ODD CHARACTERISTIC CLASSES IN ENTIRE CYCLIC HOMOLOGY AND EQUIVARIANT LOOP SPACE HOMOLOGY

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Abstract. Given a compact manifold M and $g \in C^{\infty}(M, U(l;\mathbb{C}))$ we construct a Chern character $\mathrm{Ch}^{-}(g)$ which lives in the odd part of the equivariant (entire) cyclic Chen-normalized bar complex $\underline{\mathscr{C}}(\Omega_{\mathbb{T}}(M\times\mathbb{T}))$ of M, and which is mapped to the odd Bismut-Chern character under the equivariant Chen integral map. It is also shown that the assignment $g \mapsto \mathrm{Ch}^{-}(g)$ induces a well-defined group homomorphism from the K^{-1} theory of M to the odd homology group of $\underline{\mathscr{C}}(\Omega_{\mathbb{T}}(M\times\mathbb{T}))$.

Let M be a closed Riemannian spin manifold with its Clifford multiplication

$$c : \Omega(M) \longrightarrow \mathrm{End}(S)$$

and its Dirac operator D acting in $L^2(M,S)$, and given $g \in C^{\infty}(M, U(l;\mathbb{C}))$ let D_g denote the twisted Dirac operator

$$D_g := g^{-1}Dg = D + c(g^{-1}dg),$$

considered to be acting on $L^2(M, S\otimes\mathbb{C}^l)$. Then with

$$D_{g,s} := (1-s)D + sD_g, \quad s\in[0,1],$$

the odd dimensional variant of Atiyah-Singer's index theorem states that if M is odd dimensional, then [5]

$$\frac{1}{2\pi}\int_0^1 \mathrm{Tr}\left[\dot{D}_{g,s}\exp\left(-D^2_{g,s}\right)\right] ds = \int_M \hat{A}(TM)\wedge \mathrm{ch}^{-}(g), \tag{1}$$

where $\mathrm{ch}^{-}(g) \in \Omega^{-}(M)$ denotes the odd Chern character. The left hand side of (1) is precisely the spectral flow $\mathrm{sf}(D, D_g)$ [5].

Being motivated by the considerations of Atiyah and Bismut [1, 2] for the even-dimensional case one finds that a very elegant, however purely formal, way to prove the latter formula is to assume the existence of a Duistermaat-Heckmann localization formula for the smooth loop space LM: indeed, with LM the smooth loop space, the spin structure on M induces an orientation on LM [1] and path integral formalism entails the elegant, however mathematically ill-defined, formula[1]

$$\frac{1}{2\pi}\int_0^1 \mathrm{Tr}\left[\dot{D}_{g,s}\exp\left(-D^2_{g,s}\right)\right] ds = \int_{LM} \exp(-\beta)\wedge \mathrm{Bch}^{-}(g), \tag{2}$$

[1] The even-dimensional variant of this formula is well-known [2] and the odd-dimensional case can be proved similarly [11].

where $\beta = \beta_0 + \beta_2 \in \Omega^+(LM)$ denotes the even differential form defined on smooth vector fields X, Y on LM by

$$\beta_0(X) := \int_0^1 |X_s|^2 ds, \quad \beta_2(X,Y) := \int_0^1 (\nabla X_s/\nabla s, Y_s)\, ds,$$

and where $\mathrm{Bch}^-(g) \in \Omega^-(M)$ denotes the odd Bismut-Chern character [3, 14]. Now both differential forms $\exp(-\beta)$ and $\mathrm{Bch}^-(g)$ are equivariantly closed (cf. Section 4 for the definition of the degree -1 differential P),

$$(d+P)\exp(-\beta) = 0 = (d+P)\mathrm{Bch}^-(g)$$

and so is their product. As the fixed point set of the $\mathbb{T}$-action on LM given by rotating every loop is precisely $M \subset LM$, a hypothetical Duistermaat-Heckmann localization formula immediately gives

$$\int_{LM} \exp(-\beta) \wedge \mathrm{Bch}^-(g) = \int_M \hat{A}(TM) \wedge \exp(-\beta)|_M \wedge \mathrm{Bch}^-(g)|_M,$$

as $\hat{A}(TM)$ is the inverse of the (appropriately renormalized) Euler class of the normal bundle of $M \subset LM$. This proves (1), as clearly $\exp(-\beta)|_M = 1$ and by construction $\mathrm{Bch}^-(g)|_M = \mathrm{ch}^-(g)$.
A direct implementation of the above arguments is not possible, as the right hand side of formula (2) is not well-defined for various reasons. For example, there exists no volume measure on LM, while smooth loops have Wiener measure zero, and, on the other hand, it is notoriously difficult to produce a variant of the complex $(\Omega(LM), d+P)$ if one replaces LM with the smooth Banach manifold of *continuous loops*. Nevertheless and strikingly, the above formal manipulations lead to the highly powerful machinery of hypoelliptic Dirac operators, as is explained in [3] and the references therein.
However, a possibe way out of these problems has been proposed by Getzler, Jones and Petrack (GJP) [8] [6]. In this approach, the idea is to take a model for $\Omega(LM)$ in terms of equivariant Chen integrals: this is a morphism of super complexes (cf. Section 4 below for the relevant definitions)

$$\underline{\rho} : \big(\underline{\mathscr{C}}(\Omega_{\mathbb{T}}(M \times \mathbb{T})), \underline{b} + \underline{B}\big) \longrightarrow \big(\Omega(LM), d+P\big),$$

where $\underline{\mathscr{C}}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ denotes the Chen-normalized cyclic bar complex of the DGA $\Omega_{\mathbb{T}}(M \times \mathbb{T})$. Now the GJP-program for infinite dimensional localization is as follows: here it is conjectured that the composition

$$\int_{LM} e^{-\beta} \wedge \rho(\cdot) : \underline{\mathscr{C}}(\Omega_{\mathbb{T}}(M \times \mathbb{T})) \longrightarrow \mathbb{C},$$

is mathematically well-defined (cf. [10] for first steps in this context), and that

- $\int_{LM} e^{-\beta} \wedge \rho(\cdot)$ vanishes on exact elements of $\underline{\mathscr{C}}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$,
- If $w \in \underline{\mathscr{C}}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ is closed, then one has the 'Duistermaat-Heckmann formula'

$$\int_{LM} e^{-\beta} \wedge \rho(w) = \int_M \hat{A}(TM) \wedge \pi(w),$$

where

$$\pi : \underline{\mathscr{C}}(\Omega_{\mathbb{T}}(M \times \mathbb{T})) \longrightarrow \Omega(M)$$

denotes the projection given by composition of ρ with the restriction map $\Omega(LM) \to \Omega(M)$.

If in addition one could canonically construct an element $\mathrm{Ch}^{-}(g) \in \underline{\mathscr{C}}^{-}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ such that

i) $\mathrm{Ch}^{-}(g)$ is closed
ii) $\underline{\rho}(\mathrm{Ch}^{-}(g)) = \mathrm{Bch}^{-}(g)$
iii) $\pi(\mathrm{Ch}^{-}(g)) = \mathrm{ch}^{-}(g)$,

then from the above observations we would immediately obtain a proof of (1) within the GJP-program for infinite dimensional localization. Note that in the even dimensional case such a Chern character has been constructed in [8].

