(M)$ one has

$$
\|\text{Hess}(u)\|_p \leq C(\|u\|_p + \|\Delta u\|_p).
$$
The proof of the latter result shows a deep connection between \textit{boundedness of covariant $L^p$-Riesz transforms} and $CZ(p)$: The asserted inequality follows immediately, once we have

$$\left\| \nabla (\Delta_1 + a_1)^{-1/2} d(\Delta_0 + a_1)^{-1/2} u \right\|_p \leq a_2 \| u \|_p.$$ 

But $\| d(\Delta_0 + a_1)^{-1/2} \|_{p,p} \leq C(p)$ for all $p$ is a classical result by Bakry (1987), and one can use probabilistic heat equation derivative formula by Thalmaier/F. Y. Wang (2003) for $\nabla e^{-t\Delta_1}$ to estimate $\left\| \nabla (\Delta_1 + a_1)^{-1/2} \right\|_{p,p}$ for $1 < p \leq 2$.

The volume growth assumption is subtle: It is satisfied under completeness and $\text{Ric} \geq 0$, but a negative lower bound is not enough. Indeed, if $M$ is complete with $\text{Ric} \geq (-C)(m-1)$ for some $C \geq 0$, then one has Gromov’s estimate

$$\text{vol}(B_{tr}(x)) \leq \text{vol}(B_r(x)) t^m e^{(m-1)\sqrt{C}(t-1)r} \text{ for all } t > 1, r > 0.$$ 

This inequality is sufficient, only if we can pick $C = 0$!
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Thank you!