Covariant Riesz-Transforms and the Calderon-Zygmund inequality

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Interfaces between Geometric Analysis and Mathematical Physics

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- $M$: complete Riemannian $m$-manifold
- $d : C^\infty(M) \to \Gamma_{C^\infty}(T^*M)$: exterior derivative on functions
- $d_1 : \Gamma_{C^\infty}(T^*M) \to \Gamma_{C^\infty}({\land}^2 T^*M)$: exterior derivative on 1-forms
- $T^{r,s}M \to M$: $r$-times contravariant, $s$-times covariant tensors
- $\Delta = d^\dagger d$: Laplace-Beltrami-Operator in $L^2(M)$ (e.s.a.!!)
- $\Delta_1 = d_1^\dagger d_1 + dd^\dagger$: Laplace-Beltrami-Operator in $\Gamma_{L^2}(T^*M)$ (e.s.a.!!)
- $\nabla^{r,s} : \Gamma_{C^\infty}(T^{r,s}M) \to \Gamma_{C^\infty}(T^{r,s+1}M)$: Levi-Civita (LC) connection
- $\mu$: Riemannian volume measure
Let $1 < p < \infty$. The aim of the talk is to explain the connection between path integrals and the $L^p$-boundedness of the **covariant Riesz-transform** $CRT(p)$,

\[
\forall \lambda > 0 : \quad \left\| \nabla^{0,1} (\Delta_1 + \lambda)^{-1/2} \right\|_p < \infty.
\]  

(1)

The $L^p$-boundedness of the ’usual Riesz-transform’ $RT(p)$,

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\left\| d_1 (\Delta_1 + \lambda)^{-1/2} \right\|_p < \infty
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only needs $\text{Ric} \geq -C$ for some $C > 0$ and is a (by now) classical result by Bakry (1987).

Proving $CRT(p)$ should be considerably harder than proving $RT(p)$, essentially because the Laplace-Beltrami operator commutes with the exterior differential, but not with the LC connection. In fact, $CRT(p) \Rightarrow RT(p)$ easily.
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The $L^p$-boundedness of the 'usual Riesz-transform' $RT(p)$,

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Proving $CRT(p)$ should be considerably harder than proving $RT(p)$, essentially because the Laplace-Beltrami operator commutes with the exterior differential, but not with the LC connection. In fact, $CRT(p) \Rightarrow RT(p)$ easily.
CRT($p$) plays a fundamental role in geometric analysis:

\[
\|\text{Hess}(f)\|_p = \|\nabla^{0,1} d_1 f\|_p \\
= \left\| \nabla^{0,1} (\Delta_1 + \lambda)^{-1/2} d_1 (\Delta + \lambda)^{-1/2} (\Delta + \lambda) f \right\|_p \\
\leq \left\| \nabla^{0,1} (\Delta_1 + \lambda)^{-1/2} \right\|_p \left\| d_1 (\Delta + \lambda)^{-1/2} \right\|_p \left( \|\Delta f\|_p + \lambda \|f\|_p \right), \\
\Rightarrow \|\text{Hess}(f)\|_p \leq C (\|\Delta f\|_p + \|f\|_p),
\]

the \textbf{$L^p$-Calderon-Zygmund inequality} $CZ(p)$.

- $CZ(2)$ is easily seen to hold under $\text{Ric} \geq -C$, and is false in general (G./Pigola, 2015).
- For $p \neq 2$ the inequality $CZ(p)$ is nontrivial even in $\mathbb{R}^m$; the best result so far at a full $L^p$-scale is under $|\text{Ric}| \leq C$ and $\text{inj}(M) > 0$ (G./Pigola, 2015).
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Typical applications of $CZ(p)$:

- $CZ(p)$ (& a little bit of extra work) implies the global Sobolev inequality

$$\|\text{Hess}(f)\|_p + \|df\|_p \leq C \|\Delta f\|_p + C \|f\|_p.$$  

- $CZ(p)$ (& $|\text{Riem}| \leq C$ & some extra work) implies $|\text{Hess}(f)|, \|df\| \in L^p(M)$ for weak solutions $f \in L^p(M)$ of the Poisson equation $\Delta f = h$, where $h \in L^p(M)$.