The aim of this paper is precisely to construct a canonically given element $\mathrm{Ch}^{-}(g) \in \underline{\mathscr{C}}^{-}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ satisfying the above properties i), ii), iii). In fact, our main results Theorem 5.1 and Theorem 5.3 below construct $\mathrm{Ch}^{-}(g)$ for M a compact Riemannian manifold (possibly with boundary), which satisfies i) and iii) and in addition ii) if M is closed (so that LM is a well-defined smooth Fréchet manifold). We also show in Theorem 5.1 that the assignment $g \mapsto \mathrm{Ch}^{-}(g)$ induces a well-defined group homomorphism from the K^{-1} theory of M to the odd cyclic homology group of $\underline{\mathscr{C}}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$. In fact, we show that $\mathrm{Ch}^{-}(g)$ lives in a topological subcomplex of $\underline{\mathscr{C}}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ which is defined by requiring growth conditions in the spirit of Connes' entire growth conditions [9][4]. This result suggests that $\int_{LM} e^{-\beta} \wedge \rho(\cdot)$ should actually be a continuous functional, as integration should be.

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1. Cyclic bar complex of a differential graded algebra (DGA)

In the sequel, we understand all our linear spaces to be over $\mathbb{C}$. Assume we are given a unital DGA Ω, that is,

- Ω is a unital algebra
- $\Omega = \bigoplus_{j=0}^{\infty} \Omega^j$ is graded into subspaces $\Omega^j \subset \Omega$ such that $\Omega^i \Omega^j \subset \Omega^{i+j}$ for all $i, j \in \mathbb{N}$, there is a degree $+1$ differential $d : \Omega \to \Omega$ which satisfies the graded Leibniz rule.

Of course, $\tilde{\Omega} := \Omega/\mathbb{C}$ inherits this structure canonically, and the space of chains $\mathscr{C}(\Omega)$ is defined by all sequences

$$w = (w_0, w_1, \dots) \quad \text{with} \quad w_n \in \Omega \otimes \tilde{\Omega}^{\otimes n} \quad \text{for all } n \in \mathbb{N},$$

where it is understood that $w_0 \in \Omega$. We give $\Omega \otimes \tilde{\Omega}^{\otimes n}$ the grading

$$\Omega \otimes \tilde{\Omega}^{\otimes n} = \bigoplus_{j=0}^{\infty} \bigoplus_{j_0+\cdots+j_n=j-n} \Omega^{j_0} \otimes \tilde{\Omega}^{j_1} \otimes \cdots \otimes \tilde{\Omega}^{j_n},$$

which induces a linear map

$$\Gamma : \mathscr{C}(\Omega) \longrightarrow \mathscr{C}(\Omega), \quad \Gamma w := \big((-1)^{\deg(w_0)} w_0, (-1)^{\deg(w_1)} w_1, \dots\big).$$

Since we have $\Gamma^2 = 1$, we can define a superstructure $\mathscr{C}(\Omega) = \mathscr{C}^+(\Omega) \oplus \mathscr{C}^-(\Omega)$ by setting

$$\mathscr{C}^{\pm}(\Omega) := \{w \in \mathscr{C}(\Omega) : \Gamma w = \pm w\}.$$

The following notation will be useful in the sequel:

Notation 1.1. Given $a \in \Omega \otimes \tilde{\Omega}^{\otimes n}$ we define

$$\langle a \rangle := (\dots, a, \dots) \in \mathscr{C}(\Omega)$$

to be the cochain which has a in its n-th slot and 0 anywhere else.

We have the Hochschild map of the DGA-category

$$b : \mathscr{C}(\Omega) \longrightarrow \mathscr{C}(\Omega)$$

defined on $\Omega^{j_0} \otimes \tilde{\Omega}^{j_1} \otimes \cdots \otimes \tilde{\Omega}^{j_n}$ by

$$\begin{aligned} b \langle \omega_0 \otimes \cdots \otimes \omega_n \rangle = &- \sum_{i=0}^{n} (-1)^{j_0+\ldots+j_{i-1}-i+1} \langle \omega_0 \otimes \cdots \otimes d\omega_i \otimes \cdots \otimes \omega_n \rangle \\ &- \sum_{i=0}^{n-1} (-1)^{j_0+\ldots+j_i-i} \langle \omega_0 \otimes \cdots \otimes \omega_i \omega_{i+1} \otimes \cdots \otimes \omega_n \rangle \\ &+ (-1)^{(j_n-1)(j_0+\ldots+j_{n-1}-n+1)} \langle (\omega_n \omega_0) \otimes \omega_1 \otimes \cdots \otimes \omega_{n-1} \rangle, \end{aligned}$$

and Connes' operator

$$B : \mathscr{C}(\Omega) \longrightarrow \mathscr{C}(\Omega),$$

which is defined on $\Omega^{j_0} \otimes \tilde{\Omega}^{j_1} \otimes \cdots \otimes \tilde{\Omega}^{j_n}$ by

$$B \langle \omega_0 \otimes \cdots \otimes \omega_n \rangle = \sum_{i=0}^{n} (-1)^{(\epsilon_{i-1}+1)(\epsilon_n - \epsilon_{i-1})} \langle 1 \otimes \omega_i \otimes \cdots \otimes \omega_n \otimes \omega_0 \otimes \cdots \otimes \omega_{i-1} \rangle,$$

with $\epsilon_r = j_0 + \cdots + j_r - r$. It is a well-known fact that one has

$$b^2 = 0, \quad B^2 = 0, \quad bB + bB = 0, \quad \Gamma b = -\Gamma b, \quad \Gamma B = -\Gamma B.$$

We get the short complex

$$0 \longrightarrow \mathscr{C}^+(\Omega) \xrightarrow{b+B} \mathscr{C}^-(\Omega) \xrightarrow{b+B} \mathscr{C}^+(\Omega) \longrightarrow 0, \tag{3}$$

called the *cyclic bar complex* of Ω, and the corresponding homology groups are the linear spaces given by the quotients

$$\mathsf{HC}^{\pm}(\Omega) := \frac{\{w \in \mathscr{C}^{\pm}(\Omega) : (b+B)w = 0\}}{\{v \in \mathscr{C}^{\pm}(\Omega) : v = (b+B)w \text{ for some } w \in \mathscr{C}^{\mp}(\Omega)\ \}}.$$

The subspace $\mathscr{D}(\Omega) \subset \mathscr{C}(\Omega)$ is defined to be the linear span of all $(w_0, w_1, \dots,) \in \mathscr{C}(\Omega)$ that satisfy one of the following relations:

- for all $n \in \mathbb{N}$ there exists $1 \leq r \leq n$, $f \in \Omega^0$, $\omega_0 \in \Omega$, $\omega_r \in \tilde{\Omega}$ with

$$\langle w_n \rangle = \langle \omega_0 \otimes \cdots \otimes \omega_{r-1} \otimes f \otimes \omega_{r+1} \otimes \cdots \otimes \omega_n \rangle . \tag{4}$$

- for all $n \in \mathbb{N}$ there exists $1 \leq r \leq n$, $f \in \Omega^0$, $\omega_0 \in \Omega$, $\omega_r \in \tilde{\Omega}$ with

$$\begin{aligned}\langle \omega_0 \otimes \cdots \otimes \omega_{r-1} f \otimes \omega_{r+1} \otimes \cdots \otimes \omega_n \rangle + \langle \omega_0 \otimes \cdots \otimes \omega_{r-1} \otimes df \otimes \omega_{r+1} \otimes \cdots \otimes \omega_n \rangle \\ - \langle \omega_0 \otimes \cdots \otimes \omega_{r-1} \otimes f \omega_{r+1} \otimes \cdots \otimes \omega_n \rangle .\end{aligned} \tag{5}$$

