Title of lecture course: Ausgewählte Kapitel der Mathematik (M40): Heat kernels and Brownian motion on manifolds

Prerequisites: a basic knowledge of functional analysis (spectral theorem; foundations will be repeated at the beginning of the course), a basic knowledge of manifolds (tangent spaces, vector fields)

Content: The heat kernel $p(t,x,y)$ on a Riemannian manifold $M$ is defined as the minimal fundamental solution of the heat equation

$$\partial_t p(t,x,y) = \Delta_x p(t,x,y), \quad \lim_{t \to 0^+} p(t,x,y) = \delta_y,$$

where $\Delta$ denotes the Laplace-Beltrami operator.

This object is closely related with the geometry and even the topology of $M$, and it is also the starting point of stochastic analysis on manifolds: Brownian motion on $M$ is defined to be a Markov process (possibly with explosive paths) whose transition density is given by the heat kernel.

A central result in this context states that a complete Riemannian manifold $M$ with dimension $m$ and a nonnegative Ricci curvature satisfies the so called Li-Yau heat kernel estimate

$$p(t,x,y) \leq \mu(x,\sqrt{t})^{-1} e^{-d(x,y)^2/(4t)}, \quad t > 0, x, y \in M,$$

where $\mu(x,\sqrt{t})$ denotes the volume of a ball centered in $x$ with radius $\sqrt{t}$. If furthermore $M$ has a maximal volume growth,

$$\mu(x,r) \geq Cr^m, \quad x \in M, \quad r > 0,$$

one arrives at a Gaussian upper bound of the heat kernel, which need not be satisfied on arbitrary Riemannian manifold at a large scale and thus reflects a Euclidean behaviour at $\infty$ of $M$. In addition, any complete Riemannian manifold with a nonnegative (or more generally: lower bounded) Ricci curvature is stochastically complete, meaning that Brownian motion cannot explode in a finite time.

On the other hand, complete Riemannian manifold with nonnegative Ricci curvature have the following properties: they are volume doubling and they satisfies a so called relative Faber-Krahn inequality. This raises the question, whether there is general connection between such estimates from geometric analysis and heat kernel estimates. One of the main goals of this lecture course will be to prove that in fact relative Faber-Krahn inequality is equivalent to the conjunction of volume doubling and the Li-Yau heat kernel estimate. In addition, we are going to examine some basic properties of Brownian motion on Riemannian manifolds.

A rough table of contents is:

- Spectral theorem
- Differential operators on manifolds
• Sobolev spaces on Riemannian manifolds
• Construction of the heat kernel
• Brownian motion
• Parabolic $L^2$-mean value inequality
• Characterizations of the Li-Yau heat kernel estimate