- Once one has $CZ(p)$ with a constant depending only on geometric quantities ($\text{Ric}$, inj,...), one can use it to prove $L^p$-precompactness results for sequences of Riemannian immersions $\psi_n : M_n \to \mathbb{R}^l$, $n \in \mathbb{N}$: then $\Delta \psi_n$ is essentially the mean curvature of $\psi_n$ and $\text{Hess}(\psi_n)$ its second fundamental form!
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How can we prove $CRT(p)$? Approach: reduce $CRT(p)$ to estimates for the heat semigroup on 1-forms, so that we can use probability theory.
Indeed:

- In view of the Laplace-transform

\[
\nabla^{0,1}(\Delta_1 + \lambda)^{-1/2} = \int_0^\infty \nabla^{0,1} e^{-t\Delta_1} t^{-1/2} e^{-t\lambda} dt
\]

and a highly sophisticated machinery from harmonic analysis on metric measure spaces by Auscher/Coulhon/Doung/Hofmann (2004), the estimate \( CRT(p) \) follows from the semigroup estimate \( SG(p) \)

\[
\exists C > 0 \forall t > 0 : \left\| \nabla^{0,1} e^{-t\Delta_1} \right\|_{p,p} \leq Ce^{Ct} t^{-1/2}.
\]

- Whatever it is, this machinery should be sophisticated:

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\left\| \int_0^\infty \nabla^{0,1} e^{-t\Delta_1} t^{-1/2} e^{-t\lambda} dt \right\|_{p,p} \leq C \int_0^\infty e^{Ct} t^{-1} dt = \infty...
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- Whatever it is, this machinery should be sophisticated:

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Is there a probabilistic formula for $\nabla^{0,1} e^{-t\Delta_1}$ which is explicit enough to prove $SG(p)$?

Some hope: there are probabilistic path integral formulae for $e^{-t\Delta_1}$ (‘covariant Feynman-Kac formula’), and for $d_1 e^{-t\Delta_1}$ by Bismut (1984), Elworthy/Li (1998), and Thalmaier (1997) (‘BELT formula’).
(Ω, ℙ): a probability space. Notation: \( \mathbb{E}[\cdots] = \int \cdots d\mathbb{P} \).

\( X(x) : [0, \infty) \times \Omega \rightarrow M \): Brownian motion (BM) starting from \( x \in M \); paths \( X(x)(\omega) : [0, \infty) \rightarrow M \) continuous, for all \( k \in \mathbb{N}, 0 < t_1 < \cdots < t_k, A_1, \ldots, A_k \subset M \),

\[
\mathbb{P}\{X_{t_1}(x) \in A_1, \ldots, X_{t_k}(x) \in A_k\} = \\
\int_{A_k} \cdots \int_{A_1} e^{-t_1\Delta(x, x_1)} e^{-(t_2-t_1)\Delta(x_1, x_2)} \cdots e^{-(t_k-t_{k-1})\Delta(x_{k-1}, x_k)} d\mu(x_1) \cdots d\mu(x_k).
\]

//\( r, s(x) : [0, \infty) \times \Omega \rightarrow \text{Hom}(T^r_x M, T^s_{X(x)} M) \): parallel transport along \( X(x) \) with respect to \( \nabla^{r,s} \)

\( Q(x) : [0, \infty) \times \Omega \rightarrow \text{End}(T^*_x M) \) is defined pathwise by

\[
\frac{d}{dt} Q_t(x) = -\frac{1}{2} Q_t(x)//_t^0 (x)^{-1} \text{Ric}_{T_x M}^T//_t^0 (x), \quad Q_0(x) = \text{id}_{T^*_x M}.
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\]
These processes are precisely the ingredients of the **covariant Feynman-Kac formula** (Malliavin 1978, Driver/Thalmaier 2001, G. 2012)

\[ e^{-t\Delta_1\alpha(x)} = \mathbb{E} \left[ Q_t/\mathbb{F}_{t,1}^0 (x)^{-1} \alpha(X_t(x)) \right], \]

valid for all \( \alpha \in \Gamma_{\mathcal{C}_c^\infty}(T^*M) \), \( t \geq 0, x \in M \), if Ric is (sufficiently) bounded from below.

How can one prove the covariant Feynman-Kac formula?

Using some stochastic analysis (Itô’s formula) one finds that for fixed \( t > 0 \), the process

\[ Y := Q(x)/\mathbb{F}_{t,1}^0 (x)^{-1} e^{-(t-\cdot)\Delta_1\alpha(X(\cdot))} : [0, t] \times \Omega \to T_x^*M, \]

is a so called ’local martingale’.
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On the other hand, being a local martingale, \( Y_s \) has a constant expectation in \( s \in [0, t] \), if

\[
\mathbb{E}\left[ \sup_{s \in [0, t]} |Y_s| \right] < \infty,
\]

which is the case, as under say \( \text{Ric} \geq -C \), we have \( |Q_s(x)| \leq e^{Ct} \)
\( \mathbb{P} \)-a.s. (Gronwall), and

\[
\sup_{u \in [0, t], y \in M} |e^{-u\Delta_1} \alpha(y)| < \infty \quad \text{(Kato-Simon)}.
\]