The maps Γ, b, B map $\mathscr{D}(\Omega)$ to itself, so that with

$$\mathscr{D}^{\pm}(\Omega) := \{ w \in \mathscr{D}(\Omega) : \Gamma w = \pm w \},$$

there is a short complex

$$0 \longrightarrow \mathscr{D}^{+}(\Omega) \xrightarrow{b+B} \mathscr{D}^{-}(\Omega) \xrightarrow{b+B} \mathscr{D}^{+}(\Omega) \longrightarrow 0.$$

With $\underline{\mathscr{C}}^{\pm}(\Omega) := \mathscr{C}^{\pm}(\Omega)/\mathscr{D}^{\pm}(\Omega)$, the induced quotient complex

$$0 \longrightarrow \underline{\mathscr{C}}^{+}(\Omega) \xrightarrow{b+B} \underline{\mathscr{C}}^{-}(\Omega) \xrightarrow{b+B} \underline{\mathscr{C}}^{+}(\Omega) \longrightarrow 0$$

is called the *Chen-normalized cyclic bar complex* of Ω, and the induced homology groups are denoted with $\underline{\mathsf{HC}}^{\pm}(\Omega)$. Whenever there is no danger of confusion, the equivalence class of $w \in \mathscr{C}(\Omega)$ in $\underline{\mathscr{C}}^{\pm}(\Omega)$ is denoted with the same symbol again.

2. Entire cyclic homology of a metrizable unital DGA

The following definition is motivated by the fact a locally convex space is metrizable, if and only if its topology is induced by a countable sequence of seminorms:

Definition 2.1. By a metrizable unital DGA we understand a unital DGA Ω, with a locally convex topology which is induced by a countable increasing family $\|\cdot\|_k$, $k \in \mathbb{N}$, of seminorms such that

- the differential is continuous, e.g., for every k there exists $k' \geq k$ and $C > 0$ with

$$\|d\omega\|_k \leq C \|\omega\|_{k'} \quad \text{for all } \omega \in \Omega \tag{6}$$

- the multiplication is jointly continuous, e.g., for every k there exists $k' \geq k$ and $C > 0$ with

$$\|\omega_1 \omega_2\|_k \leq C \|\omega_1\|_{k'} \|\omega_2\|_{k'} \quad \text{for all } \omega_1, \omega_2 \in \Omega \tag{7}$$

- the seminorms respect the grading, e.g.,[2]

$$\|\omega_0 + \omega_1 + \cdots\|_k = \|\omega_0\|_k + \|\omega_1\|_k + \cdots \quad \text{for all } k \in \mathbb{N},\ \omega_i \in \Omega^i \text{ for all } i \in \mathbb{N}. \tag{8}$$

[2]Note that by definition the sum in (8) is finite.

Again, $\tilde{\Omega}$ inherits the above structure canonically, and we equip the algebraic tensor product $\Omega \otimes \tilde{\Omega}^{\otimes n}$ with the induced family $\|\cdot\|_{k;n}$, $k \in \mathbb{N}$ of the π-tensor seminorms, that is, each $\|\cdot\|_{k;n}$ is defined as the smallest seminorm on $\Omega \otimes \tilde{\Omega}^{\otimes n}$ such that

$$\|\omega_0 \otimes \cdots \otimes \omega_n\|_{k;n} = \|\omega_0\|_k \cdots \|\omega_n\|_k \qquad \text{for all } \omega_0 \in \Omega, \omega_1, \dots, \omega_k \in \tilde{\Omega}.$$

Definition 2.2. The space of *entire chains* $\mathscr{C}_\epsilon(\Omega)$ is given by all $w \in \mathscr{C}(\Omega)$ with

$$\|w\|_{k,l} := \sum_{n=0}^{\infty} \frac{\|w_n\|_{k;n}}{\sqrt{n!}} l^n < \infty \qquad \text{for all } k, l \in \mathbb{N}.$$

It is easily checked that $\mathscr{C}_\epsilon(\Omega)$ becomes a locally convex space when equipped with the seminorms $\|\cdot\|_{k,l}$, $k, l \in \mathbb{N}$. Our growth conditions are modelled on the entire growth conditions for ungraded Banach algebras by Getzler/Szenes from [9]. Note that $\mathscr{C}_\epsilon(\Omega)$ is not complete (cf. Remark 4.1 below for a explanation of why in this paper we do not work with the completion of $\mathscr{C}_\epsilon(\Omega)$).

Proposition 2.3. *The operators Γ, b, B map $\mathscr{C}_\epsilon(\Omega)$ continuously to itself, in particular, with*

$$\mathscr{C}_\epsilon^{\pm}(\Omega) := \{w \in \mathscr{C}_\epsilon(\Omega) : \Gamma w = \pm w\},$$

there is a well-defined short complex

$$0 \longrightarrow \mathscr{C}_\epsilon^{+}(\Omega) \xrightarrow{b+B} \mathscr{C}_\epsilon^{-}(\Omega) \xrightarrow{b+B} \mathscr{C}_\epsilon^{+}(\Omega) \longrightarrow 0. \tag{9}$$

Proof. Fix $k, l \in \mathbb{N}$. Clearly, one has $\|\Gamma w\|_{k,l} \leq \|w\|_{k,l}$ for all $w \in \mathscr{C}_\epsilon(\Omega)$.
By the definition of a metrizable unital DGA, we may pick a constant $C'' > 0$ and a number $k'' \geq k$, such that for all $\omega \in \Omega$ one has $\|d\omega\|_k \leq C'' \|\omega\|_{k''}$. Likewise, we may pick $C' > 0$ and a number $k' \geq k$, such that for all $\omega_1, \omega_2 \in \Omega$ one has $\|\omega_1\omega_2\|_k \leq C' \|\omega_1\|_{k'} \|\omega_2\|_{k'}$. Using this and $n + 1 \leq 2^n$ it is easily checked that

$$\|bw\|_{k,l} \leq \max(C, C', 1) \|w\|_{\max(k'k''),2l} \qquad \text{for all } w \in \mathscr{C}_\epsilon(\Omega).$$

Likewise, it follows immediately that $\|Bw\|_{k,l} \leq \|1\|_k \|w\|_{k,2l}$ for all $w \in \mathscr{C}_\epsilon(\Omega)$. ■

Defining the subspace $\mathscr{D}_\epsilon(\Omega) \subset \mathscr{C}_\epsilon(\Omega)$ by $\mathscr{D}_\epsilon(\Omega) := \mathscr{D}(\Omega) \cap \mathscr{C}_\epsilon(\Omega)$, it follows automatically that the maps Γ, b, B map $\mathscr{D}_\epsilon(\Omega)$ to itself continuously, too, producing with

$$\underline{\mathscr{C}}_\epsilon^{\pm}(\Omega) := \mathscr{C}_\epsilon^{\pm}(\Omega)/\mathscr{D}_\epsilon^{\pm}(\Omega)$$

the quotient complex

$$0 \longrightarrow \underline{\mathscr{C}}_\epsilon^{+}(\Omega) \xrightarrow{b+B} \underline{\mathscr{C}}_\epsilon^{-}(\Omega) \xrightarrow{b+B} \underline{\mathscr{C}}_\epsilon^{+}(\Omega) \longrightarrow 0. \tag{10}$$

Finally we can give:

Definition 2.4. The complex (9) is called the *entire cyclic bar complex* of Ω and its homology groups are denoted with $\mathsf{HC}_\epsilon^{\pm}(\Omega)$. Likewise, the complex (10) is called the *Chen-normalized entire cyclic bar complex* of Ω and its homology groups are denoted with $\underline{\mathsf{HC}}_\epsilon^{\pm}(\Omega)$.