Therefore:

\[
e^{-t\Delta_1} \alpha(x) = \mathbb{E}[Y_0] = \mathbb{E}[Y_t] = \mathbb{E}\left[ Q_t(x) \big/ \mathbb{P}_{t}^{0,1}(x)^{-1} \alpha(X_t(x)) \right],
\]

completing the proof of the covariant Feynman-Kac formula.
Let us now prepare our attack on the path integral for $\nabla^{0,1} e^{t\Delta} \ldots$
Given a continuous process $A : [0, \infty) \times \Omega \to \mathbb{R}^1$ and a Euclidean BM $B : [0, \infty) \times \Omega \to \mathbb{R}^1$ we can define another continuous process

$$\int_{0}^{\cdot} A_s dB_s : [0, \infty) \times \Omega \to \mathbb{R}^1,$$

the Itô integral, by approximating $\int_{0}^{t} A_s dB_s(\omega)$ with 'left-point (!) Lebesgue-Stieltjes Riemann sums' (but the convergence is \textit{not} for $\mathbb{P}$-a.e. $\omega \in \Omega$).

In general, $\int_{0}^{\cdot} A_s dB_s$ will only be local martingale; however, there is the Burkholder-Davis-Gundy inequality, which states that for all $q \in [1, \infty)$, there exists $C_q < \infty$ s.t. for all $t \geq 0$,

$$\mathbb{E}\left[\sup_{s \in [0,t]} |\int_{0}^{s} A_s dB_s|^q\right] \leq C_q \mathbb{E}\left[(\int_{0}^{t} |A_s|^2 ds)^{q/2}\right] \in [0, \infty].$$
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\[
\mathbb{E} \left[ \sup_{s \in [0, t]} \left| \int_0^t A_s dB_s \right|^q \right] \leq C_q \mathbb{E} \left[ \left( \int_0^t |A_s|^2 ds \right)^{q/2} \right] \in [0, \infty].
\]
Define the section

\[
\widetilde{\text{Ric}} \in \Gamma_{C^\infty}(\text{End}(\otimes^2 T^*M)) = \Gamma_{C^\infty}(\text{End}(\text{Hom}(TM, T^*M)))
\]
on \ A \in \text{Hom}(T_xM, T_x^*M), \ v \in T_xM, \ by

\[
\widetilde{\text{Ric}}(A)(v) = \text{Ric}^T(Av) - 2\sum_{j=1}^{m} \text{Riem}^T(e_i, v)(Ae_j) \in T_x^*M,
\]
and the section

\[
\rho \in \Gamma_{C^\infty}(\text{Hom}(T^*M, \otimes^2 T^*M)) = \Gamma_{C^\infty}(\text{Hom}(T^*M, \text{Hom}(TM, T^*M)))
\]
on \ \alpha \in T_x^*M, \ v \in T_xM \ by

\[
\rho(\alpha)(v) = (\nabla^1_v \text{Ric}^T) \alpha - \sum_{j=1}^{m} (\nabla^2_{e_i} \text{Riem}^T)(e_i, v) \alpha \in T_x^*M.
\]
Define the section

$$\widetilde{\text{Ric}} \in \Gamma C_\infty (\text{End}(\otimes^2 T^* M)) = \Gamma C_\infty (\text{End}(\text{Hom}(TM, T^* M)))$$
on $A \in \text{Hom}(T_x M, T^*_x M), \ \nu \in T_x M$, by

$$\widetilde{\text{Ric}}(A)(\nu) = \text{Ric}^T(A \nu) - 2 \sum_{j=1}^{m} \text{Riem}^T(e_i, \nu)(A e_j) \in T^*_x M,$$

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$$\rho \in \Gamma C_\infty (\text{Hom}(T^* M, \otimes^2 T^* M)) = \Gamma C_\infty (\text{Hom}(T^* M, \text{Hom}(TM, T^* M)))$$
on $\alpha \in T^*_x M, \ \nu \in T_x M$ by

$$\rho(\alpha)(\nu) = (\nabla^1_v \text{Ric}^T) \alpha - \sum_{j=1}^{m} (\nabla_{e_i}^2 \text{Riem}^T)(e_i, \nu) \alpha \in T^*_x M.$$
Define

\[ \tilde{Q}(x) : [0, \infty) \times \Omega \to \text{End}(\otimes^2 T_x^*M), \]