3. Equivariant cyclic bar complex of a manifold

Assume N is a compact manifold (possibly with boundary) and denote with $\mathbb{T}$ the 1-sphere. We denote by $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ the smooth $\mathbb{T}$-invariant differential forms on $N \times \mathbb{T}$, where $\mathbb{T}$ acts trivially on N and by rotation on itself. Every element of $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ can be uniquely written in the form $\alpha + \beta \wedge \alpha_{\mathbb{T}}$ for some $\alpha, \beta \in \Omega(N)$, where $\alpha_{\mathbb{T}}$ denotes the canonical 1-form on $\mathbb{T}$. We turn $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ into a unital algebra by means of $\Omega_{\mathbb{T}}(N \times \mathbb{T}) \subset \Omega(N \times \mathbb{T})$, and give $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ the grading

$$\alpha + \beta \wedge \alpha_{\mathbb{T}} \in \Omega^j_{\mathbb{T}}(N \times \mathbb{T}) \quad \Longleftrightarrow \quad \alpha \in \Omega^j(N), \beta \in \Omega^{j+1}(N).$$

With $\partial_{\mathbb{T}}$ the canonical vector field on $\mathbb{T}$, we have the differential

$$(d + \iota_{\partial_{\mathbb{T}}})(\alpha + \beta \wedge \alpha_{\mathbb{T}}) = d\alpha + (-1)^{|\beta|}\beta + d\beta \wedge \alpha_{\mathbb{T}}, \quad \text{if } \alpha + \beta \wedge \alpha_{\mathbb{T}} \text{ is homogeneous},$$

finally turning $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ into a unital DGA. Pick now a Riemannian structure on N and consider the Levi-Civita connection ∇ acting in $\Omega(N)$. With $|\cdot|$ the fiber metric on $\wedge T^*N$, we define a family of seminorms on $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ by setting

(11)
$$\|\alpha + \beta \wedge \alpha_{\mathbb{T}}\|_k := \max_{j=0,\dots,k} \left\|\nabla^j \alpha\right\|_\infty + \max_{j=0,\dots,k} \left\|\nabla^j \beta\right\|_\infty = \max_{j=0,\dots,k} \sup_N |\nabla^j \alpha| + \max_{j=0,\dots,k} \sup_N |\nabla^j \beta|.$$

We have:

Lemma 3.1. *If N is a compact Riemannian manifold (possibly with boundary), then $\Omega_{\mathbb{T}}(N \times \mathbb{T})$ becomes a metrizable unital DGA with respect to (11).*

Proof. Let us first show that multiplication is jointly continuous: Given

$$\alpha_1 + \beta_1 \wedge \alpha_{\mathbb{T}}, \alpha_2 + \beta_2 \wedge \alpha_{\mathbb{T}} \in \Omega_{\mathbb{T}}(N \times \mathbb{T})$$

one has

$$\begin{aligned}
&\|(\alpha_1 + \beta_1 \wedge \alpha_{\mathbb{T}}) \wedge (\alpha_2 + \beta_2 \wedge \alpha_{\mathbb{T}})\|_k \\
&\leq \max_{j=0,\dots,k} \left(\left\|\nabla^j(\alpha_1 \wedge \alpha_2)\right\|_\infty + \left\|\nabla^j(\alpha_1 \wedge \beta_2)\right\|_\infty + \left\|\nabla^j(\beta_1 \wedge \alpha_2)\right\|_\infty + \left\|\nabla^j(\beta_1 \wedge \beta_2)\right\|_\infty\right) \\
&= \max_{j=0,\dots,k} \left\| \sum_{i \leq j} \binom{j}{i} \nabla^i \alpha_1 \wedge \nabla^{j-i} \alpha_2) \right\|_\infty + \max_{j=0,\dots,k} \left\| \sum_{i \leq j} \binom{j}{i} \nabla^i \alpha_1 \wedge \nabla^{j-i} \beta_2) \right\|_\infty \\
&\quad + \max_{j=0,\dots,k} \left\| \sum_{i \leq j} \binom{j}{i} \nabla^i \beta_1 \wedge \nabla^{j-i} \alpha_2) \right\|_\infty + \max_{j=0,\dots,k} \left\| \sum_{i \leq j} \binom{j}{i} \nabla^i \beta_1 \wedge \nabla^{j-i} \beta_2) \right\|_\infty \\
&\leq C_k \max_{j=0,\dots,k} \left\|\nabla^j \alpha_1\right\|_\infty \max_{j=0,\dots,k} \left\|\nabla^j \alpha_2\right\|_\infty + C_k \max_{j=0,\dots,k} \left\|\nabla^j \alpha_1\right\|_\infty \max_{j=0,\dots,k} \left\|\nabla^j \beta_2\right\|_\infty \\
&\quad + C_k \max_{j=0,\dots,k} \left\|\nabla^j \beta_1\right\|_\infty \max_{j=0,\dots,k} \left\|\nabla^j \alpha_2\right\|_\infty + C_k \max_{j=0,\dots,k} \left\|\nabla^j \beta_1\right\|_\infty \max_{j=0,\dots,k} \left\|\nabla^j \beta_2\right\|_\infty \\
&= C_k \max_{j=0,\dots,k} (\left\|\nabla^j \alpha_1\right\|_\infty + \left\|\nabla^j \beta_1\right\|_\infty) \max_{j=0,\dots,k} (\left\|\nabla^j \alpha_2\right\|_\infty + \left\|\nabla^j \beta_2\right\|_\infty) \\
&= C_k \|\alpha_1 + \beta_1 \wedge \alpha_{\mathbb{T}}\|_k \|\alpha_2 + \beta_2 \wedge \alpha_{\mathbb{T}}\|_k .
\end{aligned}$$

To show that the differential is continuous, note first that, as ∇ is torsion free, one has

$$d\omega(V_1,\dots,V_l)=(l+1)\frac{1}{l!}\sum_{\sigma\in\Sigma_l}\mathrm{sign}(\sigma)\nabla\omega(V_{\sigma(1)},\dots,V_{\sigma(l)}),$$

for all $\omega\in\Omega^l(N)$, and all vector fields $V_1,\dots,V_l$ on N. Thus given an homogeneous element $\alpha+\beta\wedge\alpha_{\mathbb{T}}$ we can estimate as follows,

$$\begin{aligned}
&\|(d+\iota_{\partial_{\mathbb{T}}})(\alpha+\beta\wedge\alpha_{\mathbb{T}})\|_k=\max_{j=0,\dots,k}\left\|\nabla^j(d\alpha+(-1)^{|\beta|}\beta)\right\|_\infty+\max_{j=0,\dots,k}\left\|\nabla^j d\beta\right\|_\infty\\
&\leq\max_{j=0,\dots,k}\left\|\nabla^j d\alpha\right\|_\infty+\max_{j=0,\dots,k}\left\|\nabla^j\beta\right\|_\infty+\max_{j=0,\dots,k}\left\|\nabla^j d\beta\right\|_\infty\\
&\leq C_{\dim(N),k}\max_{j=0,\dots,k+1}\left\|\nabla^j\alpha\right\|_\infty+C_{\dim(N),k}\max_{j=0,\dots,k+1}\left\|\nabla^j\beta\right\|_\infty\\
&=C_{\dim(N),k}\left\|\alpha+\beta\wedge\alpha_{\mathbb{T}}\right\|_{k+1},
\end{aligned}$$

completing the proof. ■

As a consequence, we get the short complexes

$$0\longrightarrow\mathscr{C}^+(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\xrightarrow{b+B}\mathscr{C}^-(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\xrightarrow{b+B}\mathscr{C}^+(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\longrightarrow 0,\tag{12}$$

$$0\longrightarrow\underline{\mathscr{C}}^+(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\xrightarrow{b+B}\underline{\mathscr{C}}^-(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\xrightarrow{b+B}\underline{\mathscr{C}}^+(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\longrightarrow 0,\tag{13}$$

$$0\longrightarrow\mathscr{C}^+_\epsilon(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\xrightarrow{b+B}\mathscr{C}^-_\epsilon(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\xrightarrow{b+B}\mathscr{C}^+(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\longrightarrow 0,\tag{14}$$

$$0\longrightarrow\underline{\mathscr{C}}^+_\epsilon(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\xrightarrow{b+B}\underline{\mathscr{C}}^-_\epsilon(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\xrightarrow{b+B}\underline{\mathscr{C}}^+_\epsilon(\Omega_{\mathbb{T}}(N\times\mathbb{T}))\longrightarrow 0.\tag{15}$$