\[ (d/dt)\tilde{Q}_t(x) = -\frac{1}{2} \tilde{Q}_t(x)(/\!/^0,2)_t^{-1} \tilde{\text{Ric}}_t /\!/^0,2 |_x, \quad \tilde{Q}_0(x) = \text{id}\otimes^2 T_x^*M, \]

\[ B(x) : [0, \infty) \times \Omega \to T_xX \quad \text{anti-dev. of } X(x) \text{ w.r.t. } \nabla^{1,0} \text{ (BM!)}, \]

and for fixed \( t > 0, \xi \in \otimes^2 T_xM \) further

\[ \ell(\zeta, t) := \frac{(t - \bullet)}{t} \zeta : [0, t] \times \Omega \to \otimes^2 T_xM, \]

\[ \ell^{(1)}(\xi, t) := -\int_0^t Q_s^T,^{-1} dB_s \tilde{Q}_s^T \ell_s(\xi, t)|_x : [0, t] \times \Omega \to T_xM, \]

\[ \ell^{(2)}(\xi, t) := \frac{1}{2} \int_0^t Q_s^T,^{-1} ((/\!/^0,2)_s^{-1} \rho(X_s) /\!/^0,2)_s^T \tilde{Q}_s^T \ell_s(\zeta, t)ds|_x \]

\[ : [0, t] \times \Omega \to T_xM. \]
Theorem (Baumgarth/G. 2018)

Assume $|\text{Riem}|, |\nabla^{1,3}\text{Riem}| \leq A$ für some $A < \infty$. Then for all $\alpha \in \Gamma_{C_c^\infty}(T^*M)$, $t > 0$, $x \in M$, $\xi \in \bigotimes^2 T_x M$ one has

$$(\nabla e^{-t\Delta_1} \alpha(x), \xi)$$

$$= -\mathbb{E} \left[ \left( Q_t(x)/\!\!/^0,1_t(x)^{-1} \alpha(X_t(x)), \ell_t^{(1)}(\xi, t) + \ell_t^{(2)}(\xi, t) \right) \right].$$

Proof: Using the Itô formula one finds (a long calculation) that the process

$$Y := \left( \tilde{Q}(x)/\!\!/^0,2(x)^{-1} \nabla e^{-(t-\bullet)\Delta_1} \alpha(X(x)), \ell(\xi, t) \right)$$

$$- \left( Q(x)/\!\!/^0,1(x)^{-1} \nabla e^{-(t-\bullet)\Delta_1} \alpha(X(x)), \ell^{(1)}(\xi, t) + \ell^{(2)}(\xi, t) \right)$$

$$: [0, t] \times \Omega \rightarrow \mathbb{R}$$

is a local martingale (without any restriction on the geometry of $M$).
Theorem (Baumgarth/G. 2018)

Assume $|\text{Riem}|, |\nabla^{1,3}\text{Riem}| \leq A$ für some $A < \infty$. Then for all $\alpha \in \Gamma_{\mathcal{C}^\infty}(T^*M)$, $t > 0$, $x \in M$, $\xi \in \otimes^2 T_xM$ one has

$$(\nabla e^{-t\Delta_1} \alpha(x), \xi) = -\mathbb{E} \left[ \left( Q_t(x)/\!/^{0,1}_t(x)^{-1} \alpha(X_t(x)), \ell^{(1)}_t(\xi, t) + \ell^{(2)}_t(\xi, t) \right) \right].$$

Proof: Using the Itô formula one finds (a long calculation) that the process

$$Y := (\tilde{Q}(x)/\!/^{0,2}(x)^{-1} \nabla e^{-(t-\bullet)\Delta_1} \alpha(X(x)), \ell(\xi, t))$$

$$- (Q(x)/\!/^{0,1}(x)^{-1} e^{-(t-\bullet)\Delta_1} \alpha(X(x)), \ell^{(1)}(\xi, t) + \ell^{(2)}(\xi, t))$$

$$: [0, t] \times \Omega \longrightarrow \mathbb{R}$$

is a local martingale (without any restriction on the geometry of $M$).
The following estimates will entail that under our assumptions on the geometry of $M$, the process $Y$ is even a martingale:

**Lemma ($\ell(j)$-estimates)**

Assume $|\text{Riem}|, |\nabla^{1,3}\text{Riem}| \leq A$ for some $A < \infty$, and let $q \in [1, \infty)$, $t > 0$, $x \in M$, $\xi \in T_x M$.