4. Equivariant Chen integrals

Let us consider a compact manifold N without boundary, and the space LN of smooth loops $\gamma:\mathbb{T}\to N$, where in the sequel we read $\mathbb{T}$ as $\mathbb{T}=[0,1]/\sim$. This becomes an infinite dimensional Fréchet manifold which is locally modelled on the Fréchet space $L\mathbb{R}^{\dim N}$ of smooth loops $\mathbb{T}\to\mathbb{R}^{\dim N}$. Then LN carries a natural smooth $\mathbb{T}$-action, given by rotating each loop, and the fixed point set of this action is precisely $N\subset LN$, embedded as constant loops. Given $\gamma\in LN$ the tangent space $T_\gamma LN$ is given by linear space of smooth vector fields on N along γ, that is,

$$T_\gamma(LN)=\left\{X\in C^\infty(\mathbb{T},N):X(t)\in T_{\gamma(t)}N\text{ for all }t\in\mathbb{T}\right\},$$

and the generator of the $\mathbb{T}$-action on LN is the vector field $\gamma\mapsto\dot\gamma$ on LN. Let ι denote the contraction with respect to the latter vector field. In the sequel, we understand $\Omega(LN)$ to be the space of sequences $(\alpha_0,\alpha_1,\dots)$ such that $\alpha_k\in\Omega^k(LN)$ for all $k\in\mathbb{N}$. For fixed $s\in\mathbb{T}$ one has the diffeomorphism

$$\phi_s:LN\longrightarrow LN,\quad\gamma\longmapsto\gamma(s+\cdot)$$

induced by the $\mathbb{T}$-action, and one gets an induced operator

$$P:\Omega(LN)\longrightarrow\Omega(LN),\quad\text{defined on }\Omega^k(LN)\text{ by }P\alpha:=\int_0^1\phi_s^*\iota\alpha\,ds.$$

Then P becomes a degree -1 derivation. In addition, there is the usual exterior derivative

$$d : \Omega(LN) \longrightarrow \Omega(LN),$$

a degree $+1$ derivation. Taking only odd/even degree forms, one gets the superstructure $\Omega = \Omega^+(LN) \oplus \Omega^-(LN)$, and we get the short complex

$$0 \longrightarrow \Omega^+(LN) \xrightarrow{d+P} \Omega^-(LN) \xrightarrow{d+P} \Omega^+(LN) \longrightarrow 0, \tag{16}$$

called the *equivariant de Rham complex of* LN. The induced homology groups are denoted with $\mathsf{H}^{\pm}_{\mathbb{T}}(LN)$.

Given $t \in \mathbb{T}$ and $\alpha \in \Omega^k(N)$ one denotes with $\alpha(t) \in \Omega^k(LN)$ the form obtained by pulling α back with respect to the evaluation map $\gamma \mapsto \gamma(t)$. With this notation at hand, one has the *equivariant Chen integral* map

$$\rho : \mathscr{C}(\Omega_{\mathbb{T}}(N \times \mathbb{T})) \longrightarrow \Omega(LN)$$

which is defined by

$$\begin{aligned} &\rho(\langle(\alpha_0 + \beta_0 \wedge \alpha_{\mathbb{T}}) \otimes \cdots \otimes (\alpha_n + \beta_n \wedge \alpha_{\mathbb{T}})\rangle) \\ &:= \int_{\{0 \leq t_1 \leq \cdots \leq t_n \leq 1\}} \alpha_0(0) \wedge (\iota\alpha_1(t_1) - \beta_1(t_1)) \wedge \cdots \wedge (\iota\alpha_n(t_n) - \beta_n(t_n))\, dt_1 \cdots dt_n. \end{aligned}$$

The map ρ is a morphism of short complexes from (12) to (16), which in turn descends to a map

$$\underline{\rho} : \underline{\mathscr{C}}(\Omega_{\mathbb{T}}(N \times \mathbb{T})) \longrightarrow \Omega(LN)$$

of short complexes from (13) to (16) (cf. [8] for these results).

Remark 4.1. It is essential to work with our definition of $\mathscr{C}_\epsilon(\Omega_{\mathbb{T}}(N \times \mathbb{T}))$ in order to be able to restrict ρ to $\mathscr{C}_\epsilon(\Omega_{\mathbb{T}}(N \times \mathbb{T}))$ and to get the induced map which is defined on $\underline{\mathscr{C}}_\epsilon(\Omega_{\mathbb{T}}(N \times \mathbb{T}))$. There seems to be no useful way to extend ρ to the completion of $\mathscr{C}_\epsilon(\Omega_{\mathbb{T}}(N \times \mathbb{T}))$, as this will lead to certain infinite series of tensor products whose image under ρ will lead to infinite series of elements of $\Omega(LN)$ having a fixed degree (noting that there seems to be no canonic way to turn $\Omega(LN)$ into a nice Fréchet space).

5. Construction of cycles in $\underline{\mathscr{C}}^-_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ and the induced cycles in $\Omega^-(LM)$

Let now M be a compact Riemannian manifold (possibly with boundary). Given $g \in C^\infty(M, U(l;\mathbb{C}))$ our aim is to construct a canonically given element

$$\mathrm{Ch}^-(g) \in \mathscr{C}^-_\epsilon(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$$

with $(\underline{b} + \underline{B})\mathrm{Ch}^-(g) = 0$. To this end, let $I := [0,1]$ and denote the canonical vector field on I with ∂_I. We denote the canonical Maurer-Cartan form on $U(l;\mathbb{C})$ by

$$\omega \in \Omega^1\big(U(l;\mathbb{C}), \mathrm{Mat}(l;\mathbb{C})\big).$$

Then for all $s \in I$ we can form the covariant derivative $d + s\omega$ on the trivial vector bundle $U(l;\mathbb{C}) \times \mathbb{C}^l \to U(l;\mathbb{C})$. Let

$$A^s \in \Omega^1\big(U(l;\mathbb{C}), \mathrm{Mat}(l;\mathbb{C})\big), \quad R^s \in \Omega^2\big(U(l;\mathbb{C}), \mathrm{Mat}(l;\mathbb{C})\big)$$

denote the connection 1-form of $d + s\omega$ and the curvature of $d + s\omega$, respectively, and

$$\mathcal{A}^s := A^s - R^s \wedge \alpha_{\mathbb{T}} \in \Omega_{\mathbb{T}}\big(U(l;\mathbb{C}) \times \mathbb{T}, \mathrm{Mat}(l;\mathbb{C})\big).$$