a) One has:

$$
\mathbb{E} \left[ \sup_{s \in [0,t]} |\ell_s^{(1)}(\xi, t) q \right]^{1/q} \leq C_{q,m} t^{-1/2} e^{tC_{A,q,m} |\xi|},
$$

b) One has:

$$
\mathbb{E} \left[ \sup_{s \in [0,t]} |\ell_s^{(2)}(\xi, t) q \right]^{1/q} \leq C e^{C_{A,m} t |\xi|}.
$$
Proof: By Gronwall

\[ |Q_s(x)|, |Q_s(x)^{-1}|, |\tilde{Q}_s(x)|, \tilde{Q}_s(x)^{-1} \leq e^{Cm_A s} \mathbb{P}\text{-f.s. for all } s \in [0, t], \]

so that

\[ \mathbb{E} \left[ \sup_{s \in [0, t]} |\ell_s^{(2)}(\xi, t)|^q \right] \leq e^{qCm_At} |\xi|^q, \]

and using the Burkholder-Davis-Gundy inequality, we find

\[ \mathbb{E} \left[ \sup_{s \in [0, t]} |\ell_s^{(1)}(\xi, t)|^q \right] \leq \mathbb{C}_{q, m} \mathbb{E} \left[ \left( \int_0^t |Q_s^{T,-1}|^2 |\tilde{Q}_s^T|^2 |\ell_s(\xi, t)|^2 \, ds \right)^{q/2} \right] \]

\[ \leq \mathbb{C}_{q, m} t^{-q/2} e^{tC_{A, q, m}} |\xi|^q, \]

completing the proof of the Lemma.
Using these results for \( q = 1 \) and
\[
\sup_{u \in [0, t], y \in M} |e^{-u \Delta^1 \alpha(y)}| < \infty, \quad \sup_{u \in [0, t], y \in M} |\nabla e^{-u \Delta^1 \alpha(y)}| < \infty \quad \text{w.l.o.g.,}
\]
we have
\[
\mathbb{E} \left[ \sup_{s \in [0, t]} |Y_s| \right] < \infty,
\]
so \( Y \) is a martingale and
\[
(\nabla e^{-t \Delta^1 \alpha(x)}(x), \xi) = \mathbb{E}[Y_0] = \mathbb{E}[Y_t]
\]
\[
= -\mathbb{E} \left[ \left( Q_t(x)/\tau_{t, 1}(x)^{-1} \alpha(X_t(x)), \ell_t^{(1)}(\xi, t) + \ell_t^{(2)}(\xi, t) \right) \right],
\]
completing the proof of the path integral formula.
Theorem (Baumgarth/G. 2018)

Assume $|\text{Riem}|, |\nabla^{1,3}\text{Riem}| \leq A$ for some $A < \infty$. Then for all $p \in (1, \infty)$ there is constant $C = C_{A,p,m} > 0$, so that for all $t > 0$ one has

$$\left\| \nabla^{0,1} e^{-t\Delta_1} \right\|_{p,p} \leq C e^{C t} t^{-1/2}.$$ 

In particular, one has $\text{CRT}(p)$ and $\text{CZ}(p)$, with constants depending only on $A, p, m$. 
Theorem (Baumgarth/G. 2018)

Assume $|\text{Riem}|, |\nabla^{1,3}\text{Riem}| \leq A$ for some $A < \infty$. Then for all $p \in (1, \infty)$ there is constant $C = C_{A,p,m} > 0$, so that for all $t > 0$ one has

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In particular, one has $\text{CRT}(p)$ and $\text{CZ}(p)$, with constants depending only on $A, p, m$. 
Proof: The formula for $\nabla^0,1 e^{-t\Delta_1} \alpha(x)$ together with

$$|Q_t(x)|, |Q_t(x)^{-1}|, |\tilde{Q}_t(x)|, \tilde{Q}_t(x)^{-1} \leq e^{C'''} t \quad \mathbb{P}\text{-f.s.},$$

Hölder for $\mathbb{E}$, and the $\ell(j)$-estimates for $q = p^*$ shows

$$|\nabla^0,1 e^{-t\Delta_1} \alpha(x)| \leq C'' e^{C'''} t \mathbb{E} [\alpha(X_t(x))]^{1/p}$$

$$= C'' e^{C'''} t \left( e^{-t\Delta} |\alpha|^p(x) \right)^{1/p},$$

so

$$\int_M |\nabla^0,1 e^{-t\Delta_1} \alpha(x)|^p d\mu(x) \leq C' e^{C'} t \int_M e^{-t\Delta} |\alpha|^p(x) d\mu(x)$$

$$\leq C' e^{C'} t \int_M |\alpha|^p(x) d\mu(x),$$

as $e^{-t\Delta}$ is a contraction in $L^r(M)$ for all $r \in [1, \infty]$ (without any assumptions on the geometry of $M$). Done!
Thank you for listening!