We set

$$A^s(g) := g^* A^s, \quad R^s_g := g^* R^s, \quad \omega_g := g^*\omega,$$

so that $A^s(g) = s\omega_g$ and by the Maurer-Cartan equation $R^s_g = (s/2)\omega_g^2$. Then we can define

$$\mathcal{A}^s(g) := A^s_g - R^s_g \wedge \alpha_{\mathbb{T}} \in \Omega_{\mathbb{T}}(M \times \mathbb{T}, \mathrm{Mat}(l;\mathbb{C})).$$

By varying s, the forms $\mathcal{A}^s(g)$ induce a form

$$\mathcal{A}(g) \in \Omega_{\mathbb{T}}(M \times I \times \mathbb{T}, \mathrm{Mat}(l;\mathbb{C}))$$

and we set

$$\mathcal{B}(g) := \iota_{\partial_I}\mathcal{A}(g) \in \Omega_{\mathbb{T}}(M \times I \times \mathbb{T}, \mathrm{Mat}(l;\mathbb{C})).$$

Then we can define

$$\mathcal{B}^s(g) \in \Omega_{\mathbb{T}}(M \times \mathbb{T}, \mathrm{Mat}(l;\mathbb{C})),$$

to be the pullback of $\mathcal{B}(g)$ with respect to the embedding

$$M \times \mathbb{T} \longrightarrow M \times I \times \mathbb{T}, \quad (x,t) \longmapsto (x,s,t).$$

In fact, by a simple calculation one finds

$$\mathcal{A}^s(g) = s\omega_g + s(1-s)\omega_g^2 \wedge \alpha_{\mathbb{T}}, \quad \mathcal{B}^s(g) = -\omega_g \wedge \alpha_{\mathbb{T}}, \tag{17}$$

so that $\mathcal{B}^s(g)$ actually does not depend on s. With these preparations, we can define an element

$$\mathrm{Ch}^-(g) = (\mathrm{Ch}^-_0(g), \mathrm{Ch}^-_1(g), \dots) \in \mathscr{C}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$$

by setting [3]

$$\mathrm{Ch}^-_n(g) := \mathrm{Tr}_n\left[\int_0^1 1 \otimes \sum_{k=1}^n \mathcal{A}^s(g)^{\otimes(k-1)} \otimes \mathcal{B}^s(g) \otimes \mathcal{A}^s(g)^{\otimes(n-k)} ds\right].$$

We refer the reader to the paper [12] by Simons and Sullivan, where a construction of the usual odd Chern character $\mathrm{ch}^-(g) \in \Omega^-(M)$ (cf. below) has been given that influenced our definition of $\mathrm{Ch}^-(g)$.

[3]Given linear spaces $V_0, \dots, V_n$, and $v^{(j)} \in \mathrm{Mat}(l; V_i)$, $j = 0, \dots, n$, the generalized trace is defined by

$$\mathrm{Tr}_n[v^{(0)} \otimes \cdots \otimes v^{(n)}] := \sum_{i_0,\dots,i_n=1,\dots l} v^{(0)}_{i_0,i_1} \otimes v^{(1)}_{i_1,i_2} \otimes \cdots \otimes v^{(n)}_{i_n,i_0}.$$

Theorem 5.1. *Let M be a compact Riemannian manifold, possibly with boundary.*
a) One has

$$\mathrm{Ch}^-(g)\in \mathscr{C}^-_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})),\quad \text{and}\quad (\underline{b}+\underline{B})\mathrm{Ch}^-(g)=0,$$

in particular, $\mathrm{Ch}^-(g)$ *induces a homology class*

$$\left[\mathrm{Ch}^-(g)\right]\in \underline{\mathsf{HC}}^-_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})).$$

b) The map

$$\mathsf{K}^{-1}(M)\longrightarrow \underline{\mathsf{HC}}^-_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})),\quad [g]\longmapsto \left[\mathrm{Ch}^-(g)\right]$$

is a well-defined group homomorphism.

Proof. a) It is easily seen that $\Gamma\mathrm{Ch}^-(g)=-\mathrm{Ch}^-(g)$. To show that

$$\mathrm{Ch}^-(g)\in \mathscr{C}^-_\epsilon(\Omega_{\mathbb{T}}(M\times\mathbb{T})),$$

set

$$C_k:=\sup_{s\in[0,1]}\max\Big(\left\|1\right\|_k,\max_{i,j=1,\dots,n}\left\|\mathcal{A}^s(g)_{ij}\right\|_k,\max_{i,j=1,\dots,n}\left\|\mathcal{B}^s(g)_{ij}\right\|_k\Big).$$

It is then easily checked that

$$\left\|\mathrm{Ch}^-(g)\right\|_{k,l}\leq \sum_{n=0}^{\infty} n\frac{(l^2C_k)^n}{\sqrt{n!}}<\infty.$$

It remains to prove

$$(b+B)\mathrm{Ch}^-(g)\in \mathscr{D}(\Omega_{\mathbb{T}}(M\times\mathbb{T})).$$

In fact,

$$B\mathrm{Ch}^-(g)\in \mathscr{D}(\Omega_{\mathbb{T}}(M\times\mathbb{T})),$$

as every $\left\langle \mathrm{Ch}^-_n(g)\right\rangle$ contains the 0-form 1 and so is of the form (4) with $f=1$. It remains to show that

$$b\mathrm{Ch}^-(g)\in \mathscr{D}(\Omega_{\mathbb{T}}(M\times\mathbb{T})).$$

In order to see the latter, let us first notice that

$$\big(b\mathrm{Ch}^-(g)\big)_n=\big(b\left\langle \mathrm{Ch}^-_n(g)\right\rangle\big)_n+\big(b\left\langle \mathrm{Ch}^-_{n+1}(g)\right\rangle\big)_n.$$

Using (17) and the explicit definition of b, we get

$$\begin{aligned}
&\big(b\left\langle \mathrm{Ch}^-_n(g)\right\rangle\big)_n\\
&=-\mathrm{Tr}_n\left[\int_0^1 1\otimes\sum_{k=1}^{n}\sum_{l=0}^{k-2}\mathcal{A}^s(g)^{\otimes l}\otimes(-s^2\omega_g^2)\otimes\mathcal{A}^s(g)^{\otimes(k-l-2)}\otimes(-\omega_g\wedge\alpha_{\mathbb{T}})\otimes\mathcal{A}^s(g)^{\otimes(n-k)}\,ds\right]\\
&\quad+\mathrm{Tr}_n\left[\int_0^1 1\otimes\sum_{k=1}^{n}\sum_{l=0}^{n-k-1}\mathcal{A}^s(g)^{\otimes(k-1)}\otimes(-\omega_g\wedge\alpha_{\mathbb{T}})\otimes\mathcal{A}^s(g)^{\otimes l}\otimes(-s^2\omega_g^2)\otimes\mathcal{A}^s(g)^{\otimes(n-k-l-1)}\,ds\right]\\
&\quad-\mathrm{Tr}_n\left[\int_0^1 1\otimes\sum_{k=1}^{n}\mathcal{A}^s(g)^{\otimes(k-1)}\otimes(\omega_g^2\wedge\alpha_{\mathbb{T}}+\omega_g)\otimes\mathcal{A}^s(g)^{\otimes(n-k)}\,ds\right],
\end{aligned}$$

and

$$\begin{aligned}
&\left(b\left\langle \mathrm{Ch}^-_{n+1}(g)\right\rangle\right)_n \\
&= -\mathrm{Tr}_n\left[\int_0^1 1\otimes\sum_{k=1}^n\sum_{l=0}^{k-2}\mathcal{A}^s(g)^{\otimes l}\otimes(+s^2\omega_g^2)\otimes\mathcal{A}^s(g)^{\otimes(k-l-2)}\otimes(-\omega_g\wedge\alpha_{\mathbb{T}})\otimes\mathcal{A}^s(g)^{\otimes(n-k)}\ ds\right] \\
&\quad + \mathrm{Tr}_n\left[\int_0^1 1\otimes\sum_{k=1}^n\sum_{l=0}^{n-k-1}\mathcal{A}^s(g)^{\otimes(k-1)}\otimes(-\omega_g\wedge\alpha_{\mathbb{T}})\otimes\mathcal{A}^s(g)^{\otimes l}\otimes(+s^2\omega_g^2)\otimes\mathcal{A}^s(g)^{\otimes(n-k-l-1)}\ ds\right] \\
&\quad - \mathrm{Tr}_n\left[\int_0^1 1\otimes\sum_{k=1}^n\mathcal{A}^s(g)^{\otimes(k-1)}\otimes(-2s\omega_g^2\wedge\alpha_{\mathbb{T}})\otimes\mathcal{A}^s(g)^{\otimes(n-k)}\ ds\right],
\end{aligned}$$

whose sum is

$$\begin{aligned}
&\mathrm{Tr}_n\left[\int_0^1 1\otimes\sum_{k=1}^n\mathcal{A}^s(g)^{\otimes(k-1)}\otimes\left(\frac{d}{ds}\mathcal{A}^s(g)\right)\otimes\mathcal{A}^s(g)^{\otimes(n-k)}\ ds\right] \\
&= \mathrm{Tr}_n\left[\int_0^1\frac{d}{ds}\left(1\otimes\mathcal{A}^s(g)^{\otimes n}\right)\ ds\right] = \mathrm{Tr}_n\left[1\otimes\mathcal{A}^1(g)^{\otimes n}\right] - \mathrm{Tr}_n\left[1\otimes\mathcal{A}^0(g)^{\otimes n}\right].
\end{aligned}$$

Thus, we finally have

$$(b\mathrm{Ch}^-(g))_n = \mathrm{Tr}_n\left[1\otimes\omega_g^{\otimes n}\right], \qquad n=1,2,\dots.$$

We now prove that

$$\left(\dots,\mathrm{Tr}_n\left[1\otimes\omega_g^{\otimes n}\right],\dots\right)\in\mathscr{D}(\Omega_{\mathbb{T}}(M\times\mathbb{T})).$$

To this end we have simply to employ the properties of the generalized trace. Indeed, for $n\geq 2$ we can write

$$\begin{aligned}
\left\langle\mathrm{Tr}_n\left[1\otimes\omega_g^{\otimes n}\right]\right\rangle &= \left\langle\mathrm{Tr}_n\left[1\otimes\omega_g\otimes\omega_g\otimes\omega_g^{\otimes(n-2)}\right]\right\rangle = -\left\langle\mathrm{Tr}_n\left[1\otimes dg^{-1}\otimes dg\otimes\omega_g^{\otimes(n-2)}\right]\right\rangle \\
&= -\left\langle\mathrm{Tr}_n\left[1\otimes dg^{-1}\otimes dg\otimes\omega_g^{\otimes(n-2)}\right]\right\rangle - \left\langle\mathrm{Tr}_{n-1}\left[g^{-1}\otimes dg\otimes\omega_g^{\otimes(n-2)}\right]\right\rangle \\
&\quad + \left\langle\mathrm{Tr}_{n-1}\left[1\otimes g^{-1}dg\otimes\omega_g^{\otimes(n-2)}\right]\right\rangle,
\end{aligned}$$

where the last two terms cancel each other because of the trace property, which is precisely of the form (5) for $f=g^{-1}$. Similarly, for $n=1$ it is sufficient to notice that

$$\left\langle\mathrm{Tr}_1\left[1\otimes\omega_g\right]\right\rangle = \left\langle\mathrm{Tr}_1\left[g^{-1}\otimes dg\right]\right\rangle,$$

which is of the form (4) with $f=g^{-1}$, completing the proof of $b\mathrm{Ch}^-(g)\in\mathscr{D}(\Omega_{\mathbb{T}}(M\times\mathbb{T}))$.
b) We have to prove the following two facts:
i) If $g,h\in C^\infty(M,U(l;\mathbb{C}))$, then one has $\mathrm{Ch}^-(g\oplus h)=\mathrm{Ch}^-(g)+\mathrm{Ch}^-(h)$.
ii) If $g_0,g_1\in C^\infty(M,U(l;\mathbb{C}))$ are connected by a smooth homotopy

$$g_\cdot\in C^\infty(M\times I,U(l;\mathbb{C})),$$

then one has

$$\mathrm{Ch}^-(g_1)-\mathrm{Ch}^-(g_0)=(\underline{b}+\underline{B})w$$

for some $w\in\mathscr{C}^+(\Omega_{\mathbb{T}}(M\times\mathbb{T}))$.
Here, property i) is an immediate consequence of the properties of the generalized trace

$\mathrm{T}r_n$ using the block diagonal form of $g \oplus h$.
To see ii), for any $t \in I$, we define the embedding

$$j_t : M \hookrightarrow M \times I, \quad x \longmapsto (x,t),$$

and $w = (w_0, w_1, \dots) \in \mathscr{C}^+(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$ by setting

$$\begin{aligned} w_n := &-\mathrm{Tr}_n \left[\int_0^1 \int_0^1 1 \otimes \sum_{k=1}^{n} \sum_{l=0}^{k-2} j_t^* \left(\mathcal{A}^s(g.)^{\otimes l} \otimes \iota_{\partial_I} \mathcal{A}^s(g.) \otimes \mathcal{A}^s(g.)^{\otimes (k-l-2)} \otimes \mathcal{B}^s(g.) \otimes \mathcal{A}^s(g.)^{\otimes (n-k)} \right) \, ds \, dt \right] \\ &+ \mathrm{Tr}_n \left[\int_0^1 \int_0^1 1 \otimes \sum_{k=1}^{n} \sum_{l=0}^{n-k-1} j_t^* \left(\mathcal{A}^s(g.)^{\otimes (k-1)} \otimes \mathcal{B}^s(g.) \otimes \mathcal{A}^s(g.)^{\otimes l} \otimes \iota_{\partial_I} \mathcal{A}^s(g.) \otimes \mathcal{A}^s(g.)^{\otimes (n-k-l-1)} \right) \, ds \, dt \right] \\ &- \mathrm{Tr}_n \left[\int_0^1 \int_0^1 1 \otimes \sum_{k=1}^{n} j_t^* \left(\mathcal{A}^s(g.)^{\otimes (k-1)} \otimes \iota_{\partial_I} \mathcal{B}^s(g.) \otimes \mathcal{A}^s(g.)^{\otimes (n-k)} \right) \, ds \, dt \right]. \end{aligned}$$

Then again it is clear that $Bw \in \mathscr{D}(\Omega_{\mathbb{T}}(M \times \mathbb{T}))$, so that $\underline{B}w = 0$. On the other hand, by using the identity

$$dj_t^* \iota_{\partial_I} \mathcal{A}^s(g.) = -j_t^* \iota_{\partial_I} d\mathcal{A}^s(g.) + \frac{\partial}{\partial t} j_t^* \mathcal{A}^s(g.),$$

and similarly for $\mathcal{B}^s$, and the same computations as in part a) we get

$$(\underline{bw} + \underline{B}w)_n = (\underline{b}w)_n = (\underline{b}w_n)_n + (\underline{b}w_{n+1})_n = \int_0^1 \frac{d}{dt} j_t^* \mathrm{Ch}^-(g.) = \mathrm{Ch}^-(g_1) - \mathrm{Ch}^-(g_0).$$

This completes the proof. ■

If M has no boundary (so that LM is a well-defined Fréchet manifold), in view of $(d+P)\underline{\rho} = \underline{\rho}(\underline{b} + \underline{B})$, we immediately get:

Corollary 5.2. *Assume M is a compact Riemannian manifold without boundary. Then for all $g \in C^\infty(M, U(l;\mathbb{C}))$ one has $(d+P)\underline{\rho}(\mathrm{Ch}^-(g)) = 0$, in particular, $\underline{\rho}(\mathrm{Ch}^-(g))$ induces a homology class $\left[\underline{\rho}(\mathrm{Ch}^-(g))\right] \in \mathsf{H}^-_{\mathbb{T}}(LM)$.*

The *odd Chern character* $\mathrm{ch}^-(g) \in \Omega^-(M)$ is the closed odd differential form defined by

$$\mathrm{ch}^-(g) := \mathrm{Tr} \left[\sum_{j=0}^{\infty} \frac{(-1)^j j!}{(2j+1)!} (g^{-1}dg)^{\wedge(2j+1)} \right].$$

We have the projection map

$$\pi : \underline{\mathscr{C}}(\Omega_{\mathbb{T}}(M \times \mathbb{T})) \longrightarrow \Omega(M)$$

which is defined as the composition of ρ with the restriction map $\Omega(LM) \to \Omega(M)$. Finally, the *odd Bismut-Chern character* is the differential form

$$\mathrm{Bch}^-(g) = (\mathrm{Bch}^-_1(g), \mathrm{Bch}^-_3(g), \dots) \in \Omega^-(LM)$$

defined by

$$\mathrm{Bch}^-_{2n-1}(g) = \mathrm{Tr} \left[\int_0^1 \int_{\{0 \le t_1 \le \dots t_n \le 1\}} \sum_{j=1}^{n} \bigwedge_{i=1}^{j-1} //^s_{t_i}(g) R^s_g(t_i) \bigwedge //^s_{t_j}(g) \dot{A}^s_g(t_j) \bigwedge_{l=j+1}^{n} //^s_{t_l}(g) R^s_g(t_l) //^s_1 dt_1 \cdots dt_n ds \right],$$

where

$$\dot{A}^s_g = dA^s_g/ds = \omega_g \in \Omega^1(M, \mathrm{Mat}(l;\mathbb{C})),$$

and where $//^s_\cdot(g)$ denotes the parallel transport with respect to the connection $d + s\omega_g$ on $M \times \mathbb{C}^l \to M$.

Theorem 5.3. *Assume M is a compact Riemannian manifold, possibly with boundary, and let $g \in C^\infty(M, U(l;\mathbb{C}))$. Then one has $\pi(\mathrm{Ch}^-(g)) = \mathrm{ch}^-(g)$, and if M has no boundary then $\mathrm{Bch}^-(g) = \rho(\mathrm{Ch}^-(g))$.*

Note that in view of Corollary 5.2, Theorem 5.3 provide a new proof of $(d+P)\mathrm{Bch}^-(g) = 0$ (see [14] for a variant of this result).

Proof of Theorem 5.3. The formula $\pi(\mathrm{Ch}^-(g)) = \mathrm{ch}^-(g)$ is a simple consequence of the definitions, once one has noticed the formula

$$\pi\left(\langle(\alpha_0 + \beta_0 \wedge \alpha_{\mathbb{T}}) \otimes \cdots \otimes (\alpha_n + \beta_n \wedge \alpha_{\mathbb{T}})\rangle\right) = \alpha_0 \wedge \cdots \wedge \alpha_n.$$

In order to see $\mathrm{Bch}^-(g) = \rho(g)$, given $t, s \in I$ define

$$V^s(g,t) \in \Omega^-(LM, \mathrm{Mat}(l;\mathbb{C}))$$

by

$$V^s_{2n+1}(g,t) = \int_{\{0\le t_1 \le \ldots t_{n+1} \le t\}} \sum_{j=1}^{n+1} \bigwedge_{i=1}^{j-1} //^s_{t_i}(g) R^s_g(t_i) \bigwedge //^s_{t_j}(g)\dot{A}^s_g(t_j) \times \bigwedge_{l=j+1}^{n+1} //^s_{t_l}(g) R^s_g(t_l) //^s_1 dt_1 \cdots dt_{n+1},$$

and the differential form

$$W^s(g,t) \in \Omega^-(LM, \mathrm{Mat}(l;\mathbb{C}))$$

by

$$W^s_{2n+1}(g,t) = \sum_{k=n+1}^{\infty} \sum_{r,j_1,\cdots,j_n=1,\text{pairwise disjoint}}^{k} \times \int_{\{0\le t_1 \le \ldots t_k \le t\}} \iota A^s_g(t_1) \cdots R^s_g(t_{j_1}) \cdots \dot{A}^s_g(t_r) \cdots R^s_g(t_{j_n}) \cdots \iota A^s_g(t_k) dt_1 \cdots dt_k.$$

Then obviously one has

$$\mathrm{Bch}^-(g) = \mathrm{Tr}\left[\int_0^1 V^s(g,t)|_{t=1} ds\right]$$

and it is easily checked from the definitions that

$$\rho(\mathrm{Ch}^-(g)) = \mathrm{Tr}\left[\int_0^1 W^s(g,t)|_{t=1} ds\right].$$

Thus it suffices to show that $W^s(g,t) = V^s(g,t)$ for all $t, s \in I$. To see this, the essential idea is to consider for every $t, s \in I$ the even form

$$X^s(g,t) = (X_0^s(g,t), X_2^s(g,t), \dots) \in \Omega^+(LM, \mathrm{Mat}(l;\mathbb{C}))$$

which is defined by

$$\begin{aligned}
&X_0^s(g,t) = //_t^s(g),\\
&\frac{d}{dt}X_{2n}^s(g,t) = X_{2n}^s(g,t)\iota A_g^s(t) + X_{2n-2}^s(g,t)R_g^s(t),\\
&X_{2n}^s(g,t)|_{t=0} = 0 \quad \text{ for all } n \geq 1,
\end{aligned}$$

and the odd form

$$Y^s(g,t) = (Y_1^s(g,t), Y_3^s(g,t), \dots) \in \Omega^-(LM, \mathrm{Mat}(l;\mathbb{C}))$$

which is defined by

$$\begin{aligned}
&\frac{d}{dt}Y_1^s(g,t) = Y_1^s(g,t)\iota A_g^s(t) + X_0^s(g,t)\dot{A}_g^s(t),\\
&\frac{d}{dt}Y_{2n+1}^s(g,t) = Y_{2n+1}^s(g,t)\iota A_g^s(t) + Y_{2n-1}^s(g,t)R_g^s(t) + X_{2n}^s(g,t)\dot{A}_g^s(t) \quad \text{ for all } n \geq 1,\\
&Y_{2n+1}^s(g,t)|_{t=0} = 0 \quad \text{ for all } n.
\end{aligned}$$

Noting that the sum that defines $W_{2n+1}^s(g,t)$ converges uniformly in t so that one can interchange d/dt with $\sum_{k=n+1}^{\infty}$, it is now easily checked that both $t \mapsto W^s(g,t)$ and $t \mapsto V^s(g,t)$ solve the IVP's which define $Y^s(g,t)$, so that

$$V^s(g,t) = W^s(g,t) = Y^s(g,t) \quad \text{ for all } t, s \in I,$$

as was claimed.

■

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